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Ph.D.THESIS

Effective Algorithms for the Study of the Degree of Algebraic Varieties in Offsetting Processes

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Y a ti, Anahí. Tú, ya lo sabes, le das a todo su sentido y su importancia.

Ara mateix
de Miquel Martí i Pol

Ara mateix enfilo aquesta agulla
amb el fil d'un propòsit que no dic
i em poso a apedaçar. Cap dels prodigis
que anunciaven taumaturgs insignes
no s'ha complert, i els anys passen de pressa.
De res a poc, i sempre amb vent de cara,
quin llarg camí d'angoixa i de silencis.
I som on som; més val saber-ho i dir-ho
i assentar els peus en terra i proclamar-nos
hereus d'un temps de dubtes i renúncies
en què els sorolls ofeguen les paraules
i amb molts miralls mig estrafem la vida.
De res no ens val l'enyor o la complanta,
ni el toc de displicent malenconia
que ens posem per jersei o per corbata
quan sortim al carrer. Tenim a penes
el que tenim i prou: l'espai d'història
concreta que ens pertoca, i un minúscul
territori per viure-la. Posem-nos
dempeus altra vegada i que se senti
la veu de tots solemnement i clara.
Cridem qui som i que tothom ho escolti.
I en acabant, que cadascú es vesteixi
com bonament li plagui, i via fora!,
que tot està per fer i tot és possible.

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Resumen

El trabajo que se presenta en esta tesis pertenece al área del Cálculo Simbólico, y en particular, al subárea de la Geometría Algebraica (Simbólica) Efectiva para curvas y superficies. Concretamente, en esta tesis se estudia la estructura de grados del polinomio multivariable que define el objeto geométrico que resulta al aplicar procesos de offsetting. Es decir, estudiamos su grado total y sus grados parciales con respecto a cada una de las variables, incluyendo la variable distancia. Para llevar a cabo este objetivo, la tesis se compone de cuatro capítulos y dos apéndices, cuya estructura se detalla a continuación:

- En el Capítulo 1 se introducen las nociones de offset genérica y de polinomio de la offset genérica, junto con sus propiedades básicas. En este capítulo se sientan las bases teóricas de nuestro objeto de estudio. En particular, se prueba la propiedad fundamental del polinomio de la offset genérica,que afirma que dicho polinomio especializa bien; es decir, para casi todo valor que se asigne a la variable distancia la especialización del polinomio ,de la offset genérica, es el polinomio que define a la offset para ese valor concreto tomado como distancia. Una vez establecida dicha conexión con la teoría clásica, se define el problema central de esta tésis, que es el problema del grado de la offset genérica. Además se presenta la notación y terminología asociadas a ese problema. Se incluyen también en este capítulo algunos lemas técnicos, que tratan sobre la aplicación de la resultante para el análisis de problemas de intersección de curvas.

- El Capítulo 2 trata del problema del grado total para la offset genérica de una curva plana. Nuestro estudio incluye el caso general en el que la curva viene dada por su ecuación implícita, y también, para curvas racionales, el caso de curvas dadas paramétricamente. En ambos casos obtenemos fórmulas eficientes para el grado total de la offset genérica. Además se presentan otras fórmulas que pueden utilizarse para el estudio teórico del grado total de la offset. En este capítulo se introducen las nociones de sistema offset-recta, curva auxiliar y puntos intrusos. Estas tres nociones juegan un papel esencial en nuestro tratamiento del problema del grado. Estas nociones se utilizan para establecer un marco común para el desarrollo de fórmulas para el grado basadas en resultantes. En el siguiente capítulo ese marco común se aplica para obtener diversas fórmulas de grado.

- El Capítulo 3 es una continuación natural del capítulo precedente. Aplicando la estrategia, métodos y lenguaje del Capítulo 2, en este capítulo se completa el análisis de la estructura de grados de la offset genérica para curvas planas. En concreto, obtenemos fórmulas para calcular cualquier grado parcial de la offset genérica, y también el grado con respecto a la variable distancia. Estas fórmulas cubren tanto el caso implícito como el caso paramétrico. Además se muestran

otras fórmulas que pueden utilizarse para el análisis teórico del problema del grado.

- El Capítulo 4 trata el problema del grado para superficies. La mayor parte del capítulo se dedica a la demostración de una fórmula de grado total para superficies racionales, dadas paramétricamente. Esta fórmula puede aplicarse siempre que la superficie generadora satisfaga cierta condición muy general. En concreto, tenemos que asumir que existe a lo sumo una cantidad finita de valores de la distancia para los que la offset de la superficie pasa por el origen (véase la Condición o Assumption 4.1, en la página 122). La fórmula requiere el cálculo de una resultante generalizada univariada, y del máximo común divisor de polinomios con coeficientes simbólicos. La sección final de este capítulo contiene un enfoque alternativo para el estudio de la estructura de grados de una superficie de revolución, independiente de los resultados previos de este capítulo. Con este enfoque se obtiene una solución completa y efectiva para el problema del grado en este caso.

- El Apéndice A contiene un resumen de las fórmulas de grado obtenidas en esta tesis. El Apéndice B muestra los resultados de algunos cálculos, correspondientes a demostraciones o ejemplos, que, por su longitud, resulta más conveniente incluir aquí.

Abstract

The research in this thesis is framed within the field of Symbolic Computation, and more specifically in the subfield of Effective (Symbolic) Algebraic Geometry of Curves and Surfaces. In particular, this thesis focuses on the study of the degree structure of the multivariate polynomial defining the geometric object generated when applying offsetting processes. That is, we study its total and partial degrees w.r.t. each variable, including the distance variable. In order to do this, the thesis is structured into four chapters and two appendixes, as follows.

- Chapter 1 presents the notions of generic offset and generic offset polynomial, with their basic properties. This chapter provides the theoretical foundation of our subject of study. In particular, we prove the fundamental property of the generic offset polynomial; i.e., that this polynomial specializes to the polynomial defining the classic offset for all, but at most finitely many values of the distance. After establishing the connection with the classical theory, we define the central problem of this thesis: the degree problem for the generic offset. We also introduce the associated notation and terminology. The chapter includes also some technical lemmas about the use of resultants for the analysis of curve intersection problems.

- Chapter 2 deals with the total degree problem for the generic offset of a plane curve. We consider the general case where the curve is given by its implicit equation and, for rational curves, we also consider the parametric representation of the curve. In both cases we provide efficient formulae for the total degree of the generic offset. Furthermore, we provide additional formulae that can be applied to obtain theoretical information about the total degree of the offset. In this chapter we will meet the notions of offset-line system, auxiliary curve and fake point. These three notions play an essential role in our approach to the degree problems studied in this thesis. We use them to develop a common framework for resultant-based degree formulae, that will be applied to several different degree problems in the following chapter.

- Chapter 3 is a natural continuation of the preceding one. In this chapter we apply the strategy, methods and language of Chapter 2, to complete the analysis of the degree problem for plane curves. Thus, we provide efficient formulae for the partial degree and the degree w.r.t the distance variable of the generic offset, both in the implicit and parametric cases. Besides, we also provide formulae that are suitable for a theoretical analysis of these degree problems.

- Chapter 4 deals with the degree problem for surfaces. The major part of this chapter is dedicated to present a total degree formula for rational surfaces, given parametrically. This formula can be applied under a very general assumption

about the surface. Namely, we need to assume that there are at most finitely many distance values for which the offset of the surface passes through the origin (see Assumption 4.1, page 122). The formula requires the computation of a univariate generalized resultant and gcds, of polynomials with symbolic coefficients. In the final section of this chapter we apply an alternative approach, independent of the previous results in this chapter, to study the offset degree structure for surfaces of revolution. We provide a complete and efficient solution for this case.

- Appendix A contains a summary of the degree formulae obtained in this thesis. Appendix B shows the results of some computations, corresponding to proofs or examples, that, due to their length, are more conveniently placed here.

Contents

Introduction.

The research in this thesis is framed within the field of Symbolic Computation, and more specifically in the subfield of Effective (Symbolic) Algebraic Geometry of Curves and Surfaces. As such, it is naturally linked with applications in the domain of Computer Aided Geometric Design. In particular, the problem we study has its origin in one of the geometrical constructions that appear in the application of Algebraic Geometry to Geometric Modelling. Algebraic Geometry provides a natural language for the mathematical description of many of the curves and surfaces that are used in Geometric Modelling. These curves and surfaces often appear in connection with geometric constructions, such as evolutes, revolution surfaces, pipe and canal surfaces, blending, offsetting, etc.

In particular, this thesis focuses on the study of the degree (total and partial w.r.t. each variable) of the multivariate polynomial defining the geometric object generated when applying offsetting processes; in Subsection 1.2.2 (page 24) we describe the degree problem in detail. So, before continuing with this introduction, let us describe, at least informally, the offsetting construction; for a detailed explanation see Chapter 1 and, more specifically, Definition 1.18 (page 17). Let $\mathcal{V}$ be an $(n-1)$-dimensional irreducible algebraic hypersurface, and let d be a non-zero base field element. Then roughly, and informally, speaking the classical offset to $\mathcal{V}$ at distance d, is the Zariski closure of the set provided by the intersection points generated as follows: for every regular point $p \in \mathcal{V}$ take the intersection of the normal line, to $\mathcal{V}$ at p, with the $(n-1)$-dimensional sphere centered at p and radius the fixed value d (see Fig 1.1, page 5 and Fig. 1.2, 6).

The classical offset construction for algebraic hypersurfaces has been, and still is, an active research subject of scientific interest. Even though the historical origins of the study of offset curves can be traced back to the work of classical geometers ([25],[26],[41]), often under the denomination of *parallel curves*, the subject received increased attention when the technological advance in the fields of Computer Assisted Design and Computer Assisted Manufacturing (CAD/CAM) resulted in a strong demand of effective algorithms for the manipulation of curves and surfaces. We quote the following from one of the two seminal papers ([17],[18]) by Farouki and Neff: *"Apart from numerical-control machining, offset curves arise in a variety of practical applications such as tolerance analysis, geometric optics, robot path-planning, and in the formulation of simple geometric procedures (growing/shrinking, blending, filleting, etc.)"*. To the applications listed by these authors we should add here some recent ones, e.g. the connection with the medial axis transform; for these, and related applications see Chapter 11 in [31], and the references contained therein.

As a result of this interest coming from the applications, many new methods and algorithms have been developed by engineers and mathematicians, and many geometric and algebraic properties of the offset construction have been studied in recent years; see, e.g.

the references [3],[4],[5],[6],[7],[8],[9],[17], [18],[23],[24],[27],[28],[35],[37],[38],[50],[51]. In addition to these references, we also refer to the thesis [2] and [49], developed within the research group of Prof. J.R. Sendra. In [49] the fundamental algebraic properties of offsets to hypersurfaces are deduced, the unirationality of the offset components are characterized, and the genus problem (for the curve case) is studied. In [2] the topological behavior of the offset curve is analyzed. So, in some sense, this thesis can be seen as a natural continuation, within this group activity, of the research on offset varieties.

With the exception of certain degenerated situations, that are indeed well known, the offset to an algebraic hypersurface is again a hypersurface (see [50]). Thus, one might answer all the problems mentioned above (parametrization expressions, genus computation, topologic types determination, degree analysis, etc), by applying the available algorithms to the resulting (offset) hypersurface. However, in most cases, this strategy results unfeasible. The reason is that the offsetting process generates a huge size increment of the data defining the offset in comparison to the data of the original variety. The challenge, therefore, is to derive information (say algebraic or geometric properties) of the offset hypersurface from the information that could be easily derived from the original (in general much simpler) hypersurface.

Framed in the above philosophy, the goal of this thesis is to provide formulas for the degree of the offset. Let us be a little bit more precise (see Subsection 1.2.2 (page 24) for a detailed description of the problem). Let $f(y_1,\ldots,y_n)$ be the defining polynomial of $\mathcal{V}$ (see above), and let us treat d as variable. Then, we introduce a new polynomial $g(d,x_1,\ldots,x_n)$ such that for almost all non-zero values d^o of d the specialization $g(d^o,x_1,\ldots,x_n)$ defines the offset to $\mathcal{V}$ at distance d^0. Such a polynomial is called the generic offset polynomial (see Definition 1.21, page 20), and the hypersurface that it defines is called the generic offset of $\mathcal{V}$ (see Def 1.18, page 17). In this situation, the goal of this thesis is to find an effective solution for computing the total degree in $\{x_1,\ldots,x_n\}$ of g as well as its partial degrees w.r.t. x_i (for $i=1,\ldots,n$) and w.r.t. d.

As possible motivations for this problem, besides its pure mathematical challenge, one might mention among others:

- Knowing the degree structure of the generic offset in advance reduces the interpolation space needed to obtain the generic offset polynomial by interpolation.
- Similarly, this knowledge can be used to identify the offset generic equation when it appears as a factor in some resultant-based computation (see e.g. [7]).
- This study can also be seen as a first step for a tropical analysis of the offset constructions (see [15]).
- Having information on the degree of the offset can be used in the manipulation of bisectors (see Section 11.1.3 in [31]; see also [12]).

In Chapters 2 and 3, our general strategy to approach the different degree problems for plane curves will be as follows: we will consider the intersection of the generic offset with a suitable pencil of lines. The choice of the pencil of lines is guided by the following principle: for a generic election of a line in the pencil and a value of the distance, the cardinal of the intersection between the offset and the line equals the degree under study; besides, we impose some additional regularity properties to these intersection points. We will use the expression offset-line system to refer to the system of equations that one obtains combining the equations for the offset construction and the equation (or equations) for the pencil of lines. Then we apply Elimination Theory in this system, in order to eliminate the variables that describe a point in the offset (and possibly, some other auxiliary variables). The result of this elimination process is used to introduce the notion of auxiliary curve. This auxiliary curve, which is constructed ad hoc for each degree problem, depends on parameters; more precisely, it depends on the choice of distance values, and on the parameters used to identify a line in the pencil. Then we show that there is a bijection between, on one hand, the offset-line intersection points, and on the other hand, some of the points in the intersection between the generating curve and the auxiliary curve. Thus we are led to consider the system of equations that describes the intersection between the generating curve and the auxiliary curve. Then we need to characterize which solutions of this second system correspond to the solutions of the offset-line system. This characterization of two types of solutions is reflected in the notions of fake and non-fake points. In general, it holds that the degree under study equals the number of non-fake points for a generic choice of value of the parameters. We show that, in each case, the multiplicity of intersection between the generating curve and the auxiliary curve at non-fake points equals one. This makes it possible to obtain the offset degrees (total, partial and w.r.t. d) by counting the generic non-fake intersection points. This is the idea behind all the degree formulae for curves. In the case of parametrically given curves, the use of this strategy benefits from the reduction of dimension due to the parametric representation of the curve. Therefore, the formulae for the parametric case are, in all cases, more efficient than their analogues for the implicit representation case.

In the case of the total degree problem for rational surfaces (in Chapter 4), the kernel of this strategy can still be applied. That is, we continue to analyze the intersection of the generic offset with a suitable pencil of lines, and we apply elimination to the corresponding offset-line system. Here we again benefit from the dimension reduction that results from the parametric representation. This helps to reduce the degree problem to a problem of intersection between plane curves. However in this case we need to consider not one, but several auxiliary curves. The characterization of non-fake points and their properties requires a harder theoretical work, compared with the case of plane curves.

There exist some (few) contributions in the literature concerning the degree problem for offset curves and surfaces. To our knowledge, the first attempt to provide a degree formula for offset curves was given by Salmon, in [41]. This formula was proved wrong in the already mentioned paper by Farouki and Neff [17]. In this paper, the authors provide a degree formula for rational curves given parametrically. They also deal separately with the case of polynomial parametrizations (see our Remark 2.41, page 75). Anton et al. provide in [7] an alternative formula (to those presented in [42] and in Chapter 2) for computing the total degree of the offsets to an algebraic curve.

Finally, let us mention that, for the case of rational surfaces, and also for surfaces of revolution, we are unaware of any existing formula for the degree of the offset.

Structure of the Thesis

The structure of the thesis is as follows:

- Chapter 1 is devoted to the notion of generic offset, and the associated degree problem. The goals of this chapter are to provide the theoretical foundation of our subject of study, and to obtain its fundamental properties. In order to do this, the chapter is structured into three sections.

  In Section 1.1 (page 9) we briefly survey the fundamental concepts related to the classical offset and the related properties that will be used in the sequel. We follow the formalism in [8] and [50], using the Incidence Diagram. Furthermore, the relation of this notion with Elimination Theory is established. In Chapters 2 and 4, when dealing with total degree problems, we will consider the intersection of the offset with a pencil of lines through the origin. Therefore, the final subsection of this section analyzes a degenerate situation for the offsetting construction that arises in this context (see Lemma 1.14, page 14).

  In Section 1.2 (page 15), we introduce the notions of generic offset and generic offset polynomial, which are central to this thesis. This section contains two main theoretical results. First, in Proposition 1.20 (page 18) we prove that the generic offset is a hypersurface. The other main theoretical result of this section is the fundamental property of the generic offset, established in Theorem 1.24 (page 21). This Theorem states that the generic offset polynomial specializes to the polynomial defining the classic offset for all but at most finitely many values of the distance. After that, we are able to define the central problem of this thesis, the degree problem for the generic offset, and the associated notation and terminology. The section concludes with the analysis of the points that appear in the offset as a result of taking the Zariski closure of the constructible sets, used to define the offset.

  Section 1.3 (page 27) contains three technical lemmas about the use of univariate resultants to study the problem of the intersection of plane algebraic curves. The first one, Lemma 1.33 (page 28) concerns the degree of the univariate resultant of two homogeneous polynomials in three variables. The purpose of the following Lemma 1.34 is to extend the use of resultants beyond the classical setting, analyzing the behavior of the resultant when some of the standard requirements are not satisfied. The last Lemma 1.35 (page 32) deals with the case when more than two curves are involved, using generalized resultants.

- Chapter 2 (page 35) is dedicated to a specific part of the offset degree problem in the case of plane algebraic curves: the analysis of the total degree problem. The main goals of this chapter are: first, to provide efficient formulae, from a computational point of view, for the total degree of the generic offset. Second, to obtain theoretical information about the total degree of the offset; allowing, e.g., to study the behavior of the degree in a given family or class of curves. In this chapter we present three different formulae for implicitly given curves, and one formula for the case of a rational curve, given parametrically. The chapter is structured into four sections.

  Section 2.1 (page 38) presents the theoretical strategy that we apply to the total degree problem, based in the analysis of the intersection of the offset with a pencil of lines through the origin. This leads to the consideration of the Offset-Line System of equations. Its solutions are analyzed in Theorem 2.5 (page 41). From this system, by –essentially– eliminating the variables that represent a point in the offset, we derive the notion of auxiliary curve (in Subsection 2.1.2). This curve depends on two parameter variables, the distance d used in the offsetting construction and a parameter k that is related with the slope of the line chosen in the pencil. Then we consider the system formed by the generating curve and the auxiliary curve, and in Theorem 2.14 we study the relation between the solution sets of these two systems. The notion of fake points (see Subsection 2.1.3 in page 50) corresponds with the invariant solutions of this system w.r.t. to the parameters (d, k).

  Section 2.2 (page 52) contains the first two total degree formulae. The first one, in Theorem 2.24 (page 56), is obtained as a consequence of Bezout Theorem, applied to the generating curve and the auxiliary curve, for any particular choice of value of (d, k), in an open subset of the space of parameters. This formula is not well suited for computation, because the open subset just mentioned is hard to describe. The second formula, in Theorem 2.27 (page 62), is based in the notion of hodograph curve, and it is a deterministic formula. Both these formulae may be used for the second goal that we have mentioned: to obtain theoretical information about the total degree of the offset. However, from the point of view of computational efficiency, these two formulae are not completely satisfactory.

  In Section 2.3 we address the computational aspect of the problem, by developing a third formula (see Theorem 2.31, page 68). This formula applies to implicitly given curves, and it requires two major computational steps: first, the computation of the univariate resultant of the polynomial defining the generating curve, and the polynomial defining the auxiliary curve (which depends on (d, k)); second, one must obtain the primitive part of that resultant w.r.t. (d, k). The key ingredients (auxiliary curve, fake points) and the properties that lead to this formula can be generalized, and these generalized versions can be found in the degree problems that we will study in later chapters. Therefore the main result of this section is Theorem 2.30 (page 65), which is presented in a language suitable for such generalizations.

In Section 2.4 (page 71), we treat the case of rational curves, given parametrically, and in Theorem 2.40 (page 75) we provide a formula for this type of curves. The formula is obtained by translating the information contained in the auxiliary curve $\mathcal{S}$ into the parameter space, and identifying the parameter values that correspond to fake points. The reduction in the dimension of the space, where the curve points are represented, implies that this formula only requires the computation of degrees and gcds of univariate polynomials. It is therefore a very efficient formula, for this specially important type of curves.

- Chapter 3 (page 79) completes the analysis of the degree problem for plane curves initiated in Chapter 2, by covering the partial degree and the degree w.r.t the distance variable of the generic offset, both in the implicit and parametric cases. As such, the strategy, methods and language of this chapter are a natural continuation of the preceding one. However, we have decided to keep them as separate chapters, partly for structural reasons (to keep some balance among chapters), partly because of the chronological sequence in which the results in these two chapters have been obtained. A total of five degree formulae are presented in this chapter, which is structured as follows:

  Section 3.1 (page 82) begins with the theoretical foundation of our strategy for the partial degree problem for implicitly given curves, introducing the Offset-Line System 3.2 (page 3.2) for this problem. Its set of solutions is discussed in Theorem 3.4 (page 84). After this is done, we proceed to obtain the auxiliary curve by using elimination techniques. Again, we consider the system formed by the generating curve and the auxiliary curves. Theorem 3.12 (88) shows the relation between the solutions of the Offset-Line System, and this second system. In this section we also describe the corresponding notion of fake points for this problem, we characterize them, and we prove some properties that are needed in the proof of the partial degree formulae in the following section.

  Section 3.2 (page 95) contains two formulae for the partial degree of the generic offset of a curve $\mathcal{C}$, given implicitly. The first one, in Theorem 3.23 (page 96) is a formula derived from Bezout's Theorem, similar to Formula 2.24 (page 56) of Chapter 2, and therefore it is more suited to the theoretical analysis of the partial degree, and not really useful for computation. Therefore, in Theorem 3.24 (page 96) we present a second, resultant based formula. Here, for the first time we take advantage of the general approach for this type of formulae that we developed in Section 2.3 of the preceding chapter. The resultant formula in this section fits neatly into that framework.

Section 3.3 (page 99) is devoted to complete the degree analysis in the implicit case, by studying the degree of the generic offset w.r.t. the distance variable d. First we describe the auxiliary curve associated with this problem (see Remark 3.27) and, in doing so, we see again the benefits of the framework that we have already established. Thus in Theorem 3.29, (page 101) we study the properties of the auxiliary curve. Then we define the fake points for this problem, we prove that they have the required properties, and finally we obtain the resultant-based formula in Theorem 3.36.

Section 3.4 (page 111) contains the formulae needed to cover the case of rational curves, given parametrically. As we did in Section 2.4 of Chapter 2, we have to translate the information of the auxiliary curves into the parameter space. We do this for the partial degree problem in Theorem 3.42 (page 115), and for the degree w.r.t. d in Theorem 3.48 (page 118). In both these cases the formulae we obtain are very efficient, requiring only the computation of degrees and gcds of univariate polynomials.

- Chapter 4 (page 121) deals with the degree problem for rational surfaces. In the first three sections, we present a total degree formula for rational surfaces, given parametrically, that applies under a very general assumption about the surface (see Assumption 4.1, page 122). The formula requires the computation of a univariate generalized resultant and gcds, of polynomials with symbolic coefficients. In the final section of this chapter we provide alternative –and simpler– degree formulae for an important class of surfaces, the surfaces of revolution. In this case, the degree problem is solved completely.

  In order to do this the strategy is based, as in previous chapters, in the study of the intersection between the offset and a pencil of lines. using the parametric representation of the surface, results in a reduction in the dimension of the space where the intersection problem is analyzed. We already saw this happening in the case of rational curves in Chapters 2 and 3. Then, eliminating the variables that represent a point in the offset, we arrive at a new system, consisting in this case of several auxiliary curves. Thus, the dimensional reduction due to the parametric representation, turns the analysis of the degree into an intersection problem of a system of plane curves, the auxiliary curves. We are led to consider their invariant solutions, that extend the notion of fake points that we have already met in previous chapters. One of the major tasks in this chapter is the characterization of these fake points, together with the verification that they possess the required multiplicity properties. These are the key ingredients in the proof of the degree formula. Departing from this line of work, in the final section of this chapter we will provide alternative –and simpler– degree formulae for an important class of surfaces, the surfaces of revolution. These formulae are derived from the case of plane curves, studied in Chapters 2 and 3, and they provide a complete and efficient solution of the degree problem for this class of surfaces.

Section 4.1 (page 125) presents our strategy for the total degree problem in the case of rational surfaces. We start with some preliminary material about parametric algebraic surfaces, their parametrizations, and the associated normal vector (Definition 4.4, page 127). After that, we study the intersection between the offset and a pencil of lines through the origin. We encode the intersection problem into the Parametric Offset-Line System 4.4 (128). The main result of this section is Theorem 4.13, (page 133), which describes the generic solution of that system.

Section 4.2 starts showing that the parametric description of the surface results in a reduction in the dimension of the space where the intersection problem is represented. Then, eliminating the variables that represent a point in the offset (and some auxiliary variables), we arrive at a new Auxiliary System 4.7 (page 137), consisting in this case of several auxiliary curves, that depends on parameters: a one dimensional parameter d that represents the offsetting distance, and a three dimensional $\bar{k} = (k_1, k_2, k_3)$ that corresponds to the choice of a line in the pencil. We analyze the relation between the solutions of the Parametric Offset-Line System and those of this new Auxiliary System (Proposition 4.16, page 139). Once more, this analysis leads to the key notion of fake points, that we identify as invariant solutions for the Affine Auxiliary System, in Proposition 4.23 (page 145).

Section 4.3 (page 146) takes the above ingredients and moves towards the proof of the degree formula. The results of this section correspond, in a certain way, with the prerequisites for the degree formulae that we have met in the statement of Theorem 2.30 (page 65) of Chapter 2. However, the situation is different –and much more complicated– because here we need to consider intersection of more than two curves, and because all the curves involved in the intersection problem depend on parameters. We begin with the projective version of the auxiliary curves introduced in the preceding section, introducing the Projective Auxiliary System 4.25 (page 150). The connection between the invariant solutions of this system and our notion of fake points is considered next. Then, we prove that the value of the multiplicity of intersection of the auxiliary curves at their non-invariant points of intersection equals one (Proposition 4.43, page 160). After that, we are ready for proof of the degree formula, in Theorem 4.45 (page 172).

Section 4.4 (page 180) is independent of the preceding results in this chapter. In it we consider the offset degree problem for the surface of revolution obtained from a plane curve $\mathcal{C}$. The construction of these surfaces from plane curves allows to connect the study of their offset degree with the results about curves in Chapters 2 and 3. We begin by formalizing the notion of surface of revolution, and we obtain some basic properties. The next step is Theorem 4.53 (page 184), where we show how to obtain the implicit equation of the revolution surface from the implicit equation of the initial curve, by a straightforward method. This is applied to the offsetting process for revolution surfaces in the fundamental Theorem 4.58 (page 4.58). This Theorem shows that the offset of a revolution surface generated by a curve is the surface of revolution generated by the offset of that curve. With this result we close the section showing how to derive degree formulae for the offset of a surface of revolution, both when the generating curve is given implicitly or parametrically.

- Appendix A (page 195) contains a summary of all the degree formulae obtained in this thesis. Appendix B (page 201) contains the results of some computations mentioned in proofs and examples. Due to their length, they have been placed here so that they do not interfere with the main text of the thesis.

Main original contributions

The main original contributions of this thesis can be summarized as follows:

- In Chapter 1:
  - The notion of generic offset $\mathcal{O}_d(\mathcal{V})$ of an algebraic hypersurface $\mathcal{V}$ is introduced in Definition 1.18 (page 17). We prove that the generic offset is indeed a hypersurface in Proposition 1.20.
  - The notion of generic offset polynomial $g(d, \bar{x})$ is introduced in Definition 1.21 (20). Its fundamental specialization property is established in Theorem 1.24 (page 21).
  - The use of resultants to resultants to study the problem of the intersection of plane algebraic curves is generalized in Lemma 1.34 (page 30) to include situations where some of the standard requirements are not satisfied.
- In Chapter 2 we provide a complete and efficient solution for the total degree problem for plane curves, given either implicitly or parametrically. More specifically, the contributions in this chapter are the following:
  - The notion of auxiliary curve $\mathcal{S}_{(d^o,k^o)}$ is introduced in Definition 2.11 (page 46). This curve is defined using the auxiliary polynomial $s \in \mathbb{C}[s, k, \bar{y}]$ (Definition 2.8, page 45). In Theorem 2.14 it is shown that for a generic choice of parameters, there is a bijection between the points in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$

and some of the points in $\mathcal{S}_{(d^o,k^o)} \cap \mathcal{C}$. These points in $\mathcal{S}_{(d^o,k^o)} \cap \mathcal{C}$ are shown to be regular points of $\mathcal{C}$, associated with the points in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$.

- The definition of fake points $\mathcal{S}_{(d^o,k^o)}$ (Definition 2.16, page 50), and the proof of their invariance (Theorem 2.19, page 50) are used to characterize precisely the points in $\mathcal{S}_{(d^o,k^o)} \cap \mathcal{C}$ associated with points in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$.
- Theorem 2.24 (page 56) provides a total degree formula for the generic offset of a plane algebraic curve, given implicitly. This formula requires the computation of the multiplicities of intersection between $\mathcal{C}$ and $\mathcal{S}_{(d^o,k^o)}$ for a generic choice of (d^o, k^o) in a Zariski open subset of the space of parameters. This formula can be used as a heuristic for the total degree computation.
- The second total degree formula, in Theorem 2.27 (page 62), is a modification of the previous one, based in the notion of hodograph curve $\mathcal{H}$ (defined in page 37). This formula does not depend on a specific choice of (d^o, k^o), and it requires the computation of the multiplicities of intersection between $\mathcal{C}$ and $\mathcal{H}$.
- The third total degree formula in Theorem 2.31 (page 68) requires the computation of a univariate resultant and gcds of polynomials with symbolic coefficients. This deterministic formula provides an efficient solution for the total degree problem in the case of implicitly given plane curves.
- In developing the resultant-based total degree formula, the notions of auxiliary curve and fake point are generalized, in order to cover the remaining degree problems, arising in Chapter 3. Thus we obtain a common framework (see Theorems 2.30, 65), which guides the construction of all the resultant-based degree formulae in this and the following chapter.
- The formula in Theorem 2.40 (page 75) determines the total degree of the generic offset of a rational curve, given parametrically. Since the formula only requires the computation of gcds of univariate polynomials with symbolic coefficients, it gives a very efficient solution for the total offset degree problem in this case. The special case of polynomial parametrizations is considered in Remark 2.41 (page 75).

• In Chapter 3 we provide formulae for the partial degree of the generic offset w.r.t. to each variable, including the distance. With these results, an efficient solution for the offset degree problem for plane curves is achieved. More specifically, the contributions in this chapter are the following:

  - In Theorem 3.23 we provide a partial degree formula for the generic offset of a plane algebraic curve, given implicitly. It uses another auxiliary curve $\mathcal{S}^1_{(d^o,k^o)}$, constructed ad hoc for this problem, and requires the computation of the multiplicities of intersection between $\mathcal{C}$ and $\mathcal{S}_{(d^o,k^o)}$ for a generic choice of (d^o, k^o) in a Zariski open subset of the space of parameters. Again, as with Formula 2.24 of Chapter 2, this formula can be used heuristically for partial degree computations.

  - Using the framework provided by Theorem 2.30 of Chapter 2 we obtain a resultant-based partial degree formula in Theorem 3.24 (page 96).
  - The formula in Theorem 3.36 (page 109) gives a solution for the problem of the offset degree w.r.t. the distance variable d, in the case of implicitly given curves. This formula is another consequence of the framework provided by Theorem 2.30 of Chapter 2.
  - Theorem 3.42 (page 115) addresses the partial degree problem for the generic offset of a rational curve, given parametrically. It provides a very efficient solution, requiring only the computation of gcds of univariate polynomials with symbolic coefficients.
  - Similarly, Theorem 3.48 (page 118) provides a solution for the problem of the degree w.r.t. to the distance for the generic offset of a rational curve, given parametrically. It also requires only the computation of gcds of univariate polynomials with symbolic coefficients.

- In Chapter 4 we provide an efficient solution for the total degree problem for rational surfaces given parametrically. For this purpose, and in this chapter, we need to require Assumption 4.1 (page 122). In addition, we present an alternative approach for surfaces of revolution, providing a complete solution of the degree problem for this class of surfaces. More specifically, the contributions in this chapter are the following:

  - The formula in Theorem 4.45 (page 172) provides a solution for the total degree problem, for algebraic surfaces given parametrically, provided that Assumption 4.1 (page 122) holds. This is, to our knowledge, the first offset degree formula for rational surfaces available in the scientific literature.
  - Theorem 4.53 (page 184) shows how to obtain the implicit equation of a surface of revolution from the implicit equation of the generating curve. Combined with well-known implicitization techniques for plane curves, this Theorem can also be used efficiently when the generating curve is given parametrically. In Theorem 4.55 this is applied to determine the degree structure of the surface of revolution from the structure of the generating curve.
  - Theorem 4.58 (page 187) proves the fundamental property about the offset of a revolution surface, showing that it agrees with the revolution surface of the offset for the same generating curve. In other word, the geometric constructions of offsetting and forming the revolution surface commute with each other.

– Propositions 4.59 –for the implicit case– and 4.61 –for the parametric case– provide an effective way to check if the generating curve satisfies the symmetry condition, that in turn allows us to link the degree structure of the offset to a revolution surface with the degree structure of the generating curve.

– The above results are used to derive Algorithm 4.62 (page 189), that computes the degree structure of the offset for the revolution surface generated by a given input curve. The input curve can be described either parametrically, or implicitly.

Publications

Most of the results mentioned above have been published and/or communicated in Conferences. More precisely, the main publications derived from this research have been:

Journal Papers:

- [42]: *Degree formulae for offset curves.* F. San Segundo and J.R. Sendra.
  Journal of Pure and Applied Algebra, 195(3):301–335, 2005.
- [46]: *Partial Degree formulae for Plane Offset Curves.* F. San Segundo and J.R. Sendra.
  Journal of Symbolic Computation, 44(6):635–654, 2009.

Conference Papers:

- *On the Degree of Offsets.* F. San Segundo, J.R. Sendra.
  IX Encuentro de Algebra Computacional y Aplicaciones (EACA 2004), Universidad de Santander, Spain, 2004. Proceedings of the conference (ISBN: 84-688-6988-0), pp. 277-281.
- *The offset degree problem for surfaces of revolution.* F. San Segundo, J. R. Sendra.
  XI Encuentro de Algebra Computacional y Aplicaciones (EACA 2008), Universidad de Granada, Spain, 2008. Proceedings of the conference, pp. 65-69.
- *Offsetting Revolution Surfaces.* F. San Segundo, J. R. Sendra.
  Seventh International Workshop on Automated Deduction in Geometry (ADG08), East China Normal University, Shanghai, China, 2008. Proceedings of the conference, pp. 154-161.

Conference Posters:

- *Offsets From the Perspective of Computational Algebraic Geometry.* F. San Segundo, J. R. Sendra, J. Sendra.
  International Symposium on Symbolic and Algebraic Computation (ISSAC 05), Chinese Academy of Sciences, Beijing, China, 2005. The abstract to this poster has been published in the ACM SIGSAM Bulletin, [47]

- *Algorithms for offset curves and surfaces* F. San Segundo, J.R.Sendra, J.Sendra. International Congress of Mathematicians (ICM2006), Madrid, Spain, 2006. This poster won the 1st. prize in Section 15 of the Poster Competition for the ICM2006.

- *On the degree of Offsets to Algebraic Curves and Surfaces.* F. San Segundo, J. R. Sendra.
  International Symposium on Symbolic and Algebraic Computation (ISSAC 07), University of Waterloo, Ontario, Canada, 2007. The abstract to this poster has been published in the ACM Communications in Computer Algebra [20].

Future lines of research

We finish this introduction, listing some open problem, related to those solved in this thesis, and that we consider as future lines of research.

1. In this thesis we prove that the generic offset equation specializes to the offset equation for all but at most finitely many values of the distance. And we will show that there are examples in which this finite set is non empty. A natural question is the characterization, and effective computation, of those values of the distance for which the specialization fails. A possible approach for this problem could be based on the existing algorithms (e.g. the DISPGB algorithm, see [29] and [30]) for the analysis of Comprehensive Gröbner Basis.

2. The degree structure of the offset to a curve gives some information about the structure of the implicit equation of the offset. This information can be used, e.g., to reduce the dimension of the space needed to obtain the implicit equation using interpolation. A far more detailed information would be to have the Newton polygon of the implicit equation of the offset. We have already obtained some results in this direction (see [15]).

3. Another natural extension of the work in this thesis is the study of the degree structure for generalized offsets (see [51]).

4. Turning to surfaces, the next step would be to obtain partial degree formulae for rational surfaces, given parametrically. In fact we already have some good hints about what the corresponding auxiliary surfaces look like. That problem, perhaps the most obviously related to the ones that have been addressed in this thesis, has been left out because of the space and time limitations of this work.

5. Finally, we should mention the problem of the offset degree structure of a general algebraic surface, given implicitly. We think about a possible multi-resultant formula for this case, but the efficiency of a such formula, even for very simple cases does not look promissory.

General Notation and Terminology

In the following pages, for the convenience of the reader, we introduce and collect the main notation and terminology that will be used throughout this thesis. At the beginning of each chapter, if necessary, we will extend/adapt the notation to that particular part.

- $\mathbb{K}$ is an algebraically closed field of characteristic zero; $\mathbb{K}^\times$ denotes $\mathbb{K} \setminus \{0_\mathbb{K}\}$
- As usual, $\mathbb{C}$ and $\mathbb{R}$ correspond to the fields of complex and real numbers, respectively.
- The n-dimensional affine space is the set $\mathbb{K}^n$, and the associated projective space will be denoted by $\mathbb{P}^n$ when no confusion arises; if necessary we will use $\mathbb{P}^n(\mathbb{K})$ to emphasize the underlying field.
- We will use $(y_1, \ldots, y_n)$ for the affine coordinates in $\mathbb{K}^n$, and $(y_0 : y_1 : \cdots : y_n)$ for the projective coordinates in $\mathbb{P}^n$, as well as the abbreviations:

  $$\bar{y} = (y_1, \ldots, y_n), \quad \bar{y}_h = (y_0 : y_1 : \cdots : y_n).$$

- In order to distinguish offset hypersurfaces from original hypersurfaces, we will also use $\bar{x} = (x_1, \ldots, x_n)$, $\bar{x}_h = (x_0 : x_1 : \cdots : x_n)$ to refer to the affine and projective space, respectively, where the offsets are, and $\bar{y}, \bar{y}_h$ as above for the original hypersurface.
- A point in $\mathbb{K}^n$ will be denoted by

  $$\bar{y}^o = (y_1^o, \ldots, y_n^o)$$

  and, correspondingly, a point in $\mathbb{P}^n$ will be denoted by

  $$\bar{y}_h^o = (y_0^o : \cdots : y_n^o)$$

  Throughout this work, we will frequently use this ^{o} superscript to indicate a particular value of a variable.

- The Zariski closure of a set $A \subset \mathbb{K}^n$ will be denoted by A^*. The projective closure of an algebraic set A will be denoted by $\overline{A}$.
- Let A be an algebraic set. We denote by $\mathrm{Sing}_a(A)$ the affine singular locus of A, and by $\mathrm{Sing}(A)$ the projective singular locus of A; i.e. the singular locus of $\overline{A}$.
- If $I \subset \mathbb{K}[\bar{x}]$ is a polynomial ideal, $\mathbf{V}(I)$ denotes the affine algebraic set defined by I; that is,

  $$\mathbf{V}(I) = \{\bar{x}^o \in \mathbb{K}^n / \forall f \in I, f(\bar{x}^o) = 0\}$$

- When we homogenize a polynomial $g \in \mathbb{K}[y_1, \ldots, y_n]$, we will use capital letters, as in $G(y_0, y_1, \ldots, y_n)$, to denote the homogenization of g w.r.t. y_0. Also, by abuse of notation, we will write $g(\bar{y})$, $G(\bar{y}_h), \mathbb{K}[\bar{y}], \mathbb{K}[\bar{y}_h]$.

- The partial derivatives w.r.t. y_i of $g(y_1, \ldots, y_n)$ and of its homogenization $G(y_0, \ldots, y_n)$ will be denoted g_i and G_i respectively, for $i = 0, \ldots, n$. The symbol ∇g (resp. ∇G) denotes the gradient vector of partial derivatives, i.e.:

$$\nabla g(\bar{y}) = (g_1, \ldots, g_n)(\bar{y}), \quad (\text{ resp. } \nabla G(\bar{y}_h) = (G_0, G_1, \ldots, G_n)(\bar{y}_h))$$

- $\mathcal{V}$ is a irreducible algebraic hypersurface in $\mathbb{K}^n$ and $f \in \mathbb{K}[\bar{y}]$ its defining polynomial; so $F(\bar{y}_h)$ is the defining polynomial of $\overline{\mathcal{V}}$. When $n = 3$ (surface case), and $n = 2$ (curve case), instead of $\mathcal{V}$, we will write Σ and $\mathcal{C}$, respectively.

- The (classical) offset at distance $d^o \in \mathbb{K}^\times$ to $\mathcal{V}$ is denoted by $\mathcal{O}_{d^o}(\mathcal{V})$ and the generic (classical) offset to $\mathcal{V}$ by $\mathcal{O}_d(\mathcal{V})$ (see Definition 1.18 in page 17). In this work, the variable d always represents the distance values.

- We denote by $g \in \mathbb{K}[d, \bar{x}]$ the generic offset equation for $\mathcal{V}$ (see Definition 1.21 in page 20). The following notation is used to denote the various notions of degree associated with the polynomial g:

  1. δ is the total degree of g w.r.t. $\bar{x}$; i.e. $\delta = \deg_{\bar{x}}(g)$.
  2. δ_i is the partial degree of g w.r.t. $\bar{x}_i$, for $i = 1, \ldots, n$; i.e. $\delta_i = \deg_{\bar{x}_i}(g)$.
  3. δ_d is the degree of g w.r.t. the distance variable d; i.e. $\delta = \deg_d(g)$.

- Given $\phi(\bar{y}), \psi(\bar{y}) \in \mathbb{K}[\bar{y}]$ we denote by $\mathrm{Res}_{y_i}(\phi, \psi)$ the univariate resultant of ϕ and ψ w.r.t. y_i, for $i = 0, \ldots, n$. And if A is a subset of the set of variables $\{y_0, \ldots, y_n\}$, we denote by $\mathrm{PP}_A(\phi)$ (resp. $\mathrm{Con}_A(\phi)$) the primitive part (resp. the content) of the polynomial ϕ w.r.t. A.

- For $n = 2$ (that is, in the case of plane curves), we will consider a generic $\mathcal{L}_k$ line through the origin, given by its implicit equation,

$$L(k, \bar{x}) : \quad x_1 - k x_2 = 0.$$

  Here k is a variable whose value determines the direction of $\mathcal{L}_k$. For $n > 2$ (in particular, in the case of surfaces), we consider a generic $\mathcal{L}_{\bar{k}}$ line through the origin, whose direction is determined by the values of a variable $\bar{k} = (k_1, \ldots, k_n)$. More precisely, for a particular value of $\bar{k}$, denoted by $\bar{k}^o$, the parametric equations of $\mathcal{L}_{\bar{k}}$ are

$$\ell_i(\bar{k}, l, \bar{x}) : \; x_i - k_i\, l = 0, \; \text{ for } i = 1, \ldots, n,$$

  where l is the parameter.

- We will keep the convention of always using the letter Δ to indicate a finite subset of values of the variable d. Accordingly, the letter Θ denotes a Zariski closed set of values of $\bar{k}$. A Zariski open subset of $\mathbb{K} \times \mathbb{K}^n$, formed by pairs of values of $(d, \bar{k})$, will be denoted by Ω. In some proofs, an open set Ω will be constructed in several steps. In these cases we will use a superscript to indicate the step in the construction. Thus, $\Omega_1^0, \Omega_1^1, \Omega_1^2$, etc. are open sets, defined in sucessive steps in the construction of Ω_1.

- A similar convention will be used for systems of equations and their solutions. A system of equations will be denoted by $\mathfrak{S}$, with sub and superscripts to distinguish between systems, and the set of solutions of the system will be denoted by Ψ, with the same choice of sub and superscripts.

- When the hypersurface $\mathcal{V}$ is parametric, we will use P to denote a rational parametrization of $\mathcal{V}$. Thus, P is given through a non-constant n-uple of rational maps in $n-1$ parameters. We will use $\bar{t} = (t_1, \ldots, t_{n-1})$ for the parameters of P and, as before, $\bar{t}^o = (t_1^o, \ldots, t_{n-1}^o)$ will stand for a particular value in $\mathbb{K}^{n-1}$ of the parameters. The parametrization P can be expressed as follows:

$$P(\bar{t}\,) = \left(\frac{P_1(\bar{t}\,)}{P_0(\bar{t}\,)}, \frac{P_2(\bar{t}\,)}{P_0(\bar{t}\,)}, \cdots, \frac{P_n(\bar{t}\,)}{P_0(\bar{t}\,)} \right) \tag{1}$$

  where $P_i \in \mathbb{K}[\bar{t}\,]$ for $i = 0, \ldots, n$, and $\gcd(P_0, P_1, \ldots, P_n) = 1$. The assumption of a common denominator for all the components of P does not represent a loss of generality.

- We will also need to consider local parametrizations of algebraic varieties. To distinguish local from (global) rational parametrizations, we will use calligraphic typeface for local parametrizations. Thus, a local parametrization will be denoted by

$$\mathcal{P}(\bar{t}) = (\mathcal{P}_1(\bar{t}), \ldots, \mathcal{P}_n(\bar{t}))$$

Chapter 1

Preliminaries and Statement of the Problem

Let us begin reviewing the classical -and informal- concepts of offset curves and surfaces. These concepts will be made precise later in this chapter (see Definition 1.2 in page 10). Let $\mathcal{C}$ be a given plane curve, and let d^o be a non-zero fixed distance. Let $\bar{p}$ be a point of $\mathcal{C}$, and let $\mathcal{L}_{\mathcal{C}}$ be the normal line to $\mathcal{C}$ at $\bar{p}$ (assume, for this informal introduction, that the normal line to $\mathcal{C}$ at $\bar{p}$ is well defined). Also, let $\bar{q}$ be any point of $\mathcal{L}_{\mathcal{C}}$ at a distance value d^o of $\bar{p}$ (there are, in principle, two possible choices for $\bar{q}$, one on each "side" of $\mathcal{C}$); equivalently, we consider the intersection points of $\mathcal{L}_{\mathcal{C}}$ with a circle centered at $\bar{p}$ and with radius d^o. The offset curve to $\mathcal{C}$ at the distance value d^o is the set $\mathcal{O}_{d^o}(\mathcal{C})$ of all the points $\bar{q}$ obtained by using this construction. This is illustrated in the Figure 1.1. The curve $\mathcal{C}$ is said to be the generating curve of $\mathcal{O}_{d^o}(\mathcal{C})$.

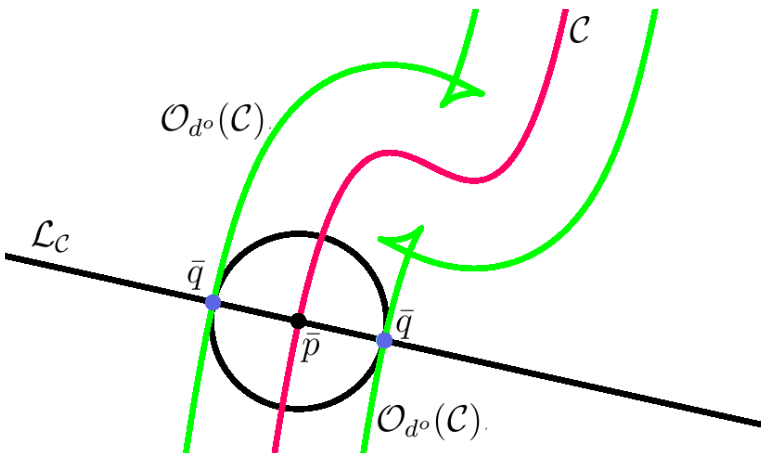

Figure 1.1: Informal Definition of Offset to a Generating Curve

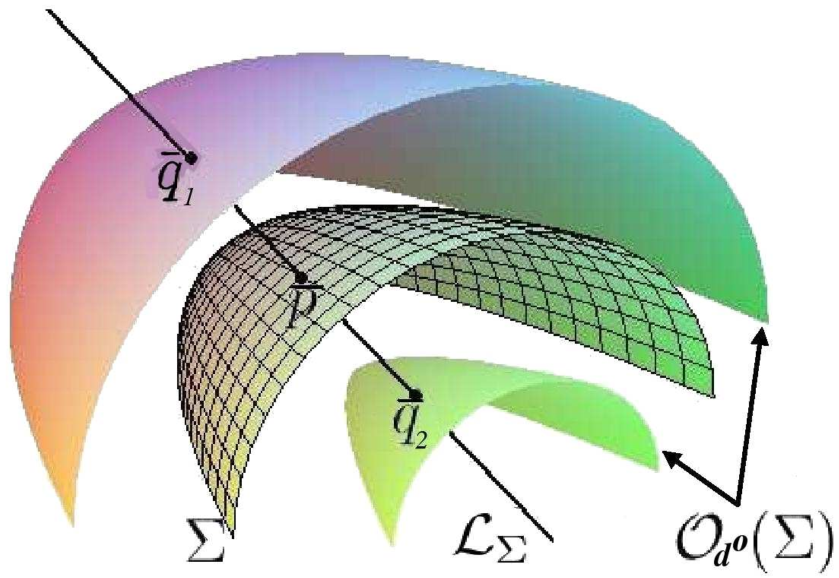

Figure 1.2: Informal Definition of Offset to a Generating Surface

In the case of a surface Σ in three-dimensional space, one can apply the same construction: at each point $\bar{p}$ of Σ, consider the normal line $\mathcal{L}_\Sigma$ to the surface. Let $\bar{q}$ be a point on that line, at a distance d^o of $\bar{p}$ (there are, again, in general two such points $\bar{q}$); equivalently, we consider the intersection points of $\mathcal{L}_\Sigma$ with a sphere centered at $\bar{p}$ and with radius d^o. The offset surface to Σ at distance value d^o, is the set $\mathcal{O}_{d^o}(\Sigma)$ of all the points $\bar{q}$ obtained by this geometric construction, illustrated in Figure 1.2. Σ is said to be the generating surface of $\mathcal{O}_{d^o}(\Sigma)$.

The above informal definition was used in the classical geometry literature, but it gives rise to several difficulties when one tries to apply it to general algebraic curves. In particular, it cannot be applied at points where the normal line is not well defined. There are another equivalent classical approaches to the definition of offset, e.g. based on the notion of envelope of a family of curves, but they are faced with similar problems.

On the other hand, offset curves and surfaces have been, and still are, widely used in applications. For example, in Computer Aided Geometric Design (CAGD), see e.g. the references [24], and [31]. In this applied context, the above informal definitions of offset curves and surfaces often result in confusing terminology, making it sometimes difficult to determine the scope of the stated results.

Thus, in our opinion, it is necessary to lay a solid foundation for the concept of offset curve and offset surface: a definition not depending on a particular computational method. When the generating curve or surface is algebraic, this formalization of the offset notion has been achieved, possibly for the first time, in [8] and [49].

The classical offset notion has been extended in several directions. The general offset (see [36]) is defined by replacing the circle or sphere, used in the classical offset definition, by a more general shape. Besides, the generalized offset (see [8] and [49]) is defined by applying a linear isometry to the normal vector of the hypersurface (the same isometry is used at all points of the hypersurface), and then performing the offset construction with this transformed vector. In this thesis, we will always work with classical offsets.

The structure of the chapter is the following:

- We first review in Section 1.1 (page 9) the fundamental concepts related to the classical offset and the related properties that will be used in the sequel. In order to do this, in Subsection 1.1.1 we follow the formalism in [8] and [50], using the Incidence Diagram 1.2 (page 10). Besides, the relation of this notion with Elimination Theory is established. In Subsection 1.1.2 (page 11) we collect several fundamental properties of the classical offset construction that we will refer to in the sequel. In the final Subsection 1.1.3 (page 13), we analyze a property related to the intersection of the classical offset with a pencil of lines through the origin. This is necessary since our strategy for addressing the offset degree problem is based precisely in the use of such a pencil of lines.

- The notions of generic offset and generic offset polynomial are introduced in Subsection 1.2.1 of Section 1.2 (page 15). These are the central objects of study in this thesis. The idea that motivates the concept of generic polynomial of the offset to $\mathcal{V}$ is to have a global expression of the offset for all (or almost all) distance values. We introduce first the notion of generic offset, which can be considered as a hypersurface that collects as level curves all the classical offsets to a given curve. This notion is introduced by considering the natural generalizations of the concepts introduced in Section 1.1, by considering the distance as a new variable; see Definition 1.21 (page 20). The fundamental property of the generic offset is established in Theorem 1.24 (page 21). In Subsection 1.2.2 (page 24) we describe the degree problem for the generic offset, which is the problem studied in this thesis. Since the classic and generic offset notions are introduced as the Zariski closure of constructible sets, in the final subsection (Subsection 1.2.3 in page 24), we address the natural problem of characterizing the points that appear in the offset as a result of this closure. This is done using the Projective Elimination Theory as described, e.g. in [14], Section 8.5.

- The final section of this chapter, Section 1.3 (page 27) contains some technical results about the use of univariate resultants to study the problem of the intersection of plane algebraic curves. The classical setting for the computation of the intersection points of two plane curves by means of resultants is well known (see for instance [10], [56] and [52]). This requires in general a linear change of coordinates. However, in this work, we need to analyze the behavior of the resultant when some of the standard requirements are not satisfied. This is the content of Lemma 1.34 (page 30). Similarly, we also need to analyze the case when more than two curves are involved, by using generalized resultants. This is done in Lemma 1.35 (page 32).

The results in this chapter, concerning the generic offset, have been published in the Journal of Symbolic Computation (see [46]).

Notation and terminology for this chapter

Alongside with the notation and terminology already introduced in page 1, in this chapter we will use the following:

- Let V be an n-dimensional vector space over $\mathbb{K}$. Two vectors $\bar{v} = (v_1, \ldots, v_n)$ and $\bar{w} = (w_1, \ldots, w_n)$ are said to be parallel if and only if

  $$v_i w_j - v_j w_i = 0, \text{ for } i, j = 1, \ldots, n$$

  In this case we write $\bar{v} \parallel \bar{w}$.

- We consider in V the symmetric bilinear form defined by:

  $$\Xi(\bar{v}, \bar{w}) = \sum_{i=1}^{n} v_i w_i$$

  which induces in V a metric vector space (see [40], [54]) with light cone of isotropy given by:

  $$L_\Xi = \{(x_1, \ldots, x_n) \in V / x_1^2 + \cdots + x_n^2 = 0\}$$

  Note that, when $\mathbb{K} = \mathbb{C}$, this is not the usual unitary space $\mathbb{C}^n$. On the other hand, when we consider the field $\mathbb{R}$, it is the usual Euclidean metric space, thus it preserves the usefulness of our results for applications. In this work, the norm $\|\bar{v}\|$ of a vector $\bar{v} \in V$ denotes a square root of $\bar{v} \cdot \bar{v}$, that is

  $$\|\bar{v}\|^2 = \Xi(\bar{v}, \bar{v}) = \sum_{i=1}^{n} v_i^2$$

  Moreover, a vector $\bar{v} \in V$ is isotropic if $\bar{v} \in L_\Xi$ (equivalently if $\|\bar{v}\| = 0$). Note that for a non-isotropic vectors there are precisely two choices of norm, which differ only by multiplication by -1.

- We denote
$$\mathrm{nor}_{(i,j)}(\bar{x},\bar{y}) = f_i(\bar{y})(x_j - y_j) - f_j(\bar{y})(x_i - y_i).$$
Recall that $f(\bar{y})$ is the defining polynomial of the irreducible hypersurface $\mathcal{V}$, and that $f_i(\bar{y})$ denotes its partial derivative w.r.t. y_i.

- The affine normal-hodograph of f is the polynomial:
$$h(\bar{y}) = f_1^2(\bar{y}) + \cdots + f_n^2(\bar{y})$$
Following our convention of notation (see page 1), H is the homogenization of h w.r.t. y_0; that is:
$$H(\bar{y}_h) = F_1^2(\bar{y}_h) + \cdots + F_n^2(\bar{y}_h)$$
H is called the projective normal-hodograph of $\mathcal{V}$. Moreover, a point $\bar{y}_h^o \in \overline{\mathcal{V}}$ (resp. $\bar{y}^o \in \mathcal{V}$) is called normal-isotropic if $H(\bar{y}_h^o) = 0$ (resp. $h(\bar{y}^o) = 0$).

- We denote by $\mathcal{V}_o$ (similarly Σ_o and $\mathcal{C}_o$) the set of non normal-isotropic affine points of $\mathcal{V}$; that is:
$$\mathcal{V}_o = \{\bar{y}^o \in \mathcal{V}/h(\bar{y}^o) \neq 0\}$$
In the rest of this work we will assume that the Zariski-open subset $\mathcal{V}_o$ is non-empty. In [49], Proposition 2, it is proved that this is equivalent to H not being a multiple of F. We denote by $\mathrm{Iso}(\mathcal{V})$ the closed set of affine normal-isotropic points of $\mathcal{V}$. Note that $\mathrm{Sing}_a(\mathcal{V}) \subset \mathrm{Iso}(\mathcal{V})$.

- If $\mathcal{K}$ is an irreducible component of an algebraic set $\mathcal{A}$, and $\mathcal{K} \subset \mathrm{Iso}(\mathcal{A})$ we will say that $\mathcal{K}$ is normal-isotropic.

Remark 1.1. *In the definition of offset (see Subsection 1.1.1 below), if the variable $\bar{y}$ represents the coordinates of a point in $\mathcal{V}$, and $\bar{x}$ represents the coordinates of a point in the offset generated by $\bar{y}$, then the normal vector to $\mathcal{V}$ at $\bar{y}$ will be required to be parallel to the vector $\bar{x} - \bar{y}$. That is, we will impose the condition*
$$\nabla f(\bar{y}) \parallel (\bar{x} - \bar{y})$$
This is equivalent to:
$$\mathrm{nor}_{(i,j)}(\bar{x},\bar{y}) = 0 \textit{ for } i,j = 1,\ldots,n.$$

1.1 Formal Definition and Basic Properties of Offset Varieties

In Subsections 1.1.1 and 1.1.2 (page 11) we provide a formal definition and some basic properties of the classical offset construction. All these results are already available in the literature (see, in particular [8] and [50]), but we prefer to include them here for ease of reference, to provide a uniform terminology and notation, and also as introductory material for our work. Then, in Subsection 1.1.3 (page 13) we study a degenerate situation, associated with the offsetting construction, that may appear when one intersects the offset with a pencil of lines through the origin (see Lemma 1.14, page 14).

1.1.1 Formal definition of classical offset

With the notation introduced above, let $d^o \in \mathbb{K}^\times$ be a fixed value, and let $\Psi_{d^o}(\mathcal{V}) \subset \mathbb{K}^{2n+1}$ be the set of solutions (in the variables $(\bar{x}, \bar{y}, u)$) of the following polynomial system:

$$\left.\begin{array}{lr}
 & f(\bar{y}) = 0 \\
\mathrm{nor}_{(i,j)}(\bar{x},\bar{y}): & f_i(\bar{y})(x_j - y_j) - f_j(\bar{y})(x_i - y_i) = 0 \\
(\text{for } i,j = 1,\ldots,n; i < j) & \\
b_{d^o}(\bar{x},\bar{y}): & (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 - (d^o)^2 = 0 \\
w(\bar{y},u): & u \cdot (\|\nabla f(\bar{y})\|^2) - 1 = 0
\end{array}\right\} \equiv \mathfrak{S}_1(d^o) \tag{1.1}$$

The first equation establishes that $\bar{y}$ is a point of $\mathcal{V}$. The equations $\mathrm{nor}_{(i,j)}$ are explained in Remark 1.1 above. The polynomial b_{d^o} defines a sphere of radius d^o centered at $\bar{y} \in \mathcal{V}$. The last equation guarantees that the said point belongs to $\mathcal{V}_o$; in particular, it is not singular. Let us consider the following offset incidence diagram:

Offset Incidence Diagram

$$\begin{tikzcd}
 & \Psi_{d^o}(\mathcal{V}) \subset \mathbb{K}^n \times \mathbb{K}^n \times \mathbb{K} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathcal{A}_{d^o}(\mathcal{V}) \subset \mathbb{K}^n & & \mathcal{V} \subset \mathbb{K}^n
\end{tikzcd} \tag{1.2}$$

where

$$\begin{cases} \pi_1 : \mathbb{K}^{2n+1} \mapsto \mathbb{K}^n \\ \pi_1(\bar{x}, \bar{y}, u) = \bar{x} \end{cases} \quad \text{and} \quad \begin{cases} \pi_2 : \mathbb{K}^{2n+1} \mapsto \mathbb{K}^n \\ \pi_2(\bar{x}, \bar{y}, u) = \bar{y} \end{cases}$$

and $\mathcal{A}_{d^o}(\mathcal{V}) = \pi_1(\Psi_{d^o}(\mathcal{V}))$.

Definition 1.2. *The* (classical) offset *to $\mathcal{V}$ at distance d^o is the algebraic set $\mathcal{A}_{d^o}(\mathcal{V})^*$ (recall that the asterisk indicates Zariski closure). It will be denoted by $\mathcal{O}_{d^o}(\mathcal{V})$.*

Remark 1.3.

1. *If there is a solution of the system 1.1 of the form $\bar{p}^o = (\bar{x}^o, \bar{y}^o, u^o)$, then we say that the point $\bar{y}^o \in \mathcal{V}_o$ and the point $\bar{x}^o \in \mathcal{O}_{d^o}(\mathcal{V})$ are* associated points.

2. *Let $I(d^o) \subset \mathbb{K}[\bar{x}, \bar{y}, u]$ be the ideal generated by the polynomials in $\mathfrak{S}_1(d^o)$; that is:*

$$I(d^o) = < f(\bar{y}), b_{d^o}(\bar{x}, \bar{y}), \mathrm{nor}_{(1,2)}(\bar{x}, \bar{y}), \ldots, \mathrm{nor}_{(n-1,n)}(\bar{x}, \bar{y}), w(\bar{y}, u) > .$$

This means that

$$\Psi_{d^o}(\mathcal{V}) = \mathbf{V}(I(d^o))$$

is the affine algebraic set defined by $I(d^o)$, and

$$\mathcal{O}_{d^o}(\mathcal{V}) = \mathbf{V}(\tilde{I}(d^o))$$

where $\tilde{I}(d^o) = I(d^o) \cap \mathbb{K}[\bar{x}]$ is the $(\bar{y}, u)$-elimination ideal of $I(d^o)$ (see [14], Closure Theorem, p. 122). In particular, this means that the offset can be computed by elimination techniques, such as Gröbner bases, resultants, characteristic sets, etc.

3. *Observe that $\pi_2(\Psi_{d^o}(\mathcal{V})) = \mathcal{V}_o$. Moreover, π_2 is a $2:1$ map over $\mathcal{V}_o$: if $\bar{p}^o = (\bar{x}^o, \bar{y}^o, u^o) \in \Psi_{d^o}(\mathcal{V})$, then the fibre $\pi_2^{-1}(\pi_2(\bar{p}^o))$ is formed precisely by the following two points:*

$$(\bar{x}^o_{\pm}, \bar{y}^o, \pm u^o)$$

where

$$\bar{x}^o_{\pm} = \bar{y}^o \pm d^o \frac{\nabla f(\bar{y}^o)}{\|\nabla f(\bar{y}^o)\|}.$$

The analysis of the fiber of π_1 is more complicated, though. We will discuss this below in connection with the degeneracy analysis of the offset.

1.1.2 Basic properties of the classical offset

In the sequel, we will refer to some properties of the classical offset construction, that we collect here for the reader's convenience.

We start with a very important geometric property regarding the normal vector of the classical offset construction.

Proposition 1.4 (Fundamental Property of the Classical Offsets)**.** *Let $\bar{y}^o \in \mathcal{V}_o$, and let $\bar{x}^o \in \mathcal{O}_{d^o}(\mathcal{V})$ be a point associated to $\bar{y}^o$. Then the normal line to $\mathcal{V}$ at $\bar{y}^o$ is also normal to $\mathcal{O}_{d^o}(\mathcal{V})$ at $\bar{x}^o$.*

Proof. See [49], Theorem 16. □

In order to prove that we can avoid degenerated situations, we will sometimes need information about the dimension of certain sets of points. The basic tools for doing this will be the incidence diagrams, analogous to 1.2 (page 10) and well known results about the dimension of the fiber of a regular map. For ease of reference, we include here a statement of one such result, in a form that meets our needs. The proof can be found in [22]

Lemma 1.5. *Let A be an affine algebraic set, and let $f : A \mapsto \mathbb{K}^p$ be a regular map. Let us denote $B = f(A)^*$. For $\bar{a}^o \in A$, let $\mu(\bar{a}^o) = \dim(f^{-1}(f(a^o)))$ Then, if $A_o \subset A$ is any irreducible component, $B_o = f(A_o)$ its image, and μ^o is the minimum value of $\mu(\bar{a}^o)$ for $\bar{a}^o \in A_o$, we have*

$$\dim(A_o) = \dim(B_o) + \mu^o$$

In particular, if there exists $a^o \in A_o$ for which $\dim(f^{-1}(f(a^o))) = 0$, then $\dim(A_o) = \dim(B_o)$.

We next analyze the number and dimension of the irreducible components of the offset.

Proposition 1.6. *$\mathcal{O}_{d^o}(\mathcal{V})$ has at most two irreducible components.*

Proof. See [49], Theorem 1. □

Proposition 1.7. *The irreducible components of $\Psi_{d^o}(\mathcal{V})$ have the same dimension as $\mathcal{V}$.*

Proof. In [49], Lemma 1, this is proved using local parametrizations. However, since, as we have seen above, π_2 is a $2:1$ map, this can also be considered a straightforward application of the preceding Lemma 1.5. □

Remark 1.8. *This implies immediately that $\mathcal{O}_{d^o}(\mathcal{V})$ has at most two irreducible components, whose dimension is less or equal than $\dim(\mathcal{V})$.*

To present the next two results, we recall some of the terminology introduced in [49]:

Definition 1.9.

1. *The offset $\mathcal{O}_{d^o}(\mathcal{V})$ is called* degenerated *if at least one of its components is not a hypersurface.*

2. *A component $\mathcal{M} \subset \mathcal{O}_{d^o}(\mathcal{V})$ is said to be a* simple component *if there exists a non-empty Zariski dense subset $\mathcal{M}_1 \subset \mathcal{M}$ such that every point of $\mathcal{M}_1$ is associated to exactly one point of $\mathcal{V}$. Otherwise, $\mathcal{M}$ is called a* special component *of the offset. Furthermore, we say that $\mathcal{O}_{d^o}(\mathcal{V})$ is* simple *if all its components are simple, and* special *if it has at least a special component (in this case, it has precisely one special component, see Proposition 1.11 below).*

Note that if the offset is degenerated, and taking into account Lemma 1.5, the map π_1 must have a non-zero dimensional fiber for some point in $\Psi_{d^o}(\mathcal{V})$.

The following two results tell us that degeneration and special components are very infrequent phenomena.

Proposition 1.10. *There is a finite set $\Delta_0 \subset \mathbb{K}$ such that if $d^o \notin \Delta_0$, then $\mathcal{O}_{d^o}(\mathcal{V})$ is not degenerated.*

Proof. See [49], Theorem 2. □

Proposition 1.11.

1. *Let $\mathcal{M}$ be an irreducible and non-degenerated component of $\mathcal{O}_{d^o}(\mathcal{V})$. Then $\mathcal{M}$ is special if and only if $\mathcal{O}_{d^o}(\mathcal{M}) = \mathcal{V}$.*
2. *$\mathcal{O}_{d^o}(\mathcal{V})$ has at least a simple component.*
3. *If $\mathcal{O}_{d^o}(\mathcal{V})$ is irreducible, then it is simple.*
4. *There is a finite set $\Delta_1 \subset \mathbb{K}$ such that, if $d^o \notin \Delta_1$ then $\mathcal{O}_{d^o}(\mathcal{V})$ is simple, and the irreducible components of $\mathcal{O}_{d^o}(\mathcal{V})$ are not contained in* Iso$(\mathcal{V})$.

Proof. See [49], Theorems 7, 8 and Corollary 6. □

The next result shows that –as expected, being a metric construction– the offset construction is invariant under rigid motions of the affine space.

Proposition 1.12. *Let $\mathcal{T}$ be a rigid motion of the affine space $\mathbb{K}^n$. Then*

$$\mathcal{T}(\mathcal{O}_{d^o}(\mathcal{V})) = \mathcal{O}_{d^o}(\mathcal{T}(\mathcal{V}))$$

Proof. See [49], Lemma 2.5 in Chapter 2. □

1.1.3 Further geometric properties of classical offsets

In the sequel we will study the intersection of the classical offset with a pencil of lines through the origin. We will see that this can lead to some degenerated situations, if the set of points of $\mathcal{V}$ where the normal line to $\mathcal{V}$ passes through the origin is *too big*. The next Lemma says that this can only happen if $\mathcal{V}$ is a sphere centered at the origin.

Remark 1.13. *The proof of Lemma 1.14 below uses the local convergence property of power series when $\mathbb{K} = \mathbb{C}$. Even though the rest of the results in this chapter apply whenever $\mathbb{K}$ is any algebraically closed field of characteristic zero, this Lemma will be used in the following chapters to obtain the degree formulae. Therefore, in the remaining chapters of this work,* we will restrict our attention to the case where $\mathbb{K} = \mathbb{C}$.

Lemma 1.14. *Let $\mathbb{K} = \mathbb{C}$, and let $\mathcal{V}_\perp \subset \mathcal{V}$ denote the set of regular points $\bar{y}^o \in \mathcal{V}$ such that the normal line to $\mathcal{V}$ at $\bar{y}^o$ is parallel to $\bar{y}^o$. If $\mathcal{V}$ is not a $(n-1)$-dimensional sphere centered at the origin, then $\mathcal{V}_\perp^*$ is a proper (possibly empty) closed subset of $\mathcal{V}$.*

Proof. Let us assume that $\mathcal{V}_\perp$ is nonempty. Let, as usual, $f(\bar{y})$ be the irreducible polynomial defining $\mathcal{V}$, and let $\tilde{\mathcal{V}}$ be the algebraic set in $\mathbb{K}^n$ defined by:

$$\begin{cases} f(\bar{y}) = 0 \\ f_i(\bar{y})y_j - f_j(\bar{y})y_i = 0 \quad (\text{for } i, j = 1, \ldots, n; i < j). \end{cases}$$

Note that this set of equations implies $\bar{y}^o \parallel \nabla f(\bar{y}^o)$ for $\bar{y}^o \in \mathcal{V}$. Then $\mathcal{V}_\perp \subset \tilde{\mathcal{V}} \subset \mathcal{V}$. Therefore, it suffices to prove that $\tilde{\mathcal{V}} \neq \mathcal{V}$. Let us suppose that $\tilde{\mathcal{V}} = \mathcal{V}$. Let

$$K(\bar{y}) = \bar{y} \cdot \nabla f(\bar{y}) = \sum_{j=1}^{n} y_j f_j(\bar{y}).$$

Then for every $\bar{y}^o \in \mathcal{V}$, using that $f_i(\bar{y}^o)y_j^o = f_j(\bar{y}^o)y_i^o$ one has that

$$f_i(\bar{y}^o)K(\bar{y}^o) = \sum_{j=1}^{n} f_i(\bar{y}^o)y_j^o f_j(\bar{y}^o) = y_i^o \sum_{j=1}^{n} f_j(\bar{y}^o)^2 = y_i^o h(\bar{y}^o),$$

for $i = 1, \ldots, n$. Now let $\bar{t} = (t_1, \ldots, t_{n-1})$ and let $\mathcal{Q}(\bar{t}) = (Q_1, \ldots, Q_n)(\bar{t})$ be a local parametrization of $\mathcal{V}$. Substituting $\mathcal{Q}$ in the above relation:

$$f_i(\mathcal{Q}(\bar{t}))K(\mathcal{Q}(\bar{t})) = Q_i(\bar{t})h(P(\bar{t}))$$

that is, $K(\mathcal{Q}(\bar{t}))\nabla f(\mathcal{Q}(\bar{t})) = h(\mathcal{Q}(\bar{t}))\mathcal{Q}(\bar{t})$. Using Prop. 2 in [49], we know that $h(\mathcal{Q}(\bar{t})) \neq 0$, and so $K(\mathcal{Q}(\bar{t})) \neq 0$. Thus:

$$\frac{h(\mathcal{Q}(\bar{t}))}{K(\mathcal{Q}(\bar{t}))} Q_i(\bar{t}) = f_i(\mathcal{Q}(\bar{t})).$$

On the other hand, since $f(\mathcal{Q}(\bar{t})) = 0$, deriving w.r.t. t_j, $(j = 1, \ldots, n-1)$ one has:

$$\sum_{i=1}^{n} f_i(\mathcal{Q}(\bar{t})) \frac{\partial Q_i(\bar{t})}{\partial t_j} = \frac{h(\mathcal{Q}(\bar{t}))}{K(\mathcal{Q}(\bar{t}))} \sum_{i=1}^{n} Q_i(\bar{t}) \frac{\partial Q_i(\bar{t})}{\partial t_j} = 0$$

From this, one concludes that

$$\frac{\partial}{\partial t_j}\left(\sum_{i=1}^{n} Q_i^2(\bar{t})\right) = 2\sum_{i=1}^{n} Q_i(\bar{t})\frac{\partial Q_i(\bar{t})}{\partial t_j} = 0$$

for $j = 1, \ldots, n-1$. This means that $\sum_{i=1}^{n} Q_i^2(\bar{t}) = c$ for some constant $c \in \mathbb{K}$. Since $\mathcal{V}$ is assumed not to be normal-isotropic, one has $c \neq 0$, and since the parametrization converges locally, we conclude that $\mathcal{V}$ equals a sphere centered at the origin. □

Remark 1.15. *Note that, if in Lemma 1.14 we consider those regular points $\bar{y}^o$ of $\mathcal{V}$ such that the normal line to $\mathcal{V}$ at $\bar{y}^o$ is parallel to the vector $\bar{y}^o - \bar{a}$ for a fixed $\bar{a} \in \mathbb{K}^n$, then $\mathcal{V}_\perp^*$ is a proper (possibly empty) closed subset of $\mathcal{V}$, unless $\mathcal{V}$ is a $(n-1)$-dimensional sphere centered at $\bar{a}$.*

A closer analysis of the proof of Lemma 1.14 shows that in fact we have also proved the following:

Corollary 1.16. *If $\mathcal{W}$ is any irreducible component of $\mathcal{V}_\perp^*$ with $\dim(\mathcal{W}) > 0$ then $\mathcal{W}$ is contained in a $(n-1)$-dimensional sphere centered at the origin. That is, there exists $d^o \in \mathbb{K}^\times$ such that if $\bar{y}^o \in \mathcal{W}$, then*

$$(y_1^0)^2 + \cdots + (y_n^0)^2 = (d^o)^2.$$

Since $\mathcal{V}_\perp^*$ has at most finitely many irreducible components, it follows that there is a finite set of distances $\{d_1^\perp, \ldots, d_p^\perp\}$ such that $\mathcal{V}_\perp^*$ is contained in the union the spheres centered at the origin and with radius $d_i^\perp$ for $i = 1, \ldots, p$.

We will use the notation $\Upsilon(\mathcal{V}_\perp) = \{d_1^\perp, \ldots, d_p^\perp\}$, and we will say that $\Upsilon(\mathcal{V}_\perp)$ is the set of critical distances of $\mathcal{V}$.

1.2 The Generic Offset and the Degree Problem

We have seen in the previous section the definition (see Definition 1.2, page 10) of the classical offset $\mathcal{O}_{d^o}(\mathcal{V})$ to a hypersurface $\mathcal{V}$ for a fixed value $d^o \in \mathbb{K}^\times$. In Subsection 1.2.1 the notion of generic offset is formally introduced, and its basic properties are derived; as we will see, many of them correspond with those of the classical offset. We will also define the generic offset polynomial (Definition 1.21, page 20), and we will analyze some fundamental properties of this polynomial. In particular, in Theorem 1.24 (page 21) we will show how this polynomial relates to the classical offset. In Subsection 1.2.2 we describe the Degree Problem for the Generic Offset, which is the basic subject of study in this thesis, and we present the associated terminology. The final Subsection 1.2.3 (page 24) aims at showing the role played by the Zariski closure operation in the offset construction, by using Projective Elimination Theory.

1.2.1 Formal definition and basic properties of the generic offset

As the distance value d^o varies, different offset varieties are obtained. The idea is to have a global expression of the offset for all (or almost all) distance values. This motivates the concept of generic polynomial of the offset to $\mathcal{V}$. This is a polynomial, depending on the distance variable d, such that for every (or almost every, see the examples below) non-zero value d^o, the polynomial specializes to the defining polynomial of the offset at that particular distance. Let us see a couple of examples that give some insight into the situation.

Example 1.17.

(a) *Using this informal definition of generic offset polynomial, and using Gröbner basis techniques, one can see that if $\mathcal{C}$ is the parabola of equation $y_2 - y_1^2 = 0$, the generic polynomial of its offset is:*
$g(d, x_1, x_2) = -48\, d^2 {x_1}^4 - 32\, d^2 {x_1}^2 {x_2}^2 + 48\, d^4 {x_1}^2 + 16\, {x_1}^6 + 16\, {x_2}^2 {x_1}^4 + 16\, d^4 {x_2}^2 - 16\, d^6 - 40\, x_2\, {x_1}^4 - 32\, {x_1}^2 {x_2}^3 + 8\, d^2 x_2\, {x_1}^2 - 32\, d^2 {x_2}^3 + 32\, d^4 x_2 + {x_1}^4 + 32\, x1^2 {x_2}^2 + 16\, {x_2}^4 - 20\, d^2 {x_1}^2 - 8\, d^2 {x_2}^2 - 8\, d^4 - 2\, x_2\, {x_1}^2 - 8\, {x_2}^3 + 8\, x_2\, d^2 + {x_2}^2 - d^2.$
In addition, and using again Gröbner basis techniques, one may check that for every distance the generic offset polynomial specializes properly (see Example 1.26 in page 22 below, for a detailed description of this example and the preceding claims).

(b) *On the other hand, the generic offset polynomial of the circle of equation $y_1^2 + y_2^2 - 1 = 0$ factors as the product of two circles of radius $1 + d$ and $1 - d$; that is:*

$$g(d, x_1, x_2) = \left(x_1^2 + x_2^2 - (1 + d)^2\right)\left(x_1^2 + x_2^2 - (1 - d)^2\right).$$

Now, observe that for $d^o = 1$, this generic polynomial gives

$$g(1, x_1, x_2) = \left(x_1^2 + x_2^2 - 2^2\right)\left(x_1^2 + x_2^2\right) = \left(x_1^2 + x_2^2 - 2^2\right)(x_1 + ix_2)(x_1 - ix_2)$$

which describes the union of a circle of radius 2, and two complex lines. This is not a correct representation of the offset at distance 1 to $\mathcal{C}$, which consists of the union of the circle of radius 2 and a point (the origin). In fact, using Gröbner basis techniques, one has that the elimination ideal $\tilde{I}(1)$ (see Remark 1.3(2), page 10) is:

$$\tilde{I}(1) = < x_2(x_1^2 + x_2^2 - 4), x_1(x_1^2 + x_2^2 - 4) > .$$

Thus, in this example we see that the generic offset polynomial does not specialize properly for $d^o = 1$. Nevertheless, for every other value of d^o the specialization is correct.

After these examples, we proceed to formally introduce the notion of generic offset and generic offset polynomial. The idea is to follow the steps in the definition of the classical offset. That is, we consider the following generalization of System $\mathfrak{S}_1(d^o)$ (see System 1.1 in page 10):

$$
\left.\begin{array}{lr}
 & f(\bar{y}) = 0 \\
\mathrm{nor}_{(i,j)}(\bar{x}, \bar{y}): & f_i(\bar{y})(x_j - y_j) - f_j(\bar{y})(x_i - y_i) = 0 \\
(\text{for } i, j = 1, \ldots, n; i < j) & \\
b(d, \bar{x}, \bar{y}): & (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 - d^2 = 0 \\
w(\bar{y}, u): & u \cdot (\|\nabla f(\bar{y})\|^2) - 1 = 0
\end{array}\right\} \equiv \mathfrak{S}_1(d) \quad (1.3)
$$

The above system will be called the Generic Offset System.

Note that here we consider d as a new variable, so that $b \in \mathbb{K}[d, \bar{x}, \bar{y}]$. Note also that the notation for this system has been chosen to make the classical offset System 1.1 appear as a specialization of this one. A solution of this system is thus a point of the form $(d^o, \bar{x}^o, \bar{y}^o, u^o) \in \mathbb{K}^{2n+2}$.

Let $\Psi(\mathcal{V}) \subset \mathbb{K} \times \mathbb{K}^n \times \mathbb{K}^n \times \mathbb{K}$ be the set of solutions of $\mathfrak{S}_1(d)$. In this case we can also consider the corresponding incidence diagram (compare with (1.2) in page 10):

Generic Offset Incidence Diagram

$$
\begin{tikzcd}
 & \Psi(\mathcal{V}) \subset \mathbb{K} \times \mathbb{K}^n \times \mathbb{K}^n \times \mathbb{K} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathcal{A}(\mathcal{V}) \subset \mathbb{K}^{n+1} & & \mathbb{K} \times \mathcal{V} \subset \mathbb{K}^{n+1}
\end{tikzcd}
\quad (1.4)
$$

where

$$
\begin{cases} \pi_1 : \mathbb{K}^{2n+2} \mapsto \mathbb{K}^{n+1} \\ \pi_1(d, \bar{x}, \bar{y}, u) = (d, \bar{x}) \end{cases} \quad \text{and} \quad \begin{cases} \pi_2 : \mathbb{K}^{2n+2} \mapsto \mathbb{K}^{n+1} \\ \pi_2(d, \bar{x}, \bar{y}, u) = (d, \bar{y}) \end{cases}
$$

and $\mathcal{A}(\mathcal{V}) = \pi_1(\Psi(\mathcal{V}))$.

Recall that we denote by $\mathcal{A}^*$ the Zariski closure of a set $\mathcal{A}$. Then one has the following definition:

Definition 1.18. *The* generic offset *to $\mathcal{V}$ is*

$$\mathcal{O}_d(\mathcal{V}) = \mathcal{A}(\mathcal{V})^* = \pi_1(\Psi(\mathcal{V}))^* \subset \mathbb{K}^{n+1}$$

Remark 1.19.

1. *Let*

$$I(d) =< f(\bar{y}), b(d,\bar{x},\bar{y}), \mathrm{nor}_{(1,2)}(\bar{x},\bar{y}), \ldots, \mathrm{nor}_{(n-1,n)}(\bar{x},\bar{y}), w(\bar{y},u) >$$

be the ideal in $\mathbb{K}[d,\bar{x},\bar{y},u]$ *generated by the polynomials in System 1.3. Note that the above definition implies that*

$$\mathcal{O}_d(\mathcal{V}) = \mathbf{V}(\tilde{I}(d))$$

where $\tilde{I}(d) = I(d) \cap \mathbb{K}[d,\bar{x}]$ *is the* $(\bar{y},u)$*-elimination ideal of* $I(d)$*.*

2. *The Closure Theorem from Elimination Theory (see e.g. Theorem 3 in page 122 of [14]) implies that the dimension of the set*

$$\mathcal{O}_d(\mathcal{V}) \setminus \pi_1(\Psi_1(\mathcal{V}))$$

is smaller than the dimension of $\mathcal{O}_d(\mathcal{V})$*. This is the set of points of the generic offset associated with singular or normal-isotropic points of* $\mathcal{V}$*.*

In the following Proposition we will see that the properties of the offset at a fixed distance, regarding its dimension and number of components (see Propositions 1.6 and 1.11 in page 12), are reflected in the generic offset. In particular, this Proposition shows that the generic offset is a hypersurface, and thus guarantees the existence of the generic polynomial (see below, Definition 1.21).

Proposition 1.20.

1. $\mathcal{O}_d(\mathcal{V})$ *has at most two components.*

2. *Each component of* $\mathcal{O}_d(\mathcal{V})$ *is a hypersurface in* $\mathbb{K}^{n+1}$*.*

Proof. (Adapted from Lemma 1, Theorem 1 and Theorem 2 in [49]). We begin by showing that if K is a component of $\Psi_1(\mathcal{V})$, then $\dim(K) = n$. Thus

$$\dim(\Psi_1(\mathcal{V})) = n \tag{1.5}$$

Let $\psi^o = (d^o, \bar{x}^o, \bar{y}^o, u^o) \in K$. Then, $\bar{y}^o \in \mathcal{V}$ is a regular point of $\mathcal{V}$. Let $\mathcal{P}(\bar{t})$, with $\bar{t} = (t_1, \ldots, t_{n-1})$, be a local parametrization of $\mathcal{V}$ at $\bar{y}^o$, with $\mathcal{P}(\bar{t}^o) = \bar{y}^0$. Then, it holds that one of the local parametrizations defined by:

$$\mathcal{P}^{\pm}(d,\bar{t}) = \left(d,\, \mathcal{P}(\bar{t}) \pm d\frac{\nabla f(\mathcal{P}(\bar{t}))}{\|\nabla f(\mathcal{P}(\bar{t}))\|},\, \mathcal{P}(\bar{t}),\, \frac{1}{\|\nabla f(\mathcal{P}(\bar{t}))\|^2}\right)$$

parametrizes $\Psi_1(\mathcal{V})$ locally at ψ^o (we choose sign so that $\mathcal{P}^{\pm}(\bar{t}^o) = \psi^o$). Since $(d, \mathcal{P}(\bar{t}))$ parametrizes $\mathbb{K} \times \mathcal{V}$, we get that $(d,\bar{t})$ are algebraically independent, and so $\dim(K) = n$.

Now we can prove the first statement of the proposition. Since the number of components of $\mathcal{O}_d(\mathcal{V})$ is at most the number of components of $\Psi_1(\mathcal{V})$, one just only has to prove that $\Psi_1(\mathcal{V})$ has at most two components. Let us suppose that Γ_1, Γ_2 y Γ_3 are three different components of $\Psi_1(\mathcal{V})$ and let $Z = \pi_2(\Gamma_1) \cap \pi_2(\Gamma_2) \cap \pi_2(\Gamma_3)$, where π_2 is the projection of the incidence diagram 1.4. Then, it holds that $\dim(Z) = n$. Observe that if $\dim(Z) < n$ then $\dim(\mathcal{V} \setminus Z) = \dim(\bigcup_{i=1}^{3}(\mathcal{V} \setminus \pi_2(\Gamma_i))) = n$. Hence, at least one of the sets $\mathcal{V} \setminus \pi_2(\Gamma_i)$ is of dimension n, which is impossible since $\pi_2(\Gamma_i)$ are constructible sets of dimension n. On the other hand, it holds that $\dim(Z \cap \pi_2(\Gamma_i \cap \Gamma_j)) < n$ for $i < j$. Then

$$Z \setminus \bigcup_{i \neq j} (Z \cap \pi_2(\Gamma_i \cap \Gamma_j)) \neq \emptyset.$$

Now, take $\bar{p} = (d^o, \bar{y}^o) \in Z \setminus \bigcup_{i \neq j}(Z \cap \pi_2(\Gamma_i \cap \Gamma_j))$, then $\pi_2^{-1}(\bar{p}) = \{\bar{q}_1, \bar{q}_2, \bar{q}_3\}$ where $\bar{q}_i \neq \bar{q}_j$ for $i < j$, which is impossible since the mapping π_2 is $(2:1)$ on $\pi_2(\Psi_1(\mathcal{V}))$.

Finally we can prove statement 2 in the proposition. We analyze the dimension of the tangent space to a component of the generic offset. Let $(d^o, \bar{y}^o) \in \pi_2(\Psi_1(\mathcal{V}))$, such that the two points $(d^o, \bar{x}_1^o), (d^o, \bar{x}_2^o) \in \mathcal{O}_d(\mathcal{V})$ generated by $(d^o, \bar{y}^o)$ satisfy that the dimension of their tangent spaces is the dimension of the corresponding component of $\mathcal{O}_d(\mathcal{V})$. Let $u^o = \dfrac{1}{\|\nabla f(\bar{y}^o)\|^2}$, and let $\mathcal{P}(\bar{t})$ be a local parametrization of $\mathcal{V}$ at $\bar{x}^o$. Then, it holds that :

$$\tilde{\mathcal{P}}^+(d, \bar{t}) = \left(d,\, \mathcal{P}(\bar{t}) + d\frac{\nabla f(\mathcal{P}(\bar{t}))}{\|\nabla f(\mathcal{P}(\bar{t}))\|}\right) \quad \text{and} \quad \tilde{\mathcal{P}}^-(d, \bar{t}) = \left(d,\, \mathcal{P}(\bar{t}) - d\frac{\nabla f(\mathcal{P}(\bar{t}))}{\|\nabla f(\mathcal{P}(\bar{t}))\|}\right)$$

parametrize locally $\mathcal{O}_d(\mathcal{V})$ at $(d^o, \bar{x}_1^o)$, and $(d^o, \bar{x}_2^o)$. In this situation, let $Q^{\pm}$ be as above, and consider the following map:

$$\begin{array}{ccccccccc} \psi^+ : & \mathbb{K}^n & \longrightarrow & \Psi_1(\mathcal{V}) & \xrightarrow{\varphi^+} & \mathcal{A}(\mathcal{V}) & \xhookrightarrow{i} & \mathbb{K}^{n+1} \\ & (d, \bar{t}) & \longrightarrow & \mathcal{P}^+(d, \bar{t}) & \longrightarrow & \tilde{\mathcal{P}}^+(d, \bar{t}) & \longrightarrow & \tilde{\mathcal{P}}^+(d, \bar{t}). \end{array}$$

Similarly, we define ψ^- and φ^-. Now consider the following homomorphism, defined by the differential $d\psi^+$ (similarly for $d\psi^-$), between the tangent space to $\Psi_1(\mathcal{V})_1$ at $(d^o, \bar{x}_1^o, \bar{y}^o, u^o)$ and the tangent space $\mathcal{T}_{(d^o, \bar{x}_1^o)}$ to $\mathcal{A}(\mathcal{V})_1$ at $(d^o, \bar{x}_1^o)$, where $\Psi_1(\mathcal{V})_1$ and $\mathcal{A}(\mathcal{V})_1$ denote the component of $\Psi_1(\mathcal{V})$ and $\mathcal{A}(\mathcal{V})$ containing the points $(d^o, \bar{x}_1^o, \bar{y}^o, u^o)$ and $(d^o, \bar{x}_1^o)$, respectively. Then one has that

$$\dim(\mathcal{A}(\mathcal{V})_1) \geq \dim(\mathcal{T}_{(d^o, \bar{x}_1^o)}) \geq \dim(\mathrm{Im}(d\varphi^+)) = \mathrm{rank}(\mathcal{J}_{\varphi^+}),$$

where $\mathcal{J}_{\varphi^+}$ denotes the jacobian matrix of φ^+. Furthermore, by Equation 1.5 at the beginning of this proof, one has that

$$n = \dim(\Psi_1(\mathcal{V})_1) \geq \dim(\mathcal{A}(\mathcal{V})_1) \geq \mathrm{rank}(\mathcal{J}_{\varphi^+}).$$

On the other hand, if we take any point of the form $(0, \bar{t}^o) \in \mathbb{K}^n$, that is, with $d^o = 0$, we must get $\operatorname{rank}(\mathcal{J}_{\varphi^+}) = n$ at that point; otherwise, one would conclude that the rank of the jacobian of $\mathcal{P}(\bar{t})$ is smaller than $n-1$, which is impossible since $\mathcal{V}$ is a hypersurface. □

As a first consequence of this Proposition, $\mathcal{O}_d(\mathcal{V})$ is defined by a polynomial $g(d, \bar{x}) \in \mathbb{K}[d, \bar{x}]$ (see [53], p.69, Theorem 3). Thus, we arrive at the following definition:

Definition 1.21. *The* generic offset polynomial *is the defining polynomial of the hypersurface $\mathcal{O}_d(\mathcal{V})$. In the sequel,* we denote *by $g(d, \bar{x})$ the generic offset equation.*

The first property of the generic offset polynomial that we study regards its factorization:

Lemma 1.22. *The generic offset polynomial is primitive w.r.t. $\bar{x}$*

Proof. Suppose, on the contrary, that $g(d, \bar{x})$ has a non-constant factor in $\mathbb{K}[d]$. That is

$$g(d, \bar{x}) = A(d)\tilde{g}(d, \bar{x})$$

Let $d^o \neq 0$ be any root of $A(d)$. Then the hypersurface $\mathcal{Z}$ in $\mathbb{K} \times \mathbb{K}^n$ defined by $d = d^o$ is contained in $\mathcal{O}_d(\mathcal{V})$. Taking Remark 1.19(2) (page 18) into account, one has that there is an open non-empty subset of $\mathcal{Z}$ contained in $\pi_1(\Psi_1(\mathcal{V}))$. This in turn implies that there is an open subset $\tilde{\mathcal{Z}}$ of $\mathbb{K}^n$ such that if $\bar{x}^o \in \tilde{\mathcal{Z}}$, then $\bar{x}^o \in \mathcal{O}_{d^o}(\mathcal{V})$ (the classical offset at distance d^o). This is a contradiction, since we know that $\mathcal{O}_{d^o}(\mathcal{V})$ has dimension less or equal to $n-1$. Thus, we are left with the case when $A(d)$ is a power of d. The argument must be different in this case, since the classical offset is only defined for $d^o \in \mathbb{K}^\times$. However, the reasoning is similar: we conclude that there is an open non-empty subset $\tilde{\mathcal{Z}}_0$ of $\mathbb{K}^n$ such that if $\bar{x}^o \in \tilde{\mathcal{Z}}_0$, then the system ("classical offset system for distance 0")

$$\left.\begin{array}{l} f(\bar{y}) = 0 \\ f_i(\bar{y})(x_j^o - y_j) - f_j(\bar{y})(x_i^o - y_i) = 0 \\ (\text{for } i, j = 1, \ldots, n; i < j) \\ (x_1^o - y_1)^2 + \cdots + (x_n^o - y_n)^2 = 0 \\ u \cdot (\|\nabla f(\bar{y})\|^2) - 1 = 0 \end{array}\right\} \equiv \mathfrak{S}_1(d)$$

has solutions. Now, if $(\bar{y}^o, u^o)$ is a solution of this system then

1. $\nabla f(\bar{y}^o)$ is not isotropic,

2. $\bar{y}^o - \bar{x}^o$ is isotropic,

3. and $\nabla f(\bar{y}^o)$ is parallel to $\bar{y}^o - \bar{x}^o$,

Thus one has that $\bar{y}^o - \bar{x}^o = 0$. But since $\bar{x}^o$ runs through an open subset of $\mathbb{K}^n$, this contradicts the fact that $\mathcal{V}$ is a hypersurface. □

Remark 1.23.

1. *Observe that the polynomial g may be reducible (recall the example of the circle) but by construction it is always square-free. Moreover, by Proposition 1.20 (page 18) and Lemma 1.22, g is either irreducible or factors into two irreducible factors not depending only on d.*

2. *We will also call $g(d, \bar{x}) = 0$ the* generic offset equation *of $\mathcal{V}$.*

The following theorem gives the fundamental property of the generic offset.

Theorem 1.24. *For all but finitely many exceptions, the generic offset polynomial specializes properly. That is, there exists a finite (possibly empty) set $\Delta_2 \subset \mathbb{K}$ such that if $d^o \notin \Delta_2$, then*

$$g(d^o, \bar{x}) = 0$$

is the equation of $\mathcal{O}_{d^o}(\mathcal{V})$.

Proof. Let $G(d)$ be a reduced Gröbner basis of $I(d)$ w.r.t. an elimination ordering that eliminates $(\bar{y}, u)$. Then, up to multiplication by a non-zero constant, $G(d) \cap \mathbb{K}[d, \bar{x}]$ is a Gröbner basis of $\tilde{I}(d)$. Proposition 1.20 above shows that $\tilde{G}(d) = G(d) \cap \mathbb{K}[d, \bar{x}] =< \nu(d)g(d, \bar{x}) >$, where $\nu(d)$ is a non-zero polynomial, depending only on d (see the Remark preceding this proof). But then (see [14], exercise 7, page 283) there is a finite (possibly empty) set $\Delta_2^1 \subset \mathbb{K}$ such that for $d^o \notin \Delta_2^1$, $G(d)$ specializes well to a Gröbner basis of $I(d^o)$ (defined in Remark 1.3, page 10). It follows that, since $\tilde{I}(d^o) = I(d^o) \cap \mathbb{K}[\bar{x}]$, then $\tilde{G}(d^o) = \{\nu(d^o)g(d^o, \bar{x})\}$ is a Gröbner basis of $\tilde{I}(d^o)$. In particular, if Δ_2^2 is the finite set of zeros of $\nu(d)$, then for $d^o \notin \Delta_2 = \Delta_2^1 \cup \Delta_2^2$, and $d^o \neq 0$, one has that $g(d^o, \bar{x})$ is the equation for $\mathcal{O}_{d^o}(\mathcal{V})$. □

For future reference, we collect in the following corollary all the information about the –finite– set of *bad* distances that appear in the offsetting construction.

Corollary 1.25. *There is a finite set $\Delta \subset \mathbb{K}^\times$ such that for $d^o \notin \Delta$, the following hold:*

(1) (non degeneracy): $\mathcal{O}_{d^o}(\mathcal{V})$ is not degenerated.

(2) (simplicity): $\mathcal{O}_{d^o}(\mathcal{V})$ is simple.

(3) (good specialization): if $g(d, \bar{x}) = 0$ is the generic offset polynomial, $g(d^o, \bar{x}) = 0$ is the equation of $\mathcal{O}_{d^o}(\mathcal{V})$.

(4) (degree invariance):

$$\deg_{\bar{x}}(\mathcal{O}_d(\mathcal{V})) = \deg_{\bar{x}}(\mathcal{O}_{d^o}(\mathcal{V})), \quad \deg_{x_i}(\mathcal{O}_d(\mathcal{V})) = \deg_{x_i}(\mathcal{O}_{d^o}(\mathcal{V})) \textit{ for } i = 1, \ldots, n.$$

Proof. Take $\Delta^1 = \Delta_0 \cup \Delta_1 \cup \Delta_2$, with Δ_0 as in Proposition 1.10, Δ_1 as in Proposition 1.11(4) and Δ_2 as in Theorem 1.24 above. Furthermore, let $p(d)\bar{x}^\mu$ be a term of $g(d, \bar{x})$ of maximal degree w.r.t. $\bar{x}$. That is, $\bar{\mu} = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$, with $\sum \mu_i = \deg_{\bar{x}}(g)$, where $p(d) \in \mathbb{C}[d]$ is a non-zero polynomial. Then take:

$$\Delta^{\bar{x}} = \Delta \cup \{d^o \in \mathbb{C} \,|\, p(d^o) = 0\},$$

and similarly, for $i = 1, \ldots, n$ construct $\Delta^{\bar{x}_i}$, by considering a term of $g(d, \bar{x})$ of maximal degree w.r.t x_i. Finally, taking

$$\Delta = \Delta^1 \cup \Delta^{\bar{x}} \cup \Delta^{\bar{x}_1} \cdots \cup \Delta^{\bar{x}_n},$$

our claim holds. □

Let us see a first example of a generic offset polynomial.

Example 1.26. *For the parabola $\mathcal{C}$ with defining polynomial $f(y_1, y_2) = y_2 - y_1^2$, the generic offset system turns into:*

$$\left.\begin{array}{lr} & f(\bar{y}) = y_2 - y_1^2 \\ \mathrm{nor}_{(1,2)}(\bar{x}, \bar{y}): & -2y_1(x_2 - y_2) - (x_1 - y_1) = 0 \\ b(d, \bar{x}, \bar{y}): & (x_1 - y_1)^2 + (x_2 - y_2)^2 - d^2 = 0 \\ w(\bar{y}, u): & u \cdot (\|4y_1^2 + 1\|^2) - 1 = 0 \end{array}\right\}$$

Computing a Gröbner elimination basis of $I(d) =< f, \mathrm{nor}_{(1,2)}, b, w >$, we obtain (with the notation in the proof of Theorem 1.24):

$$G(d) = \{g(d, \bar{x}), \chi_1(d, \bar{x}), \ldots, \chi_8(d, \bar{x})\}$$

where:
$g(d, \bar{x}) = 16\,{x_1}^6 + 16\,{x_1}^4{x_2}^2 - 40\,{x_1}^4x_2 - 32\,{x_1}^2{x_2}^3 + (-48\,d^2 + 1)\,{x_1}^4 + (-32\,d^2 + 32)\,{x_1}^2{x_2}^2 + 16\,{x_2}^4 + (8\,d^2 - 2)\,{x_1}^2x_2 + (-32\,d^2 - 8)\,{x_2}^3 + (48\,d^4 - 20\,d^2)\,{x_1}^2 + (16\,d^4 - 8\,d^2 + 1)\,{x_2}^2 + (32\,d^4 + 8\,d^2)\,x_2 - 16\,d^6 - 8\,d^4 - d^2$
and
$\chi_1(d, \bar{x}) = 12\,d^2u{x_1}^2 + 16\,d^2u{x_2}^2 - 4\,d^2ux_2 + (-12\,d^4 + d^2)\,u - 4\,{x_1}^4 + 8\,{x_1}^2x_2 + 8\,d^2{x_1}^2 - 4\,{x_2}^2 + 4\,d^2x_2 - 4\,d^4 + 3\,d^2$
$\chi_2(d, \bar{x}) = 64\,d^2u{x_2}^3 - 48\,d^2u{x_1}^2 - 16\,d^2u{x_2}^2 + 28\,d^2ux_2 + (-60\,d^4 - 3\,d^2)\,u - 64\,{x_1}^4x_2 + 128\,{x_1}^2{x_2}^2 + 128\,d^2{x_1}^2x_2 - 64\,{x_2}^3 + 36\,d^2{x_1}^2 + 112\,d^2{x_2}^2 + (-64\,d^4 + 36\,d^2)\,x_2 - 36\,d^4 + 3\,d^2$
$\chi_3(d, \bar{x}) = 12\,y_2 - 16\,u{x_1}^2 - 16\,u{x_2}^2 + 8\,ux_2 + (16\,d^2 - 1)\,u - 8\,x_2 + 1$
$\chi_4(d, \bar{x}) = 12\,y_1x_2 + (-12\,d^2 - 3)\,y_1 - 8\,y_2x_1x_2 - 14\,y_2x_1 + 8\,{x_1}^3 + 8\,x_1{x_2}^2 - 6\,x_1x_2 +$

$(-8\,d^2+3)\,x_1$
$\chi_5(d,\bar{x}) = 3\,y_1x_1 + 2\,y_2x_2 - y_2 - 2\,x_1{}^2 - 2\,x_2{}^2 + 2\,d^2$
$\chi_6(d,\bar{x}) = 12\,d^2u^2 - 4\,ux_1{}^2 - 16\,ux_2{}^2 - 4\,ux_2 + (4\,d^2-1)\,u + 4\,x_2 + 1$
$\chi_7(d,\bar{x}) = 12\,d^2y_1u + 8\,y_2ux_1x_2 + 14\,y_2ux_1 - 3\,y_1 - 8\,ux_1{}^3 - 8\,ux_1x_2{}^2 + 6\,ux_1x_2 + (8\,d^2+3)\,ux_1$
$\chi_8(d,\bar{x}) = y_1{}^2 + y_2{}^2 - 2\,y_1x_1 - 2\,y_2x_2 + x_1{}^2 + x_2{}^2 - d^2.$
In particular,

$$G(d) \cap \mathbb{K}[d,\bar{x}] =< g(d,\bar{x}) > .$$

And so $g(d,\bar{x})$ is the generic offset polynomial for the parabola $\mathcal{C}$.
This Gröbner basis has been computed considering the generators of $I(d)$ as polynomials in $\mathbb{K}(d)[\bar{x},\bar{y},u]$. This means that we have relationships of the form:

$$g(d,\bar{x}) = a_1(d)f(\bar{x}) + a_2(d)\mathrm{nor}_{(1,2)}(\bar{x},\bar{y}) + a_3(d)b(d,\bar{x},\bar{y}) + a_4(d)w(\bar{y},u)$$

and for $i = 1,\ldots,8$:

$$\chi_i(d,\bar{x}) = b_{i1}(d)f(\bar{x}) + b_{i2}(d)\mathrm{nor}_{(1,2)}(\bar{x},\bar{y}) + b_{i3}(d)b(d,\bar{x},\bar{y}) + b_{i4}(d)w(\bar{y},u)$$

where $a_1,\ldots,a_4,b_{11},\ldots,b_{84} \in \mathbb{K}(d)$. The result in Exercise 7, in page 283 of [14] indicates that the Gröbner basis specializes well for all values d^o such that none of the denominators of the rational functions a_i and b_{ij} vanish at d^o. In this particular example, one may compute these rational functions and check that they are all constant. Therefore, specializing $g(d,\bar{x})$ provides the offset equation for every non-zero value of d. The computations in this example were obtained with the computer algebra system Singular (see [21]). We do not include here the details of the computations, because of obvious space limitations.

The following result, about the dependence on d of the generic offset polynomial, is an easy consequence of Theorem 1.24 above. In fact, Theorem 1.24 implies that there are infinitely many values d^o such that $g(d^o,\bar{x})$ is the polynomial of $\mathcal{O}_{d^o}(\mathcal{V})$ and, simultaneously, $g(-d^o,\bar{x})$ is the polynomial of $\mathcal{O}_{-d^o}(\mathcal{V})$. But, because of the symmetry in the construction, the offsets $\mathcal{O}_{d^o}(\mathcal{V})$ and $\mathcal{O}_{-d^o}(\mathcal{V})$ are exactly the same algebraic set. Thus, it follows that for infinitely many values of d^o it holds that up to multiplication by a non-zero constant:

$$g(d^o,\bar{x}) = g(-d^o,\bar{x}).$$

Hence, we have proved the following proposition:

Proposition 1.27. *The generic offset polynomial belongs to $\mathbb{K}[\bar{x}][d^2]$. That is, it only contains even powers of d.*

1.2.2 The degree problem for the generic offset

Now we are in a position to describe the main goals of this work. Let $g \in \mathbb{K}[d, \bar{x}]$ be, as before, the generic offset polynomial for $\mathcal{V}$. Then:

- The total degree problem consists of finding formulae to compute the total degree of g in the variables $\bar{x}$. We denote this total degree by δ.
- The partial degree problem consists of finding formulae to compute the partial degree of g w.r.t. x_i, for $i = 1, \ldots, n$. We denote each of these partial degrees as δ_i; i.e. $\delta_i = \deg_{x_i}(g)$.
- The distance degree problem consists of finding formulae to compute the degree of g w.r.t. the variable d. We denote this degree by δ_d.
- Finally, the –complete– offset degree problem refers to the problem of finding the whole set of values $\{\delta, \delta_1, \ldots, \delta_n, \delta_d\}$.

For a fixed value of the distance, the total (resp. partial in x_i) degree of the defining polynomial of $\mathcal{O}_{d^o}(\mathcal{V})$ will be denoted by δ^o (resp. δ_i^o). Thus, for $d^o \notin \Delta$ (Δ as in Corollary 1.25), we have $\delta^o = \delta, \delta_i^o = \delta_i$ for $i = 1, \ldots, n$. Note that Proposition 1.27 implies that δ_d is always even.

We will illustrate the above statements with an example.

Example 1.28. *In Example 1.26,see page 22, we have seen that for the parabola $\mathcal{C}$ with defining polynomial $f(y_1, y_2) = y_2 - y_1^2$, the generic offset polynomial is (we have underlined those terms that are relevant for the degree problem):*

$$g(d, \bar{x}) = \underline{16\, {x_1}^6} + 16\, {x_1}^4 {x_2}^2 - 40\, {x_1}^4 x_2 - 32\, {x_1}^2 {x_2}^3 + (-48\, d^2 + 1)\, {x_1}^4 + (-32\, d^2 + 32)\, \underline{{x_1}^2 {x_2}^2} + \underline{16\, {x_2}^4} + (8\, d^2 - 2)\, {x_1}^2 x_2 + (-32\, d^2 - 8)\, {x_2}^3 + (48\, d^4 - 20\, d^2)\, {x_1}^2 + (16\, d^4 - 8\, d^2 + 1)\, {x_2}^2 + (32\, d^4 + 8\, d^2)\, x_2 - \underline{16\, d^6} - 8\, d^4 - d^2$$

Thus, in this example the solution of the degree problem is:

$$\delta = 6,\ \delta_1 = 6,\ \delta_2 = 4, \delta_d = 6$$

1.2.3 The generic offset and projective elimination

As we have already mentioned, for $d^o \in \mathbb{K}^\times$, every solution $(\bar{x}^o, \bar{y}^o, u^o)$ of the system 1.1 (page 10) gives a point $\bar{x}^o$ on $\mathcal{O}_{d^o}(\mathcal{V})$, associated to the non normal-isotropic point $\bar{y}^o$ on $\mathcal{V}$. But, conversely, in general not every point on the offset corresponds to a solution of this system, since $\mathcal{O}_{d^o}(\mathcal{V})$ is the Zariski closure of $\mathcal{A}_{d^o}(\mathcal{V})$. To have a better picture, with the terminology of the generic offset incidence diagram 1.4 (page 17), we would like to have some more information about the fiber $\pi_1^{-1}(d^o, \bar{x}^o)$ for $(d^o, \bar{x}^o) \in \mathcal{O}_d(\mathcal{V}) \setminus \mathcal{A}(\mathcal{V})$.

It is well known in Elimination Theory that, in this situation, the points in $\mathcal{O}_{d^o}(\mathcal{V}) \setminus \mathcal{A}_{d^o}(\mathcal{V})$ come from the solutions at *infinity* of the system 1.1, where infinity in this context refers to the projective space associated to the variables being eliminated (see e.g. [14], Section 8.5). To make this observation more precise and useful, we need to consider the projectivization of some of the notions we have already introduced in an affine context. However, it turns out that the computation of the homogenization of the corresponding projective ideals is too difficult to be useful for our purposes. We will work instead with a smaller ideal, whose zero set gives under projection a superset of the generic offset. For this purpose, recall that

$$I(d) = < f(\bar{y}), b(d, \bar{x}, \bar{y}), \mathrm{nor}_{(i,j)}(\bar{x}, \bar{y}), w(\bar{y}, u) >,$$

and consider the ideal (in $\mathbb{K}[d, \bar{x}, \bar{y}, u]$)

$$I_1(d) = < f(\bar{y}), b(d, \bar{x}, \bar{y}), \mathrm{nor}_{(i,j)}(\bar{x}, \bar{y}) > \subset I(d).$$

Then for the $(\bar{y}, u)$-elimination ideals

$$\tilde{I}_1(d) = I_1(d) \cap \mathbb{K}[d, \bar{x}] \text{ and } \tilde{I}(d) = I(d) \cap \mathbb{K}[d, \bar{x}]$$

one has

$$\tilde{I}_1(d) \subset \tilde{I}(d)$$

and therefore $\mathcal{O}_d(\mathcal{V}) = \mathbf{V}(\tilde{I}(d)) \subset \mathbf{V}(\tilde{I}_1(d))$ (see Remark 1.19 in page 18). Thus, we may obtain some information about $\mathcal{O}_d(\mathcal{V})$ by studying $\mathbf{V}(\tilde{I}_1(d))$. To study this set, we observe that $\mathbf{V}(\tilde{I}_1(d)) = (\pi_1(\tilde{\mathcal{A}}(\mathcal{V})))^*$, where $\tilde{\mathcal{A}}(\mathcal{V}) \subset \mathbb{K} \times \mathbb{K}^n \times \mathbb{K}^n \times \mathbb{K}$ is the set of solutions of the following system:

$$\text{(for } i, j = 1, \ldots, n; i < j) \qquad \left.\begin{array}{r} f(\bar{y}) = 0 \\ f_i(\bar{y})(x_j - y_j) - f_j(\bar{y})(x_i - y_i) = 0 \\ (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 - d^2 = 0 \end{array}\right\} \equiv \mathfrak{S}_2(d) \quad (1.6)$$

and π_1 is as in System 1.4, page 17.

Remark 1.29. *In general, the set $\mathbf{V}(\tilde{I}_1(d))$ is different from $\mathcal{O}_d(\mathcal{V})$. In particular, for every $d^o \in \mathbb{K}^\times$, the spheres of radius d^o centered at every affine singularity of $\mathcal{V}$ are included in $\mathbf{V}(\tilde{I}_1(d)) \setminus \mathcal{O}_d(\mathcal{V})$.*

Next we consider the system of equations obtained by homogenizing w.r.t y_0 the variables $\bar{y}$ in System $\mathfrak{S}_2(d)$. Let $F(\bar{y}_h), \mathrm{Nor}_{(i,j)}(\bar{x}, \bar{y}_h)$ and $B(d, \bar{x}, \bar{y}_h)$ be the homogenization w.r.t y_0 of $f(\bar{y}), \mathrm{nor}_{(i,j)}(\bar{x}, \bar{y})$ and $b(d, \bar{x}, \bar{y})$ respectively. Then we consider the system:

$$\left.\begin{array}{lr} & F(\bar{y}_h) = 0 \\ \mathrm{Nor}_{(i,j)}(\bar{x}, \bar{y}_h): & F_i(\bar{y}_h)(y_0x_j - y_j) - F_j(\bar{y}_h)(y_0x_i - y_i) = 0 \\ (\text{for } i, j = 1, \ldots, n; i < j) & \\ B(d, \bar{x}, \bar{y}_h): & (y_0x_1 - y_1)^2 + \cdots + (y_0x_n - y_n)^2 - y_0^2 d^2 = 0 \end{array}\right\} \equiv \mathfrak{S}_2^h(d) \tag{1.7}$$

as well as the ideal $J(d)$ generated by $\{F(\bar{y}_h), \mathrm{Nor}_{(i,j)}(\bar{x}, \bar{y}_h), B(d, \bar{x}, \bar{y}_h)\}$ over the ring $\mathbb{K}[d, \bar{x}, \bar{y}_h]$. The ideal $J(d)$ should not be confused with the $\bar{y}$-homogenization $I_1^h(d)$ of the ideal $I_1(d)$. In fact one has $J(d) \subset I_1^h(d)$, but equality is not guaranteed. This relation is carried onto the elimination ideals, so that

$$J(d) \cap \mathbb{K}[d, \bar{x}] \subset \tilde{I}_1(d).$$

In Proposition 1.30 below we will make precise the ideas about solutions at infinity at the beginning of this subsection. Let $\mathcal{J}(\mathcal{V}) = \mathbf{V}(J(d)) \subset \mathbb{K}^{n+1} \times \mathbb{P}^n$ be the set of solutions of $\mathfrak{S}_2^h(d)$ in $\mathbb{K}^{n+1} \times \mathbb{P}^n$. Then we have this the projective incidence diagram:

Generic Offset Projective Incidence Diagram

$$\begin{tikzcd} & \mathcal{J}(\mathcal{V}) \subset \mathbb{K}^{n+1} \times \mathbb{P}^n \arrow[dl, "\pi_1^h"'] \arrow[dr, "\pi_2^h"] & \\ \pi_1^h(\mathcal{J}(\mathcal{V})) \subset \mathbb{K}^{n+1} & & \pi_2^h(\mathcal{J}(\mathcal{V})) \subset \mathbb{P}^n \end{tikzcd} \tag{1.8}$$

where $\begin{cases} \pi_1^h : \mathbb{K}^{n+1} \times \mathbb{P}^n \mapsto \mathbb{K}^{n+1} \\ \pi_1^h(d, \bar{x}, \bar{y}_h) = (d, \bar{x}) \end{cases}$ and $\begin{cases} \pi_2^h : \mathbb{K}^{n+1} \times \mathbb{P}^n \mapsto \mathbb{P}^n \\ \pi_2^h(d, \bar{x}, \bar{y}_h) = (\bar{y}_h) \end{cases}$

In this situation, one has the following result:

Proposition 1.30. *Since* $\mathbf{V}(I_1^h(d)) \subset \mathbf{V}(J(d)) = \mathcal{J}(\mathcal{V})$, *then*

$$\mathcal{O}_d(\mathcal{V}) \subset \pi_1^h(\mathcal{J}(\mathcal{V})).$$

Proof. Note that

$$\mathcal{O}_d(\mathcal{V}) = \mathbf{V}(\tilde{I}(d)) \subset \mathbf{V}(\tilde{I}_1(d)) = \pi_1^h(\mathbf{V}(I_1^h(d))) \subset \pi_1^h(\mathcal{J}(\mathcal{V})).$$

The equality $\mathbf{V}(\tilde{I}_1(d)) = \pi_1^h(\mathbf{V}(I_1^h(d)))$ is a standard result of Projective Elimination Theory, see e.g. Corollary 10 in page 394 of [14], combined with Theorem 3 in page 191 of the same authors. □

Let us note that the computation of the ideal $I_1^h(d)$ would give a more precise description of the generic offset. Unfortunately, $I_1^h(d)$ is hard to obtain from $I(d)$. But nevertheless, the relationship $\mathcal{O}_d(\mathcal{V}) \subset \pi_1^h(\mathcal{J}(\mathcal{V}))$ allows us to draw some conclusions about the generic offset from the ideal $J(d)$. The first conclusion is that, for every $(d^o, \bar{x}^o) \in \mathcal{O}_d(\mathcal{V})$, there is $\bar{y}_h^o \in \mathbb{P}^n$, such that $(d^o, \bar{x}^o, \bar{y}_h^o) \in \mathcal{J}(\mathcal{V})$. This leads to a generalization of the notion of associated points (see Remark 1.3 in page 10); note that $\pi_2^h(\mathcal{J}(\mathcal{V})) = \overline{\mathcal{V}}$.

Definition 1.31. *If $(d^o, \bar{x}^o) \in \mathcal{O}_d(\mathcal{V})$ and $(d^o, \bar{x}^o, \bar{y}_h^o) \in \mathcal{J}(\mathcal{V})$, we will say that $\bar{x}^o$ and $\bar{y}_h^o \in \overline{\mathcal{V}}$ are* associated points at the distance *d^o.*

We have seen (Remark 1.19(2), page 18) that the dimension of the set $\mathcal{O}_d(\mathcal{V})\backslash\pi_1(\Psi_1(\mathcal{V}))$ is smaller than the dimension of $\mathcal{O}_d(\mathcal{V})$. We can now give a more detailed description of this set. Thus, let $(d^o, \bar{x}^o) \in \mathcal{O}_d(\mathcal{V}) \setminus \pi_1(\Psi_1(\mathcal{V}))$, with $d^o \neq 0$, and let us suppose that $\bar{x}^o$ is associated with $\bar{y}_h^o$ at d^o. Then we have the following three possibilities:

1. $\bar{y}_h^o$ is an affine singularity of $\overline{\mathcal{V}}$.

2. let us suppose that $\bar{y}_h^o$ is an affine (thus $y_0^o = 1$) normal-isotropic point of $\overline{\mathcal{V}}$, and it is not a singularity. Then $\nabla F(\bar{y}_h^o)$ is a non-zero isotropic vector, and the $\mathrm{Nor}_{(i,j)}$ equations of $\mathfrak{S}_2^h(d)$ imply that $(y_0^o x_1^o - y_1^o, \ldots, y_n^o x_1^o - y_n^o)$ is also isotropic. Therefore, from $B(d^o, \bar{x}^o, \bar{y}_h^o) = 0$ one has that $y_0^o d^o = d^o = 0$. Since $d^o \in \mathbb{K}^\times$, one gets a contradiction. Therefore, $\bar{x}^o$ cannot be associated with affine non-singular normal isotropic points of $\mathcal{V}$.

3. Finally, let us suppose that $\bar{y}_h^o \in \overline{\mathcal{V}}$ is a point at infinity; that is, with $y_0^o = 0$. Then the last equation in $\mathfrak{S}_2^h(d)$ gives $y_1^2 + \cdots + y_n^2 = 0$. Thus, $\bar{y}_h^o$ must be isotropic. Furthermore, the equations $\mathrm{Nor}_{(i,j)}$ then imply that $\bar{y}_h^o$ is also normal-isotropic.

The set of points $\bar{y}_h^0 \in \overline{\mathcal{V}}$ lying at infinity, and being simultaneously isotropic and normal-isotropic is empty for many hypersurfaces $\mathcal{V}$. It is convenient to summarize the above results in the form of a proposition:

Proposition 1.32. *If the system*

$$\begin{cases} F(0, y_1, \ldots, y_n) = 0 \\ F_i(0, y_1, \ldots, y_n) y_j - F_j(0, y_1, \ldots, y_n) y_i = 0, \ (\textit{for } i, j = 1, \ldots, n; i < j) \\ y_1^2 + \cdots + y_n^2 = 0 \end{cases} \tag{1.9}$$

has no non-zero solutions, every point in the generic offset with $d^o \neq 0$ is associated with affine (regular) non normal-isotropic points of $\mathcal{V}$ or with affine singularities of $\mathcal{V}$.

1.3 Intersection of Curves and Resultants

In the following chapters we will show that in the planar curve case and in the parametric surface case, we can translate the offset degree problem into of a suitably constructed planar curves intersection problem. In this section we gather some results about the planar curves intersection problem to be used in the sequel.

It is well known that the intersection points of two plane curves, without common components, as well as their multiplicity of intersection, can be computed by means of resultants. For this, a suitable preparatory change of coordinates may be required (see for instance, [10], [56] and, for a modern treatment of the subject, [52]). In this work, for reasons that will turn out to be clear in subsequent sections and chapters, we need to analyze the behavior of the resultant factors, and their correspondence with multiplicities of intersection, when some of the standard requirements are not satisfied. Similarly, we also need to analyze the case when more than two curves are involved.

More precisely, we present three technical lemmas. The first one, Lemma 1.33, is devoted to the degree of the univariate resultant w.r.t. y_0 of two homogeneous polynomials in $\mathbb{K}[y_0, y_1, y_2]$. This lemma extends the well known result about the degree of the resultant of two homogeneous polynomials (see e.g. Theorem 10.9 in page 30 of [56]), to the case in which the polynomials are not necessarily general in the variable used for computing the resultant. The lemma addresses the common situation in which one has the affine equations of two plane curves, with degrees m, n respectively. Then, if they are homogenized w.r.t. y_0, it is not always the case that the resulting homogeneous polynomials have degrees in y_0 equal to m and n respectively. Thus, it is convenient to be able to relate the degree of the resultant with the degrees m and n. We will use this lemma to analyze the behavior of the degree of the resultant of two polynomials, when specializing some parameters in those polynomials.

The second lemma, Lemma 1.34, shows that, under certain conditions, the multiplicity of intersection is reflected in the factors appearing in the resultant, even though the curves are not properly set. In particular, the requirement that no two intersection points lie on a line through the origin can be relaxed, obtaining in this case the total multiplicity of intersection along that line. We will explain the role played by this lemma when we will discuss the total degree formula for plane curves in Chapter 2.

The third lemma, Lemma 1.35, is a generalization of Corollary 1 in [33]. It shows that generalized resultants can be used to study the intersection points of a finite family of curves. This last lemma will be applied in Chapter 4 to the case of surfaces.

Lemma 1.33. *Let*

$$\begin{cases} F(y_0, y_1, y_2) = a_n(y_1, y_2) + a_{n-1}(y_1, y_2)y_0 + a_{n-2}(y_1, y_2)y_0^2 + \cdots + a_{n-k}(y_1, y_2)y_0^k \\ G(y_0, y_1, y_2) = b_m(y_1, y_2) + b_{m-1}(y_1, y_2)y_0 + b_{m-2}(y_1, y_2)y_0^2 + \cdots + b_{m-p}(y_1, y_2)y_0^p \end{cases}$$

be two homogeneous polynomials, with $\gcd(F, G) = 1$, *and* $k, p > 0$, *where* a_i, b_i *are homogeneous polynomials of degree* i *in* y_1, y_2, *and such that* $a_{n-k}b_{m-p} \neq 0$. *Then it holds that*

$$\deg_{\{y_1,y_2\}}(\mathrm{Res}_{y_0}(F, G)) = np + mk - kp.$$

Proof. Let us denote by $R(y_1, y_2) = \mathrm{Res}_{y_0}(F, G)$. The Sylvester matrix $(s_{i,j})_{1\le i,j\le p+k}$ of F and G w.r.t. y_0 is

$$\begin{pmatrix}
a_{n-k} & & & & b_{m-p} & & & \\
a_{n-(k+1)} & a_{n-k} & & & b_{m-(p+1)} & b_{m-p} & & \\
 & a_{n-(k+1)} & \ddots & & & b_{m-(p+1)} & \ddots & \\
\vdots & & \ddots & a_{n-k} & \vdots & & \ddots & b_{m-p} \\
 & \vdots & & a_{n-(k+1)} & & \vdots & & b_{m-(p+1)} \\
a_n & & & & b_m & & & \\
 & a_n & & \vdots & & b_m & & \vdots \\
 & & \ddots & & & & \ddots & \\
 & & & a_n & & & & b_m
\end{pmatrix}$$

Thus, the (i, j)-element $s_{i,j}$ of this matrix is 0 or it is a homogeneous polynomial of degree d_{ij}, where:

$$d_{ij} = \begin{cases} (n-k) + i - j & \text{if } 1 \le j \le p \\ (m-p) + i - (j-p) & \text{if } p+1 \le j \le p+k \end{cases}$$

Therefore, $R(y_1, y_2)$ is a sum of products of the form

$$\prod s_{\sigma(1)1} s_{\sigma(1)1} \cdots s_{\sigma(p+k)(p+k)}$$

where σ is a permutation of the set $\{1, \ldots, (p+k)\}$.

Since $\gcd(F, G) = 1$, the resultant is a non-zero sum of homogeneous polynomials of degree

$$\begin{aligned}
&\sum_{j=1}^{p+k} d_{\sigma(j)j} = \sum_{j=1}^{p} d_{\sigma(j)j} + \sum_{j=p+1}^{p+k} d_{\sigma(j)j} = \\
&\sum_{j=1}^{p} (n-k) + \sigma(j) - j + \sum_{j=p+1}^{p+k} (m-p) + \sigma(j) - (j-p) = \\
&= (n-k)p + (m-p)k + pk + \textstyle\sum_{j=1}^{p+k} \sigma(j) - \sum_{j=1}^{p+k} j = np + km - kp
\end{aligned}$$

□

As we have mentioned in the introduction to this section, the multiplicity of intersection of two projective plane curves can be read at the resultant of their defining polynomials. In fact, this is often used to define the multiplicity of intersection. More precisely (see [52]), let $\mathcal{C}_1$ and $\mathcal{C}_2$ be projective plane curves, without common components, such that $(1:0:0) \notin \mathcal{C}_1 \cup \mathcal{C}_2$, and $(1:0:0)$ does not belong to any line connecting two points in $\mathcal{C}_1 \cap \mathcal{C}_2$. Let $F(y_0, y_1, y_2)$, resp. $G(y_0, y_1, y_2)$, be the defining polynomials of $\mathcal{C}_1$, resp. $\mathcal{C}_2$. Let $\bar{y}_h^o = (y_0^o : y_1^o : y_2^o) \in \mathcal{C}_1 \cap \mathcal{C}_2$, and let

$$R(y_1, y_2) = \mathrm{Res}_{y_0}(F, G)$$

Then the multiplicity of intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$ at $\bar{y}_h^o$, denoted by $\mathrm{mult}_{\bar{y}_h^o}(\mathcal{C}_1, \mathcal{C}_2)$, equals the multiplicity of the corresponding factor $(y_2^o y_1 - y_1^o y_2)$ in $R(y_1, y_2)$. However, in the following Lemma we see how the multiplicity of intersection of two curves on a line through the origin can be read in the resultant, under certain circumstances, even though the curves are not properly set. This lemma can be seen as a generalization of Theorem 5.3, page 111 in [56].

Lemma 1.34. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two projective algebraic plane curves without common components, given by the homogeneous polynomials $F(y_0, y_1, y_2)$ and $G(y_0, y_1, y_2)$, respectively. Let $p_1, \ldots, p_k$ be the intersection points, different from $(1 : 0 : 0)$, of $\mathcal{C}_1$ and $\mathcal{C}_2$ lying on the line of equation $\beta y_1 - \alpha y_2 = 0$. Then the factor $(\beta y_1 - \alpha y_2)$ appears in $\mathrm{Res}_{y_0}(F, G)$ with multiplicity equal to*

$$\sum_{i=1}^{k} \mathrm{mult}_{p_i}(\mathcal{C}_1, \mathcal{C}_2)$$

Proof. Using the well known additivity property of the multiplicity of intersection (see e.g. [19], section 3.3, or [52], page 39), w.l.o.g., we can assume that the polynomials F and G in the statement of the lemma are irreducible. Furthermore, if $\deg_{y_0}(F) = \deg_{y_0}(G) = 0$ the claim holds directly, since in this case $\mathrm{Res}_{y_0}(F, G) = 1$ and the sum in the statement is zero. Note that the only intersection point in this case is $(1 : 0 : 0)$. If $\deg_{y_0}(F) = 0$ and $\deg_{y_0}(G) > 0$, then $\mathcal{C}_1$ is a line (since we are assuming that F is irreducible). W.l.o.g., by means of a suitable change of coordinates, we can assume that in this situation, $F(\bar{y}_h) = y_2$, and that $(1 : 0 : 0)$ does not belong to $\mathcal{C}_1 \cap \mathcal{C}_2$. Then,

$$\mathrm{Res}_{y_0}(F, G) = y_2^{\deg_{y_0}(G)}.$$

In this case, there are no intersection points of $\mathcal{C}_1$ and $\mathcal{C}_2$ lying on the line of equation $\beta y_1 - \alpha y_2 = 0$, unless $\beta = 0$ (and $\alpha \neq 0$). Thus, the points $p_1, \ldots, p_k$ in the statement of the lemma correspond to the solutions of $G_0(y_0, y_1) = G(y_0, y_1, 0) = 0$, and we have

$$\sum_{i=1}^{k} \mathrm{mult}_{p_i}(\mathcal{C}_1, \mathcal{C}_2) = \deg_{y_0}(G_0).$$

Let us write

$$G(\bar{y}_h) = \sum_{i=0}^{\deg_{y_2}(G)} G_i(y_0, y_1) y_2^i,$$

where, for $i = 0, \ldots, \deg_{y_2}(G)$, G_i is a form of degree $\deg(G) - i$ and, by assumption (since $(1 : 0 : 0) \notin \mathcal{C}_1 \cap \mathcal{C}_2$), y_1 does not divide G_0. Therefore $\deg_{y_0}(G_0) = \deg_{y_0}(G)$ and the claim follows.
Thus, for the rest of the proof, we assume that $\deg_{y_0}(F) > 0, \deg_{y_0}(G) > 0$, and both F and G are irreducible. By means of a suitable linear change of coordinates (linear in y_1, y_2, and leaving y_0 unaffected) we may transform the line $\beta y_1 - \alpha y_2 = 0$ onto the line $y_1 = 0$. Under this change of coordinates the multiplicity of the factors in the resultant, and the multiplicity of intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$ are both preserved. Therefore, w.l.o.g., we assume that $\alpha = 0, \beta = 1$. Thus, if we write

$$R(y_1, y_2) := \mathrm{Res}_{y_0}(F, G) = y_1^{\ell} H(y_1, y_2)$$

where $\gcd(y_1, H) = 1$, one has to prove that

$$\ell = \sum_{i=1}^{k} \mathrm{mult}_{p_i}(\mathcal{C}_1, \mathcal{C}_2)$$

Let

$$f(y_0, y_1) = F(y_0, y_1, 1), \quad g(y_0, y_1) = G(y_0, y_1, 1)$$

Note that f (similarly for g) is constant if and only if F depends only on y_2. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are irreducible, and have no common components, only one of them can be in such a situation. In this case, we can use another change of coordinates to transform the line $\beta y_1 - \alpha y_2 = 0$ onto the line $y_2 = 0$, and then we repeat the proof below with the roles of y_1 and y_2 interchanged.

Let $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$ be the affine curves defined in the (y_0, y_1)-plane by f and g, respectively. Since the point at infinity (w.r.t. y_2) of the line $y_1 = 0$ is $(1 : 0 : 0)$, one deduces that $\{p_1, \ldots, p_k\}$ correspond exactly, by means of the linear change of coordinates, to the affine (now w.r.t. y_0) intersection points of $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$ lying on the line $y_1 = 0$.

Let $\mathbb{K}((y_1))$ be the (algebraically closed) field of formal Puiseux series in y_1. We will consider f and g as polynomials in $\mathbb{K}((y_1))[y_0]$. Let

$$p_1, \ldots, p_k, q_{k+1}, \ldots, q_j$$

be the intersection points of $\tilde{\mathcal{C}}_1$ with the line $y_1 = 0$, and let

$$p_1, \ldots, p_k, r_{k+1}, \ldots, r_m$$

be the intersection points of $\tilde{\mathcal{C}}_2$ with the line $y_1 = 0$, where $q_i \neq r_j$ for every i and j.

Then, the polynomials f and g can be expressed as

$$\begin{cases} f(y_1, y_0) = c_1 \prod_{\alpha} (y_0 - \bar{y}_\alpha) \prod_{\alpha'} (y_0 - \bar{y}_{\alpha'}) \\ g(y_1, y_0) = c_2 \prod_{\beta} (y_0 - \bar{y}_\beta) \prod_{\beta'} (y_0 - \bar{y}_{\beta'}) \end{cases}$$

where $c_1, c_2, \bar{y}_\alpha, \bar{y}_{\alpha'}, \bar{y}_\beta, \bar{y}_{\beta'} \in \mathbb{K}((y_1))$, c_1, c_2 are non-zero, $\bar{y}_\alpha$ and $\bar{y}_\beta$ correspond to places centered at some p_i, $\bar{y}_{\alpha'}$ correspond to places centered at some q_i, and finally $\bar{y}_{\beta'}$ correspond to places centered at some r_i.

It follows that

$$\mathrm{Res}_{y_0}(f,g) = c_1^{\deg_{y_0}(g)} c_2^{\deg_{y_0}(f)} \prod_{(\alpha,\beta)} (\bar{y}_\alpha - \bar{y}_\beta) \prod_{(\alpha',\beta)} (\bar{y}_{\alpha'} - \bar{y}_\beta) \prod_{(\alpha,\beta')} (\bar{y}_\alpha - \bar{y}_{\beta'}) \prod_{(\alpha',\beta')} (\bar{y}_{\alpha'} - \bar{y}_{\beta'}).$$

Thus, one has that

1. the order of $\prod_{(\alpha,\beta)}(\bar{y}_\alpha - \bar{y}_\beta)$ equals the sum of the multiplicities of intersection of $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$ at the points p_i.
2. the order of $\prod_{(\alpha',\beta)}(\bar{y}_{\alpha'} - \bar{y}_\beta)$, $\prod_{(\alpha,\beta')}(\bar{y}_\alpha - \bar{y}_{\beta'})$, $\prod_{(\alpha',\beta')}(\bar{y}_{\alpha'} - \bar{y}_{\beta'})$ is 0.

Therefore the order of $\mathrm{Res}_{y_0}(f,g)$ is $\displaystyle\sum_{i=1}^{k} \mathrm{mult}_{p_i}(\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2)$. Now, the points p_i correspond precisely to the points of intersection between $\mathcal{C}_1$ and $\mathcal{C}_2$ of the form $(c:0:1)$. This includes every intersection point of $\mathcal{C}_1$ and $\mathcal{C}_2$ on the line $y_1 = 0$ with the exception of $(1:0:0)$. On the other hand, taking into account that $\mathrm{Res}_{y_0}(f,g) = R(y_1, 1)$ one has that $\mathrm{Res}_{y_0}(f,g) = y_1^\ell H(y_1, 1)$, where y_1 does not divide $H(y_1,1)$. Hence,

$$\ell = \sum_{i=1}^{k} \mathrm{mult}_{p_i}(f,g).$$

□

The following Lemma is a generalization of Corollary 1 in [33]. It shows that generalized resultants can be used to study the intersection points of a finite family of curves.

Lemma 1.35. *Let $\mathcal{C}_0, \ldots, \mathcal{C}_m$ be the projective plane curves, defined by the homogeneous polynomials $F_0, \ldots, F_m \in \mathbb{K}[\bar{t}_h]$, respectively. Let us suppose that the following hold:*

(i) *$F_1, \ldots, F_m$ have positive degree in t_0.*

(ii) $\deg_{\bar{t}_h}(F_1) = \cdots = \deg_{\bar{t}_h}(F_m)$.

(iii) $\gcd(F_1, \ldots, F_m) = 1$.

Let us denote:

$$F(\bar{c}, \bar{t}_h) = c_1 F_1(\bar{t}^h) + \cdots + c_m F_m(\bar{t}^h)$$

and let

$$R(\bar{c},\bar{t}) = \mathrm{Res}_{t_0}\left(F_0(\bar{t}^h), F(\bar{c},\bar{t}_h)\right),$$

(note that by (iii)*, $R(\bar{c},\bar{t})$ is not identically zero). Finally, let $\mathrm{lc}_{t_0}(F_0) \in \mathbb{C}[\bar{t}]$ and $\mathrm{lc}_{t_0}(F) \in \mathbb{C}[\bar{c},\bar{t}]$ denote, respectively, the leading coefficients w.r.t. t_0 of F_0 and F.*

If $\bar{t}^o = (t_1^o, t_2^o) \in \mathbb{K}^2 \setminus \{\bar{0}\}$ is such that $\mathrm{Cont}_{\bar{c}}(R)(\bar{t}^o) = 0$ and

$$\mathrm{lc}_{t_0}(F_0)(\bar{t}^o)\cdot \mathrm{lc}_{t_0}(F)(\bar{c},\bar{t}^o) \neq 0,$$

there exists t_0^o such that $\bar{t}_h^o = (t_0^o : t_1^o : t_2^o) \in \bigcap_{i=0}^m \mathcal{C}_i$.

Proof. First, observe that if $\deg_{t_0}(F_0) = 0$, then $\mathrm{lc}_{t_0}(F_0) = F_0$ and $R(\bar{c},\bar{t}) = F_0^{\deg_{\bar{t}_0}(F_1)}$. Thus, in this case the lemma holds trivially, since there is no $\bar{t}^o \in \mathbb{K}^2 \setminus \{\bar{0}\}$ satisfying the hypothesis of the lemma. Thus, w.l.o.g., in the rest of the proof, we assume that $\deg_{t_0}(F_0) > 0$.

Since $\mathrm{lc}_{t_0}(F)(\bar{c},\bar{t}^o) \neq 0$, there exists an open set $\Phi \subset \mathbb{K}^m$ such that if $\bar{c}^o = (c_1^o, \ldots, c_m^o) \in \Phi$, the leading coefficient w.r.t. t_0 of $F(\bar{c}^o, \bar{t}^o, t_0) \in \mathbb{C}[t_0]$ is $\mathrm{lc}_{t_0}(F)(\bar{c}^o,\bar{t}^o)$, and it is non-zero. Therefore, by the Extension Theorem, (see [14], page 159), there exists $\zeta(\bar{c}^o) \in \mathbb{K}$ (which, in principle, could depend on $\bar{c}^o$) such that

$$F_0(\zeta(\bar{c}^o), t_1^o, t_2^o) = F(\zeta(\bar{c}^o), t_1^o, t_2^o) = 0.$$

We claim that there is $t_0^o \in \mathbb{K}$ (not depending on $\bar{c}^o$), such that

$$F_0(t_0^o, t_1^o, t_2^o) = F_1(t_0^o, t_1^o, t_2^o) = \cdots = F_m(t_0^o, t_1^o, t_2^o) = 0.$$

To see this note that, since $\mathrm{lc}_{t_0}(F_0) \neq 0$, there is a non-empty finite set of solutions of the following equation in t_0:

$$F_0(t_0, t_1^o, t_2^o) = 0.$$

Let $\zeta_1, \ldots, \zeta_p$ be the solutions. If

$$F_1(\zeta_j, t_1^o, t_2^o) = \cdots = F_m(\zeta_j, t_1^o, t_2^o) = 0$$

holds for some $j = 1, \ldots, p$, then it suffices to take $t_0^o = \zeta_j$. Let us suppose that this is not the case, and we will derive a contradiction. Then there exists an open set $\Phi_1 \subset \Phi$, such that if $\bar{c}^o \in \Phi_1$, then

$$F(\zeta_j, t_1^0, t_2^0) = c_1^o F_1(\zeta_j, t_1^0, t_2^0) + \cdots + c_m^o F_m(\zeta_j, t_1^0, t_2^0) \neq 0$$

for every $j = 1, \ldots, p$. This means that, for $\bar{c}^o \in \Phi_1$, there is no solution of:

$$\begin{cases} F_0(t_0, \bar{t}^o) = 0 \\ F(\bar{c}^o, t_0, \bar{t}^o) = 0 \end{cases}$$

Since the resultant specializes properly in Φ, this implies that: $R(\bar{c}^o, \bar{t}^o) \neq 0$. But, denoting

$$M(\bar{t}) = \mathrm{Cont}_{\bar{c}}\left(R(\bar{c}, \bar{t})\right), \quad \text{and } N(\bar{c}, \bar{t}) = \mathrm{PP}_{\bar{c}}\left(R(\bar{c}, \bar{t})\right),$$

we have

$$R(\bar{c}^o, \bar{t}^o) = M(\bar{t}^o) N(\bar{c}^o, \bar{t}^o) = 0$$

because, by hypothesis $M(\bar{t}^o) = 0$. This contradiction proves the result. □

Chapter 2

Total Degree Formulae for Plane Curves

Let $\mathcal{C}$ be an irreducible affine plane curve over $\mathbb{C}$ (see Remark 1.13 in page 14 for the reasons underlying the restriction $\mathbb{K} = \mathbb{C}$). In this chapter we deal with the problem of giving formulae that provide explicitly the total degree of its generic offset $\mathcal{O}_d(\mathcal{C})$ (the definitions and terminology specific to this chapter are introduced below in this introduction). We treat the general case, and therefore we provide offset degree formulae for algebraic curves, non-necessarily rational given either implicitly or, in the rational case, also parametrically.

More precisely, in this chapter we present three different formulae for the case of curves given implicitly, and one formula for the case of a rational curve, when we are given a parametrization. This last formula provides a simplified extension to the one in [17], requiring only gcds of univariate polynomials, easily derived from the parametrization.

The first formula appears in Section 2.2 (see page 52), and it is based on an auxiliary curve, called $\mathcal{S}_{(d^o,k^o)}$, that is defined for all pairs (d^o, k^o) in a non-empty Zariski open subset of $\mathbb{C}^2$. This formula is used theoretically, although one may consider an heuristic algorithm from it. The second formula (see page 57) is based on the hodograph curve $\mathcal{H}$ associated to the original curve $\mathcal{C}$, and expresses the offset degree by means of the degree of $\mathcal{C}$ and the multiplicity of intersection of $\mathcal{C}$ and $\mathcal{H}$ at their intersection points, that turn to be the affine singularities and the intersection points at infinity.

The third formula that we present (in Section 2.3, page 63) is based on the resultant of the defining polynomial of the original curve and the polynomial defining generically the auxiliary curve $\mathcal{S}$. In later chapters we will find analogous situations, in which the scheme of *elimination process, fake points, resultant-based formula* is repeated. Therefore, the proof of the formula in Section 2.3 is presented in a framework suitable for all these analogous situations.

The two last mentioned formulae provide deterministic algorithms. The resultant based formula requires taking the primitive part w.r.t. (d, k) of a univariate resultant over $\mathbb{C}[d, k][y_0, y_1, y_2]$. The computations for the hodograph based formula stay over $\mathbb{C}[y_0, y_1, y_2]$. However, from the point of view of efficiency and ease of implementation, the resultant-based formula is in fact our formula of choice in actual computations. However, the proof of this formula uses the fact that a line through the origin meets $\overline{\mathcal{C}}$ in a finite number of points. This is clearly not always the case when $\overline{\mathcal{C}}$ is itself a line through the origin, and in fact the resultant-based formula can not be applied to this case. Note that since the degree of the offset to any line is known, namely 2, this does not restrict in practice the applicability of the formula. This is the reason for the following assumption:

Assumption 2.1. In Section 2.3 of this chapter, we assume that $\mathcal{C}$ is not a line through the origin.

The fourth formula (in Section 2.4, page 71) deals with the case of a rational generating curve, given parametrically. This formula only requires the computation of degrees and gcds of univariate polynomials, that are easily constructed from the parametrization of the curve.

The structure of the chapter is the following:

- In Section 2.1, we describe the general theoretical strategy (in Subsection 2.1.1), and we prove some technical results that will be used throughout this –and the following– chapters. We also introduce the auxiliary curve $\mathcal{S}$ (Subsection 2.1.2), with its main properties, and finally, we present the notion of fake points (Subsection 2.1.3), and prove their invariance.

- In Section 2.2 we establish the first two total degree formulae. The first one, involving $\mathcal{S}$, in Subsection 2.2.1, can not be applied easily to compute the offset degree of a concrete example. In order to overcome this problem, in Subsection 2.2.2 we obtain a second, deterministic formula, based on the notion of hodograph curve.

- The third formula, in Section 2.3, takes the computational efficiency one step further, by developing a resultant based formula.

- In the last section, Section 2.4, we show how these ideas particularize to the rational case, with an additional gain in simplification and efficiency for this case, and we will give a new formula where only univariate gcd's are used.

The results in this chapter have been published in the Journal of Pure and Applied Algebra (see [42]).

Notation and terminology for this chapter

In this chapter, we will adapt some of the notational conventions introduced in page 1 to the case of curves:

- Since $n = 2$, then $\bar{x} = (x_1, x_2), \bar{y} = (y_1, y_2)$, while their homogeneous counterparts are $\bar{x}_h = (x_0, x_1, x_2), \bar{y}_h = (y_0, y_1, y_2)$.

- The symbol $\mathcal{C}$ denotes an irreducible plane algebraic curve, defined over $\mathbb{C}$ by the irreducible polynomial $f(\bar{y}) \in \mathbb{C}[\bar{y}]$.

- For a plane curve there is only one polynomial $\mathrm{nor}_{(i,j)}(\bar{x}, \bar{y})$ (these were introduced with the notation in page 8), namely $\mathrm{nor}_{(1,2)}(\bar{x}, \bar{y})$. Therefore, in this chapter we will abbreviate this by simply writing $\mathrm{nor}(\bar{x}, \bar{y})$; i.e.:

$$\mathrm{nor}(\bar{x}, \bar{y}) = \mathrm{nor}_{(1,2)}(\bar{x}, \bar{y})$$

- The hodograph of $\mathcal{C}$ is the polynomial $h(y_1, y_2) = f_1^2 + f_2^2$, and it defines an affine curve, the hodograph curve of $\mathcal{C}$, denoted by $\mathcal{H}$. Moreover, $H(\bar{y}_h) = F_1^2(\bar{y}_h) + F_2^2(\bar{y}_h)$.

- When $\mathcal{C}$ is rational, we will use t as a parametrization parameter, and $P(t)$ will denote a proper parametrization of $\mathcal{C}$, given by

$$P(t) = \left(\frac{X(t)}{W(t)}, \frac{Y(t)}{W(t)} \right)$$

  where $X, Y, W \in \mathbb{C}[t]$, and $\gcd(X, Y, W) = 1$. Note that the requirement of a proper parametrization does not restrict the applicability of this work, since there are efficient algorithms for the proper reparametrization problem for curves (see, e.g. [52], page 188). Even if the starting point is a real parametrization, it is also possible to obtain a real proper reparametrization (see, e.g. [39])

- $\mathcal{O}_d(\mathcal{C})$ denotes the generic offset of $\mathcal{C}$. Furthermore, $g \in \mathbb{C}[d, x_1, x_2]$ denotes, as usual, the generic offset polynomial, and δ denotes the total degree in $\bar{x}$ of $\mathcal{O}_d(\mathcal{C})$.

- We will consider a pencil of lines through the origin, denoted by $\mathcal{L}_k$, with equation:

$$L(k, \bar{x}) : \quad x_1 - kx_2 = 0.$$

  As usual, a particular value of the slope variable k will be denoted by k^o, and the corresponding line is $\mathcal{L}_{k^o}$.

2.1 General Strategy

Our approach to the degree problem is based on Bézout's Theorem. We consider the degree as the number of intersection points of $\mathcal{O}_{d^o}(\mathcal{C})$, counted properly, with a generic line $\mathcal{L}$. Nevertheless, since the implicit equation of $\mathcal{O}_{d^o}(\mathcal{C})$ is not known, we will compute the number of intersection points in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}$ indirectly, by counting the points in $\mathcal{C}$ that, in a $1:1$ correspondence, generate the points in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}$.

More precisely, in Subsection 2.1.1 we will show that it suffices to consider the intersection with the lines in the pencil of lines through the origin $\mathcal{L}_k$, presented in the introduction of this chapter. This is done by analyzing the solution set of the Offset-Line System 2.2 (page 39); that is, the system obtained adjoining the equation of the pencil of lines through the origin to the generic offset system. Theorem 2.5 is the main result in this subsection. It shows that for a particular choice of (d^o, k^o) we can use the line $\mathcal{L}_{k^o}$ to try to obtain $\delta^o = \deg_{\bar{x}}(\mathcal{O}_{d^o}(\mathcal{C}))$. Note that, according to the degree invariance property established in Corollary 1.25(4) (page 21), if the value of $\delta^o = \deg_{\bar{x}}(\mathcal{O}_{d^o}(\mathcal{C}))$ is invariant in a certain open set of values of d, then it coincides with the generic degree $\delta = \deg_{\bar{x}}(\mathcal{O}_d(\mathcal{C}))$ for all values of d in that open set. Then, in Subsection 2.1.2 (page 44) we eliminate the variables $\bar{x}$ from the System 2.2. As a result of that elimination process, we switch our attention from the points $\bar{x}_i^o$ in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$ to the associated points $\bar{y}_i^o$ in the curve $\mathcal{C}$. And we identify these associated points as being intersection points of $\mathcal{C}$ with a certain auxiliary curve $\mathcal{S}_{(d^o,k^o)}$ (see Theorem 2.14, page 47). However, as we will see, $\mathcal{C} \cap \mathcal{S}_{(d^o,k^o)}$ usually contains other points besides these. This justifies the introduction, in Subsection 2.1.3 (page 50), of the notion of fake and non-fake intersection points between $\mathcal{C}$ and $\mathcal{S}_{(d^o,k^o)}$. The most useful and important property of these points is their invariance, shown in Theorem 2.19 (page 50).

2.1.1 The offset-line system

We recall, for the convenience of the reader, that for $d^o \in \mathbb{C}^\times$ the offset system (see System 1.1 in page 10) for $\mathcal{C}$ turns into:

$$\left.\begin{array}{lr} & f(\bar{y}) = 0 \\ \operatorname{nor}(\bar{x},\bar{y}): & f_2(\bar{y})(x_1 - y_1) - f_1(\bar{y})(x_2 - y_2) = 0 \\ b_{d^o}(\bar{y},\bar{x}): & (x_1 - y_1)^2 + (x_2 - y_2)^2 - (d^o)^2 = 0 \\ w(\bar{y},u): & u \cdot (f_1^2(\bar{y}) + f_2^2(\bar{y})) - 1 = 0 \end{array}\right\} \equiv \mathfrak{S}_1(d^o) \qquad (2.1)$$

As usual, considering d as a variable, we will denote by $\mathfrak{S}_1(d)$ the generic version of System 2.1. As we have already discussed in Chapter 1, every solution $(\bar{x}^o, \bar{y}^o, u^o)$ of this system gives a point $\bar{x}^o$ on $\mathcal{O}_{d^o}(\mathcal{C})$, associated to the non normal-isotropic point $\bar{y}^o$ on $\mathcal{C}$, but not every point on the offset corresponds to a solution of this system. Note however that, in the case of curves, there are only finitely many points in $\mathcal{O}_{d^o}(\mathcal{C})$ that are not associated to regular points on $\mathcal{C}$ (see Remark 1.19 in page 18).

Now, we are interested in those solutions of System 2.1 lying on the line $\mathcal{L}_{k^o}$. That is, we want to analyze the solutions of:

$$\left.\begin{array}{lr} & f(\bar{y})=0 \\ nor(\bar{x},\bar{y}): & f_2(\bar{y})(x_1-y_1)-f_1(\bar{y})(x_2-y_2)=0 \\ b_{d^o}(\bar{y},\bar{x}): & (x_1-y_1)^2+(x_2-y_2)^2-(d^o)^2=0 \\ w(\bar{y},u): & u\cdot(f_1^2(\bar{y})+f_2^2(\bar{y}))-1=0 \\ L(k^o,\bar{x}): & x_1-k^o x_2=0 \end{array}\right\} \equiv \mathfrak{S}_2(d^o,k^o) \tag{2.2}$$

The generic version of System 2.1 (with d and k considered as variables) will be denoted by $\mathfrak{S}_2(d,k)$, and we will refer to it as the Offset-Line System. In Theorem 2.5 we will prove that, for all but finitely many values $(d^o,k^o)\in\mathbb{C}^\times\times\mathbb{C}$, there are δ solutions of System 2.2, corresponding to δ different affine intersection points of $\mathcal{L}_{k^o}$ and $\mathcal{O}_{d^o}(\mathcal{C})$. In order to do that, we first need some technical results.

When intersecting the offset with a line through the origin, a special situation arises if that line is parallel to the normal at a point where the offset and the line intersect: in this case, two of the intersection points between $\mathcal{O}_{d^o}(\mathcal{C})$ and $\mathcal{L}_{k^o}$ can be associated to the same point in $\mathcal{C}$. This is a situation we want to avoid, and the following lemma is the basic step to do this.

Lemma 2.2. *If $\mathcal{C}$ is not a circle centered at the origin, there is a finite, possibly empty, subset $\Theta_1\subset\mathbb{C}$ such that for $k^o\notin\Theta_1$ the system*

$$\begin{cases} f(\bar{y})=0 \\ f_1(\bar{y})-k^o f_2(\bar{y})=0 \\ y_1-k^o y_2=0 \end{cases} \tag{2.3}$$

has no solution.

Proof. Let $\bar{y}^o=(y_1^o,y_2^o)$ be a solution of System 2.3. Then, from the last two equations one gets that :

$$y_1^o f_2(\bar{y}^o)-y_2^o f_1(\bar{y}^o)=0.$$

Thus $\bar{y}^o\in\mathcal{C}\cap\mathcal{G}$, where $\mathcal{G}$ is the curve given by $y_1f_2(\bar{y})-y_1f_2(\bar{y})$. Suppose that $y_1f_2(\bar{y})-y_1f_2(\bar{y})$ does not vanish identically on $\mathcal{C}$. Then, by Bézout's theorem, there is a finite number of points $\bar{y}_1^o,\ldots,\bar{y}_m^o\in\mathcal{G}\cap\mathcal{C}$. Let $k_1,\ldots,k_m$ be the values of k such that $\bar{y}_i^o$ lies on the line $y_1-k_iy_2=0$. In this case we can take $\Theta_1=\{k_1,\ldots,k_m\}$. Now suppose that $y_1f_2(\bar{y})-y_2f_1(\bar{y})$ vanishes identically on $\mathcal{C}$. Since $\mathcal{C}$ is irreducible, then

$$y_1f_2(\bar{y})-y_2f_1(\bar{y})=\lambda f(\bar{y})$$

for some constant λ (note that $\deg(y_1 f_2(\bar{y}) - y_2 f_1(\bar{y})) \leq \deg(f)$, and that λ might be zero). This, with the terminology of Lemma 1.14 in page 14, means that $\mathcal{C}_\perp = \mathcal{C} \setminus \mathrm{Sing}_a(\mathcal{C})$. Applying that lemma we conclude that $\mathcal{C}$ is a circle centered at the origin. □

Applying Lemma 2.2 one may derive the following result on offset curves, that states that the origin $\bar{0}$ belongs to $\mathcal{O}_{d^o}(\mathcal{C})$ for at most finitely many distance values.

Proposition 2.3. *There exists a finite subset Δ_1 of $\mathbb{C}$ such that, for $d^o \notin \Delta_1$ the origin does not belong to $\mathcal{O}_{d^o}(\mathcal{C})$.*

Proof. In the case of a circle centered at the origin the result follows directly. Let us assume that $\mathcal{C}$ is not a circle centered at the origin, and let us also assume that the origin $\bar{0}$ belongs to $\mathcal{O}_{d^o}(\mathcal{C})$ for infinitely many values d^o. Using Proposition 1.12 in page 13, w.l.o.g. we can assume that $\bar{0}$ does not belong to $\mathcal{C}$. Then consider the pencil $\mathcal{L}_k$ of lines passing trough $\bar{0}$, with equation $y_1 - ky_2 = 0$. First, we exclude those values of k for which the corresponding line passes through a singular point of $\mathcal{C}$. This excludes a finite number of lines (here we use the fact that $\bar{0} \notin \mathcal{C}$, and that $\mathcal{C}$ is irreducible and hence does not have multiple components). Since each such line contains finitely many points of $\mathcal{C}$, we are excluding at most finitely many distance values; more precisely, those values $d^o = \pm\|\bar{p}^o\|$, where $\bar{p}^o \in \mathrm{Sing}_a(\mathcal{C})$. Now, for the remaining values of k, we consider on each line the intersection points with $\mathcal{C}$. These intersection points $\bar{y}^o$ are regular points of $\mathcal{C}$, and the non-zero vector $\nabla f(\bar{y}^o)$ is normal to $\mathcal{C}$ at $\bar{y}^o$. If this normal vector points to the origin (this implies that $f_1 - kf_2 = 0$ at $\bar{y}^o$) then $\bar{0} \in \mathcal{O}_{d^o}(\mathcal{C})$ for $d^o = \pm\|\bar{p}^o\|$. However, our assumption implies that this must be indeed the case for infinitely many values of d. Therefore this happens for infinitely many points and infinitely many values k^o (note that each line contains at most a finite number of points of $\mathcal{C}$). Thus, every such point $\bar{y}^o$ and value k^o give a solution of the System 2.3 in Lemma 2.2. But Lemma 2.2 shows that this is a contradiction. Therefore we conclude that $\bar{0} \in \mathcal{O}_{d^o}(\mathcal{C})$ only for those values d^o in a finite subset of $\mathbb{C}$. Let Δ_1 be the complement of this set. □

Remark 2.4.

1. *We want to emphasize that this proposition does not hold in a non-algebraic context. For example, the "offset" to the analytic curve with implicit equation $y^3 - \sin(x) = 0$, passes through the origin for infinitely many values of d. In fact, for this curve, all the offsets with values of d equal to $k\pi$, for $k \in \mathbb{Z}$, pass through the origin. This is illustrated in Figure 2.1, where the curve (in red) and some offsets passing through the origin (for $k = 1, 2, 3$) are shown.*

2. *Note that the proposition is not true if, instead of the origin of $\mathbb{C}^2$, or any other affine point, we consider a point at infinity. For instance, all the offsets to the line given by $y_1 = 0$, namely $y_1^2 - (d^o)^2 = 0$, pass through the point $(0 : 0 : 1)$.*

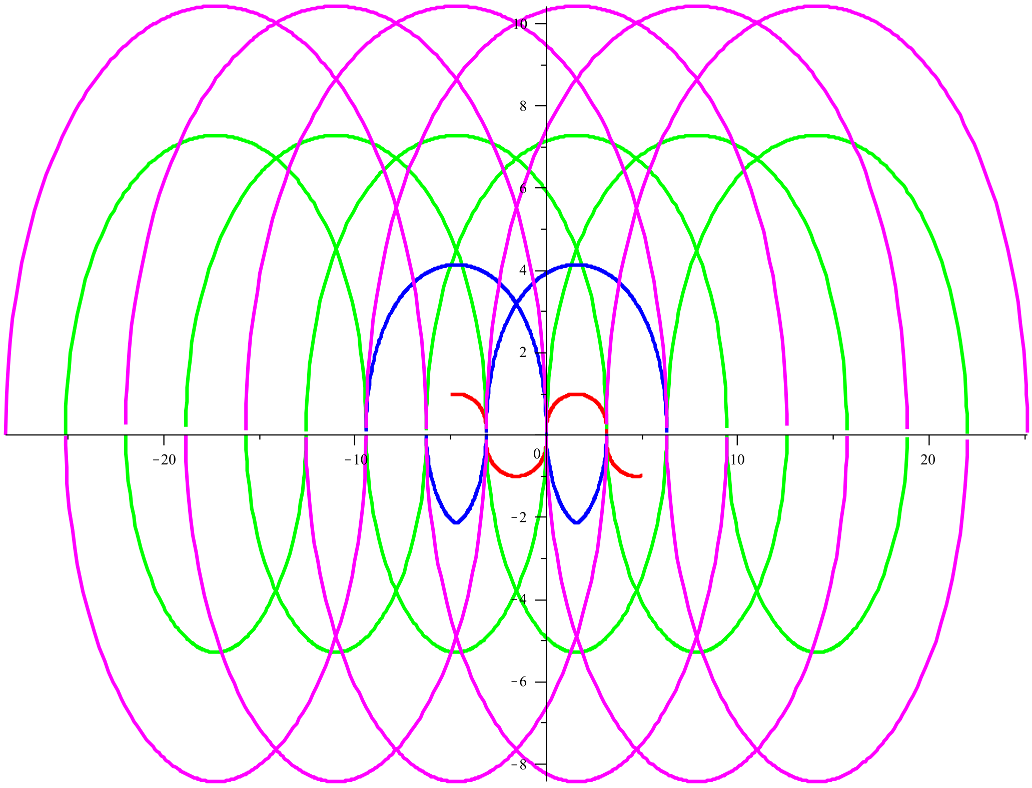

Figure 2.1: A smooth curve with infinitely many offsets through the origin

From the previous lemmas one may prove the following theorem:

Theorem 2.5. *There exists a non-empty Zariski open subset Ω_0 of $\mathbb{C}^2$ such that for $(d^o, k^o) \in \Omega_0$, the set of solutions of system $\mathfrak{S}_2(d^o, k^o)$ (see System 2.2 in page 39) is finite and contains δ solutions that correspond to δ different affine intersection points $\bar{x}_1^o, \ldots, \bar{x}_\delta^o$ of $\mathcal{L}_{k^o}$ and $\mathcal{O}_{d^o}(\mathcal{C}) \setminus \{\bar{0}\}$, generated by δ different affine regular points $\bar{y}_1^o, \ldots, \bar{y}_\delta^o$ on $\mathcal{C}$. Moreover, if $(d^o, k^o) \in \Omega_0$, then $k^o \neq \pm i$, and all the points in $\mathcal{L}_{k^o} \cap \mathcal{C}$ are affine non normal-isotropic points of $\mathcal{C}$.*

Proof. The last claim in the theorem is a simple consequence of the fact that $\mathcal{C}$ has finitely many normal-isotropic points, and finitely many points at infinity. Besides, the two lines $\mathcal{L}_{k^o}$, with $k^o = \pm i$, define a closed subset of $\mathbb{C}^2$. Thus, there is an open subset Ω_0^0 such that, if $(d^o, k^o) \in \Omega_0^0$, then $1 + (k^o)^2 \neq 0$, and the line $\mathcal{L}_{k^o}$ meets $\mathcal{C}$ only in affine non normal-isotropic points. In order to prove the rest of the claims in the theorem, we distinguish the following cases:

(a) If $\mathcal{C}$ is a circle of radius r, a simple algebraic manipulation shows that the theorem holds for $(d, k) \in \Omega_0 = (\mathbb{C}^\times \setminus \{r\}) \times \mathbb{C}$.

(b) If $\mathcal{C}$ is a line, with equation $y_2 - a_1 y_1 - a_2 = 0$ ($a_1, a_2 \in \mathbb{C}$), the theorem holds for $(d^o, k^o) \in \Omega_0 = (\mathbb{C}^\times) \times (\mathbb{C} \setminus \{a_1\})$. For the lines of the form $y_2 = a_2$ the theorem holds for $(d^o, k^o) \in \Omega_0 = (\mathbb{C}^\times) \times \mathbb{C}$.

(c) So let us assume that $\mathcal{C}$ is not a circle or a line, and let $g \in \mathbb{C}[d, \bar{x}]$ be the squarefree polynomial that defines the generic offset. We know (see Remark 1.23 on page 21) that g has at most two factors. We will first show that $\tilde{g}(d, k, x_2) = g(d, kx_2, x_2)$ is also squarefree. In order to do this, consider the map $\phi : \mathbb{C}^3 \mapsto \mathbb{C}^3$ defined by

$$\phi(d, k, x_2) = (d, kx_2, x_2).$$

Since ϕ is birational, it follows that, $\mathcal{M} = \phi^{-1}(\mathcal{O}_d(\mathcal{C}))$ has the same number of components as $\mathcal{O}_d(\mathcal{C})$ (here we use the fact that $\mathcal{C}$ is not a line; therefore, none of the components of $\mathcal{O}_d(\mathcal{C})$ is one of the planes defined by $k = 0$ or $x_2 = 0$). Let us first suppose that g is irreducible. Therefore $\mathcal{M}$ is irreducible. Let $m(x_2, d, k)$ be the irreducible polynomial defining $\mathcal{M}$. Since $\tilde{g}(d, k, x_2) = g(d, kx_2, x_2)$ vanishes on $\mathcal{M}$, then

$$\tilde{g}(d, k, x_2) = g(d, kx_2, x_2) = (m(d, k, x_2))^p$$

for some $p \in \mathbb{N}$. But then, formally changing $k = \frac{x_1}{x_2}$, we have:

$$g(d, x_1, x_2) = (m(d, \frac{x_1}{x_2}, x_2))^p = \left(\frac{m_1(d, x_1, x_2)}{m_2(x_2)}\right)^p$$

for some $m_1 \in \mathbb{C}[d, x_1, x_2]$, and $m_2 \in \mathbb{C}[x_2]$. And so

$$(m_2(x_2))^p g(d, x_1, x_2) = (m_1(d, x_1, x_2))^p$$

But, since g is irreducible, it follows that $p = 1$, and so

$$\tilde{g}(d, k, x_2) = m(d, k, x_2)$$

proves that $\tilde{g}$ is irreducible, and hence, squarefree. Suppose know that g factors as $g = g_1 g_2$ for two irreducible polynomials $g_1, g_2 \in \mathbb{C}[d, x_1, x_2]$. Then $\mathcal{M}$ has two components, and we can write

$$\tilde{g}(d, k, x_2) = g(d, kx_2, x_2) = (m_1(d, k, x_2))^{p_1}(m_2(d, k, x_2))^{p_2}$$

where m_1, m_2 are irreducible, and $p_1, p_2 \in \mathbb{N}$. Again, using the formal substitution $k = \frac{x_1}{x_2}$ and the uniqueness of the irreducible factorization we obtain that $p_1 = 1, p_2 = 1$ and $m_1 = g_1, m_2 = g_2$. Therefore,

$$\tilde{g}(d, k, x_2) = m_1(d, k, x_2) m_2(d, k, x_2)$$

and $\tilde{g}$ is squarefree.

Now, since $\tilde{g}$ is squarefree, $\mathrm{Dis}_{x_2}(\tilde{g}(x_2, d, k))$ is a non-identically zero polynomial. Thus, it defines either the empty set (if it is constant) or a curve $\mathcal{D}_1$ in $\mathbb{C}^2$. Let $\Omega_0^1 = \Omega_0^0$ in the first case, and $\Omega_0^1 = \Omega_0^0 \setminus \mathcal{D}_1$ in the second.
Next, write

$$g(d, \bar{x}) = \sum_{i=0}^{\delta} g_i(d, \bar{x}),$$

where g_i is homogeneous of degree i in $\bar{x}$, and $g_\delta \neq 0$. Thus, $g_\delta(d, kx_2, x_2) = x_2^\delta g_\delta(d, k, 1)$ where $g_\delta(d, k, 1) \neq 0$. Therefore, $g(d, 1, k)$ defines either the empty set (if it is a constant) or a curve $\mathcal{D}_2$ in $\mathbb{C}^2$. Let $\Omega_0^2 = \Omega_0^1$ or $\Omega_0^2 = \Omega_0^1 \setminus \mathcal{D}_2$, respectively.
Besides, by Corollary 1.25 in page 21, we know that there is only a finite set of *bad* distances, Δ, such that for $d^o \notin \Delta$, the equation of $\mathcal{O}_{d^o}(\mathcal{C})$ is $g(d^o, x_1, x_2) = 0$. Let $\mathcal{D}_3$ be the union of the lines with equations $d = d^*$ for $d^* \in \Delta$. We define $\Omega_0^3 = \Omega_0^2 \setminus \mathcal{D}_3$.
Let now $\bar{y}^o = (y_1^o, y_2^o)$ be one of the finitely many affine normal-isotropic points of $\mathcal{C}$ (note that $\mathcal{C}$ is irreducible). We consider the polynomial

$$C_{\bar{y}^o}(d, k, x_2) = (kx_2 - y_1^o)^2 + (x_2 - y_2^o)^2 - d^2,$$

as well as the resultant:

$$R_{\bar{y}^o}(d, k) = \mathrm{Res}_{x_2}(\tilde{g}(d, kx_2, x_2), C_{\bar{y}^o}(d, k, x_2)).$$

This resultant vanishes identically only if both polynomials have a common factor in x_2. But $C_{\bar{y}^o}$ is irreducible, and ϕ is birational, so $C(\phi(d, k, x_2))$ is irreducible too. Hence, this could only happen if, for every $d^o \notin \Delta$, $\mathcal{O}_{d^o}(\mathcal{C})$ contains a circle of radius d^o centered at $\bar{y}^o$. This would imply that $\mathcal{C}$ is itself a circle centered at $\bar{y}^o$, which is impossible since $\bar{y}^o \in \mathcal{C}$. Thus, $R_{\bar{y}^o}$ is not zero, and it defines either the empty set or a curve in $\mathbb{C}^2$. Let $\mathcal{D}_4$ be the curve (or empty set) obtained as the union of such curves or empty sets, for all the possible points $\bar{y}^o$. We define $\Omega_0^4 = \Omega_0^3 \setminus \mathcal{D}_4$. Now, if Θ_1 is the open set obtained in Lemma 2.2, page 39, we define $\Omega_0^5 = \Omega_0^4 \cap (\mathbb{C}^\times \times \Theta_1)$. Next, define $\Omega_0^6 = \Omega_0^5 \cap (\Delta_1 \times \mathbb{C})$, where Δ_1 is the open set in Proposition 2.3, page 40.
Then, for $(d^o, k^o) \in \Omega_0^6$, the following properties hold:

1. $g(d^o, \bar{x})$ is the defining polynomial of $\mathcal{O}_{d^o}(\mathcal{C})$, because of the construction of Ω_0^3.
2. $g(d^o, k^o x_2, x_2)$ is a polynomial in x_2 of degree δ (the leading coefficient of $\tilde{g}$ w.r.t. x_2 does not vanish because of the construction of Ω_0^2), with δ different roots (because of the construction of Ω_0^1). Every root x_2^o of this polynomial corresponds to a point $\bar{x}^o = (k^o x_2^o, x_2^o) \in \mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$. This gives δ different points $\bar{x}_1^o, \ldots, \bar{x}_\delta^o$.

3. No point in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$ can be associated with a singularity of $\mathcal{C}$, because of the construction of Ω_0^4. Therefore, each one of the δ points $\bar{x}_i^o$, that we have obtained above, must be associated to a regular point $\bar{y}_i^o \in \mathcal{C}$. Thus, we have constructed δ solutions of the System 2.2

4. The generating points $\bar{y}_i^o \in \mathcal{C}$ are different. In fact, suppose that $\bar{x}_j^o, \bar{x}_k^o$ are associated to the same point $\bar{y}_i^o$. This can only occur if $\bar{y}_i^o \in \mathcal{L}_{k^o}$ and the normal vector to $\mathcal{C}$ at $\bar{y}_i^o$ is parallel to $\mathcal{L}_{k^o}$. Hence, $(f_1 - k^o f_2)$ vanishes at $\bar{y}_i^o$. Therefore $\bar{y}_i^o$ is a solution of the System 2.3 in Lemma 2.2, and that is impossible because of the construction of Ω_0^5.

5. None of the points $\bar{x}_i^o$ is the origin, because of the construction of Ω_0^6.

Thus, we can take $\Omega_0 = \Omega_0^6$ as the open set in the statement of the theorem. □

In the proof of Theorem 2.5 we have shown that it is possible to choose an open non-empty subset of values (d^o, k^o) for which none of the points in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$ is associated with an affine normal-isotropic point of $\mathcal{C}$. In fact, there is nothing special about the set $\mathrm{Sing}_a(\mathcal{C})$ in that part of the proof: the only property we used is that the set $\mathrm{Sing}_a(\mathcal{C})$ is finite. In the following sections, we will sometimes need to avoid certain finite subsets of $\mathcal{C}$. We state this in the following lemma, for ease of reference.

Lemma 2.6. *Let Ω_0 be as in Theorem 2.5. If $\mathcal{X} \subset \mathcal{C}$ is a finite set, there exists an open non-empty subset $\Omega_0^{\mathcal{X}} \subset \Omega_0$ such that, if $(d^o, k^o) \in \Omega_0^{\mathcal{X}}$, then none of the points in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$ is associated with a point in $\mathcal{X}$.*

Proof. See the proof of Theorem 2.5, page 41. □

Remark 2.7. *In Lemma 4.10 of Chapter 4 (see page Lemma 131), we will meet a similar situation in the case of surfaces. The proof of Lemma 4.10 can be extended to the case of curves (and in general, hypersurfaces). Thus, it can be used as a replacement for the previous Lemma 2.6. However, for the sake of completeness and independence, we prefer to keep this lemma and its proof.*

2.1.2 The auxiliary curve for total degree

In Theorem 2.5 we have seen that, if δ is the total offset degree in $\bar{x}$, there is an open set $\Omega_0 \subset \mathbb{C}^2$ such that, if $(d^o, k^o) \in \Omega_0$, then there are precisely δ points $\bar{x}_j^o$ in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}, i = 1, \ldots, \delta$ which are associated to regular affine points $\bar{y}_i^o$ in $\mathcal{C}$, and the correspondence $\bar{x}_i^o \to \bar{y}_i^o$ is a bijection. The strategy now is to eliminate the variables $\bar{x}$ from the above System 2.2, in order to obtain information about δ through the solutions in $\bar{y}$ of the resulting system. This means that we switch our attention from the points $\bar{x}_i^o$ in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$ to the associated points $\bar{y}_i^o$ in the curve $\mathcal{C}$. In order to do that we will identify these associated points as intersection points of $\mathcal{C}$ with a certain auxiliary curve $\mathcal{S}_{(d^o,k^o)}$. The purpose of this section is to define that curve, analyze some of its properties, and show how the set $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$ relates to the set $\mathcal{C} \cap \mathcal{S}_{(d^o,k^o)}$.

First, we will define the polynomial that, as we will show, defines $\mathcal{S}_{(d^o,k^o)}$ for (d^o, k^o) in a certain open set. We recall that $h(\bar{y})$ is the hodograph of $\mathcal{C}$, and $H(\bar{y}_h)$ its homogenization w.r.t. y_0.

Definition 2.8. *The* auxiliary polynomial $s \in \mathbb{C}[d, k, \bar{y}]$ *is defined as:*

$$s(d, k, \bar{y}) = h(\bar{y})(y_1 - ky_2)^2 - d^2(f_1(\bar{y}) - kf_2(\bar{y}))^2$$

and its $\bar{y}$-homogenization w.r.t. y_0 is denoted by $S(d, k, \bar{y}_h)$; that is:

$$S(d, k, \bar{y}_h) = H(\bar{y})(y_1 - ky_2)^2 - d^2y_0^2(F_1(\bar{y}) - kF_2(\bar{y}))^2$$

We will now analyze the degree of S. Recall that $n = \deg_{\bar{y}_h}(F)$.

Lemma 2.9. *The degree of $S(d, k, \bar{y}_h)$ in $\bar{y}_h$ is $2n$.*

Proof. The polynomial S can be expressed as:

$$S(d, k, \bar{y}_h) = S_1(d, k, \bar{y}_h) + S_2(d, k, \bar{y}_h) + S_3(d, k, \bar{y}_h)$$

where:

$$\begin{cases} S_1(d, k, \bar{y}_h) = ((y_1 - ky_2)^2 - d^2y_0^2)F_1^2(\bar{y}_h) \\ S_2(d, k, \bar{y}_h) = ((y_1 - ky_2)^2 - d^2y_0^2k^2)F_2^2(\bar{y}_h) \\ S_3(d, k, \bar{y}_h) = 2d^2y_0^2kF_2(\bar{y}_h)F_1(\bar{y}_h) \end{cases}$$

Now let $m = \max(\deg_{y_1}(F), \deg_{y_2}(F))$. We distinguish four different cases:

1. If $m = n = \deg_{y_1}(F)$ then F may be written as

   $$F(y_0, y_1, y_2) = c_1y_1^n + c_2y_1^{n-1}y_2 + c_3y_1^{n-1}y_0 + K(y_0, y_1, y_2),$$

   where $c_1, c_2, c_3 \in \mathbb{C}$, $c_1 \neq 0$, and $K(y_0, y_1, y_2)$ is a homogeneous polynomial with $\deg(K) \leq n, \deg_{y_1}(K) \leq n - 2$. Therefore:

   $$\begin{aligned} F_1(y_0, y_1, y_2) &= c_1ny_1^{n-1} + c_2(n-1)y_1^{n-2}y_2 + c_3(n-1)y_1^{n-2}y_0 + K_1, \\ F_2(y_0, y_1, y_2) &= c_2y_1^{n-1} + K_2 \end{aligned}$$

   where K_1, K_2 denote the partial derivatives of K with respect to y_1, y_2 respectively. Note that $\deg_{y_1}(F_1) = n - 1$. On the other hand, $\deg_{y_1}(F_2) = n - 1$ if $c_2 \neq 0$ and $\deg_{y_1}(F_2) \leq n-2$ if $c_2 = 0$. Let us first assume that $c_2 = 0$. Then it is easy to see that $\deg_{y_1}(S_1) = 2n$, while $\deg_{y_1}(S_2) \leq 2n-2$ and $\deg_{y_1}(S_3) \leq 2n-2$. Hence, one gets $\deg_{y_1}(S) = 2n$ in this case. Now, assume that $c_2 \neq 0$. Then the leading terms of S_1, S_2, S_3 w.r.t. y_1 are $c_1^2n^2y_1^{2n}, c_2^2y_1^{2n}, 2d^2kc_1c_2ny_0^2y_1^{2n-2}$ respectively. Thus, in this case, the leading term of S w.r.t. y_1 is $(c_1^2n^2 + c_2^2)y_1^{2n}$, and therefore $\deg_{y_1}(S) = 2n$

2. If $m = n = \deg_{y_2}(F)$ a similar reasoning shows that S has degree $2n$.

3. If $m = \deg_{y_1}(F)$ with $0 < m < n$, then F may be written as

$$F(y_0, y_1, y_2) = c_1 y_1^m y_2^{n-m} + K(y_0, y_1, y_2)$$

where $c \neq 0$, and K is a homogeneous polynomial with $\deg(K) = n, \deg_{y_1}(K) \leq m - 1$. Thus:

$$\begin{aligned} F_1(y_0, y_1, y_2) &= c_1 m y_1^{m-1} y_2^{n-m} + K_1(y_0, y_1, y_2), \\ F_2(y_0, y_1, y_2) &= c_1 (n - m) y_1^m y_2^{n-m-1} + K_2(y_0, y_1, y_2) \end{aligned}$$

Note that $\deg_{y_1}(K_1) \leq m - 2, \deg_{y_1}(K_2) \leq m - 1$. Hence, the leading terms of S_1, S_2, S_3 w.r.t. y_1 are, respectively,

$$\begin{array}{l} c_1^2 m^2 y_1^{2m} y_2^{2n-2m}, \quad c_1^2 (n - m)^2 y_1^{2m+2} y_2^{2n-2m-2}, \\ 2d^2 k y_0^2 c_1^2 m(n - m) y_1^{2m-1} y_2^{2n-2m-1}. \end{array}$$

Therefore the leading term of S w.r.t. y_1 is $c_1^2 (n - m)^2 y_1^{2m+2} y_2^{2n-2m-2}$, and hence the degree of S is $2n$.

4. Finally, if $m = \deg_{y_2}(F)$ with $0 < m < n$, a similar reasoning shows that S has degree $2n$. □

Remark 2.10.

1. Since $S(d, k, \bar{y}_h)$ is the homogenization w.r.t. y_0 of $s(d, k, \bar{y})$, the degree of $s(d, k, \bar{y})$ in $\bar{y} = (y_1, y_2)$ is also $2n$.

2. Let Ω_0 be the open subset in Theorem 2.5. Since $S(d, k, \bar{y}_h)$ is a non-zero polynomial, homogeneous in $\bar{y}_h$, there is obviously an open subset $\Omega_1 \subset \Omega_0$ such that, for $(d^o, k^o) \in \Omega_1$, the specialization $S(d^o, k^o, \bar{y}_h)$ is a non-zero polynomial, which is the y_0-homogenization of $s(d^o, k^o, \bar{y})$, with $\deg_{\bar{y}}(s(d^o, k^o, \bar{y})) = 2n$.

This remark leads to the following definition:

Definition 2.11. *Let Ω_1 be the open subset in the preceding remark. Then, for every $(d^o, k^o) \in \Omega_1$ we define the* auxiliary curve $\mathcal{S}_{(d^o,k^o)}$ *to $\mathcal{C}$ as the affine plane curve defined over $\mathbb{C}$ by the polynomial $s(d^o, k^o, \bar{y})$.*

The following lemma shows how the elimination of the variables $\bar{x}$ in the System 2.2 leads naturally to the auxiliary curve.

Lemma 2.12. *Let us consider*

$$\begin{cases} \widehat{\mathrm{nor}}(\bar{x}, \bar{y}, \hat{f}_1, \hat{f}_2) = \hat{f}_2\,(x_1 - y_1) - \hat{f}_1\,(x_2 - y_2) \\ b(d, \bar{x}, \bar{y}) = (x_1 - y_1)^2 + (x_2 - y_2)^2 - d^2 \\ L(k, \bar{x}) = x_1 - k x_2 \end{cases}$$

as polynomials in $\mathbb{C}[d, k, \bar{x}, \bar{y}, \hat{f}_1, \hat{f}_2]$ *(here* $\hat{f}_1, \hat{f}_2$ *are new variables that replace the partial derivatives in* $\mathrm{nor}(\bar{x}, \bar{y})$*). Let* $I =< \widehat{\mathrm{nor}}, b, L >$ *be the ideal generated by these polynomials. If* $J = I \cap \mathbb{C}[d, k, \bar{y}, \hat{f}_1, \hat{f}_2]$ *is the* $\bar{x}$*-elimination ideal of* I*, then* $J =< \hat{s}(d, k, \bar{y}, \hat{f}_1, \hat{f}_2) >$*, where*

$$\hat{s}(d, k, \bar{y}, \hat{f}_1, \hat{f}_2) = (\hat{f}_1^2 + \hat{f}_2^2)(y_1 - ky_2)^2 - d^2(\hat{f}_1 - k\hat{f}_2)^2$$

Proof. This can be obtained by a standard Gröbner basis computation. $\square$

Remark 2.13. *The computation shows that*

$$\hat{s}(d, k, \bar{y}, \hat{f}_1, \hat{f}_2) = \nu_1 \widehat{\mathrm{nor}} + \nu_2 b + \nu_3 L \tag{2.4}$$

where:

$$\begin{cases} \nu_1 = -k^2\hat{f}_1^2 + 2k\hat{f}_1\hat{f}_2 - \hat{f}_2^2 \\ \nu_2 = -x_1\hat{f}_2 + x_2k^2\hat{f}_1 - x_2k\hat{f}_2 + x_2\hat{f}_1 - k^2\hat{f}_1y_2 - k^2\hat{f}_2y_1 - 2k\hat{f}_1y_1 + 2k\hat{f}_2y_2 + \hat{f}_1y_2 + \hat{f}_2y_1 \\ \nu_3 = \hat{f}_1^2(2ky_1 - x_1k - x_2) + \hat{f}_2^2(x_2 - 2y_2 + ky_1) + 2x_1\hat{f}_1\hat{f}_2 - x_2k\hat{f}_1\hat{f}_2 + k\hat{f}_1\hat{f}_2y_2 - 2\hat{f}_1\hat{f}_2y_1 \end{cases}$$

Note, in particular, that the formal identity 2.4 implies that if $(d^o, k^o, \bar{x}^o, \bar{y}^o, u^o)$ *is a solution of System 2.2 (page 39), then* $s(d^o, k^o, \bar{y}^o) = 0$*.*

In the following theorem we show, as announced in the introduction of this subsection, how the set $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$ relates to the set $\mathcal{C} \cap \mathcal{S}_{(d^o,k^o)}$. In Figure 2.2 we illustrate, for the case of a parabola, the intuitive geometric role played by the auxiliary curve $\mathcal{S}_{(d^o,k^o)}$, with a particular choice of (d^o, k^o). In that figure, the intersection points $\bar{y}_i^o \in \mathcal{C} \cap \mathcal{S}_{(d^o,k^o)}$ with real coordinates are shown, together with the associated points $\bar{x}_i^o \in \mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$.

Theorem 2.14. *Let* $(d^o, k^o) \in \Omega_1$*, where* Ω_1 *is the open subset in Remark 2.10, page 46. It holds that:*

1. *If* $(\bar{x}^o, \bar{y}^o, u^o)$ *is a solution of the System 2.2 corresponding to an intersection point* $\bar{x}^o \in \mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$ *then* $\bar{y}^o \in \mathcal{C} \cap \mathcal{S}_{(d^o,k^o)}$*, and* $f_1(\bar{y}^o) - k^o f_2(\bar{y}^o) \neq 0$*.*

2. *Conversely, let* $\bar{y}^o = (y_1^o, y_2^o) \in \mathcal{C} \cap \mathcal{S}_{(d^o,k^o)}$ *be such that*

$$f_1(\bar{y}^o) - k^o f_2(\bar{y}^o) \neq 0.$$

Then,

(2.1) $\bar{y}^o$ *is a non normal-isotropic point of* $\mathcal{C}$*.*

(2.2) Let $\bar{x}^o = (x_1^o, x_2^o)$*, where:*

$$x_2^o = \frac{f_1(\bar{y}^o)y_2^o - f_2(\bar{y}^o)y_1^o}{f_1(\bar{y}^o) - k^o f_2(\bar{y}^o)}, \quad x_1^o = k^o x_2^o, \quad u^o = \frac{1}{h(\bar{y}^o)}.$$

$(\bar{x}^o, \bar{y}^o, u^o)$ *is a solution of the System 2.2, and* $\bar{x}^o \in \mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$ *is associated to* $\bar{y}^o$ *in the sense of Remark 1.3 (page 10).*

Proof. In order to prove statement (1), let $\bar{x}^o = (x_1^o, x_2^o)$ and $\bar{y}^o = (y_1^o, y_2^o)$. We use the last equation in System 2.2 to eliminate x_1. One obtains:

$$\left\{ \begin{array}{l} f(y_1^o, y_2^o) = 0 \\ (k^o x_2^o - y_1^o)^2 + (x_2^o - y_2^o)^2 - (d^o)^2 = 0 \\ -f_2(\bar{y}^o)(k^o x_2^o - y_1^o) + f_1(\bar{y}^o)(x_2^o - y_2^o) = 0 \end{array} \right. \tag{2.5}$$

From the last equation we get:

$$(f_1(\bar{y}^o) - k^o f_2(\bar{y}^o))x_2^o = f_1(\bar{y}^o)y_2^o - f_2(\bar{y}^o)y_1^o$$

Let us see that $f_1(\bar{y}^o) - k^o f_2(\bar{y}^o) \neq 0$. Indeed, if $f_1(\bar{y}^o) - k^o f_2(\bar{y}^o) = 0$, then one also has $f_1(\bar{y}^o)y_2^o - f_2(\bar{y}^o)y_1^o = 0$. Therefore $(v_1, v_2) = (f_1(\bar{y}^o), f_2(\bar{y}^o))$ is a solution of the homogeneous linear system $\{v_1 y_2^o - v_2 y_1^o = 0, v_1 - k^o v_2 = 0\}$. Moreover, by Theorem 2.5, we know that $\bar{y}^o$ is not a singular point of $\mathcal{C}$, and hence the solution $(f_1(\bar{y}^o), f_2(\bar{y}^o))$ is non-trivial. Thus the determinant of the linear system, namely $y_1^o - k^o y_2^o$, is zero. This implies that $\bar{y}^o$ is a solution of the System, 2.3), which is impossible because $(d^o, k^o) \in \Omega_0$, and hence $k^o \in \Theta_1$ (see Lemma 2.2 and the construction of Ω_0^5 in the proof of Theorem 2.5, page 41).
Finally, we still have to prove that $\bar{y}^o \in \mathcal{C} \cap \mathcal{S}_{(d^o,k^o)}$. The first equation in System 2.2 directly implies that $p \in \mathcal{C}$. And Remark 2.13 in page 47 proves that $\bar{y}^o \in \mathcal{S}_{(d^o,k^o)}$.

Let us prove statement (2). If we suppose $h(\bar{y}^o) = 0$, then from $s(d^o, k^o, \bar{y}^o) = 0$, and since $d^o \neq 0$, one has $f_1(\bar{y}^o) - k^o f_2(\bar{y}^o) = 0$. This is a contradiction, and so $h(\bar{y}^o) \neq 0$. Thus, the point $(\bar{x}^o, \bar{y}^o, u^o)$ is well defined, and clearly satisfies the first and the two last equations of System 2.2. Now, using that $s(d^o, k^o, \bar{y}^o) = 0$, one has that

$$(x_1^o - y_1^o)^2 + (x_2^o - y_2^o)^2 = \left(k^o \frac{f_1(\bar{y}^o)y_2^o - f_2(\bar{y}^o)y_1^o}{f_1(\bar{y}^o) - k^o f_2(\bar{y}^o)} - y_1^o \right)^2 + \left(\frac{f_1(\bar{y}^o)y_2^o - f_2(\bar{y}^o)y_1^o}{f_1(\bar{y}^o) - k^o f_2(\bar{y}^o)} - y_2^o \right)^2$$

$$= \frac{f_1^2(\bar{y}^o)(k^o y_2^o - y_1^o)^2 + f_2^2(\bar{y}^o)(k^o y_2^o - y_1^o)^2}{(f_1(\bar{y}^o) - k^o f_2(\bar{y}^o))^2} = \frac{(f_1^2(\bar{y}^o) + f_2^2(\bar{y}^o))(k^o y_2^o - y_1^o)^2}{(f_1(\bar{y}^o) - k^o f_2(\bar{y}^o))^2} = (d^o)^2$$

and therefore $(\bar{x}^o, \bar{y}^o, u^o)$ also verifies satisfies the second equation in the System 2.2. Finally, let us see that the third equation is also satisfied. Observe that

$$(f_1(\bar{y}^o) - k^o f_2(\bar{y}^o))x_2^o = f_1(\bar{y}^o)y_2^o - f_2(\bar{y}^o)y_1^o$$

and so $-f_2(\bar{y}^o)(k^o x_2^o - y_1^o) + f_1(\bar{y}^o)(x_2^o - y_2^o) = -f_2(\bar{y}^o)(x_1^o - y_1^o) + f_1(\bar{y}^o)(x_2^o - y_2^o) = 0$. □

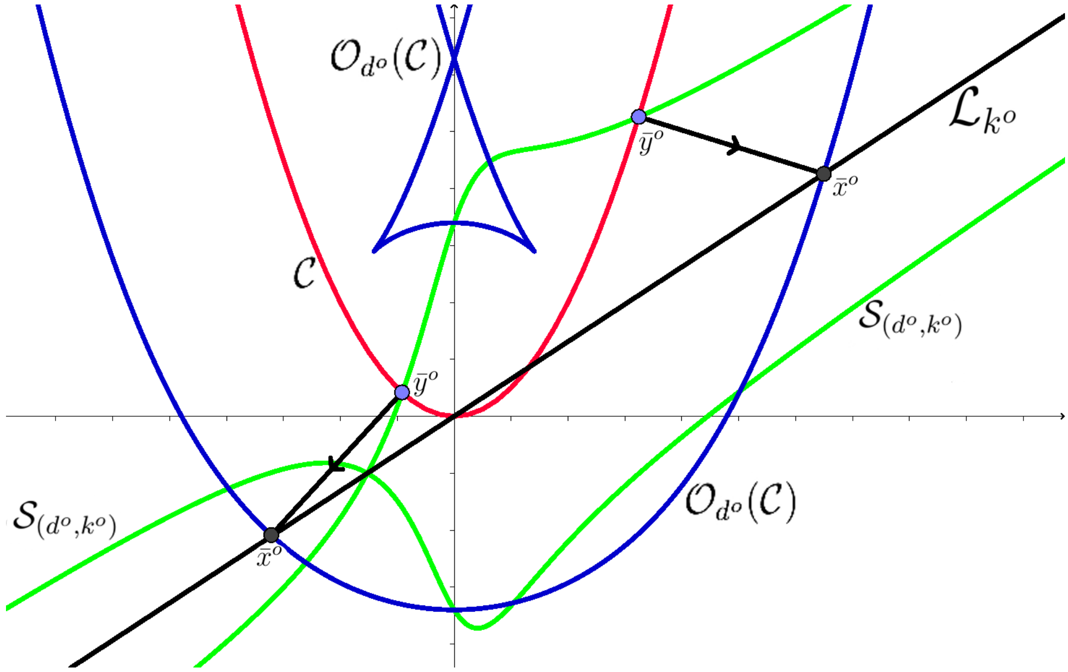

Figure 2.2: The auxiliary curve $\mathcal{S}_{(d^o,k^o)}$ for a parabola. In the figure, the curve $\mathcal{C}$ is pictured in red, $\mathcal{S}_{(d^o,k^o)}$ in green, $\mathcal{O}_{d^o}(\mathcal{C})$ appears in blue and $\mathcal{L}_{k^o}$ in black. The intersection points $\bar{y}_i^o \in \mathcal{C} \cap \mathcal{S}_{(d^o,k^o)}$ with real coordinates are shown as solid blue dots, each of them connected with an arrow to the corresponding associated point $\bar{x}_i^o \in \mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$, shown as a solid black dot.

Remark 2.15. *In Theorem 2.14, if $(\bar{x}^o, \bar{y}^o, u^o)$ is a solution of System 2.2 with $f_1(\bar{y}^o) - k^o f_2(\bar{y}^o) \neq 0$, then $\bar{y}^o$ is a point in $\mathcal{C} \cap \mathcal{S}_{(d^o,k^o)}$. Of course, $\mathcal{C} \cap \mathcal{S}_{(d^o,k^o)}$ may contain other points besides these. For example, consider the following two situations:*

1. $\mathrm{Sing}_a(\mathcal{C}) \subset \mathcal{C} \cap \mathcal{S}_{(d^o,k^o)}$*, but $f_1 - k^o f_2$ vanishes at the affine singularities.*

2. *Since we are working projectively, new intersection points of $\overline{\mathcal{C}}$ and $\overline{\mathcal{S}_{(d^o,k^o)}}$ at infinity may be introduced. But, by Theorem 2.5, we know hat these new intersection points at infinity do not correspond to points on $\mathcal{O}_{d^o}(\mathcal{C})$.*

In the sequel, we will find out how many of the points in $\mathcal{C} \cap \mathcal{S}_{(d^o,k^o)}$ are not associated to points in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$. To count them we will use Bezout's theorem. In the following subsection we analyze this type of points, that we will call fake intersection points.

2.1.3 Fake points (total degree case)

In the next definition, we introduce the notion of fake and non-fake intersection points.

Definition 2.16. *Let $(d^o, k^o) \in \Omega_1$, where Ω_1 is as in Remark 2.10, page 46.*

- *A point in $\overline{\mathcal{C}} \cap \overline{\mathcal{S}}_{(d^o,k^o)}$ not associated to a point in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$ will be called a* fake (intersection) point of $\overline{\mathcal{C}}$ and $\overline{\mathcal{S}}_{(d^o,k^o)}$, for the total degree problem.
- *A point in $\overline{\mathcal{C}} \cap \overline{\mathcal{S}}_{(d^o,k^o)}$ that is associated to a point in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$ will be called a* non-fake (intersection) point of $\overline{\mathcal{C}}$ and $\overline{\mathcal{S}}_{(d^o,k^o)}$, for the total degree problem.

We denote by $\mathcal{F}_{(d^o,k^o)}$ the set of all fake intersection points of $\overline{\mathcal{C}}$ and $\overline{\mathcal{S}}_{(d^o,k^o)}$. In addition, we decompose the set $\mathcal{F}_{(d^o,k^o)}$ in two subsets, denoted *by $\mathcal{F}^{\infty}_{(d^o,k^o)}$ and by $\mathcal{F}^{a}_{(d^o,k^o)}$ the set of fake points at infinity and the set of affine fake points, respectively.*

The set of fake points appears, in this definition, to depend on the choice of (d^o, k^o). However, in Theorem 2.19 we will show that $\mathcal{F}_{(d^o,k^o)}$ is in fact independent from (d^o, k^o).

As a preliminary step in the proof of that theorem, the following corollary (whose proof is immediate from Theorem 2.14, page 47) gives a first characterization of the set of fake points.

Corollary 2.17. *Let $(d^o, k^o) \in \Omega_1$. The fake intersection points of $\overline{\mathcal{C}}$ and $\overline{\mathcal{S}}_{(d^o,k^o)}$ are those points in $\overline{\mathcal{C}} \cap \overline{\mathcal{S}}_{(d^o,k^o)}$ satisfying the equation:*

$$y_0^2(F_1(\bar{y}_h) - k^o F_2(\bar{y}_h))^2 = 0.$$

Remark 2.18. *Note, in particular, that for $(d^o, k^o) \in \Omega_1$ all the points at infinity of $\overline{\mathcal{C}} \cap \overline{\mathcal{S}}_{(d^o,k^o)}$ are fake points, hence a non-fake point is always an affine point.*

Now, we are ready to state and prove the next theorem:

Theorem 2.19. *Let $(d^o, k^o) \in \Omega_1$, where Ω_1 is as in Remark 2.10, page 46.*

(1) $\mathcal{F}^{\infty}_{(d^o,k^o)}$ *is the set of intersection points at infinity of $\overline{\mathcal{C}}$ and $\overline{\mathcal{H}}$.*

(2) $\mathcal{F}_{(d^o,k^o)} \subset \overline{\mathcal{C}} \cap \overline{\mathcal{H}}$.

(3) $\mathcal{F}^{a}_{(d^o,k^o)} = \mathrm{Sing}_a(\mathcal{C})$.

(4) $\mathcal{F} = \displaystyle\bigcap_{(d^o,k^o)\in\Omega_1} \left(\overline{\mathcal{S}^1_{(d^o,k^o)}} \cap \overline{\mathcal{C}}\right)$.

Proof.

(1) Let $\bar{y}_h^o \in \mathcal{F}^{\infty}_{(d^o,k^o)}$. Then $y_0^o = 0$, and substituting in S we get:

$$(F_1^2(\bar{y}_h^o) + F_2^2(\bar{y}_h^o))(y_1^o - k^o y_2^o)^2 = 0$$

Now, observe that the factor $(y_1^o - k^o y_2^o)$ can not vanish, since $(d^o, k^o) \in \Omega_1$, and therefore all the intersections of $\mathcal{C}$ and $\mathcal{L}_{k^o}$ are affine (see the last claim in Theorem 2.14). Therefore one has:

$$H(\bar{y}_h^o) = F_1^2(\bar{y}_h^o) + F_2^2(\bar{y}_h^o) = 0$$

On the other hand, if $\bar{y}_h^o \in \overline{\mathcal{C}} \cap \overline{\mathcal{H}}$ is a point at infinity, with $y_0^o = 0$, upon substitution one gets that $S(d^o, k^o, \bar{y}_h^o) = 0$, and therefore $\bar{y}_h^o \in \overline{\mathcal{C}} \cap \overline{\mathcal{S}_{(d^o,k^o)}}$.

(2) By (1) we know that $\mathcal{F}^{\infty}_{(d^o,k^o)} \subset \overline{\mathcal{C}} \cap \overline{\mathcal{H}}$. Let us see that $\mathcal{F}^{a}_{(d^o,k^o)} \subset \overline{\mathcal{C}} \cap \overline{\mathcal{H}}$. Take $\bar{y}_h^o \in \mathcal{F}^{a}_{(d^o,k^o)}$. Then, since:

$$s(d^o, k^o, \bar{y}^o) = (f_1^2(\bar{y}^o) + f_2^2(\bar{y}^o))(y_1^o - k^o y_2^o)^2 - (d^o)^2(f_1(\bar{y}^o) - k^o f_2(\bar{y}^o))^2 = 0$$

and we have seen in Theorem 2.14 that an affine fake intersection satisfies

$$f_1(\bar{y}^o) - k^o f_2(\bar{y}^o) = 0.$$

this implies that:

$$(f_1^2(\bar{y}^o) + f_2^2(\bar{y}^o))(y_1^o - k^o y_2^o)^2 = 0.$$

And we know that $y_1^o - k^o y_2^o \neq 0$ holds, because our choice of k^o (recall Lemma 2.2 in page 39) excludes the possibility of simultaneously having

$$f(y_1^o, y_2^o) = 0, \quad f_1(\bar{y}^o) - k^o f_2(\bar{y}^o) = 0, \quad y_1^o - k^o y_2^o = 0.$$

Therefore, all affine fake intersections are also intersections of $\mathcal{C}$ with $\mathcal{H}$.

(3) By (3) we know that $\mathcal{F}^{a}_{(d^o,k^o)} \subset \mathcal{C} \cap \mathcal{H}$. Thus, if $\bar{y}^o \in \mathcal{F}^{a}_{(d^o,k^o)}$, then $f_1(\bar{y}^o)^2 + f_2(\bar{y}^o)^2 = 0$. On the other hand, by Theorem 2.14, one has that $f_1(\bar{y}^o) - k^o f_2(\bar{y}^o) = 0$. Therefore, $(1 + (k^o)^2) f_2(\bar{y}^o)^2 = 0$. Thus, if $f_2(\bar{y}^o) \neq 0$, we conclude that $1 + (k^o)^2 = 0$, contradicting the last part of Theorem 2.5, page 41. Therefore $f_2(\bar{y}^o) = 0$, and hence $f_1(\bar{y}^o) = 0$; i.e., $\bar{y}^o$ is an affine singularity of $\mathcal{C}$.

(4) The "$\subset$" inclusion is clear. Let us prove the other inclusion. Considering S as a polynomial in $\mathbb{C}[\bar{y}_h][d, k]$, its coefficients are:

$$(F_1^2 + F_2^2) y_1^2,\ (F_1^2 + F_2^2) y_1\, y_2,\ (F_2^2 + F_1^2) y_2^2,\ y_0^2 F_1^2,\ y_0^2 F_1 F_2, \text{ and } y_0^2 F_2^2.$$

If

$$\overline{y}_h^o \in \bigcap_{(d^o,k^o)\in\Omega_1} \left(\overline{\mathcal{S}^1_{(d^o,k^o)}} \cap \overline{\mathcal{C}}\right)$$

it follows that, in particular,

$$(y_0^o)^2 F_1(\overline{y}_h^o)^2 = 0,\ (y_0^o)^2 F_2(\overline{y}_h^o)^2 = 0.$$

Thus, $y_0^o = 0$ or $F_1(\overline{y}_h^o) = F_2(\overline{y}_h^o) = 0$ holds. In any case, using (3), and Remark 2.18, the point belongs to $\mathcal{F}$. □

Remark 2.20.

1. *From this Theorem one deduces that, although the curve $\mathcal{S}_{(d^o,k^o)}$ is defined depending on $(d^o,k^o) \in \Omega_1$ (see Definition 2.11), the set $\mathcal{F}_{(d^o,k^o)}$ does not depend on the choice of $(d^o,k^o) \in \Omega_1$. Therefore, from now, we will simply* denote by *$\mathcal{F}$ (resp. $\mathcal{F}^a$, $\mathcal{F}^\infty$) the set of fake points for the total degree problem (resp. affine fake points and fake points at infinity), for any choice of $(d^o,k^o) \in \Omega_1$.*

2. *Suppose that the affine origin $(1:0:0) \in \overline{\mathcal{C}} \cap \overline{\mathcal{S}}_{(d^o,k^o)}$. Then, since $y_1 - ky_2 = 0$ holds at $(1:0:0)$, the equation $S(d^o,k^o,1,0,0) = 0$ implies that $(F_1 - k^oF_2)(1,0,0) = (f_1 - k^of_2)(1,0,0) = 0$. Therefore, for $(d^o,k^o) \in \Omega_1$, if the origin is in $\overline{\mathcal{C}} \cap \overline{\mathcal{S}}_{(d^o,k^o)}$, it is always fake.*

The following proposition, which is easily derived from the above results, is the bridge that, combined with the degree invariance (see Corollary 1.25(4), page 21), leads to the degree formulae in the following sections.

Proposition 2.21. *Let Ω_1 be as in Remark 2.10, page 46. For $(d^o,k^o) \in \Omega_1$:*

$$\#\left(\left(\overline{\mathcal{S}_{(d^o,k^o)}} \cap \overline{\mathcal{C}}\right) \setminus \mathcal{F}\right) = \delta.$$

Proof. See Theorem 2.5 (page 41), and Theorem 2.14 (page 47). □

2.2 First Total Degree Formulae

In this section we will derive the first two formulae for the total degree for the total degree $\delta = \deg_{\bar{x}}(g(d,\bar{x}))$. In order to do that, we will use the results of the previous section, especially Theorem 2.14 (page 47). More precisely, in the first subsection we present a formula based on Bezout's Theorem, applied to analyze the intersection of $\overline{\mathcal{C}}$ with the auxiliary curve $\overline{\mathcal{S}_{(d^o,k^o)}}$, for $(d^o,k^o) \in \Omega_1$, where Ω_1 is the open subset of $\mathbb{C}^2$ in Remark 2.10 (page 46). As a first step, to prove that Bezout's Theorem can be applied, we will see that, for $(d^o,k^o) \in \Omega_1$, $\overline{\mathcal{C}}$ and $\overline{\mathcal{S}_{(d^o,k^o)}}$ do not have common components, and we will also see that the multiplicity of intersection of $\overline{\mathcal{C}}$ and $\overline{\mathcal{S}_{(d^o,k^o)}}$ at every non-fake intersection point is one. From these results, and the properties of the auxiliary curve already obtained, we will derive the first formula.

In most cases, given the defining polynomial of a curve $\mathcal{C}$, the formula obtained in the first subsection cannot be applied easily to compute δ for $\mathcal{C}$. The basic problem is that the use of the formula requires the choice of a value (d^o, k^o) in a certain open subset. However, in Subsection 2.2.2 (page 57), we will show that all the information about multiplicity of intersection between $\mathcal{S}$ and $\mathcal{C}$ can be obtained from the hodograph curve $\mathcal{H}$ of $\mathcal{C}$. From this result we will derive a second, deterministic formula (Theorem 2.27 in page 62), that does not require a particular choice of values of the variables (d, k). In order to do that, we will prove that $\sum_{\bar{y}^o \in \mathcal{F}} \mathrm{mult}_{\bar{y}_h^o}(\mathcal{C}, \mathcal{S})$ can be computed by analyzing the multiplicity of intersection of $\mathcal{C}$ and the hodograph at the fake points. This analysis requires to distinguish, as we will see, between the fake points at infinity, and the affine fake points.

2.2.1 Total degree formula involving the auxiliary curve

The following two lemmas analyze the intersection of $\overline{\mathcal{C}}$ with the auxiliary curve $\overline{\mathcal{S}_{(d^o,k^o)}}$, for $(d^o, k^o) \in \Omega_1$.

Lemma 2.22. *Let Ω_1 be as in Remark 2.10. Then, for $(d^o, k^o) \in \Omega_1$, $\overline{\mathcal{C}}$ and $\overline{S_{(d^o,k^o)}}$ have no common component.*

Proof. Let us assume that $\overline{\mathcal{C}}$ and $\overline{S_{(d^o,k^o)}}$ have a common component. Since F is irreducible, one has that there exists a polynomial $K(\bar{y}_h)$ (depending on (d^o, k^o)) such that:

$$S(d^o, k^o, \bar{y}_h) = F(\bar{y}_h)K(\bar{y}_h)$$

Thus $S(d^o, k^o, \bar{y}_h)$ would vanish on every point of $\overline{\mathcal{C}}$. If there were infinitely many points in $\overline{\mathcal{C}} \cap \overline{\mathcal{S}_{(d^o,k^o)}}$ with $F_1 - k^o F_2 \neq 0$, this would imply that there are infinitely many affine points in $\mathcal{C} \cap \overline{\mathcal{S}_{(d^o,k^o)}}$ with $f_1 - k^o f_2 \neq 0$. Then, Theorem 2.14 (page 47) would give an infinite number of affine intersections between the line $x_1 - k^o x_2 = 0$ and the offset. This contradicts the finiteness of the set of solutions of System $\mathfrak{S}_2((d^o, k^o))$ in Theorem 2.5 (page 41). Thus, we may assume that $f_1(\bar{y}^o) - k^o f_2 \bar{y}^o = 0$ for finitely many points $\bar{y}^o \in \mathcal{C}$. Substituting any of these points in the auxiliary polynomial gives:

$$(f_1^2(\bar{y}^o) + f_2^2(\bar{y}^o))(y_1^o - k^o y_2^o) = 0$$

However, since $(d^o, k^o) \in \Omega_1$, one has that there is only a finite number of solutions of System 2.3. Therefore, we conclude that there are infinitely many points on $\mathcal{C}$ where $f_1^2 + f_2^2$ and $f_1 - k^o f_2 = 0$ vanish simultaneously. This implies that $(1 + (k^o)^2)f_2^2 = 0$ for infinitely many points in $\mathcal{C}$. Thus, since $k^o \neq \pm i$, because $(d^o, k^o) \in \Omega_1$, we get that $f_1 = 0, f_2 = 0$ for infinitely many points on $\mathcal{C}$, which is impossible. $\square$

In the following lemma we analyze, for $(d^o, k^o) \in \Omega_1$, the multiplicity of intersection of $\overline{\mathcal{C}}$ and $\overline{S_{(d^o,k^o)}}$ at a non-fake intersection point. Recall (see Remark 2.18, page 50) that non-fake points are always affine.

Lemma 2.23. *For* $(d^o, k^o) \in \Omega_1$, *if* $\bar{y}^o \in \overline{\mathcal{C}} \cap \overline{\mathcal{S}_{(d^o,k^o)}} \setminus \mathcal{F}$, *then* $\mathrm{mult}_{\bar{y}^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}_{(d^o,k^o)}}) = 1$.

Proof. By Theorem 2.19 and by Remark 2.18 (page 50) we know that $\bar{y}^o$ is an affine regular point of $\mathcal{C}$. Therefore, there is only one branch of $\mathcal{C}$ passing through $\bar{y}^o$. Let

$$\mathcal{P}(t) = (\mathcal{P}_1(t), \mathcal{P}_2(t))$$

be a place of $\mathcal{C}$ centered at $\bar{y}^o$. Then the multiplicity of intersection $\mathrm{mult}_{\bar{y}^o}(\mathcal{C}, \mathcal{S}_{(d^o,k^o)})$ is equal to the order in t of $s(d^o, k^o, \mathcal{P}(t))$. Let $\bar{x}^o$ be the point in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$ associated with $\bar{y}^o$. Our construction implies that $\mathrm{mult}_{\bar{x}^o}(\mathcal{O}_{d^o}(\mathcal{C}), \mathcal{L}_{k^o}) = 1$. Thus it is enough to show that $\mathrm{mult}_{\bar{y}^o}(\mathcal{C}, \mathcal{S}_{(d^o,k^o)}) = \mathrm{mult}_{\bar{x}^o}(\mathcal{O}_{d^o}(\mathcal{C}), \mathcal{L}_{k^o})$. The proof will proceed as follows:

- first, we compute the formal power series $s(d^o, k^o, \mathcal{P}(t))$.
- Afterwards, we use $\mathcal{P}(t)$ to determine a place $\mathcal{Q}(t)$ of $\mathcal{O}_{d^o}(\mathcal{C})$ centered at $\bar{x}^o$, and then we obtain $L(k^o, \mathcal{Q}(t))$.

Therefore, the proof will be completed if we can show that $\mathrm{ord}(L(k^o, \mathcal{Q}(t))) = \mathrm{ord}(s(d^o, k^o, \mathcal{P}(t)))$.
Let

$$\begin{cases} f_1(\mathcal{P}(t)) = v_1 + \alpha t + \cdots \\ f_2(\mathcal{P}(t)) = v_2 + \beta t + \cdots \end{cases}$$

for some v_1, v_2, α, β, where $f_1(\bar{y}^o) = v_1$, $f_2(\bar{y}^o) = v_2$. This means that the tangent vector to $\mathcal{C}$ at $\bar{y}^o$ is $(-v_2, v_1)$ and so, there exists λ such that the place $\mathcal{P}(t)$ can be expressed in the form:

$$\mathcal{P}(t) := \begin{cases} \mathcal{P}_1(t) = y_1^o - \lambda v_2 t + \cdots \\ \mathcal{P}_2(t) = y_2^o + \lambda v_1 t + \cdots \end{cases}$$

The following notation will be useful in the rest of the proof:

$$\begin{array}{lll} T_0 = v_1^2 + v_2^2, & T_1 = v_1\alpha + v_2\beta, & T_2 = y_1^o - k y_2^o, \\ T_3 = v_1 - k v_2, & T_4 = v_2 + k v_1, & T_5 = \alpha - k\beta. \end{array}$$

Note that, since $(d^o, k^o) \in \Omega_1$, then T_0, T_2, T_3 are not identically zero. Now, substituting $\mathcal{P}(t)$ into the polynomial $s(d^o, k^o, \bar{y})$ leads to a power series whose zero-order term $A_0(t)$ must vanish (because $\bar{y}^o \in \mathcal{S}$). This term is:

$$A_0(t) = \left({v_1}^2 + {v_2}^2\right)\left(y_1^o - k^o y_2^o\right)^2 - (d^o)^2\left(v_1 - k v_2\right)^2 = T_0 T_2^2 - (d^o)^2 T_3^2 = 0$$

Therefore we get that:

$$(d^o)^2 = \frac{T_0 T_2^2}{T_3^2}.$$

Thus, the first-order term $A_1(t)$ of $s(d^o, k^o, \mathcal{P}(t))$ is:

$$\begin{aligned}
&A_1(t) = 2\Big(({v_1}^2 + {v_2}^2)\,(y_1^o - k^o y_2^o)\,(-\lambda\, v_2 - k^o \lambda\, v_1) + (v_1\alpha + v_2\beta)\,(y_1^o - k^o y_2^o)^2 \\
&- (d^o)^2\,(v_1 - k^o v_2)\,(\alpha - k^o\beta)\Big) = \\
&= 2\,(-\lambda T_0 T_2 T_4 + T_1 T_2^2 - (d^o)^2 T_3 T_5) = 2\left(-\lambda T_0 T_2 T_4 + T_1 T_2^2 - \frac{T_0 T_2^2}{T_3^2} T_3 T_5\right) = \\
&2\frac{T_2}{T_3}\,(-\lambda T_0 T_3 T_4 + T_1 T_2 T_3 - T_0 T_2 T_5)
\end{aligned}$$

Next, using $\mathcal{P}(t)$, we generate a place $\mathcal{Q}(t)$ of $\mathcal{O}_{d^o}(\mathcal{C})$ centered at $\bar{x}^o$, where $(d^o)^2 = T_0 T_2^2/T_3^2$. First, note that, since p is a regular point, $v_1^2 + v_2^2 \neq 0$. Therefore the order of

$$f_1^2(\mathcal{P}(t)) + f_2^2(\mathcal{P}(t)) = (v_1^2 + v_2^2) + 2(v_1\alpha + v_2\beta)t + \cdots$$

is zero. Thus, it is a unit, and hence

$$\frac{1}{\sqrt{f_1^2(\mathcal{P}(t)) + f_2^2(\mathcal{P}(t))}}$$

can be expressed as the following formal power series.

$$\frac{1}{\sqrt{f_1^2(\mathcal{P}(t)) + f_2^2(\mathcal{P}(t))}} = \frac{1}{\sqrt{v_1^2 + v_2^2}} - \frac{v_1\alpha + v_2\beta}{(v_1^2 + v_2^2)^{3/2}} t + \cdots$$

And so:

$$\begin{cases}
\dfrac{f_1(\mathcal{P}(t))}{\sqrt{f_1^2(\mathcal{P}(t)) + f_2^2(\mathcal{P}(t))}} = \dfrac{v_1}{\sqrt{T_0}} + \left(\dfrac{\alpha}{\sqrt{T_0}} - \dfrac{T_1\, v_1}{{T_0}^{3/2}}\right) t + \cdots \\
\dfrac{f_2(\mathcal{P}(t))}{\sqrt{f_1^2(\mathcal{P}(t)) + f_2^2(\mathcal{P}(t))}} = \dfrac{v_2}{\sqrt{T_0}} + \left(\dfrac{\beta}{\sqrt{T_0}} - \dfrac{T_1 v_2}{{T_0}^{3/2}}\right) t + \cdots
\end{cases}$$

Therefore $\mathcal{Q}(t)$ is one of the two places:

$$\mathcal{Q}^{\pm}(t) = (\mathcal{Q}_1^{\pm}(t), \mathcal{Q}_2^{\pm}(t)) = \mathcal{P}(t) \pm d^o \frac{(f_1(\mathcal{P}(t)), f_2(\mathcal{P}(t)))}{\sqrt{f_1^2(\mathcal{P}(t)) + f_2^2(\mathcal{P}(t))}}$$

and so:

$$\begin{cases}
\mathcal{Q}_1^{\pm}(t) = \left(y_1^o \pm \dfrac{d^o v_1}{\sqrt{T_0}}\right) + \left(-\lambda\, v_2 \pm \dfrac{d^o\alpha}{\sqrt{T_0}} \mp \dfrac{d^o\, T_1\, v_1}{{T_0}^{3/2}}\right) t + \cdots \\
\mathcal{Q}_2^{\pm}(t) = \left(y_2^o \pm \dfrac{d^o v_2}{\sqrt{T_0}}\right) + \left(\lambda\, v_1 \pm \dfrac{d^o\beta}{\sqrt{T_0}} \mp \dfrac{d^o\, T_1\, v_2}{{T_0}^{3/2}}\right) t + \cdots
\end{cases}$$

Substituting $\mathcal{Q}^{\pm}(t)$ in the equation of the line $\mathcal{L}_{k^o}$ one has:

$$\mathcal{Q}_1^{\pm}(t) - k^o\mathcal{Q}_2^{\pm}(t) = \left(y_1^o \pm \frac{d^o v_1}{\sqrt{T_0}} - k^o\left(y_2^o \pm \frac{d^o v_2}{\sqrt{T_0}}\right)\right) +$$
$$t\left(-\lambda\, v_2 \pm \frac{d^o\alpha}{\sqrt{T_0}} \mp \frac{d^o T_1 v_1}{{T_0}^{3/2}} - k^o\left(\lambda\, v_1 \pm \frac{d^o\beta}{\sqrt{T_0}} \mp \frac{d^o\, T_1 v_2}{{T_0}^{3/2}}\right)\right) + \cdots =$$
$$\left(T_2 \pm d^o\frac{T_3}{\sqrt{T_0}}\right) + t\left(-\lambda T_4 \pm \frac{d^o\, T_5}{\sqrt{T_0}} \mp \frac{d^o\, T_1\, T_3}{T_0^{3/2}}\right) + \cdots$$

Now, since $\mathrm{mult}_{\bar{x}^o}(\mathcal{O}_{d^o}(\mathcal{C}), \mathcal{L}_{k^o}) = 1$, one has that

$$B_0 = T_2 \pm d^o\frac{T_3}{\sqrt{T_0}} = 0, \text{ and } B_1 = \left(-\lambda T_4 \pm \frac{d^o\, T_5}{\sqrt{T_0}} \mp \frac{d^o\, T_1\, T_3}{T_0^{3/2}}\right) \neq 0$$

Therefore

$$d^o = \mp\frac{T_2\sqrt{T_0}}{T_3}$$

Substituting the above equality in B_1 one gets

$$B_1 = \left(-\lambda T_4 - \frac{T_2\, T_5}{T_3} + \frac{T_1\, T_2\, T_3}{T_0\, T_3}\right) = \frac{1}{T_0\, T_3}\left(-\lambda T_0\, T_3\, T_4 - T_0\, T_2\, T_5 + T_1\, T_2\, T_3\right);$$

note that this result does not depend on the previous choice of sign. We observe that $2T_0\, T_2 A_1 = B_1$. Thus, since T_2, T_0 are both non-zero, one has that $A_1 \neq 0$ and hence $\mathrm{mult}_{\bar{y}^o}(\mathcal{C}, \mathcal{S}_{(d^o,k^o)}) = 1$. □

Applying the previous lemmas one may derive the following:

Theorem 2.24 (First total degree formula for offset curves). *Let Ω_1 be as in Remark 2.10, page 46. Then, if (d^o, k^o) in Ω_1, it holds that:*

$$\boxed{\delta = \deg_{\bar{x}}(\mathcal{O}_d(\mathcal{C})) = \deg_{\bar{x}}(g(d,\bar{x})) = 2\left(\deg_{\bar{y}}(\mathcal{C})\right)^2 - \sum_{\bar{y}_h^o\in\mathcal{F}} \mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}_{(d^o,k^o)}}) \qquad (2.6)}$$

Proof. By Lemma 2.22 (page 53) and Lemma 2.9 (page 45) (see also Remark 2.10, page 46), and by Bézout's Theorem, we know that for $(d^o, k^o) \in \Omega_1$,

$$\deg(\mathcal{C})\deg(\mathcal{S}_{(d^o,k^o)}) = 2\left(\deg(\mathcal{C})\right)^2 = \sum_{\bar{y}_h^o\in\overline{\mathcal{C}}\cap\overline{\mathcal{S}_{(d^o,k^o)}}} \mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}_{(d^o,k^o)}}).$$

We decompose this sum as follows:

$$\sum_{\bar{y}_h^o\in\overline{\mathcal{C}}\cap\overline{\mathcal{S}_{(d^o,k^o)}}} \mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}_{(d^o,k^o)}}) = \sum_{\bar{y}_h^o\in\overline{\mathcal{C}}\cap\overline{\mathcal{S}_{(d^o,k^o)}}\setminus\mathcal{F}} \mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}_{(d^o,k^o)}}) + \sum_{\bar{y}_h^o\in\mathcal{F}} \mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}_{(d^o,k^o)}}).$$

Now, by Proposition 2.21 (page 52), the cardinal of the set $\overline{\mathcal{C}} \cap \overline{\mathcal{S}_{(d^o,k^o)}} \setminus \mathcal{F}$ is δ. And by Lemma 2.23 (page 54), for $\bar{y}_h^o \in \overline{\mathcal{C}} \cap \overline{\mathcal{S}_{(d^o,k^o)}} \setminus \mathcal{F}$, we have $\mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}_{(d^o,k^o)}}) = 1$. Therefore:

$$2\,(\deg(\mathcal{C}))^2 = \delta + \sum_{\bar{y}_h^o \in \mathcal{F}} \mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}_{(d^o,k^o)}}),$$

and the formula holds. □

2.2.2 Total degree formula involving the hodograph

The main drawback of the degree formula obtained in Theorem 2.24 is that it requires the use of a value (d^o, k^o) in Ω_1, and this open set can be computationally hard to determine for a particular curve. However, in this subsection, we will overcome this problem, by showing that all the information about multiplicity of intersection between $\mathcal{S}$ and $\mathcal{C}$ is encoded in the hodograph curve $\mathcal{H}$ of $\mathcal{C}$. Therefore, we will be able to state a new, deterministic formula (see Theorem 2.27 in page 62), where no particular choice of values of the variables (d, k) appear.

In order to do that, the basic idea consists in proving that $\sum_{\bar{y}^o \in \mathcal{F}} \mathrm{mult}_{\bar{y}_h^o}(\mathcal{C}, \mathcal{S})$ can be computed by analyzing the multiplicity of intersection of $\mathcal{C}$ and the hodograph at the fake points. In the following we see how the fake intersection points of $\overline{\mathcal{C}}$ and $\overline{\mathcal{S}}$, i.e. the points in $\mathcal{F}$, and their multiplicities are related to the intersection points of $\overline{\mathcal{C}}$ and $\overline{\mathcal{H}}$. In this analysis we will see that the fake points at infinity, and the affine fake points have a slightly different behavior. For this reason, we will treat them separately. From Theorem 2.19 (page 50) we know that the points in $\mathcal{F}$ can be determined by computing the intersection points of $\overline{\mathcal{C}}$ and $\overline{\mathcal{H}}$ at infinity, plus the affine singularities of $\overline{\mathcal{C}}$. However, in order to adapt the degree formula in Theorem 2.24, we still have to analyze how $\mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}})$ and $\mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{H}})$ are related, for $\bar{y}_h^o \in \mathcal{F}$. In order to do that, we distinguish between affine fake points, and fake points at infinity. The next lemma shows the behavior of the multiplicity of intersection for points in $\mathcal{F}^a$. Recall that these points are the affine singularities of $\mathcal{C}$.

Lemma 2.25. *Let Ω_1 be as in Remark 2.10, page 46. There is an open subset $\Omega_2^1 \subset \Omega_1$ such that, if (d^o, k^o) in Ω_2^1, and $\bar{y}^o \in \mathcal{F}^a$, then:*

$$\mathrm{mult}_{\bar{y}^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}_{(d^o,k^o)}}) = \mathrm{mult}_{\bar{y}^o}(\overline{\mathcal{C}}, \overline{\mathcal{H}}).$$

Proof. Note that, if $\mathcal{C}$ is a line, then $\mathcal{F}^a = \emptyset$, and in this case the proof is trivial. Thus let us assume that $\mathcal{C}$ is not a line. Let $\mathcal{P}(t) = (\mathcal{P}_1(t), \mathcal{P}_2(t))$ be any place of $\mathcal{C}$ centered at $\bar{y}^o$. We will show that $\mathrm{ord}(h(\mathcal{P}(t))) = \mathrm{ord}(s(d^o, k^o, \mathcal{P}(t)))$, from where one deduces the result. Since $f(\mathcal{P}(t)) = 0$, taking the derivative of this expression with respect to t one gets:

$$f_1(\mathcal{P}(t))\mathcal{P}_1'(t) + f_2(\mathcal{P}(t))\mathcal{P}_2'(t) = 0$$

Let us assume, w.l.o.g., that $\mathrm{ord}(\mathcal{P}_1') \leq \mathrm{ord}(\mathcal{P}_2')$. Then the above equality gives that (note that if $\mathcal{P}_1'(t)$ is identically zero, then $\mathcal{C}$ is the line given by $y_1 = y_1^o$ and, we have excluded lines):

$$f_1(\mathcal{P}(t)) = -\frac{\mathcal{P}_2'(t)}{\mathcal{P}_1'(t)} f_2(\mathcal{P}(t))$$

Now let

$$\begin{cases} \mathcal{P}_1(t) = y_1^o + \lambda_1 t^{r_1} + \cdots \\ \mathcal{P}_2(t) = y_2^o + \mu_1 t^{s_1} + \cdots \end{cases}$$

be the formal power series defining $\mathcal{P}(t)$. Then

$$\begin{cases} \mathcal{P}_1'(t) = \lambda_1 r_1 t^{r_1-1} + \cdots \\ \mathcal{P}_2'(t) = \mu_1 s_1 t^{s_1-1} + \cdots \end{cases}$$

where $r_1 \leq s_1$, $\lambda_1\mu_1 \neq 0$. And using this in the quotient one has that:

$$f_1(\mathcal{P}(t)) = -\frac{\mathcal{P}_2'(t)}{\mathcal{P}_1'(t)} f_2(\mathcal{P}(t)) = -\frac{\mu_1 s_1 t^{s_1-1} + \cdots}{\lambda_1 r_1 t^{r_1-1} + \cdots} f_2(\mathcal{P}(t))$$

Note that the order of the series in the numerator is lower or equal to the power in the denominator. After dividing both numerator and denominator by this power, we get a series in the denominator that is a unit in the ring of formal power series. This means that we may write:

$$f_1(\mathcal{P}(t)) = -c(t) f_2(\mathcal{P}(t))$$

where $c(t) \in \mathbb{C}((t))$ is a formal power series whose order is $s_1 - r_1$.

Now, using this expression, we substitute $\mathcal{P}(t)$ in the polynomials h and s , defining $\mathcal{H}$ and $\mathcal{S}$ respectively, to get:

$$h(\mathcal{P}(t)) = f_1(\mathcal{P}(t))^2 + f_2(\mathcal{P}(t))^2 = (1 + c(t)^2) f_2(\mathcal{P}(t))^2$$

In this situation, we observe that $1 + c(t)^2 \neq 0$, since otherwise $\mathcal{C}$ and $\mathcal{H}$ would have infinitely many common points, which is impossible because $\mathcal{C}$ is assumed to be non normal-isotropic. Therefore, it holds that

$$\mathrm{ord}(h(\mathcal{P}(t)))) = \mathrm{ord}(1 + c^2) + 2\,\mathrm{ord}(f_2(\mathcal{P}(t))) = 2\,\mathrm{ord}(f_2(\mathcal{P}(t))).$$

Since $\bar{y}^o$ is a singularity of $\mathcal{C}$, we know that $f_2(\bar{y}^o) = 0$. Hence:

$$\mathrm{ord}(f_2(\mathcal{P}(t))) \geq 1$$

On the other hand, one has:

$$s(d^o, k^o, \mathcal{P}(t)) = h(\mathcal{P}(t))(\mathcal{P}_1(t) - k^o\mathcal{P}_2(t))^2 - (d^o)^2(f_1(\mathcal{P}(t)) - k^o f_2(\mathcal{P}(t)))^2$$

and $L(k^o, \bar{y}^o) = y_1^o - k^o y_2^o = \mathcal{P}_1(0) - k^o \mathcal{P}_2(0) \neq 0$ (recall the last claim in Theorem 2.5, page 41). Therefore, upon substitution of $\mathcal{P}(t)$ into the equation of $\mathcal{L}_{k^o}$, a power series of order 0 is obtained. Thus, $\operatorname{ord}(h(\mathcal{P}(t))L(k^o, \mathcal{P}(t))) = \operatorname{ord}(h(\mathcal{P}(t)))$, while substituting $\mathcal{P}(t)$ into $f_1 - k^o f_2$ one gets that:

$$(f_1(\mathcal{P}(t)) - k^o f_2(\mathcal{P}(t)))^2 = (c(t) - k^o)^2 f_2(\mathcal{P}(t))^2$$

Therefore:

$$s(d^o, k^o, \mathcal{P}(t)) = \left((1 + c^2)L(k^o, \mathcal{P}(t))^2 - (d^o)^2(c - k^o)^2\right) f_2(\mathcal{P}(t))^2$$

and the term in parenthesis is a power series whose term of order 0 can only vanish for a finite set of values k^o. Let Ω_2^1 be the intersection of Ω_1 with the complementary of this finite set. Therefore, for $k^o \in \Omega_2^1$ one has that:

$$\operatorname{ord}(h(\mathcal{P}(t))) = \operatorname{ord}(s(d^o, k^o, \mathcal{P}(t)))$$

If $\operatorname{ord}(\mathcal{P}_1') > \operatorname{ord}(\mathcal{P}_2')$ the above discussion can be repeated with the roles of f_1 and f_2 interchanged. $\square$

Once the multiplicity of intersection at $\mathcal{F}^a$ has been studied, we proceed to analyze the points in $\mathcal{F}^\infty$. As we have already mentioned, the result is slightly different.

Lemma 2.26. *Let Ω_1 be as in Remark 2.10, page 46. There is an open subset $\Omega_2 \subset \Omega_1$ such that, if (d^o, k^o) in Ω_2, and $\bar{y}_h^o \in \mathcal{F}^\infty$, then it holds that*

1. *If $y_2^o \neq 0$, $\operatorname{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}_{(d^o,k^o)}}) = \min\left(\operatorname{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{H}}), \operatorname{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, y_0^2 F_1^2)\right)$*
2. *If $y_2^o = 0$, $\operatorname{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}_{(d^o,k^o)}}) = \min\left(\operatorname{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{H}}), \operatorname{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, y_0^2 F_2^2)\right)$*

Proof. We prove (1). A similar reasoning can be applied to prove (2). Thus, let us assume that $y_2^o \neq 0$. Now, let $P(t) = (\mathcal{P}_0(t) : \mathcal{P}_1(t) : 1)$ be a place of $\overline{\mathcal{C}}$ centered at $\bar{y}_h^o$. We observe that

$$\operatorname{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{H}}) = \operatorname{ord}(H(\mathcal{P}(t)))$$

and that

$$\operatorname{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, y_0^2 F_1^2) = 2\operatorname{ord}\big(\mathcal{P}_0(t)) + 2\operatorname{ord}(F_1(\mathcal{P}(t))\big),$$

while

$$\operatorname{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}_{(d^o,k^o)}}) = \operatorname{ord}\big(S(d^o, k^o, \mathcal{P}(t))\big)$$

Reasoning as in Lemma 2.25, and assuming that $\operatorname{ord}(\mathcal{P}_1') \leq \operatorname{ord}(\mathcal{P}_0')$, one gets that:

$$F_1(\mathcal{P}(t)) = -\frac{\mathcal{P}_0'(t)}{\mathcal{P}_1'(t)} F_0(\mathcal{P}(t)) = c(t) F_0(\mathcal{P}(t)),$$

where $c(t)$ is a formal power series whose order is

$$\operatorname{ord}(\mathcal{P}_0'(t)) - \operatorname{ord}(\mathcal{P}_1'(t)).$$

Since F is a homogeneous polynomial, by Euler's identity, and taking into account that $F(\mathcal{P}(t)) = 0$, one has that:

$$0 = \mathcal{P}_0(t)F_0(\mathcal{P}(t)) + \mathcal{P}_1(t)F_1(\mathcal{P}(t)) + 1 \cdot F_2(\mathcal{P}(t)).$$

And so

$$F_2(\mathcal{P}(t)) = -(\mathcal{P}_1(t)F_1(\mathcal{P}(t)) + \mathcal{P}_0(t)F_0(\mathcal{P}(t))) = -(\mathcal{P}_1(t)c(t) + \mathcal{P}_0(t))F_0(\mathcal{P}(t))$$

Replacing this in $H(\mathcal{P}(t)) = F_1^2(\mathcal{P}(t)) + F_2^2(\mathcal{P}(t))$ leads to:

$$F_1^2(\mathcal{P}(t)) + F_2^2(\mathcal{P}(t)) = c(t)^2F_0^2(\mathcal{P}(t)) + (\mathcal{P}_1(t)c(t) + \mathcal{P}_0(t))^2F_0^2(\mathcal{P}(t)) = \\ \left(c(t)^2 + (\mathcal{P}_1(t)c(t) + \mathcal{P}_0(t))^2\right) F_0^2(\mathcal{P}(t)).$$

Now we observe that $\operatorname{ord}(\mathcal{P}_1(t)c(t)) \geq \operatorname{ord}(c(t))$ and $\operatorname{ord}(\mathcal{P}_0(t)) \geq \operatorname{ord}(c(t))$. Therefore,

$$\operatorname{ord}(H(\mathcal{P}(t))) \geq 2\operatorname{ord}(c(t)) + 2\operatorname{ord}(F_0(\mathcal{P}(t))).$$

In Example 2.29 one may check that this inequality might in fact be strict.

Next, we substitute $\mathcal{P}(t)$ into $S(d^o, k^o, \bar{y}_h)$. Reasoning as in the affine case, one has that $\bar{y}_h^o$ does not belong to $\mathcal{L}_{k^o}$, and so $\operatorname{ord}\left(H(\mathcal{P}(t))L(k^o, \mathcal{P}(t))^2\right) = \operatorname{ord}(H(\mathcal{P}(t)))$. Let us analyze the term obtained when we replace $\bar{y}^h$ with $\mathcal{P}(t)$ in $y_0^2(F_1(\bar{y}_h) - k^oF_2(\bar{y}_h))^2$. We get:

$$\mathcal{P}_0^2(t)(F_1(\mathcal{P}(t)) - k^oF_2(\mathcal{P}(t)))^2 = \mathcal{P}_0^2(t)\big(c(t)F_0(\mathcal{P}(t)) + k^o(\mathcal{P}_1(t)c(t) + \mathcal{P}_0(t))F_0(\mathcal{P}(t))\big)^2 = \\ \mathcal{P}_0^2(t)(c(t) + k^o(\mathcal{P}_1(t)c(t) + \mathcal{P}_0(t)))^2F_0(\mathcal{P}(t))^2.$$

There is, therefore, an open set $\Omega_2^2 \subset \Omega_2^1$ (see Lemma 2.25, page 57), such that for $(d^o, k^o) \in \Omega_3^2$, the factor $(c(t) + k^o(\mathcal{P}_1(t)c(t) + \mathcal{P}_0(t)))$ has the same order as $c(t)$, because the $\operatorname{ord}(\mathcal{P}_1(t)c(t)) \geq \operatorname{ord}(c(t))$, and $\operatorname{ord}(\mathcal{P}_0(t)) \geq \operatorname{ord}(c(t))$. So we have seen that, if $\operatorname{ord}(\mathcal{P}_1'(t)) \leq \operatorname{ord}(\mathcal{P}_0'(t))$, then:

$$\begin{cases} \operatorname{ord}(H(\mathcal{P}(t))) \geq 2\operatorname{ord}(c(t)) + 2\operatorname{ord}(F_0(\mathcal{P}(t))) \\ \operatorname{ord}(\mathcal{P}_0^2(t)(F_1(\mathcal{P}(t)) - k^oF_2(\mathcal{P}(t)))^2) = 2\operatorname{ord}(\mathcal{P}_0(t)) + 2\operatorname{ord}(c(t)) + 2\operatorname{ord}(F_0(\mathcal{P}(t))) \end{cases}.$$

Therefore the order of $S(d^o, k^o, \mathcal{P}(t))$ depends of the relative position between the orders of $H(\mathcal{P}(t))$ and $\mathcal{P}_0^2(t)(F_1(\mathcal{P}(t)) - k^oF_2(\mathcal{P}(t)))^2$. In most cases the order of $H(\mathcal{P}(t))$ is $2\operatorname{ord}(c(t)) + 2\operatorname{ord}(F_0(\mathcal{P}(t)))$ and so this is also the order of $S(d^o, k^o, \mathcal{P}(t))$. But, for some curves, cancelations occur in this series, and then the order of $S(d^o, k^o, \mathcal{P}(t))$ is controlled by $\mathcal{P}_0^2(t)(F_1(\mathcal{P}(t)) - k^oF_2(\mathcal{P}(t)))^2$. Observe that, even when both $H(\mathcal{P}(t))$ and $\mathcal{P}_0^2(t)(F_1(\mathcal{P}(t)) - k^oF_2(\mathcal{P}(t)))^2$ have the same order, cancelation among them can only occur at certain values k^o. Thus, there is an open set $\Omega_2^3 \subset \Omega_2^2$, such that for $(d^o, k^o) \in \Omega_2^3$, one has

$$\mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{F}}, \overline{\mathcal{S}}) = \min(\mathrm{ord}(H(\mathcal{P}(t))), \mathrm{ord}(\mathcal{P}_0^2(t)(F_1(\mathcal{P}(t)) - k^o F_2(\mathcal{P}(t)))^2)).$$

Now observe that

$$F_1(\mathcal{P}(t)) = c(t) F_0(\mathcal{P}(t))$$

gives

$$\mathrm{ord}(c(t)) = \mathrm{ord}(F_1(\mathcal{P}(t))) - \mathrm{ord}(F_0(\mathcal{P}(t)))$$

and so:

$$\begin{cases} \mathrm{ord}(H(P(t))) \geq 2\,\mathrm{ord}(F_1(\mathcal{P}(t))) \\ \mathrm{ord}\big(\mathcal{P}_0^2(t)(F_1(\mathcal{P}(t)) - k^o F_2(\mathcal{P}(t)))^2\big) = 2\,\mathrm{ord}(\mathcal{P}_0(t)) + 2\,\mathrm{ord}(F_1(\mathcal{P}(t))) \end{cases}.$$

from where one deduces the result, when $\mathrm{ord}(\mathcal{P}_1'(t)) \leq \mathrm{ord}(\mathcal{P}_0'(t))$.

Next, suppose that $\mathrm{ord}(\mathcal{P}_0'(t)) < \mathrm{ord}(\mathcal{P}_1'(t))$. Then we would get that

$$\begin{cases} F_0(\mathcal{P}(t)) = c(t) F_1(\mathcal{P}(t)) \\ F_2(\mathcal{P}(t)) = -\mathcal{P}_1(t) F_1(\mathcal{P}(t)) - \mathcal{P}_0(t) F_0(\mathcal{P}(t)) \end{cases}$$

and so

$H(\mathcal{P}(t)) = F_1^2(\mathcal{P}(t)) + (-\mathcal{P}_1(t) F_1(\mathcal{P}(t)) - c(t)\mathcal{P}_0(t) F_1(\mathcal{P}(t)))^2 =$

$F_1^2(\mathcal{P}(t))\,(1 + (\mathcal{P}_1(t) + c(t)\mathcal{P}_0(t))^2)\,.$

Thus $\mathrm{ord}(H(\mathcal{P}(t))) \geq 2\,\mathrm{ord}(F_1(\mathcal{P}(t)))$. Substituting $\mathcal{P}(t)$ in $y_0^2(F_1(\bar{y}_h) - k^o F_2(\bar{y}_h))^2$ we get:

$\mathcal{P}_0(t)^2(F_1(\mathcal{P}(t)) - k^o(-\mathcal{P}_1(t) F_1(\mathcal{P}(t)) - \mathcal{P}_0(t) F_1(\mathcal{P}(t))))^2 =$

$\mathcal{P}_0^2(t) F_1^2(\mathcal{P}(t))(1 - k^o(\mathcal{P}_1(t) + \mathcal{P}_0(t) c(t)))^2.$

Thus, there is an open set $\Omega_2^4 \subset \Omega_2^3$, such that for $(d^o, k^o) \in \Omega_2^4$, one has

$$\mathrm{ord}\big(\mathcal{P}_0^2(t)(F_1(\mathcal{P}(t)) - k^o F_2(\mathcal{P}(t)))^2\big) = 2\,\mathrm{ord}(\mathcal{P}_0(t)) + 2\,\mathrm{ord}(F_1(\mathcal{P}(t))),$$

because $(1 - k^o(\mathcal{P}_1(t) + \mathcal{P}_0(t) c(t)))$ is of order 0. These are the same results for $\mathrm{ord}(H(\mathcal{P}(t)))$ and $\mathrm{ord}\big(\mathcal{P}_0^2(t)(F_1(\mathcal{P}(t)) - k^o F_2(\mathcal{P}(t)))^2\big)$ that we obtained before, and therefore, taking $\Omega_2 = \Omega_2^4$, the result holds. □

Applying Lemmas 2.25 and 2.26 in combination with Theorem 2.24, one deduces the following formula for the total offset degree, in terms of the hodograph curve.

Theorem 2.27 (Total degree formula based on the hodograph curve). *For $\bar{y}_h^o \in \mathcal{F}^\infty$, let*

$$A_{\bar{y}_h^o} = \begin{cases} \min\left(\text{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{H}}), \text{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, y_0^2 F_1^2)\right) & \text{if } \bar{y}_2^o \neq 0 \\ \min\left(\text{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{H}}), \text{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, y_0^2 F_2^2)\right) & \text{if } \bar{y}_2^o = 0 \end{cases}$$

Then, if $g(d, \bar{x})$ is the generic offset polynomial of $\mathcal{C}$, the following formula holds:

$$\boxed{\delta = \deg_{\bar{x}}(\mathcal{O}_d(\mathcal{C})) = \deg_{\bar{x}}(g(d,\bar{x})) = 2\deg_{\bar{y}}(\mathcal{C}) - \sum_{\bar{y}^o \in \mathcal{F}^a} \text{mult}_{\bar{y}^o}(\overline{\mathcal{C}}, \overline{\mathcal{H}}) - \sum_{\bar{y}_h^o \in \mathcal{F}^\infty} A_{\bar{y}_h^o} \quad (2.7)}$$

Remark 2.28. *If we denote by $\overline{\mathcal{C}} \cap_\infty \overline{\mathcal{H}}$ the set of intersection points of $\overline{\mathcal{C}}$ and $\overline{\mathcal{H}}$ at infinity, then, using Theorem 2.19, the formula in Theorem 2.27 can be rewritten as:*

$$\delta = \deg_{\bar{x}}(g(d,\bar{x})) = 2\deg_{\bar{y}}(\mathcal{C}) - \sum_{\bar{y}^o \in \text{Sing}_a(\overline{\mathcal{C}})} \text{mult}_{\bar{y}^o}(\overline{\mathcal{C}}, \overline{\mathcal{H}}) - \sum_{\bar{y}_h^o \in \overline{\mathcal{C}} \cap_\infty \overline{\mathcal{H}}} A_{\bar{y}_h^o}$$

Theorem 2.27 rises the natural question on whether $A_{\bar{y}_h^o}$ can be taken as $\text{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{H}})$, and therefore whether $\overline{\mathcal{H}}$ can be taken as a substitute of $\overline{\mathcal{S}}$ at infinity, when computing multiplicities of intersection. Most of the examples seem to point that this is the case. However, the next example shows that, in general, this is not true.

Example 2.29 (The lemniscate). *We consider the lemniscate $\mathcal{C}$ given by :*

$$f(y_1, y_2) = (y_1^2 + y_2^2)^2 - 2y_1^2 + 2y_2^2$$

See Figure 2.3; in this figure, the generating curve $\mathcal{C}$ is pictured with a red line and the offset curves are pictured in blue. The generic offset to $\mathcal{C}$ has degree 12*, and is given by:*

$g(d,\bar{x}) = x_1^{12} + x_2^{12} + 6x_1^{10}x_2^2 + 15x_1^8x_2^4 + 20x_1^6x_2^6 + 15x_1^4x_2^8 + 6x_1^2x_2^{10} - 6x_1^{10}d^2 - 30x_1^8x_2^2d^2 - 60x_1^6x_2^4d^2 - 60x_1^4x_2^6d^2 - 30x_1^2x_2^8d^2 - 6x_2^{10}d^2 + 15x_1^8d^4 + 60x_1^6x_2^2d^4 + 90x_1^4x_2^4d^4 + 60x_1^2x_2^6d^4 + 15x_2^8d^4 - 20x_1^6d^6 - 60x_1^4x_2^2d^6 - 60x_1^2x_2^4d^6 - 20x_2^6d^6 + 15x_1^4d^8 + 30x_1^2x_2^2d^8 + 15x_2^4d^8 - 6x_1^2d^{10} - 6x_2^2d^{10} + d^12 - 6x_1^{10} - 18x_1^8x_2^2 - 12x_1^6x_2^4 + 12x_1^4x_2^6 + 18x_1^2x_2^8 + 6x_2^{10} + 24x_1^8d^2 + 48x_1^6x_2^2d^2 - 48x_1^2x_2^6d^2 - 24x_2^8d^2 - 36x_1^6d^4 - 36x_1^4x_2^2d^4 + 36x_1^2x_2^4d^4 + 36x_2^6d^4 + 24x_1^4d^6 - 24x_2^4d^6 - 6x_1^2d^8 + 6x_2^2d^8 + 13x_1^8 + 4x_1^6x_2^2 - 18x_1^4x_2^4 + 4x_1^2x_2^6 + 13x_2^8 - 46x_1^6d^2 - 42x_1^4x_2^2d^2 - 42x_1^2x_2^4d^2 - 46x_2^6d^2 + 45x_1^4d^4 + 42x_1^2x_2^2d^4 + 45x_2^4d^4 - 4x_1^2d^6 - 4x_2^2d^6 - 8d^8 - 12x_1^6 + 20x_1^4x_2^2 - 20x_1^2x_2^4 + 12x_2^6 + 44x_1^4d^2 - 44x_2^4d^2 - 40x_1^2d^4 + 40x_2^2d^4 + 4x_1^4 - 8x_1^2x_2^2 + 4x_2^4 - 16x_1^2d^2 - 16x_2^2d^2 + 16d^4$

This polynomial has been computed by Gröbner basis elimination techniques, using the computer algebra system Singular. Let us compare this with the results of the hodograph-based degree formula. The singularities of $\mathcal{C}$ are three double points: the affine origin

$\bar{y}_h^o = (1:0:0)$ *and the cyclic points at infinity,* $\bar{y}_h^{\pm} = (0:1:\pm i)$*; one can check that the only intersection points of* $\overline{\mathcal{C}}$ *and* $\overline{\mathcal{H}}$ *at infinity are* $\bar{y}_h^{\pm}$*. Therefore* $\mathcal{F}_\infty = \{\bar{y}_h^{\pm}\}$ *and* $\mathcal{F}^a = \{\bar{y}_h^o\}$*. In addition, it is easy to check that*

$$\operatorname{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{H}}) = 4 \textit{ and } \operatorname{mult}_{\bar{y}_h^{\pm}}(\overline{\mathcal{C}}, \overline{\mathcal{H}}) = 10.$$

Therefore, if we use $\overline{\mathcal{H}}$ *as a substitute for* $\overline{\mathcal{S}}$ *at these points, we would deduce that:*

$$\delta = 2n^2 - \operatorname{mult}_{\bar{y}_h^o}(\overline{\mathcal{F}}, \overline{\mathcal{H}}) - \operatorname{mult}_{\bar{y}_h^+}(\overline{\mathcal{F}}, \overline{\mathcal{H}}) - \operatorname{mult}_{\bar{y}_h^-}(\overline{\mathcal{F}}, \overline{\mathcal{H}}) = 2\cdot 4^2 - 4 - 10 - 10 = 8,$$

and this is not the correct answer. *But if we use* $y_0^2F_1^2$ *instead of* $\mathcal{H}$ *(note that* $y_2^{\pm} \neq 0$*) we would get:*

$$\operatorname{mult}_{\bar{y}_h^{\pm}}(\overline{\mathcal{F}}, y_0^2F_1^2) = 8,$$

and hence

$$A_{\bar{y}_h^{\pm}} = \min\left(\operatorname{mult}_{\bar{y}_h^{\pm}}(\overline{\mathcal{F}}, \overline{\mathcal{H}}), \operatorname{mult}_{\bar{y}_h^{\pm}}(\overline{\mathcal{F}}, y_0^2F_1^2)\right) = \min(10, 8) = 8.$$

Therefore, our formula gives the right value for δ*:*

$$\delta = 2n^2 - \operatorname{mult}_{p_1}(F, H) - A_{p_+} - A_{p_-} = 2\cdot 4^2 - 4 - 8 - 8 = 12.$$

This phenomenon with the lemniscate is related with the places of $\overline{\mathcal{C}}$ *at the cyclic points. For instance, at* $\bar{y}_h^+ = (0:1:i)$ *the curve* $\overline{\mathcal{C}}$ *has two branches, and a place for one of the branches of* $\overline{\mathcal{C}}$ *at* $\bar{y}_h^+$ *is:*

$$\mathcal{P}(t) = (\mathcal{P}_0(t) : \mathcal{P}_1(t) : \mathcal{P}_2(t)) = \left(-t : 1 + t + \frac{1}{8}t^3 - \frac{1}{8}t^4 + \frac{15}{128}t^5 + \cdots : i\right)$$

and then we get

$$\begin{cases} H(\mathcal{P}(t)) = 32t^5 + \cdots \\ S(d^o, k^o, \mathcal{P}(t)) = \big(64(d^o)^2(k^o)^2 + 128i(d^o)^2k^o - 64(d^o)^2\big)t^4 + \cdots \\ \mathcal{P}_0^2(t)F_1^2(\mathcal{P}(t)) = 64t^4 + \cdots \end{cases}$$

so that (generically in (d^o, k^o)*) the order of* $S(d^o, k^o, \mathcal{P}(t))$ *is not equal to the order of* $H(\mathcal{P}(t))$*, but it equals the order of* $\mathcal{P}_0^2(t)F_1^2(\mathcal{P}(t))$*.*

2.3 Total Degree Formula Involving Resultants

As we have already discussed, the advantage of the hodograph-based formula in Theorem 2.27 (page 62), compared to the first formula in Theorem 2.24 (page 56), is that

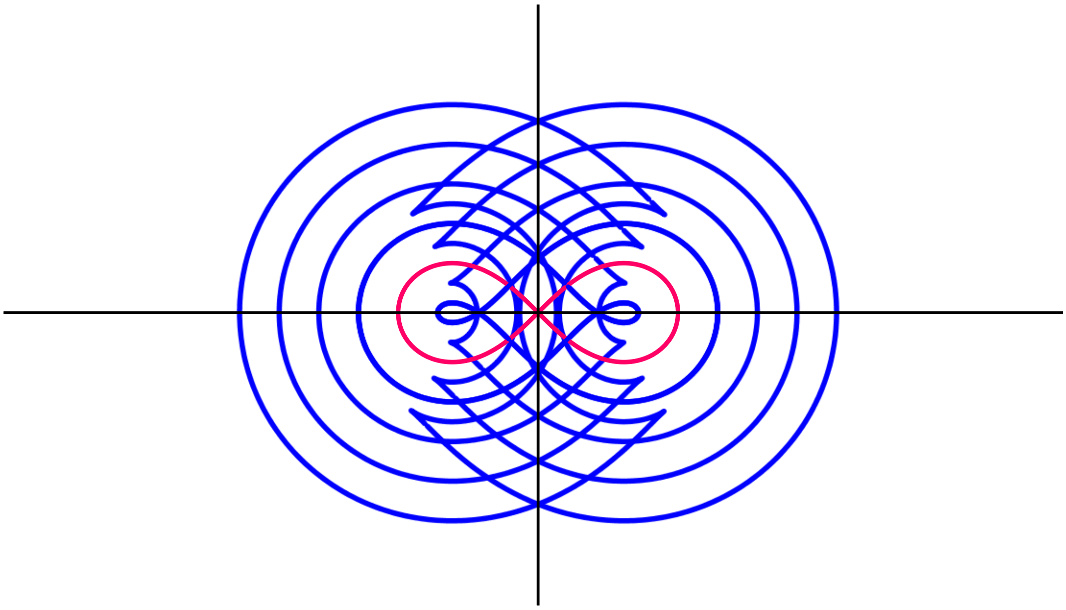

Figure 2.3: A lemniscate and some of its offset curves.

it is deterministic: it does not depend on a particular choice of values for the variables (d, k) in Ω_1 (the open set in Remark 2.10, page 46); of course, if one would know a description of the open set Ω_1 that allows to check computationally whether $(d^o, k^o) \in \Omega_1$ or, even better, to compute points $(d^o, k^o) \in \Omega_1$, then the formula in Theorem 2.27 would be deterministic too. The hodograph-based formula is, therefore, well suited for the theoretical analysis of the offset degree, e.g. in a family of curves. However, from the computational point of view, that formula requires the computation of the set $\mathcal{F}$, and the multiplicities of intersection described in Theorem 2.27. Thus, it would be computationally convenient to have a formula that does not require the set $\mathcal{F}$ to be obtained. In this section we will state and prove one such formula, in Theorem 2.31.

In Section 1.3 (page 27) we already mentioned that the univariate resultant is a natural tool to compute the intersection multiplicities between two projective plane curves. Since our previous formulae in this chapter reduce the generic offset total degree computation to a multiplicity of intersection, this new degree formula is based on a resultant computation. The lemmas in Section 1.3, specially Lemma 1.33 and Lemma 1.34, will play a fundamental role in the proof of this formula. Besides, the proof of Theorem 2.30 uses the Assumption 2.1 (page 36) that $\mathcal{C}$ is not a line through the origin. This in turn implies that the resultant-based formula can not be applied in that case. However, since the degree of the offset to any line is known, namely 2, this poses no practical restriction on the applicability of the formula.

Later, in Sections 3.2 (page 95) and 3.3 (page 99) of Chapter 3, when analyzing the problem of the partial degrees and the degree in d of the generic offset, we will find other situations which involve the intersection of $\mathcal{C}$ with auxiliary curves, depending on parameters, that play the rôle that S plays here, and a concept of fake and non-fake intersection points with properties analogous to those described in the previous results. The next result shows how those properties of an auxiliary curve can be used to establish a degree formula. We will give here (in Theorem 2.30) a general formulation, a common framework, in order to apply this same result to all those situations.

As we have said, we want to consider generic auxiliary curves depending on parameters, but the nature of these parameters varies from one degree problem to another. In the statement of the next theorem we use the variables $\bar{\omega} = (\omega_1, \ldots, \omega_p)$ to represent these parameters. As usual, a particular value of these variables will be denoted by $\bar{\omega}^o$. The polynomial defining the auxiliary curve is then $Z(\bar{\omega}, \bar{y})$, and the set of fake points associated with this problem, denoted by $\mathcal{F}_{\mathcal{Z}}$, is defined as the set of invariant solutions of

$$F(\bar{y}_h) = Z(\bar{\omega}, \bar{y}_h) = 0$$

w.r.t. $\bar{\omega}$, as $\bar{\omega}$ takes values in a certain non-empty Zariski-open subset $\Omega \subset \mathbb{C}^p$ (see hypothesis (3) of Theorem 2.30 for a precise definition). Here, as usual, $F(\bar{y}_h)$ is the form defining the projective closure $\overline{\mathcal{C}}$ of $\mathcal{C}$.

Theorem 2.30. *Let $\mathcal{C}$ be an irreducible affine plane curve, not being a line, and let $Z(\bar{\omega}, \bar{y}_h) \in \mathbb{C}[\bar{\omega}, \bar{y}_h]$ be homogeneous in $\bar{y}_h$ and depending on y_0. Let us suppose that there exists an open set $\Omega \subset \mathbb{C}^p$ such that, for $\bar{\omega}^o \in \Omega$ the following hold:*

1. $\deg_{\bar{y}_h}(Z(\bar{\omega}^o, \bar{y}_h)) = \deg_{\bar{y}_h}(Z(\bar{\omega}, \bar{y}_h))$. *Let $\mathcal{Z}_{\bar{\omega}^o}$ be the plane curve defined by $Z(\bar{\omega}^o, \bar{y}_h)$ (note that $Z(\bar{\omega}^o, \bar{y}_h)$ is non-constant).*

2. *$\mathcal{Z}_{\bar{\omega}^o}$ and $\mathcal{C}$ do not have common components.*

3. *Let*
$$\mathcal{F}_{\mathcal{Z}} = \bigcap_{\bar{\omega}^o \in \Omega} \left(\overline{\mathcal{C}} \cap \overline{\mathcal{Z}_{\bar{\omega}^o}}\right).$$
Then, for every $\bar{y}_h^o \in \left(\overline{\mathcal{Z}_{\bar{\omega}^o}} \cap \overline{\mathcal{C}}\right) \setminus \mathcal{F}_{\mathcal{Z}}$, we require that $\operatorname{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{Z}_{\bar{\omega}^o}}) = 1$.

4. *Let $Z(\bar{\omega}, \bar{y}_h)$ be considered as an element of $(\mathbb{C}[\bar{y}_h])[\bar{\omega}]$, so that one has:*
$$Z(\bar{\omega}, \bar{y}_h) = \sum_{\alpha} Z_\alpha(\bar{y}_h) \bar{\omega}^\alpha$$
for some $Z_\alpha(\bar{y}_h) \in \mathbb{C}[\bar{y}_h]$. Let $\mathcal{Z}_\alpha$ be the closed set defined by $Z_\alpha(\bar{y}_h)$. Then it holds that:
$$\bigcap_{\alpha} (\overline{\mathcal{C}} \cap \overline{\mathcal{Z}_\alpha}) \subset \mathcal{F}_{\mathcal{Z}}.$$

5. $(1:0:0) \notin \left(\overline{\mathcal{Z}_{\bar{\omega}^o}} \cap \overline{\mathcal{C}}\right) \setminus \mathcal{F}_{\mathcal{Z}}$

Then, there exists a non-empty open subset $\Omega_\star \subset \Omega$ such that for $\bar{\omega}^o \in \Omega_\star$:

$$\#\big((\overline{\mathcal{Z}_{\bar{\omega}^o}} \cap \overline{\mathcal{C}}) \setminus \mathcal{F}_{\mathcal{Z}}\big) = \deg_{\bar{y}}\big(\mathrm{PP}_{\bar{\omega}}\left(\mathrm{Res}_{y_0}(F(\bar{y}_h), Z(\bar{\omega}, \bar{y}_h))\right)\big).$$

Proof. We denote by $R(\bar{\omega}, \bar{y}) = \mathrm{Res}_{y_0}(F(\bar{y}_h), Z(\bar{\omega}_h, \bar{y}_h))$. Let $R(\bar{\omega}, \bar{y})$ factor as

$$R(\bar{\omega}, \bar{y}) = M(\bar{y})N(\bar{\omega}, \bar{y},)$$

where M and N are the content and primitive part of R w.r.t. $\bar{\omega}$, respectively. Then M and N are homogeneous polynomials in $\bar{y}$, and $M \in \mathbb{C}[\bar{y}], N \in \mathbb{C}[\bar{\omega}][\bar{y}]$. This implies that M factors over $\mathbb{C}$ in linear factors, namely:

$$M = \prod_{i=1}^{r}(\beta_i y_1 - \alpha_i y_2),$$

with $(\alpha_i, \beta_i) \in \mathbb{C}^2 \setminus \{\bar{0}\}$ for $i = 1, \ldots, r$.

We observe that the leading coefficient $L(\bar{\omega}_h, \bar{y})$ of $Z(\bar{\omega}_h, \bar{y}_h)$ w.r.t. y_0 is a non-zero polynomial in $\mathbb{C}[\bar{\omega}][\bar{y}]$. If L does not depend on $\bar{\omega}$ or any coefficient of L w.r.t. $\{\bar{y}\}$ is a non-zero constant we take $\Gamma^0 = \emptyset$; otherwise we take Γ^0 as the intersection of all curves in $\mathbb{C}^2$ defined by each non-constant coefficient of L w.r.t. $\{\bar{y}\}$. Let $\Omega_\star^1 = \Omega \setminus \Gamma^0$. Since F does not depend on $\bar{\omega}$, for every $\bar{\omega}^o \in \Omega_\star^1$, both leading coefficients of F and $Z(\bar{y}_H, \bar{\omega}^o)$ w.r.t. y_0 do not vanish. In particular, this implies that the resultant specializes properly. That is, if

$$Z^o(\bar{y}_h) = Z(\bar{\omega}^o, \bar{y}_h) \text{ and } R_0(\bar{y}) = \mathrm{Res}_{y_0}(F(\bar{y}_h), Z^o(\bar{y}_h)),$$

then for $\bar{\omega}^o \in \Omega_\star^1$ one has:

$$R_0(\bar{y}) = M(\bar{y})N(\bar{\omega}^o, \bar{y}).$$

By Lemma 1.33 (page 28), and because of the construction of $\Omega_\star^1$ and hypothesis (1), we observe that R and R_0 have the same degree. Hence the degree of $N(\bar{\omega}, \bar{y})$ and $N_0(\bar{y}) = N(\bar{\omega}^o, \bar{y})$ is also the same. Moreover, since N_0 is a homogeneous polynomial in $\bar{y}$, it can be factored as

$$N_0(\bar{y}) = \prod_{j=1}^{s}(\beta'_j y_1 - \alpha'_j y_2),$$

with $(\alpha'_j, \beta'_j) \in \mathbb{C}^2 \setminus \{\bar{0}\}$ for $j = 1, \ldots, s$. Thus

$$R_0(\bar{y}) = M(\bar{y}) \cdot N_0(\bar{y}) = \prod_{i=1}^{r}(\beta_i y_1 - \alpha_i y_2) \prod_{j=1}^{s}(\beta'_j y_1 - \alpha'_j y_2)$$

In this situation, for $\bar{\omega}^o \in \Omega_\star^1$ let $\mathcal{B}_{\bar{\omega}^o} = \left(\overline{\mathcal{Z}_{\bar{\omega}^o}} \cap \overline{\mathcal{C}}\right) \setminus \mathcal{F}_{\mathcal{Z}}$ (these are the non-fake points for the chosen value ω^o). Then, since $\deg(N) = \deg(N_0)$, the proof ends if we find a non-empty open subset $\Omega_\star \subset \Omega_\star^1$ such that

$$\#(\mathcal{B}_{\bar{\omega}^o}) = \deg(N_0) \text{ for } \bar{\omega}^o \in \Omega_\star$$

We start the construction of $\Omega_\star$. First, we prove that there exists a non-empty open subset $\Omega_\star^2 \subset \Omega_\star^1$ such that, if $\bar{\omega}^o \in \Omega_\star^2$, then $\gcd(N_0, M) = 1$. Indeed, first we observe that, because of their construction, $\gcd(N, M) = 1$. Now, for each factor $(\beta_i y_1 - \alpha_i y_2)$ of M, we consider the polynomial $N(\bar{\omega}, \alpha_i, \beta_i)$. This polynomial is not identically zero because $\gcd(N, M) = 1$. Then we set $\Omega_\star^2 = \Omega_\star^1 \setminus (\Gamma_1 \cup \cdots \cup \Gamma_r)$, where Γ_i is the curve in $\mathbb{C}^2$ defined by $N(\bar{\omega}, \alpha_i, \beta_i)$.

Now we prove the existence of a non-empty open subset $\Omega_\star^3 \subset \Omega_\star^2$ such that for $\bar{\omega}^o \in \Omega_\star^3$ the projective lines $\overline{\mathcal{L}_i}$, defined by the equations $\beta_i y_1 - \alpha_i y_2 = 0$, do not contain points of $\mathcal{B}_{\bar{\omega}^o}$; recall that $\beta_1 y_1 - \alpha_i y_2$ is one of the factors of M. For this purpose, observe that $\overline{\mathcal{L}_i}$ meets $\overline{\mathcal{C}}$ in a finite number of points (recall that by assumption $\mathcal{C}$ is irreducible and it is not a line).

Let, for $i = 1, \ldots, r$

$$\Xi_i = \left(\overline{\mathcal{C}} \cap \overline{\mathcal{L}_i}\right) \setminus \mathcal{F}_{\mathcal{Z}}.$$

and let $\bar{y}_h^o \in \Xi_i$. Then the polynomial $Z(\bar{\omega}, \bar{y}_h^o)$ is not identically zero. Otherwise, it would imply that all coefficients of $Z(\bar{\omega}, \bar{y}_h,)$ w.r.t. $\bar{\omega}$ vanish at $\bar{y}_h^o$ and, by hypothesis (5), that implies

$$\bar{y}_h^o \in \bigcap_\alpha (\overline{\mathcal{C}} \cap \overline{\mathcal{Z}_\alpha}) \subset \mathcal{F}_{\mathcal{Z}},$$

which is impossible. Then, if $\Gamma_{\bar{y}_h^o}$ is the curve in $\mathbb{C}^2$ defined by $Z(\bar{\omega}, \bar{y}_h^o)$, let

$$\Omega_\star^3 = \Omega_\star^2 \setminus (\bigcup_{i=1}^r \bigcup_{\bar{y}_h^o \in \Xi_i} \Gamma_{\bar{y}_h^o}).$$

This is an open set because each Ξ_i is finite for $i = 1, \ldots, r$. Let us see that $\Omega_\star^3$ satisfies the requirements. Let $\bar{\omega}^o \in \Omega_\star^3$, and assume that there exists $\bar{y}_h^o \in [\overline{\mathcal{L}_i} \cap \overline{\mathcal{Z}_{\bar{\omega}^o}} \cap \overline{\mathcal{C}}] \setminus \mathcal{F}_{\mathcal{Z}}$. Then, $\bar{y}_h^o \in \Xi_i$. Now, since $\bar{y}_h^o \in \overline{\mathcal{Z}(\bar{\omega}^o)}$, one has that $Z(\bar{\omega}^o, \bar{y}_h^o) = 0$, which is a contradiction since, by construction, $\bar{\omega}^o \notin \Gamma_{\bar{y}_h^o}$.

Finally, the last open subset is constructed. Let $K(\bar{y})$ be the leading coefficient of $F(\bar{y}_h)$ w.r.t. y_0. Note that $K \in \mathbb{C}[\bar{y}]$ is homogeneous. Then, we choose a non-empty Zariski open subset $\Omega_\star^4 \subset \Omega_\star^3$ such that for every $\bar{\omega}^o \in \Omega_\star^4$ it holds that $\gcd(N_0, K) = 1$. For this purpose, let K factor as

$$K = \prod_{i=1}^m (\beta_i'' y_1 - \alpha_i'' y_2),$$

with $(\alpha_i'', \beta_i'') \in \mathbb{C}^2 \setminus \{\bar{0}\}$ for $i = 1, \ldots, m$. We consider the polynomials $N(\omega, \alpha_i'', \beta_i'')$. These polynomials are not identically zero, because otherwise it would imply (note that N is homogeneous in $\bar{y}$) that N has a factor $(\beta_i'' y_1 - \alpha_i'' y_2)$, and N is primitive w.r.t. $\bar{\omega}$. Then, we consider

$$\Omega_\star^4 = \Omega_\star^3 \setminus (\Phi_1 \cup \cdots \cup \Phi_n),$$

where Φ_i is the curve in $\mathbb{C}^2$ defined by $N(\omega, \alpha_i'', \beta_i'')$. Let us see that $\Omega_\star^4$ satisfies the requirements. Suppose that $\bar{\omega}^o \in \Omega_\star^4$ and that there exists a factor $\Lambda = \beta_j' y_1 - \alpha_j' y_2$ of $N_0 = N(\bar{\omega}^o, \bar{y})$ such that $\gcd(\Lambda, K) \neq 0$. Then, there exists $i \in \{1, \ldots, m\}$ such that $\Lambda = \beta_i'' y_1 - \alpha_i'' y_2$. Thus $N(\omega^o, \alpha_i'', \beta_i'') = 0$. That is, $\omega^o \in \Psi_i$ which is a contradiction.

Now, we take $\Omega_\star = \Omega_\star^4$, and we prove that for every $\bar{\omega}^o \in \Omega_\star$, $\#(\mathcal{B}_{\bar{\omega}^o}) = \deg(N_0)$:

(a) Let us see that if $\bar{y}_h^o \in \mathcal{F}_{\mathcal{Z}} \setminus \{(0:0:1)\}$ then $(y_2^o y_1 - y_1^o y_2)$ divides M. Indeed: $\bar{y}_h^o \in \overline{\mathcal{Z}_{\bar{\omega}^o}} \cap \overline{\mathcal{C}}$ for every $\bar{\omega}^o \in \Omega_\star$. Thus, $R_0(\bar{\omega}^o, \bar{y}^o) = 0$ for every $\bar{\omega}^o \in \Omega_\star$. Since the resultant specializes properly in Ω^*, because of the construction of $\Omega_\star^1$, then $R(\bar{\omega}, \bar{y}^o) = M(\bar{y}^o) N(\bar{\omega}, \bar{y}^o)$ vanishes on $\Omega_\star$. Moreover, $N(\bar{\omega}, \bar{y}^o)$ cannot vanish on $\Omega_\star$, since otherwise it would imply that $(y_2^o y_1 - y_1^o y_2)$ divides N, and N is primitive w.r.t. $\bar{\omega}$. Thus, $M(\bar{y}^o) = 0$.

(b) Let us see that every linear factor of N_0 (for every $\bar{\omega}^o \in \Omega_\star$) generates a point in $\mathcal{B}_{\omega^o}$. Indeed: let $(y_2^o y_1 - y_1^o y_2)$ divide N_0 then, because of the construction of $\Omega_\star^4$, there exists y_0^o such that $\bar{y}_h^o = (y_0^o : y_1^o : y_2^o) \in \overline{\mathcal{Z}_{\bar{\omega}^o}} \cap \overline{\mathcal{C}}$. Note that $\bar{y}_h^o \neq (1:0:0)$. Now, taking into account (a), and because of the construction of $\Omega_\star^2$, one has that $\bar{y}_h^o \in \mathcal{B}_{\bar{\omega}^o}$.

(c) Let us see that every point in $\mathcal{B}_{\bar{\omega}^o}$ (for every $\bar{\omega}^o \in \Omega_\star$) generates a factor in N_0. Indeed, let $\bar{y}_h^o \in \mathcal{B}_{\bar{\omega}^o}$, then by hypothesis (5) $A = (y_2^o y_1 - y_1^o y_2) \neq 0$. Thus, A divides R_0, and because of the construction of $\Omega_\star^3$, A does not divide M. Therefore, A divides N_0.

Now $\#(\mathcal{B}_{\bar{\omega}^o}) = \deg(N_0)$ follows from Lemma 1.34 (page 30), from (b), (c), from hypothesis (4), and because $\gcd(M, N_0) = 1$ in $\Omega_\star$. □

Using Theorem 2.30 (page 65), we obtain a deterministic formula for the total degree, requiring the computation of a univariate resultant and gcds.

Theorem 2.31. *Let $\mathcal{C}$ not be a line through the origin. The following formula holds:*

$$\boxed{\delta = \deg(\mathcal{O}_d(\mathcal{C})) = \deg_{\{y_1,y_2\}}\left(\mathrm{PP}_{\{d,k\}}\left(\mathrm{Res}_{y_0}(F(\bar{y}_h), S(d,k,\bar{y}_h))\right)\right) \qquad (2.8)}$$

Proof. In order to prove the theorem, we apply Theorem 2.30 to $\mathcal{C}$, $Z(\bar{\omega}, \bar{y}_h) = S(d,k,\bar{y}_h)$, where $\bar{\omega} = (d,k)$, and $\Omega = \Omega_1$, where Ω_1 is as in Remark 2.10 (page 46). We check that all the hypothesis are satisfied:

- $\mathcal{C}$ is irreducible and it is not a line by assumption.
- Recall that S can be written as:

  $$S(d,k,\bar{y}_h) = H(\bar{y})(y_1 - ky_2)^2 - d^2 y_0^2 (F_1(\bar{y}) - kF_2(\bar{y}))^2.$$

  Thus, since $F_1^2 + F_2^2$ and F_1^2 are not identically zero, S depends on y_0, and hypothesis (1) holds.
- Hypothesis (2) in Theorem 2.30 follows from Lemma 2.22 (page 53).
- Hypothesis (3) and (4) follow from Theorem 2.19(4) (page 50) and from Lemma 2.23 (page 54).
- Hypothesis (5) follows from Remark 2.20 (page 52).

Then, Theorem 2.30 implies that there exists a non-empty open $\Omega_\star \subset \Omega_1$ such that for $(d^o, k^o) \in \Omega_\star$

$$\#([\overline{\mathcal{S}_{(d^o,k^o)}} \cap \overline{\mathcal{C}}] \setminus \mathcal{F}) = \deg_{\{\bar{y}\}}\left(\mathrm{PP}_{\{d,k\}}\left(\mathrm{Res}_{y_0}(F(\bar{y}_h), S(d,k,\bar{y}_h))\right)\right)$$

Now the theorem follows from Proposition 2.21 (page 52). $\square$

In the following example we illustrate the use of the resultant based formula in Theorem 2.31 (page 68).

Example 2.32 (The Cayley Sextic). *We consider the Cayley Sextic $\mathcal{C}$ with homogeneous implicit equation:*

$$f(\bar{y}) = 4(y_1^2 + y_2^2 - y_1 y_0)^3 - 27 y_0^2 (y_1^2 + y_2^2)^2 = 0.$$

See Figure 2.4, where the curve $\mathcal{C}$ is depicted in red, and some of its offset curves in blue. The homogeneous auxiliary polynomial is given by:
$S(d,k,\bar{y}_h) = ((24\,y_1^5 + 48\,y_1^3 y_2^2 - 60\,y_1^4 y_0 + 24\,y_2^4 y_1 - 72\,y_1^2 y_2^2 y_0 - 60\,y_1^3 y_0^2 - 12\,y_2^4 y_0 - 84\,y_1\,y_2^2 y_0^2 - 12\,y_1^2 y_0^3)^2 + (24\,y_1^4 y_2 + 48\,y_2^3 y_1^2 - 48\,y_1^3 y_2\,y_0 + 24\,y_2^5 - 48\,y_2^3 y_1\,y_0 - 84\,y_1^2 y_2\,y_0^2 - 108\,y_2^3 y_0^2)^2)(y_1 - ky_2)^2 - d^2 y_0^2 (24\,y_1^5 + 48\,y_1^3 y_2^2 - 60\,y1^4 y_0 + 24\,y_2^4 y_1 - 72\,y_1^2 y2^2 y_0 - 60\,y_1^3 y_0^2 - 12\,y_2^4 y_0 - 84\,y_1\,y_2^2 y_0^2 - 12\,y_1^2 y_0^3 - k(24\,y_1^4 y_2 + 48\,y_2^3 y_1^2 - 48\,y_1^3 y_2\,y_0 + 24\,y_2^5 - 48\,y_2^3 y_1\,y_0 - 84\,y_1^2 y_2\,y0^2 - 108\,y_2^3 y_0^2))^2$.
And then one gets, for the resultant in Theorem 2.31:
$\mathrm{Res}_{y_0}(F(\bar{y}_h), S(d,k,\bar{y}_h)) = -406239826673664\,y_1^4 (y_1^2 + y_2^2)^{20} y_2^4 (-12\,y_1^3 y_2^3 k^3 d^2 + 36\,y_1^3 y_2^3 k^3 d + 36\,y_1\,{y_2}^5 k^3 d - 24\,y_1\,y_2^5 k^3 d^2 + 12\,{y_1}^5 y_2\,k^3 d^2 + 4\,y_1\,y_2^5 k^3 d^3 - 4\,y_1^5 y_2\,k^3 d^3 + 3\,y_1^6 d^3 k^2 - 81\,y_2^2 y_1^4 d k^2 - 15\,y_2^2 y_1^4 d^3 k^2 - 4\,y_1^3 y_2^3 k^3 - y_1^6 d^3 - 9\,y_1^6 d - 12\,y_2\,y_1^5 k - 18\,d^2 k^2 y_2^6 - 54\,y_2^5 y_1\,kd + 48\,y_2^5 y_1\,kd^2 - 12\,y_2^5 y_1\,d^3 k + 66\,y_2^2 y_1^4 d^2 k^2 - 24\,y_2^2 y1^4 d^2 - 60\,y_2\,y_1^5 kd^2 + 4\,y_1^6 + 3\,y_2^6 d^3 k^2 - 54\,y_2^4 y_1^2 dk^2 + 27\,y_2^4 y_1^2 d + 5\,y_2^4 d^3 y1^2 - 15\,y_2^4 y_1^2 d^3 k^2 - 30\,y2^4 y_1^2 d^2 - 12\,y_2^3 y_1^3 kd^2 - 12\,y_1^6 d^2 k^2 + 54\,y_2\,y_1^5 kd + 60\,{y_2}^4 y_1^2 k^2 d^2 + 12\,y_2\,y_1^5 d^3 k + 27\,k^2 y_2^6 d + 6\,y_1^6 d^2 - {y_2}^6 d^3 + 18\,y_2^2 y_1^4 d + 5\,y_2^2 d^3 y_1^4 + 12\,y_2^2 y_1^4 k^2) \cdot (12\,y_1^3 y_2^3 k^3 d^2 +$

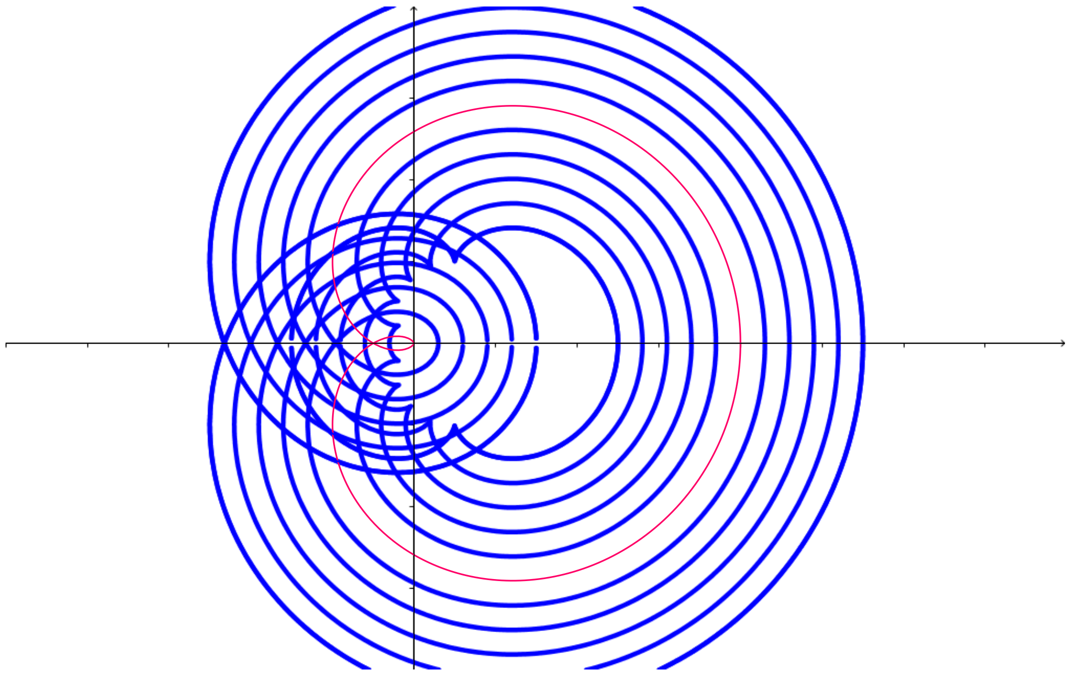

Figure 2.4: The Cayley Sextic and some of its offset curves.

$36\, y_1^3 y_2^3 k^3 d + 36\, y_1\, y_2^5 k^3 d + 24\, y1\, y_2^5 k^3 d^2 - 12\, y_1^5 y_2\, k^3 d^2 + 4\, y_1\, y_2^5 k^3 d^3 - 4\, y_1^5 y_2\, k^3 d^3 + 3\, y_1^6 d^3 k^2 - 81\, y_2{}^2 y_1^4 d k^2 - 15\, y_2^2 y_1^4 d^3 k^2 + 4\, y_1^3 y_2^3 k^3 - y_1^6 d^3 - 9\, y_1{}^6 d + 12\, y_2\, y_1^5 k + 18\, d^2 k^2 y2^6 - 54\, y_2^5 y_1\, kd - 48\, y_2^5 y_1\, kd^2 - 12\, y_2^5 y_1\, d^3 k - 66\, y_2^2 y1^4 d^2 k^2 + 24\, y_2^2 y_1^4 d^2 + 60\, y2\, y_1^5 k d^2 - 4\, y_1^6 + 3\, y_2^6 d^3 k^2 - 54\, y_2^4 y_1^2 d k^2 + 27\, y_2^4 y1^2 d + 5\, y_2^4 d^3 y_1^2 - 15\, y_2^4 y_1^2 d^3 k^2 + 30\, y_2^4 y_1^2 d^2 + 12\, y_2^3 y_1^3 k d^2 + 12\, y_1^6 d^2 k^2 + 54\, y_2\, y_1^5 k d - 60\, y_2^4 y_1^2 k^2 d^2 + 12\, y_2\, y_1^5 d^3 k + 27\, k^2 y_2^6 d - 6\, y_1^6 d^2 - y_2^6 d^3 + 18\, y_2^2 y_1^4 d + 5\, y_2^2 d^3 y_1^4 - 12\, y_2^2 y_1^4 k^2)$.

From this expression one can check that $\mathrm{PP}_{\{d,k\}}\left(\mathrm{Res}_{y_0}(F(\bar{y}_h), S(d,k,\bar{y}_h))\right)$ *is the product of the two last factors appearing above. Thus, using the formula one concludes that:*

$$\delta = \deg(\mathcal{O}_d(\mathcal{C})) = \deg_{\{y_1,y_2\}}\left(\mathrm{PP}_{\{d,k\}}\left(\mathrm{Res}_{y_0}(F(\bar{y}_h), S(d,k,\bar{y}_h))\right)\right) = 12.$$

The generic offset polynomial for $\mathcal{C}$*, obtained using elimination techniques is the following –reducible– polynomial:*

$g(d,\bar{x}) = (-16\, x_2^2 x_1^2 d^2 + 16\, x_2^2 x_1\, d^2 - 15\, x_1^4 + 4\, d^4 x_1^2 - 4\, x_1^3 + d^4 - 8\, x_1^4 d^2 + 16\, x_1^3 d^2 + 30\, x_2^2 d^2 - 48\, x_2^2 x_1^2 d + 4\, x_2^6 - 12\, x_1^5 + 22\, x_1^2 d^2 - 24\, x_2^2 x_1^3 + 12\, x_1^2 x_2^4 - 12\, x_2^4 x_1 + 4\, x_1^6 - 42\, x_1^2 x_2^2 - 4\, x_1\, d^4 - 24\, x_2^4 d + 12\, x_1^4 x_2^2 + 4\, x_2^2 d^4 - 27\, x_2^4 - 12\, x_1\, d^2 - 24\, x_1^4 d + 24\, x_1^2 d^3 - 20\, x_1\, d^3 + 12\, x_1\, x_2^2 d + 12\, x_1^2 d + 24\, x_2^2 d^3 + 12\, x_1^3 d - 8\, d^2 x_2^4 + 4\, d^3) \cdot (-16\, x_2^2 x_1^2 d^2 + 16\, x_2^2 x_1\, d^2 - 15\, x_1^4 + 4\, d^4 x_1^2 - 4\, x_1^3 + d^4 - 8\, x_1^4 d^2 + 16\, x_1^3 d^2 + 30\, x_2^2 d^2 + 48\, x_2^2 x_1^2 d + 4\, x_2^6 - 12\, x_1^5 + 22\, x_1^2 d^2 - 24\, x_2^2 x_1^3 + 12\, x_1^2 x_2^4 - 12\, x_2^4 x_1 + 4\, x_1^6 - 42\, x_1^2 x_2^2 - 4\, x_1\, d^4 + 24\, x_2^4 d + 12\, x_1^4 x_2^2 + 4\, x_2^2 d^4 - 27\, x_2^4 - 12\, x_1\, d^2 + 24\, x_1^4 d - 24\, x_1^2 d^3 + 20\, x_1\, d^3 - 12\, x_1\, x_2^2 d - 12\, x_1^2 d - 24\, x_2^2 d^3 - 12\, x_1^3 d - 8\, d^2 x_2^4 - 4\, d^3)$

and this confirms the result of the formula.

2.4 Total Degree Formula for Rational Curves

The formulae derived in the previous sections are valid for arbitrary irreducible algebraic plane curves, whether rational or not. In this section, we see how a similar reasoning can be adapted for the particular case of rational plane curves given parametrically. There are, at least, two advantages associated with doing this: first, having this formula at our disposal means that there is no need of implicitization for computing the offset degree. Second, the parametric representation of the curve implies a reduction in the dimension of the space where the curve points are represented, This, in turn, results in a simplification of the computational effort required by the degree formula. As we will see, this formula only requires the computation of degrees and gcds of univariate polynomials. This is particularly relevant because of the high importance of parametric curves in the applications, e.g. to CAGD.

In order to do this, the idea is simply to translate the information contained in the auxiliary curve $\mathcal{S}$ into the parameter space. The result of this approach is a univariate auxiliary polynomial, see Definition 2.36 (page 72) that contains precisely that information. Analyzing its invariant solutions leads to the offset degree for parametrically given curves (in Theorem 2.40, page 75). That formula provides an easy to apply alternative to the formula presented in [17]. We also show in this section how this formula applies to the case of polynomial parametrizations, and in this case our formula coincides precisely with the one in [17].

To be more precise, let

$$P(t) = \left(\frac{X(t)}{W(t)}, \frac{Y(t)}{W(t)}\right)$$

be a rational parametrization of a curve $\mathcal{C}$, with $\gcd(X, Y, W) = 1$. Since there is a general proper reparametrization algorithm for rational curves (see e.g. [52], page 193), we assume, without loss of generality, that P is a proper parametrization.

From the parametrization P one can derive a special normal vector to $\mathcal{C}$, as follows. First, we consider the polynomials:

$$\begin{cases} A_1(t) = -(W(t)Y'(t) - W'(t)Y(t)) \\ A_2(t) = W(t)X'(t) - W'(t)X(t), \end{cases}$$

where the prime denotes the derivative w.r.t. t. Note that only one of the polynomials $A_i(t)$ can be identically zero, and in this case $\mathcal{C}$ is a (horizontal or vertical) line. Let $G = \gcd(A_1, A_2)$.

Definition 2.33. *We call*

$$N(t) = \left(\frac{A_1(t)}{G(t)}, \frac{A_2(t)}{G(t)}\right)$$

the associated normal vector *of the parametrization $P(t)$. We denote the components of N by $N(t) = (N_1(t), N_2(t))$.*

Definition 2.34. *The* parametric hodograph *of the parametrization P is defined as*

$$H_P(t) = N_1^2(t) + N_2^2(t).$$

Remark 2.35. *The associated normal vector of P has the following properties:*

- $N_i \in \mathbb{C}[t]$ *for $i = 1, 2$.*
- *If $A_1(t)$ (resp. $A_2(t)$) vanishes identically, then $N(t) = (0, 1)$ (resp $N(t) = (1, 0)$). This case appears if and only if $\mathcal{C}$ is a parallel line to one of the coordinate axis.*
- $\gcd(N_1(t), N_2(t)) = 1$, *because of the construction. In particular, $N(t^o) \neq (0, 0)$ for every $t^o \in \mathbb{C}$. Moreover, there are some $\mu \in \mathbb{N}$ and $Q(t) \in \mathbb{C}[t]$, with* $\gcd(Q(t), W(t)) = 1$, *such that*

$$f_i(P(t)) = f_i\left(\frac{X(t)}{W(t)}, \frac{Y(t)}{W(t)}\right) = \frac{Q(t)}{(W(t))^\mu} N_i(t) \text{ for } i = 1, 2.$$

Recall that $f_i(\bar{y})$ for $i = 1, 2$, are the partial derivatives of $f(\bar{y})$, the irreducible polynomial defining $\mathcal{C}$. Note that the polynomial $Q(\bar{t})$ introduced above is not identically zero. Otherwise, one has $f_i(P(\bar{t})) = 0$ for $i = 1, 2, 3$, and this implies that $f(\bar{y})$ is a constant polynomial, which is a contradiction.

To obtain the degree formula for rational curves we replace $\bar{y}_h$ with $P(t)$ in the auxiliary polynomial $s(d, k, \bar{y})$. One gets:

$$s(d, k, P(t)) = h(P(t))\left(\frac{X(t)}{W(t)} - k\frac{Y(t)}{W(t)}\right)^2 - d^2\big(f_1(P(t)) - kf_2(P(t))\big)^2$$

and taking into account the above remark:

$$s(d, k, P(t)) = \frac{Q^2(t)}{(W(t))^{2\mu+2}}\big(H_P(t)\,(X(t) - kY(t))^2 - d^2W^2(t)\,(N_1(t) - kN_2(t))^2\big).$$

This leads to the following definition:

Definition 2.36. *The* parametric auxiliary polynomial *associated with the parametrization $P(t)$ of $\mathcal{C}$ is*

$$s_P(d, k, t) = H_P(t)\,(X(t) - kY(t))^2 - d^2W^2(t)\,(N_1(t) - kN_2(t))^2\,.$$

We consider $s_P(d, k, t)$ as a polynomial in $\mathbb{C}[t][d, k]$, and write it as follows:

$$\begin{aligned} s_P(d, k, t) &= H_P(t)X^2(t) + H_P(t)Y^2(t)k^2 - 2H_P(t)X(t)Y(t)k \\ &- W^2(t)N_1^2(t)d^2 - W^2(t)N_2^2(t)d^2k^2 + 2W^2(t)N_1(t)N_2(t)d^2k \end{aligned}$$

To connect this with our analysis of fake points in the preceding sections, we look at the content of s_P w.r.t. (d, k). We denote:

$$U(t) = \gcd\left(H_P(t), W^2(t)\right).$$

Recall that $\mathrm{Con}_{(d,k)}$ denotes the content of a polynomial w.r.t. the variables (d, k).

Lemma 2.37. $\mathrm{Con}_{(d,k)}\left(s_P(d, k, t)\right) = U(t)$

Proof. Taking into account that $\gcd(X, Y, W) = 1$ and $\gcd(N_1, N_2) = 1$, one has that:

$\mathrm{Con}_{(d,k)}\left(s_P(d, k, t)\right) =$

$\gcd\left(H_P(t)X^2(t), H_P(t)Y^2(t), H_P(t)X(t)Y(t), W^2(t)N_1^2(t), W^2(t)N_2^2(t), W^2(t)N_1(t)N_2(t)\right) =$

$\gcd\left(H_P(t)\gcd\left(X(t), Y(t)\right)^2, W^2(t)\gcd\left(N_1(t), N_2(t)\right)^2\right) =$

$\gcd\left(H_P(t)\gcd\left(X(t), Y(t)\right)^2, W^2(t)\right) =$

$\gcd\left(H_P(t), W^2(t)\right)\gcd\left(X(t), Y(t), W(t)\right)^2 = \gcd\left(H_P(t), W^2(t)\right).$ □

Proposition 2.38. *Let Ω_1 be as in Remark 2.10, page 46. There is an open subset $\Omega_4 \subset \Omega_1$ such that, for $(d^o, k^o) \in \Omega_4$:*

(1) $\left(\mathcal{S}_{(d^o,k^o)} \cap \mathcal{C}\right) \setminus \mathcal{F} \subset P(\mathbb{C})$

(2) There is a one-to-one correspondence between the points of $\left(\mathcal{S}_{(d^o,k^o)} \cap \mathcal{C}\right) \setminus \mathcal{F}$ and the values $t^o \in \mathbb{C}$ that verify $s_P(d^o, k^o, t^o) = 0$ and $U(t^o) \neq 0$.

Proof. We start by constructing the set Ω_4. Let $A \subset \mathbb{C}$ be the finite set of roots of $Q(t)$ (see Remark 2.35, page 72). Since $\gcd(W, Q) = 1$, if $t^o \in A$, then $W(t^o) \neq 0$. Thus, the set $\mathcal{A}_1 = P(A)$ is a well-defined finite subset of $\mathcal{C}$. Besides, the set $\mathcal{A}_2 = \mathcal{C} \setminus P(\mathbb{C})$ is also a finite subset of $\mathcal{C}$. Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and let $\Omega_0^{\mathcal{A}}$ be the set provided by Lemma 2.6, page 44, when one takes $\mathcal{X} = \mathcal{A}$. We consider $\Omega_4^0 = \Omega_0^{\mathcal{A}} \cap \Omega_1$. Note that $W(t^o) \neq 0$ implies $U(t^o) \neq 0$.

Next, since $\gcd(W, Q) = 1$, if $Q(t^o) = 0$, then $W(t^o) \neq 0$. Thus, since the set of roots of Q is finite and $\gcd(N_1, N_2) = 1$ (see Remark 2.35, page 72), there exists a non-empty open $\Omega_4^1 \subset \Omega_4^0$ such that, for $(d^o, k^o) \in \Omega_4^1$, $Q(t^o) = 0$ implies $s_P(d^o, k^o, t^o) \neq 0$. Similarly, if $W(t^o) = 0$ but $U(t^o) \neq 0$, then $H_P(t^o) \neq 0$ and

$$s_P(d^o, k, t^o) = H_P(t^o)(X(t^o) - kY(t^o))^2.$$

Thus, since $\gcd(X, Y, W) = 1$, it follows that there is an open nonempty subset $\Omega_4^2 \subset \Omega_4^1$ such that for $(d^o, k^o) \in \Omega_4^2$, if $W(t^o) = 0$ but $U(t^o) \neq 0$, then $s_P(d^o, k^o, t^o) \neq 0$.

Using that P is a proper parametrization, we can choose $\Omega_4^3 \subset \Omega_4^2$ such that for $(d^o, k^o) \in \Omega_4^3$, if t_1^o, t_2^o are roots of $s_P(d^o, k^o, t)$ with $U(t_1^o)U(t_2^o) \neq 0$ and $P(t_1^o) = P(t_2^o)$, then $t_1^o = t_2^o$.

Finally, set $\Omega_4 = \Omega_4^3$. We will prove that for $(d^o, k^o) \in \Omega_4$, statements (1) and (2) hold.

(1) Let us assume that $\bar{y}^o \in \big(\mathcal{S}_{(d^o,k^o)} \cap \mathcal{C}\big) \setminus \mathcal{F}$ (recall that all the points in $\big(\mathcal{S}_{(d^o,k^o)} \cap \mathcal{C}\big) \setminus \mathcal{F}$ are affine, see Remark 2.18, page 50). This implies (see Corollary 2.17, page 50 and Theorem 2.14, page 47) that $\bar{y}^o$ is associated with $\bar{x}^o \in \mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$. Because of the construction of Ω_4^0, this implies that there exists $t^o \in \mathbb{C}$ with $P(t^o) = \bar{y}^o$. So, (1) holds.

(2) Let us suppose first that $\bar{y}^o \in \big(\mathcal{S}_{(d^o,k^o)} \cap \mathcal{C}\big) \setminus \mathcal{F}$. We have proved in the preceding paragraph that there is $t^o \in \mathbb{C}$ such that $P(t^o) = \bar{y}^o$. In particular $W(t^o) \neq 0$, and so $U(t^o) \neq 0$. Since $\bar{y}^o \notin \mathrm{Sing}_a(\mathcal{C})$, one concludes from Remark 2.35 (page 72) that $Q(\bar{t}^o) \neq 0$. Then, from

$$s(d^o, k^o, \bar{y}^o) = s(d^o, k^o, P(t^o)) = \frac{Q^2(t^o)}{(W(t^o))^{2\mu+2}} s_P(d^o, k^o, t^o) = 0 \qquad (2.9)$$

one has $s_P(t^o) = 0$. Conversely, let us suppose that $t^o \in \mathbb{C}$ and $s_P(d^o, k^o, t^o) = 0$, with $U(t^o) \neq 0$. The construction of Ω_4^1, resp. Ω_4^2, guarantees that in this case $Q(t^o) \neq 0$, resp. $W(t^o) \neq 0$. Therefore, $\bar{y}^o = P(t^o)$ is a well defined affine point of $\mathcal{C}$, and $s(d^o, k^o, \bar{y}^o) = 0$ follows again from the above equality 2.9. Besides, since

$$f_i(\bar{y}^o) = \frac{Q(t^o)}{(W(t^o))^{\mu}} N_i(t^o) \text{ for } i = 1, 2,$$

and $Q(t^o) \neq 0$, then $\bar{y}^o \notin \mathrm{Sing}_a(\mathcal{C})$. Therefore, $P(t^o) \in \big(\mathcal{S}_{(d^o,k^o)} \cap \mathcal{C}\big) \setminus \mathcal{F}$. Finally, the construction of Ω_4^3 implies that the correspondence is one-to-one. Thus, (2) holds. $\square$

The final tool we need for the degree formula is the following lemma about the multiplicity of intersection of two algebraic curves, where one of the curves is a rational curve, given parametrically. This result follows from the interpretation of the multiplicity of intersection by means of places. Nevertheless, for the sake of completeness, we provide a proof.

Lemma 2.39. *Let $\mathcal{C}_1$ be a rational curve, with a proper parametrization $P(t) = (X(t)/W(t), Y(t)/W(t))$, and let $\mathcal{C}_2$ be an algebraic plane curve, with defining polynomial $\varphi \in \mathbb{C}[y_1, y_2]$. Let $\bar{y}^o \in \mathcal{C}_1 \cap \mathcal{C}_2$, and let us assume that* $\mathrm{mult}_{\bar{y}^o}(\mathcal{C}_1, \mathcal{C}_2) = 1$, *and that $\bar{y}^o \in P(\mathbb{C})$, with $\bar{y}^o = P(t^o)$ for $t^o \in \mathbb{C}$ (note that t^o is uniquely defined). If we denote:*

$$\sigma(t) = W(t)^{\deg_{\bar{y}}(\varphi)} \varphi\left(\frac{X(t)}{W(t)}, \frac{Y(t)}{W(t)}\right)$$

then $\sigma(t)$ is a polynomial, and t^o is a simple root of $\sigma(t)$.

Proof. W.l.o.g., let us assume that $t^o = 0$, and let $\nu = \deg_{\bar{y}}(\varphi)$. Let $\varphi(t) = \sum_{i=0}^{\nu} \varphi_i(\bar{y})$, where φ_i is the homogeneous part of φ of degree i. Then

$$\sigma(t) = W(t)^{\nu} \varphi\left(\frac{X(t)}{W(t)}, \frac{Y(t)}{W(t)}\right) = \sum_{i=0}^{\nu} W(t)^{\nu-i}\varphi_i\left(X(t), Y(t)\right)$$

shows that $\sigma(t)$ is a polynomial. Besides, since $W(t^o) \neq 0$, $W(t)$ is a unit in $\mathbb{C}((t))$. Thus $P(t) = (X(t)/W(t), Y(t)/W(t))$ determines in a natural way a local parametrization $\mathcal{P}(t)$ of $\mathcal{C}_1$ at $\bar{y}^o$. And, since $\mathrm{mult}_{\bar{y}^o}(\mathcal{C}_1, \mathcal{C}_2) = 1$, we have $\mathrm{ord}(\varphi(\mathcal{P}(t))) = 1$. Therefore $\mathrm{ord}(\sigma(t)) = \mathrm{ord}(W(t)^{\nu}\, \varphi(\mathcal{P}(t))) = 1$. □

We are now ready to state and prove an offset degree formula for parametric curves:

Theorem 2.40. *The following formula holds:*

$$\boxed{\delta = \deg(\mathcal{O}_d(\mathcal{C})) = \deg_t\left(\mathrm{PP}_{(d,k)}(s_P(d,k,t))\right) = \deg_t(s_P(d,k,t)) - \deg_t(U(t)) \quad (2.10)}$$

Proof. First note that there is an open set of values (d^o, k^o) for which

$$\deg_t(s_P(d,k,t)) = \deg_t(s_P(d^o,k^o,t))$$

Thus, it suffices to prove that for $(d^o, k^o) \in \Omega_3$, (with Ω_4 as in Proposition 2.38, page 73)

$$\delta = \deg(\mathcal{O}_d(\mathcal{C})) = \deg_t(s_P(d^o,k^o,t)) - \deg_t(U(t))$$

Now, since $\Omega_4 \subset \Omega_1$, one has that for $(d^o, k^o) \in \Omega_4$, $\#\big(\big(\mathcal{S}_{(d^o,k^o)} \cap \mathcal{C}\big) \setminus \mathcal{F}\big) = \delta$ (see Proposition 2.21, page 52). By Proposition 2.38, this implies that there are precisely δ different roots of $s_P(k^o, d^o, t)$ which are not roots of $U(t)$. Besides, by Lemma 2.23 (page 54), and Lemma 2.39, these roots are simple. □

Remark 2.41 (The special case of polynomial parametrizations)**.** *If the parametrization $P(t)$ is polynomial, then $W(t) = 1$ in the above discussion. This implies that, when constructing the associated normal vector we have:*

$$(A_1(t), A_2(t)) = (-Y'(t), X'(t))$$

Therefore $G(t) = \gcd(A_1, A_2) = \gcd(-Y'(t), X'(t))$ and we define, as before:

$$N(t) = \left(\frac{A_1(t)}{G(t)}, \frac{A_2(t)}{G(t)}\right)$$

Notice that now $U(t) = \gcd\left(H_P(t), W^2(t)\right) = 1$, and so s_P is primitive w.r.t. to (d,k). Besides, in this case, it is easy to see that

$$\deg_t(H_P(t)) = 2\max\left(\deg_t(X'(t)), \deg_t(Y'(t))\right) - 2\deg_t(\gcd\left(X'(t), Y'(t)\right)) =$$

$$2\max\left(\deg_t(X(t)),\deg_t(Y(t))\right) - 2 - 2\deg_t(\gcd\left(X'(t),Y'(t)\right))$$

and therefore

$$\deg_t(s_P(d,k,t)) = 2\max\left(\deg_t(X(t)),\deg_t(Y(t))\right) + \deg_t(H(t)) =$$

$$4\max\left(\deg_t(X(t)),\deg_t(Y(t))\right) - 2\deg_t(\gcd\left(X'(t),Y'(t)\right)) - 2$$

This coincides with the result described in [17] for the polynomial case.

To finish this section we will see some examples of the use of Formula in Theorem 2.40.

Example 2.42 (The cardioid)**.** *Consider the cardioid given by the parametrization:*

$$\mathcal{P}(t) = \left(\frac{-1024t^3}{(1+16t^2)^2}, \frac{-128t^2(4t-1)(4t+1)}{(1+16t^2)^2}\right).$$

See Figure 2.5, where the curve $\mathcal{C}$ is depicted in red, and some of its offset curves in blue. To apply the formula in Theorem 2.40 to this curve we begin by computing the associated normal vector

$$N = (48t^2 - 1, 4t(16t^2 - 3)).$$

The parametric auxiliary polynomial associated with $P(t)$ is given by
$s_P(d,k,t) = (1+16\,t^2)^3(-1024\,t^3 - k(-2048\,t^4 + 128\,t^2))^2 - d^2(256\,t^4 + 32\,t^2 + 1)^2(48\,t^2 - 4\,kt(-3+16\,t^2) - 1)^2$
and it holds that $\deg_t(s_P(d,k,t)) = 14$. *We then compute*

$$U(t) = \gcd(H_p, W^2) = \gcd((1+16t^2)^3, (1+16t^2)^4) = (1+16t^2)^3,$$

and using the formula in Theorem 2.40 we get that $\delta = \deg(\mathcal{O}_d(\mathcal{C})) = 14 - 6 = 8$.
In fact, in this example it is possible to compute the generic offset polynomial for $\mathcal{C}$, by using elimination techniques. One has:
$g(d,\bar{x}) = -x_1^8 - 4x_1^6x_2^2 + 3x_1^6d^2 - 6x_1^4x_2^4 + 9x_1^4x_2^2d^2 - 3x_1^4d^4 - 4x_1^2x_2^6 + 9x_1^2x_2^4d^2 - 6x_1^2x_2^2d^4 + x_1^2d^6 - x_2^8 + 3x_2^6d^2 - 3x_2^4d^4 + x_2^2d^6 - 16x_1^6x_2 - 48x_1^4x_2^3 + 36x_1^4x_2d^2 - 48x_1^2x_2^5 + 72x_1^2x_2^3d^2 - 24x_1^2x_2d^4 - 16x_2^7 + 36x_2^5d^2 - 24x_2^3d^4 + 4x_2d^6 + 32x_1^6 + 48x_1^4d^2 - 96x_1^2x_2^4 + 240x_1^2x_2^2d^2 - 84x_1^2d^4 - 64x_2^6 + 192x_2^4d^2 - 132x_2^2d^4 + 4d^6 + 256x_1^4x_2 + 256x_1^2x_2^3 + 64x_1^2x_2d^2 + 256x_2^3d^2 - 320x_2d^4 - 256x_1^4 + 512x_1^2d^2 - 256d^4$,
and this confirms the above result.

Example 2.43 (The Nodal Cubic)**.** *We consider the nodal cubic $\mathcal{C}$ (Descartes Folium) given by the parametrization:*

$$P(t) = \left(\frac{t}{1+t^3}, \frac{t^2}{1+t^3}\right).$$

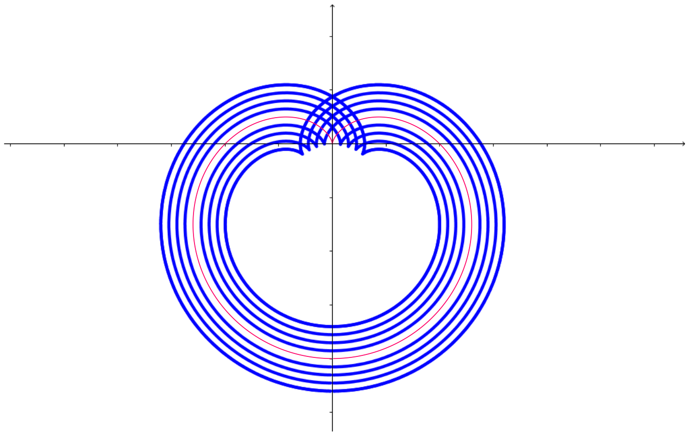

Figure 2.5: Some offset curves of the cardioid.

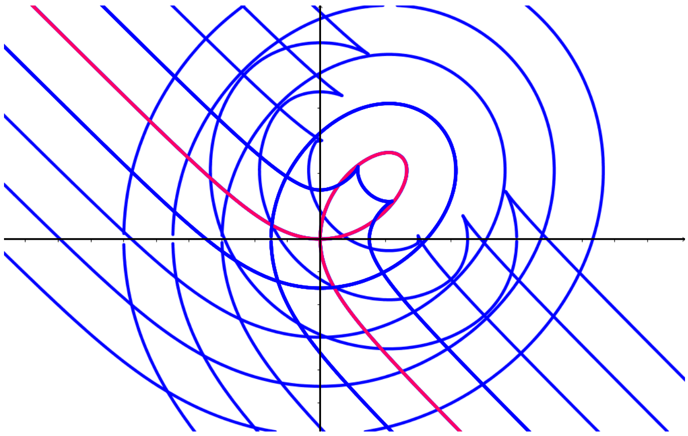

Figure 2.6: A Nodal Cubic and some of its offset curves.

See Figure 2.6, where the curve $\mathcal{C}$ is depicted in red, and some of its offset curves in blue. Let us apply the formula in Theorem 2.40 to this curve.The associated normal vector is:

$$N = (t(t^3-2), 1-2t^3).$$

Thus, the parametric auxiliary polynomial associated with $P(t)$ is given by
$s_P(d,k,t) = (4\,t^2 - 4\,t^5 + t^8 + 1 - 4\,t^3 + 4\,t^6)(t - kt^2)^2 - d^2(1+t^3)^2(t(-2+t^3) + k(-1+2\,t^3))^2.$
and it holds that $\deg_t(s_P(d,k,t)) = 14$. Since, in this example, $H_p(t) = 4\,t^2 - 4\,t^5 + t^8 + 1 - 4\,t^3 + 4\,t^6$, $W(t) = (1+t^3)^2$, and $U = \gcd(H_p, W) = 1$, one concludes using the formula in Theorem 2.40 that $\delta = \deg(\mathcal{O}_d(\mathcal{C})) = 14$.
In fact, the generic offset polynomial for $\mathcal{C}$, obtained using elimination techniques (see Appendix B, page 204), confirms this result.

Chapter 3

Partial Degree Formulae for Plane Curves

The topic of this chapter is the natural continuation of the preceding one: the computation of the partial degree of the generic offset polynomial w.r.t. each variable, including the distance one. Combining the results of this and the previous chapter, one has a complete and efficient solution to the degree problem for plane algebraic curves (see Subsection 1.2.2, page 24), both in the parametric and implicit case. The strategy applied is also common between these two chapters. In each case, we look for the adequate auxiliary curve, in the sense that we have already met in Chapter 2. The mission of the auxiliary curve is to take the degree information from the offset, which is not available, to the generating curve. Each degree problem, in the implicit case, is therefore encoded into an intersection problem between the curve $\mathcal{C}$ and a suitable auxiliary curve, that depends on parameters; the nature of these parameters also varies from one problem to another. Furthermore, the construction of these elimination curves is guided by Elimination Theory.

After the appropriate parameters have been identified, and the right choice of auxiliary curve has been done, the notion of fake points appears naturally. These are the invariant solutions of the system formed by the generating curve and the auxiliary curve. Invariance, in this context, means that these point are solutions for all values of the parameters in a non-empty Zariski open subset of the space of parameters. We have to show, in each degree problem, that these fake points can be considered as artifacts introduced by the elimination process, and that they have to be removed, in order to obtain the right value of the degree. Their invariance is, of course, the key ingredient that makes this removal process possible. As we shall see, a crucial step in this process is the proof that the remaining points, the non-fake ones, are counted properly in the intersection; that is, with multiplicity equal to one.

To complete the process, the framework developed in Theorem 2.30 (page 65) was designed to derive a deterministic, as well as efficient, degree formula from this setting of auxiliary curves and non-fake points with the right multiplicities. In this Chapter we will have two opportunities of seeing this framework giving, as a result, the desired formulae. More precisely, Theorems 3.24 (page 96) and 3.36 (page 109) are both of them proved as application of the common framework Theorem 2.30.

The above observations apply to the case of curves that are given implicitly. In the case of rational curves, given parametrically, the strategy is to translate in each case the information of the corresponding auxiliary curve into the parameter space. Doing this we not only obtain a degree formula for the parametric representation; the decrease of dimension comes along with an additional gain in efficiency, since the formulae that we obtain in this case only require the computation of gcds of univariate polynomials.

In general, in this chapter, when dealing with the partial degree problem, we will only discuss in detail how to compute δ_1, the partial degree in x_1 of the generic offset equation $g(d, \bar{x})$. This implies no loss of generality, since simply exchanging the variables x_1 and x_2 allows to compute δ_2. Nevertheless, in the sequel, we will give more details where it becomes necessary.

In Chapter 2 we were forced in some cases to exclude from our consideration some especially simple types of curves, that we use as ingredients of our strategy, or that are closely related with the offset construction (see Assumption 2.1 in page 36). Here we meet a similar situation, that leads to the following:

Assumption 3.1. *In this chapter we* exclude *the case where $\mathcal{C}$ is a horizontal or vertical line. Note that, in particular, this implies that we can assume $\delta_i > 0$ in all cases.*

However, since the generic offset equation to a line is well known, this exclusion poses no real restriction on the applicability of the degree formulae that we will derive. In fact, one may check that the only horizontal or vertical line for which the formulae in this chapter are not applicable are precisely those given by $y_1 = 0$ and $y_2 = 0$, the coordinate axes.

The structure of the chapter is the following:

- In Section 3.1 (page 82) we begin the analysis of the partial degree problem for implicitly given curves. In Subsection 3.1.1 we introduce the Offset-Line System 3.2 (page 83) for this problem, and the notion of y_2-ramification point of $\mathcal{C}$. The main result of this subsection is Theorem 3.4 (page 84), which describes the set of solutions of System 3.2 for a generic choice of (d^o, k^o). In Subsection 3.1.2 (page 86) we obtain by elimination the auxiliary curve $\mathcal{C}\cap\mathcal{S}^1(d^o, k^o)$ for the δ_1-problem. Then, Theorem 3.12 (page 88) shows the relation between the solutions of the Offset-Line System, and some points in $\mathcal{C}\cap\mathcal{S}^1(d^o, k^o)$. Finally, in Subsection 3.1.3 we study the fake points associated with this problem, proving their invariance w.r.t. (d, k) (in Corollary 3.22, page 95). Propositions 3.20 and 3.21 (page 90) contain the properties that are needed in the proof of the partial degree formulae in the following section.

- Section 3.2 (page 95) presents two formulae for the partial degree of the generic offset of a curve $\mathcal{C}$, given by its implicit equation. Subsection 3.2.1 describes, in Theorem 3.23 (page 96), a degree formula directly derived from Bezout's Theorem. This formula is not deterministic. Therefore, in Subsection 3.2.2 (page 96) we present a second, deterministic formula, that only requires a univariate resultant and gcd computations.

- To complete the degree analysis in the implicit case, in Section 3.3 (page 99) we study δ_d, the degree of g w.r.t. d. In Subsection 3.3.1 we describe the auxiliary curve associated with this problem (see Remark 3.27) and, in Theorem 3.29, (page 101) we prove that this auxiliary curve has the desired properties. We also define the corresponding notion of fake points (Definition 3.31, page 105). Finally, in Proposition 3.35 (page 106) we present the prerequisites needed for the proof of the degree formula. The resultant-based formula appears in Subsection 3.3.2 (page 108), in Theorem 3.36.

- In the final Section 3.4 (page 111) we extend our analysis of δ_1, δ_2 and δ_d, to include the case of curves given parametrically. The situation is similar to Section 2.4 (page 71) of Chapter 2. In Subsection 3.4.1 (page 112) this is done for the partial degree problem, introducing the parametric auxiliary polynomial in Definition 3.38 (page 112). The partial degree formula appears in Theorem 3.42 (page 115). Then, in Subsection 3.4.2 (page 117) we consider the degree w.r.t. d. The auxiliary polynomial is described in Definition 3.44 (page 117), and the degree formula appears in Theorem 3.48 (page 118).

The results in this chapter have been published in the Journal of Symbolic Computation, [46].

Notation and terminology for this chapter

Most of the notation for Chapter 2, introduced in page 8, is used also in this chapter. We only need to point out some differences in notation between this and the previous chapter:

- We will consider a pencil of horizontal lines through the origin, denoted by $\mathcal{L}_k$, with equation:
$$L(k, \bar{x}): \quad x_2 - k = 0.$$
A particular value of the variable k will be denoted by k^o, and the corresponding horizontal line is $\mathcal{L}_{k^o}$.

- We keep the notation for systems and their solution sets introduced in page 1. However, the subscripts and superscripts used to identify these systems and solution sets do not extend from Chapter 2 to the present chapter. That is, System $\mathfrak{S}_2(d^o, k^o)$ of Chapter 2 (page 35) does not necessarily coincide with System $\mathfrak{S}_2(d^o, k^o)$ in this chapter. If necessary, we will mention the corresponding chapter to avoid confusion.

3.1 General Strategy for the Partial Degree

In this section we develop the theoretical structure underlying our approach to the partial degree problem. Starting with the analysis of the corresponding Offset-Line System 3.2 (page 83) in Subsection 3.1.1, we will introduce the notion of y_2-ramification point of $\mathcal{C}$. These points play for the partial degree a similar role as the points in $\mathcal{C}_\perp$ (see Lemmas 1.14, page 14, and 2.3, page 39) played for the total degree formulae in Chapter 2. Then, in Theorem 3.4 (page 84), which is the main result of this subsection, we describe the set of solutions of System 3.2 for a generic choice of (d^o, k^o). In Subsection 3.1.2 (page 86) we introduce the auxiliary curve $\mathcal{C} \cap \mathcal{S}^1(d^o, k^o)$ for the δ_1-problem, obtained by eliminating $\bar{x}$ (and the auxiliary variable u) from System 3.2. Theorem 3.12 (page 88) shows the relation between the solutions of the Offset-Line System, and some points in $\mathcal{C} \cap \mathcal{S}^1(d^o, k^o)$, again for a generic choice of (d^o, k^o). Finally, in Subsection 3.1.3 we study the fake points associated with this problem. We prove their invariance w.r.t. (d, k) (see Corollary 3.22, page 95). In Propositions 3.20 and 3.21 (page 90), we obtain the properties of the fake points that are relevant for the proof of the partial degree formulae in the following section.

3.1.1 The offset-line system for partial degree

As we have said in the introduction, to address the partial degree problem, we will analyze the number of intersection points between a generic horizontal line and the offset of $\mathcal{C}$ at a generic distance. Let therefore

$$L(k, \bar{x}) : x_2 - k = 0$$

be the equation of a generic horizontal line $\mathcal{L}_k$. We will show that for any choice of (d^o, k^o) in a certain non-empty open subset of $\mathbb{C}^\times \times \mathbb{C}$, δ_1 equals the number of points in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$. As in the case of the total degree, then we compute the number of points in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$ indirectly, by using an auxiliary curve to count the points in $\mathcal{C}$ that, in a 1:1 correspondence, generate the points in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$.

Recall that the generic offset system for plane curves (page 38) is:

$$\left.\begin{array}{lr} & f(\bar{y})=0 \\ \operatorname{nor}(\bar{x},\bar{y}): & f_2(\bar{y})(x_1-y_1)-f_1(\bar{y})(x_2-y_2)=0 \\ b_d(\bar{x},\bar{y}): & (x_1-y_1)^2+(x_2-y_2)^2-d^2=0 \\ w(\bar{y},u): & u\cdot(f_1^2(\bar{y})+f_2^2(\bar{y}))-1=0 \end{array}\right\}\equiv \mathfrak{S}_1(d) \tag{3.1}$$

For our present purposes, we add the equation of the line $\mathcal{L}_k$. That is, we consider the following system:

$$\left.\begin{array}{lr} & f(\bar{y})=0 \\ b(d,\bar{x},\bar{y}): & (x_1-y_1)^2+(x_2-y_2)^2-d^2=0 \\ \operatorname{nor}(\bar{x},\bar{y}): & f_2(\bar{y})(x_1-y_1)-f_1(\bar{y})(x_2-y_2)=0 \\ w(\bar{y},u): & u\cdot(f_1^2(\bar{y})+f_2^2(\bar{y}))-1=0 \\ L(k,\bar{x}): & x_2-k=0 \end{array}\right\}\equiv \mathfrak{S}_2(d,k) \tag{3.2}$$

As usual, for $(d^o,k^o)\in\mathbb{C}^\times\times\mathbb{C}$ we denote by $\mathfrak{S}_2(d^o,k^o)$ the specialization of $\mathfrak{S}_2(d,k)$. We also denote by $\Psi_2(d^o,k^o)$ the set of solutions of $\mathfrak{S}_2(d^o,k^o)$.

The following theorem provides the theoretical foundation of our strategy, by establishing the 1:1 correspondence between the points in $\mathcal{O}_{d^o}(\mathcal{C})\cap\mathcal{L}_{k^o}$, and the points in $\mathcal{C}$ that generate them. We recall that a ramification point of a curve is a point on the curve where at least one of the partial derivatives of the implicit equation vanishes. In our case, since we are setting y_2 equal to a constant to analyze the partial degree δ_1, we restrict our attention to this variable.

Definition 3.2. *A point $\bar{y}^o\in\mathcal{C}$ will be called a y_2-ramification point of $\mathcal{C}$ if:*

$$f_2(\bar{y}^o)=\frac{\partial f}{\partial y_2}(\bar{y}^o)=0.$$

The set of y_2-ramification points of $\mathcal{C}$ will be denoted by $\operatorname{Ram}_2(\mathcal{C})$. The y_1-ramification points, and the set $\operatorname{Ram}_1(\mathcal{C})$, are defined similarly.

Remark 3.3. *Recall (see Assumption 3.1, page 80) that we assume that $\mathcal{C}$ is irreducible, and it is not a horizontal or vertical line. Under these assumptions, the set of y_2-ramification points of $\mathcal{C}$ is (empty) or finite; the same holds, of course, for the y_1-ramification points. This is a consequence of Bezout's Theorem, applied to the system: $f(\bar{y})=f_2(\bar{y})=0$. Since $\mathcal{C}$ is irreducible, and the degree can only decrease under derivation, if the system has infinitely many solutions, it follows that $f_2(\bar{y})\equiv 0$. That is, $\mathcal{C}$ is a vertical line.*

Note that if $\bar{y}^o$ is a non-normal isotropic point in $\mathcal{C}$, and $\bar{x}^o\in\mathcal{L}_{k^o}\cap\mathcal{O}_{d^o}(\mathcal{C})$ is associated with $\bar{y}^o$, then there exists $u^o\in\mathbb{C}^\times$ such that $(u^o,\bar{x}^o,\bar{y}^o)\in\Psi_2(d^o,k^o)$. However, when $\bar{y}^o$ is a y_2-ramification point of $\mathcal{C}$, the two points associated with $\bar{y}^o$ in $\mathcal{O}_{d^o}(\mathcal{C})$ belong to $\mathcal{L}_{k^o}$ (see Figure 3.1, where this situation is illustrated for an ellipse –in red– and

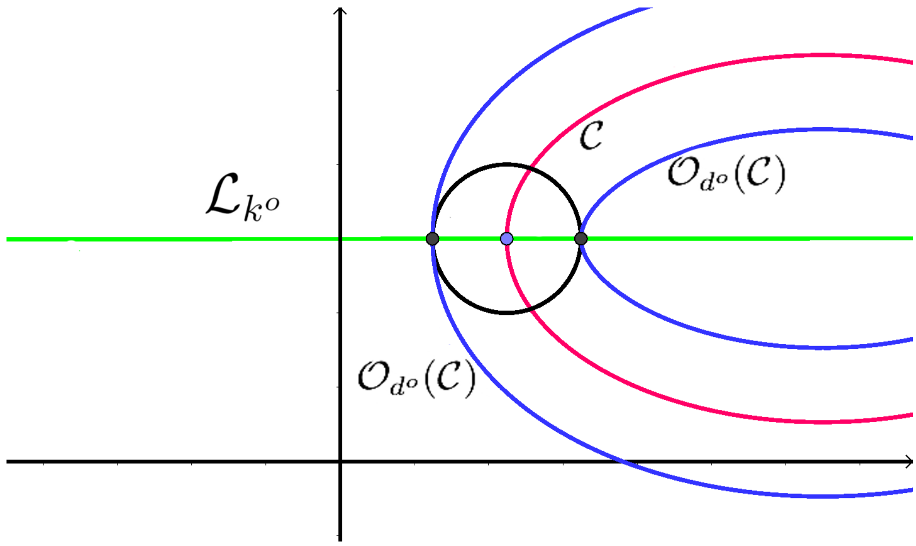

Figure 3.1: The situation associated with y_2-ramification points

its offset –in blue. The horizontal line $\mathcal{L}_{k^o}$ –in green– in this example is precisely the normal to $\mathcal{C}$ at their intersection points). This situation must be avoided to obtain a 1:1 correspondence between the $\bar{y}^o \in \mathcal{C}$ and the $\bar{x}^o \in \mathcal{L}_{k^o} \cap \mathcal{O}_{d^o}(\mathcal{C})$ that they generate. The next theorem shows that this can be done by restricting the values of (d, k) to a certain non-empty open subset.

Theorem 3.4. *There exists a non-empty Zariski open subset Ω_0 of $\mathbb{C}^2$ such that for $(d^o, k^o) \in \Omega_0$ the following hold:*

1. *If $(u^o, \bar{x}^o, \bar{y}^o) \in \Psi_2(d^o, k^o)$, then $\bar{y}^o$ is a (non-normal-isotropic and) non-y_2-ramification point of $\mathcal{C}$.*
2. *$\#\Psi_2(d^o, k^o) = \delta_1$.*
3. *There are no two different elements of $\Psi_2(d^o, k^o)$ with the same value of $\bar{y}^o$.*

Proof. Let us consider the generic offset equation as a polynomial in $\mathbb{C}[d, x_2][x_1]$, by writing:

$$g(d, x_1, x_2) = \sum_{i=0}^{\delta_1} g_i(d, x_2) x_1^i,$$

where g_{δ_1} is not identically zero. Observe that by assumption $\delta_1 > 0$. Thus, the set of solutions of $g_{\delta_1}(d, k) = 0$ is either empty, or a curve $\mathcal{D}_1$ in $\mathbb{C}^2$. We define $\Omega_0^1 = \mathbb{C}^2 \setminus \mathcal{D}_1$.

Let $\Upsilon = \{d_1, \ldots, d_m\}$ be the finite set of distances in Theorem 1.24 (page 21). Thus, for $d^o \notin \Upsilon$, the equation of $\mathcal{O}_{d^o}(\mathcal{C})$ is $g(d^o, x_1, x_2) = 0$. Let $\mathcal{D}_2$ be the (empty set or) curve defined by the union of the lines with equations $d = d_i$ for $d_i \in \Upsilon$. We define $\Omega_0^2 = \Omega_0^1 \setminus \mathcal{D}_2$. Then, for $(d^o, k^o) \in \Omega_0^2$,

$$g(d^o, x_1, k^o) = \sum_{i=1}^{\delta_1} g_i(d^o, k^o) x_1^i$$

is a polynomial in x_1 of degree δ_1 (the leading coefficient does not vanish because of the construction of Ω_0^1). Now, since g is square-free (see Remark 1.23, page 21), the discriminant $\mathrm{Dis}_{x_1}(g(d, x_1, k))$ is a non-identically zero polynomial in (d, k). Thus, it defines (the empty set or) a curve $\mathcal{D}_3$ in the $\mathbb{C}^2$. We define $\Omega_0^3 = \Omega_0^2 \setminus \mathcal{D}_3$.

Let now $\bar{y}^o \in \mathrm{Ram}_2(\mathcal{C}) \cup \mathrm{Iso}(\mathcal{C})$. Recall that $\mathrm{Iso}(\mathcal{C})$ is the (finite) set of normal-isotropic points of $\mathcal{C}$, and note that $\mathrm{Ram}_2(\mathcal{C}) \cup \mathrm{Iso}(\mathcal{C})$ is a finite set (see Remark 3.3, page 83). The construction now follows closely the construction of the set Ω_4^0, in Theorem 2.5 of Chapter 2 (page 41). We compute the following resultant between the generic offset polynomial and the equation of a d-circle centered at $\bar{y}^o$ (recall that $\bar{y}^o = (y_1^o, y_2^o)$ is a point in $\bar{y}^o \in \mathrm{Ram}_2(\mathcal{C}) \cup \mathrm{Iso}(\mathcal{C})$).

$$R_{\bar{y}^o}(d, k) = \mathrm{Res}_{x_1}(g(d, x_1, k), (x_1 - y_1^o)^2 + (k - y_2^o)^2 - d^2).$$

This resultant vanishes identically only if both polynomials have a common factor in x_2. But the polynomial $(x_1 - y_1^o)^2 + (k - y_2^o)^2 - d^2$ is irreducible (see the analogous situation for the construction of Ω_0^4 in the proof of Theorem 2.5, page 41). Hence, this could only happen if, for every $d^o \notin \Delta$, $\mathcal{O}_{d^o}(\mathcal{C})$ contains a circle of radius d^o centered at $\bar{y}^o$. This would imply that $\mathcal{C}$ is itself a circle centered at $\bar{y}^o$, which is impossible since $\bar{y}^o \in \mathcal{C}$. Therefore, $R_{\bar{y}^o}$ defines a curve in $\mathbb{C}^2$. Let $\mathcal{D}_4$ be the curve obtained as the union of such curves for all points $\bar{y}^o \in \mathrm{Ram}_2(\mathcal{C}) \cup \mathrm{Iso}(\mathcal{C})$. We define $\Omega_0^4 = \Omega_0^3 \setminus \mathcal{D}_4$. Then, for $(d^o, k^o) \in \Omega_0^4$, no intersection point of $\mathcal{O}_{d^o}(\mathcal{C})$ and $\mathcal{L}_{k^o}$ can be associated with a normal-isotropic or y_2-ramification point of $\mathcal{C}$.

Since $\mathrm{Ram}_2 \mathcal{C}$ is finite, we can exclude those values of k such that the line $x_2 = k$ passes through one of those y_2-ramification points. Let $\mathcal{D}_5$ be the finite union of such lines, and define $\Omega_0^5 = \Omega_0^4 \setminus \mathcal{D}_5$.

Let $\Omega_0 = \Omega_0^5$ and $(d^o, k^o) \in \Omega_0$. Then, because of the construction of Ω_0^2, we know that $g(d^o, x_1, x_2)$ is the equation of $\mathcal{O}_{d^o}(\mathcal{C})$. Besides, the equation

$$g(d^o, x_1, k^o) = \sum_{i=1}^{\delta_1} g_i(d^o, k^o) x_1^i = 0$$

has exactly δ_1 different roots because of the construction of Ω_0^1 and Ω_0^3. Every solution of this equation represents an affine intersection point of $\mathcal{O}_{d^o}(\mathcal{C})$ and $\mathcal{L}_{k^o}$. Moreover, because of the choice of Ω_0^4, these points are associated to non-normal-isotropic and non-y_2-ramification points of $\mathcal{C}$. This proves statements (1) and (2) of the theorem.

Finally, if two different elements of $\Psi_2(d^o, k^o)$ have the same value $\bar{y}^o$, then $\mathcal{L}_{k^o}$ must be normal to $\mathcal{C}$ at $\bar{y}^o$. This implies that $\bar{y}^o$ is a y_2-ramification point of $\mathcal{C}$, contradicting the construction of Ω_0^5. This proves statement (3). □

Remark 3.5.

1. *Note that (because of the construction of Ω_0^5 in the above proof), for $k^o \in \Omega_0$ the line $\mathcal{L}_{k^o}$ does not contain y_2-ramification points of $\mathcal{C}$.*

2. *Note also, that Theorem 3.4(2) (page 84) implies that, for $(d^o, k^o) \in \Omega_0$ and $\bar{y}^o \in \mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$, one has*

$$\operatorname{mult}_{\bar{y}^o}(\mathcal{O}_{d^o}(\mathcal{C}), \mathcal{L}_{k^o}) = 1$$

In Section 3.4, and in order to prove the degree formulae for the parametric case, we will need to avoid certain finite subset $\mathcal{X} \subset \mathcal{C}$. The situation is analogous to Lemma 2.6 of Chapter 2 (page 44), but we can not use directly that lemma: it depends on the pencil of lines through the origin used in Chapter 2. However, we have just seen how to do this, in the proof of Theorem 3.4 (page 84), for the case when $\mathcal{X} = \operatorname{Ram}_2(\mathcal{C}) \cup \operatorname{Iso}(\mathcal{C})$ (see the construction of Ω_0^4). And the same argument applies for any finite set. Thus we have proved the following:

Lemma 3.6. *Let Ω_0 be as in Theorem 3.4. If $\mathcal{X} \subset \mathcal{C}$ is a finite set, there exists an open non-empty subset $\Omega^0_{\mathcal{X}} \subset \Omega^0$ such that, if $(d^o, k^o) \in \Omega^0_{\mathcal{X}}$, then none of the points in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$ is associated with a point in $\mathcal{X}$.*

Proof. See the paragraph preceding the statement of the lemma. □

The strategy now is to eliminate $\bar{x}$ and u from the system $\mathfrak{S}_2(d, k)$, in order to obtain information about δ_1 through the solutions in $\bar{y}$ of the resulting system. This means that we switch our attention from the points in $\mathcal{O}_{d^o} \cap \mathcal{L}_{k^o}$ to the associated points in $\mathcal{C}$. In order to do that we will identify these associated points as intersection points of $\mathcal{C}$ with a certain auxiliary curve $\mathcal{S}$.

3.1.2 The auxiliary curve for partial degree

The auxiliary curve mentioned at the end of the previous section is obtained computing a Gröbner basis to eliminate $\bar{x}$ and u in the system $\mathfrak{S}_2(d, k)$. More precisely, one has the following lemma, which is analogous to Lemma 2.12 (page 46).

Lemma 3.7. *We consider*

$$\begin{cases} \widehat{\mathrm{nor}}(\bar{x}, \bar{y}, \hat{f}_1, \hat{f}_2) = \hat{f}_2\,(x_1 - y_1) - \hat{f}_1\,(x_2 - y_2) \\ b(d, \bar{x}, \bar{y}) = (x_1 - y_1)^2 + (x_2 - y_2)^2 - d^2 \\ L(k, \bar{x}) = x_2 - k \end{cases}$$

as polynomials in $\mathbb{C}[d, k, \bar{x}, \bar{y}, \hat{f}_1, \hat{f}_2]$ *($\hat{f}_1, \hat{f}_2$ are new variables that replace the partial derivatives in* $\mathrm{nor}(\bar{x}, \bar{y})$*). Let* $I =< \widehat{\mathrm{nor}}, b, L >$ *be the ideal generated by these polynomials. If* $J = I \cap \mathbb{C}[d, k, \bar{y}, \hat{f}_1, \hat{f}_2]$ *is the* $\bar{x}$*-elimination ideal of* I*, then* $J =< \hat{s}^1(d, k, \bar{y}, \hat{f}_1, \hat{f}_2) >$*, where*

$$\hat{s}^1(d, k, \bar{y}, \hat{f}_1, \hat{f}_2) = (\hat{f}_1^2 + \hat{f}_2^2)(y_2 - k)^2 - d^2 \hat{f}_2^2(\bar{y})$$

Proof. This is a standard Gröbner basis computation □

Remark 3.8. *In fact, it can be easily checked that*

$$\hat{s}^1(d, k, \bar{y}, \hat{f}_1, \hat{f}_2) = \nu_1(\bar{y}) b(d, \bar{x}, \bar{y}) + \nu_2(\bar{x}, \bar{y}) \widehat{\mathrm{nor}}(\bar{x}, \bar{y}, \hat{f}_1, \hat{f}_2) + \nu_3(\bar{x}, \bar{y}) L(k, \bar{x})$$

where

$$\begin{cases} \nu_1(\hat{f}_2) = -\hat{f}_2^2 \\ \nu_2(\bar{x}, \bar{y}, \hat{f}_1, \hat{f}_2) = \hat{f}_1\,(x_2 - y_2) + \hat{f}_2\,(x_1 - y_1) \\ \nu_3(k, \bar{x}, \bar{y}, \hat{f}_1, \hat{f}_2) = (\hat{f}_2^2 + \hat{f}_1^2)\,(2y_2 - x_2 - k) \end{cases}$$

This leads to the following definition:

Definition 3.9. *Let* s^1 *be the polynomial:*

$$s^1(d, k, \bar{y}) = h(\bar{y})(y_2 - k)^2 - d^2 f_2^2(\bar{y}),$$

where $h(\bar{y}) = f_1^2(\bar{y}) + f_2^2(\bar{y})$ *is the hodograph of* $\mathcal{C}$*. For every* $(d^o, k^o) \in \mathbb{C}^\times \times \mathbb{C}$*, the* auxiliary curve $\mathcal{S}^1_{(d^o,k^o)}$ to $\mathcal{C}$ for the x_1-partial degree problem *is the affine plane curve defined over* $\mathbb{C}$ *by the polynomial* $s^1(d^o, k^o, \bar{y})$*. The polynomial* $s^1(d, k, \bar{y})$ *will be called the* auxiliary polynomial to $\mathcal{C}$ for the x_1-partial degree problem

Let us consider the following system:

$$\mathfrak{S}_3(d, k) \equiv \begin{cases} f(\bar{y}) = 0 \\ s^1(d, k, \bar{y}) = 0 \end{cases} \tag{3.3}$$

For $(d^o, k^o) \in \mathbb{C}^\times \times \mathbb{C}$ we denote by $\mathfrak{S}_3(d^o, k^o)$ the specialization of $\mathfrak{S}_2(d, k)$, and we denote by $\Psi_3(d^o, k^o)$ the set of solutions of $\mathfrak{S}_3(d^o, k^o)$.

Remark 3.10. *The auxiliary polynomial in Definition 3.9 addresses the x_1 partial degree problem. The* auxiliary polynomial to $\mathcal{C}$ for the x_2-partial degree problem *is:*

$$s^2(d,k,\bar{y}) = h(\bar{y})(y_1-k)^2 - d^2 f_1^2(\bar{y}),$$

Remark 3.11. *Let us suppose that $\bar{y}^o \in \mathcal{C} \setminus \mathrm{Ram}_2(\mathcal{C})$, and $s^1(d^o,k^o,\bar{y}^o) = 0$. Then, since $d^o \in \mathbb{C}^\times$, it follows that $\bar{y}^o \notin \mathrm{Iso}(\mathcal{C})$.*

The following theorem shows the relation between the solution sets $\Psi_2(d^o,k^o)$ and $\Psi_3(d^o,k^o)$.

Theorem 3.12. *Let Ω_0 be as in Theorem 3.4, and let $(d^o,k^o) \in \Omega_0$.*

(a) If $(u^o,\bar{x}^o,\bar{y}^o) \in \Psi_2(d^o,k^o)$, then $\bar{y}^o \in \Psi_3(d^o,k^o) \setminus \mathrm{Ram}_2(\mathcal{C})$.

(b) Conversely, if $\bar{y}^o \in \Psi_3(d^o,k^o) \setminus \mathrm{Ram}_2(\mathcal{C})$, then there exist $\bar{x}^o \in \mathbb{C}^2$ and $u^o \in \mathbb{C}^\times$ such that $(u^o,\bar{x}^o,\bar{y}^o) \in \Psi_2(d^o,k^o)$.

Proof.

(a) Lemma 3.7 and Remark 3.8 (page 87) imply that $\bar{y}^o \in \Psi_3(d^o,k^o)$. By Theorem 3.4(1) (page 84), one has $\bar{y}^o \notin \mathrm{Ram}_2(\mathcal{C})$.

(b) Let $\bar{y}^o \in \Psi_3(d^o,k^o) \setminus \mathrm{Ram}_2(\mathcal{C})$. Then $\bar{y}^o \notin \mathrm{Iso}(\mathcal{C})$ (recall Remark 3.11), and so

$$\bar{x}^o = \left(\frac{-f_1(\bar{y}^o)y_2^o + f_2(\bar{y}^o)y_1^o + f_1(\bar{y}^o)k^o}{f_2(\bar{y}^o)}, k^o\right) \text{ and } u^o = \frac{1}{f_1^2(\bar{y}^o)+f_2^2(\bar{y}^o)} \quad (3.4)$$

are well defined. Substituting $(u^o,\bar{x}^o,\bar{y}^o)$ in $\mathfrak{S}_2(d^o,k^o)$ one sees that $(u^o,\bar{x}^o,\bar{y}^o) \in \Psi_2(d^o,k^o)$. □

Example 3.13. *Let $\mathcal{C}$ be the ellipse defined by the equation*

$$\frac{y_1^2}{4} + \frac{y_2^3}{9} = 1.$$

Then we get

$$s^1(d,k,\bar{y}) = (324y_1^2 + 64y_2^2)(y_2-k)^2 - 64d^2y_2^2$$

In Figure 3.2 we illustrate the role played by the auxiliary curve $\mathcal{S}^1_{(d^o,k^o)}$ for this example.

Remark 3.14. *Note that $\Psi_3(d^o,k^o)$ may contain other points besides those appearing in the theorem. For example, every affine singularity of $\mathcal{C}$ is also a point of $\Psi_3(d^o,k^o)$. But the theorem shows a 1:1 correspondence between $\Psi_2(d^o,k^o)$ and the points in $\Psi_3(d^o,k^o) \setminus$ $\mathrm{Ram}_2(\mathcal{C})$. In particular, for $(d^o,k^o) \in \Omega_0$:*

$$\delta_1 = \#\big(\Psi_3(d^o,k^o) \setminus (\mathrm{Ram}_2(\mathcal{C}))\big).$$

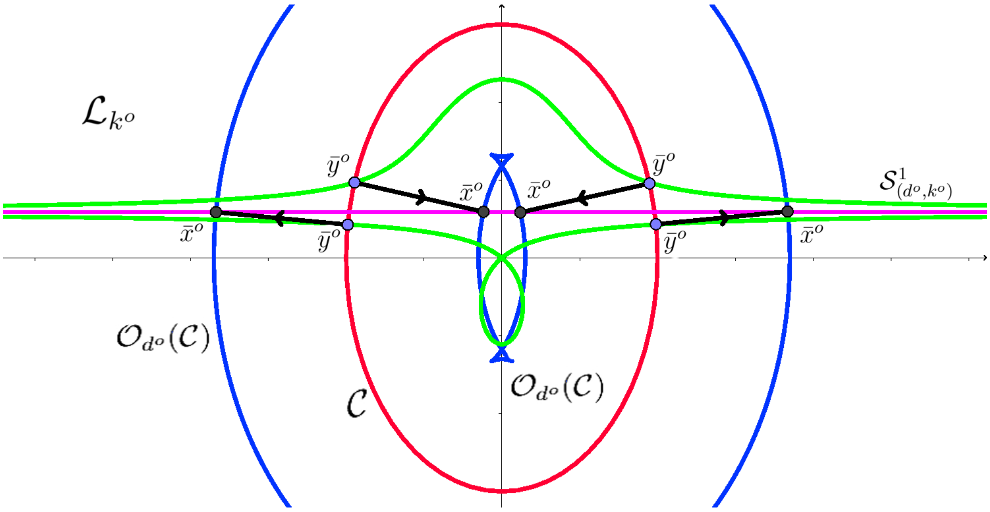

Figure 3.2: *The auxiliary curve $\mathcal{S}^1_{(d^o,k^o)}$ for the ellipse of Example 3.13. In the figure, the curve $\mathcal{C}$ is pictured in red, $\mathcal{S}^1_{(d^o,k^o)}$ in green, $\mathcal{O}_{d^o}(\mathcal{C})$ appears in blue and $\mathcal{L}_{k^o}$ in magenta. The intersection points $\bar{y}^o_i \in \mathcal{C} \cap \mathcal{S}^1_{(d^o,k^o)}$ with real coordinates are shown as solid blue dots, each of them connected with an arrow to the corresponding associated point $\bar{x}^o_i \in \mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$, shown as a solid black dot.*

3.1.3 Fake points (partial degree case)

The results in the preceding subsection show that we can use standard techniques, such as those provided by Bezout's Theorem, to analyze $\Psi_3(d^o, k^o) = \mathcal{C} \cap \mathcal{S}^1_{(d^o,k^o)}$. In order to do this, we have to ensure the following: first, we are going to consider all the intersection points of $\mathcal{C}$ and $\mathcal{S}^1_{(d^o,k^o)}$, and so we have to treat the problem projectively. Thus, we consider the projective closures of the curves, and we denote them by $\overline{\mathcal{C}}$ and $\overline{\mathcal{S}^1_{(d^o,k^o)}}$, respectively. Secondly, $\overline{\mathcal{C}} \cap \overline{\mathcal{S}^1_{(d^o,k^o)}}$ may contain also points that do not correspond to points in $\Psi_2(d^o, k^o)$, and we need to distinguish them. This fact motivates the following definition.

Definition 3.15. *Let Ω_0 be as in Theorem 3.4, and let $(d^o, k^o) \in \Omega_0$.*

1. *The affine points of $\overline{\mathcal{C}} \cap \overline{\mathcal{S}^1_{(d^o,k^o)}}$ that are not y_2-ramification points of $\mathcal{C}$ are called* non-fake points for the x_1-partial degree problem.

2. *The remaining points of $\overline{\mathcal{C}} \cap \overline{\mathcal{S}^1_{(d^o,k^o)}}$ are called* fake points for the x_1-partial degree problem.

We denote by $\mathcal{F}_1$ the set of all fake points.

Remark 3.16.

1. *Observe that, by definition, any point of* $\overline{\mathcal{C}} \cap \overline{\mathcal{S}^1_{(d^o,k^o)}}$ *at infinity is fake.*
2. *Although* $\mathcal{F}_1$ *seems to depend on the choice of* $(d^o, k^o) \in \Omega_0$*, in the next proposition we show that it is in fact invariant. Nevertheless, the set of non-fake points does depend on* (d, k)*.*

Since we are working projectively, we denote by F, F_1, F_2 and S^1 the homogenization w.r.t. y_0 of the polynomials f, f_1, f_2 and s, respectively. We denote, as usual, $\bar{y}_h = (y_0 : y_1 : y_2)$. Observe that:

$$S^1(d, k, \bar{y}_h) = (F_2^2(\bar{y}_h) + F_1^2(\bar{y}_h))(y_2 - ky_0)^2 - y_0^2 d^2 F_2^2(\bar{y}_h). \tag{3.5}$$

Proposition 3.17. $\bar{y}_h^o \in \mathcal{F}_1$ *if and only if* $\bar{y}_h^o \in \overline{\mathcal{C}}$ *and either*

1. $\bar{y}_h^o$ *is affine and singular or*
2. $\bar{y}_h^o$ *is* $(0 : 1 : 0)$ *or*
3. $\bar{y}_h^o$ *is at infinity satisfying* $F_1^2(\bar{y}_h^o) + F_2^2(\bar{y}_h^o) = 0$*.*

Proof. Let $\bar{y}_h^o \in \mathcal{F}_1$. Then there exists $(d^o, k^o) \in \Omega_0$ (Ω_0 as in Theorem 3.4), such that $\bar{y}_h^o \in \overline{\mathcal{C}} \cap \overline{\mathcal{S}^1_{(d^o,k^o)}}$ and either $y_0^o \neq 0$ and $F_2(\bar{y}_h^o) = 0$, or $y_0^o = 0$. If $y_0^o = 0$, since $S^1(d^o, k^o, \bar{y}_h^o) = 0$ one has that $H(\bar{y}_h^o) y_2^o = 0$, and hence either $\bar{y}_h^o = (0 : 1 : 0)$ or $\bar{y}_h^o$ is at infinity and it is isotropic. On the other hand, if $y_0^o \neq 0$ and $F_2(\bar{y}_h^o) = 0$, since $\bar{y}_h^o \in \overline{\mathcal{S}^1_{(d^o,k^o)}}$ one has that $F_1(\bar{y}_h^o)(y_2^o - k^o y_0^o) = 0$. Now, because of the construction of Ω_0 (see Remark 3.5, page 86), $y_2^o - k^o y_0^o \neq 0$. Therefore, $\bar{y}_h^o$ is affine and singular.

Conversely, if $\bar{y}_h^o \in \overline{\mathcal{C}}$ and it satisfies any of the three conditions in the statement of the proposition, then $\bar{y}_h^o \in \overline{\mathcal{S}^1_{(d^o,k^o)}}$. Thus, by Definition 3.15 the implication holds. □

From the above characterization, the next corollary follows.

Corollary 3.18. *The set* $\mathcal{F}_1$ *is finite, and does not depend on* $\{d, k\}$*.*

Remark 3.19. *Let* $(d^o, k^o) \in \Omega_0$*, and let* $\bar{y}_h^o \in \Psi_3(d^o, k^o)$ *be a non-fake point (in particular,* $\bar{y}_h^o$ *is affine). Then necessarily* $y_2^o - k^o \neq 0$ *(see the proof of Proposition 3.17).*

The following proposition, which is analogous to Proposition 2.21 (page 52), leads to the partial degree formulae in the following sections.

Proposition 3.20. *If* $(d^o, k^o) \in \Omega_0$*, then*

$$\delta_1 = \#\big((\overline{\mathcal{S}^1_{(d^o,k^o)}} \cap \overline{\mathcal{C}}) \setminus \mathcal{F}_1\big).$$

Proof. Proposition 3.17 and Definition 3.15 imply that

$$\Psi_3(d^o,k^o)\setminus(\mathrm{Ram}_2(\mathcal{C}))=\left(\overline{\mathcal{S}^1_{(d^o,k^o)}}\cap\overline{\mathcal{C}}\right)\setminus\mathcal{F}_1$$

and so (recall Remark 3.14, page 88), one has:

$$\delta_1=\#\left((\overline{\mathcal{S}^1_{(d^o,k^o)}}\cap\overline{\mathcal{C}})\setminus\mathcal{F}_1\right).$$

□

In order to apply Bézout's Theorem we need to prove that $\overline{\mathcal{C}}$ and $\overline{\mathcal{S}^1_{(d^o,k^o)}}$ do not have common components, and we have to analyze the multiplicity of intersection of $\overline{\mathcal{C}}$ and $\overline{\mathcal{S}^1_{(d^o,k^o)}}$ at the non-fake points. This is the content of the following proposition. The items in this proposition will be used to prove the degree formulae in the next section.

Proposition 3.21. *There exists a non-empty open subset $\Omega_1\subset\Omega_0$, where Ω_0 is as in Theorem 3.4, such that for every $(d^o,k^o)\in\Omega_1$ the following hold:*

(1) $\deg(\mathcal{S}^1_{(d^o,k^o)})=2\deg(\mathcal{C})$,

(2) $\mathcal{C}$ *and* $\mathcal{S}^1_{(d^o,k^o)}$ *have no common component,*

(3) if $\bar{y}_h^o\in\Psi_3(d^o,k^o)\setminus\mathcal{F}_1$*, then* $\mathrm{mult}_{\bar{y}_h^o}(\mathcal{C},\mathcal{S}^1_{(d^o,k^o)})=1$.

(4) Let $S^1(d,k,\bar{y}_h)$ *be considered as an element of* $(\mathbb{C}[\bar{y}_h])[d,k]$*:*

$$S^1(d,k,\bar{y}_h)=S_{2,0}(\bar{y}_h)d^2+S_{0,2}(\bar{y}_h)k^2+S_{0,1}(\bar{y}_h)k+S_{0,0}(\bar{y}_h)$$

where:

$$\begin{cases}S_{2,0}(\bar{y}_h)=-F_2^2y_0^2\\ S_{0,2}(\bar{y}_h)=(F_2^2+F_1^2)y_0^2,\\ S_{0,1}(\bar{y}_h)=-2(F_2^2+F_1^2)y_2y_0,\\ S_{0,0}(\bar{y}_h)=(F_2^2+F_1^2)y_2^2,\end{cases}$$

and let $\mathcal{S}_\alpha$ be the curve defined by $S_\alpha(\bar{y}_h)$, where α is any of the subscripts $(2,0),(0,2),(0,1),(0,0)$*. Then it holds that:*

$$\bigcap_\alpha(\overline{\mathcal{C}}\cap\overline{\mathcal{S}_\alpha})\subset\mathcal{F}_1.$$

(5) $(1:0:0)\notin\left(\overline{\mathcal{C}}\cap\overline{\mathcal{S}^1_{(d^o,k^o)}}\right)\setminus\mathcal{F}_1$

Proof.

(1) We have $S^1(d,k,\bar{y}_h) = H(\bar{y}_h)(y_2-ky_0)^2 - y_0^2d^2F_2^2$. The form $y_0^2d^2F_2^2$ has degree $2n$ in $\bar{y}_h$ (recall that $n = \deg(\mathcal{C})$), and the form $(F_2^2+F_1^2)(\bar{y}_h)(y_2-ky_0)^2$ has degree less or equal than $2n$ in $\bar{y}_h$. Thus $\deg_{\bar{y}_h}(S^1(d,k,\bar{y}_h)) = 2n$. Now, specializing in (d^o,k^o), the degree could only drop if the two forms were to become identical. That is generically impossible, since d does not appear in the first one and k does not appear in the second one. Thus, there exists $\Omega_1^1 \subset \Omega_0$ such that, for $(d^o,k^o) \in \Omega_1^1$, $\deg(\mathcal{S}^1_{(d^o,k^o)}) = 2\deg(\mathcal{C})$.

(2) Let us see that for $(d^o,k^o) \in \Omega_0$, $\overline{\mathcal{C}}$ and $\overline{\mathcal{S}^1_{(d^o,k^o)}}$ have no common components. Assume that they do. Then, since F is irreducible, there exists $K(\bar{y}_h) \in \mathbb{C}[\bar{y}_h]$ such that

$$S^1(d^o,k^o,\bar{y}_h) = K(\bar{y}_h)F(\bar{y}_h).$$

Now, we will see that then F_2 vanishes on almost all point of $\mathcal{C}$. That implies that $\mathcal{C}$ is a vertical line, which is impossible by assumption. Indeed, if there were infinitely many points in $\overline{\mathcal{C}} \cap \overline{\mathcal{S}^1_{(d^o,k^o)}}$ with $F_2 \neq 0$, this would imply infinitely many affine points in $\mathcal{C} \cap \mathcal{S}^1_{(d^o,k^o)}$ with $f_2 \neq 0$. Then Theorems 3.4 and 3.12 would give an infinite number of affine intersections between the line $x_2 - k^o = 0$ and the offset, which is impossible; note that if $\mathcal{O}_{d^o}(\mathcal{C})$ contains a line, then $\mathcal{C}$ is a line.

(3) Let $(d^o,k^o) \in \Omega_0$, and let $\bar{y}_h^o \in \Psi_3(d^o,k^o) \setminus \mathcal{F}_1$. Thus, $\bar{y}^o$ is an affine regular point of $\mathcal{C}$. Therefore, there is only one branch of $\mathcal{C}$ passing through $\bar{y}^o$. Let $\bar{x}^o$ be the point in $\mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$ associated with $\bar{y}^o$ (see Theorem 3.12 (page 88) for the existence of $\bar{x}^o$). Besides, we know (recall Remark 3.5, page 86) that, $\mathrm{mult}_{\bar{y}^o}(\mathcal{O}_{d^o}(\mathcal{C}), \mathcal{L}_{k^o}) = 1$. Thus it would be enough to prove that

$$\mathrm{mult}_{\bar{y}^o}(\mathcal{C}, \mathcal{S}^1_{(d^o,k^o)}) = \mathrm{mult}_{\bar{y}^o}(\mathcal{O}_{d^o}(\mathcal{C}), \mathcal{L}_{k^o}).$$

The rest of the proof of (3) will proceed as follows:

- First, we consider a place $\mathcal{P}(t) = (y_1(t), y_2(t))$ of $\mathcal{C}$ centered at $\bar{y}^o$, and we compute $s^1(d^o,k^o,\mathcal{P}(t))$. Note that the order of this formal power series is $\mathrm{mult}_{\bar{y}^o}(\mathcal{C}, \mathcal{S}^1_{(d^o,k^o)})$.
- Second, we use $\mathcal{P}(t)$ to obtain a place $\mathcal{Q}(t)$ of $\mathcal{O}_{d^o}(\mathcal{C})$ centered at $\bar{x}^o$, and we obtain $L(k^o, \mathcal{Q}(t))$. Note that the order of this formal power series is $\mathrm{mult}_{\bar{x}^o}(\mathcal{O}_{d^o}(\mathcal{C}), \mathcal{L}_{k^o})$.
- Finally we prove that $\mathrm{ord}\,(L(k^o,\mathcal{Q}(t))) = \mathrm{ord}(s^1(d^o,k^o,\mathcal{P}(t)))$.

Let

$$\begin{cases} f_1(\mathcal{P}(t)) = v_1 + \alpha t + \cdots \\ f_2(\mathcal{P}(t)) = v_2 + \beta t + \cdots \end{cases}$$

for some $v_1, v_2, \alpha, \beta \in \mathbb{C}$, where $f_1(\bar{y}^o) = v_1, f_2(\bar{y}^o) = v_2$. This means that the tangent vector to $\mathcal{C}$ at $\bar{y}^o$ is $(-v_2, v_1)$ and so, there exists $\lambda^o \in \mathbb{C}^\times$ such that the place $\mathcal{P}(t)$ can be expressed in the form:

$$\mathcal{P}(t) : \begin{cases} y_1 = y_1^o - \lambda^o v_2 t + \cdots \\ y_2 = y_2^o + \lambda^o v_1 t + \cdots \end{cases}$$

The notation $T_0 = \sqrt{v_1^2 + v_2^2}$ and $T_1 = v_1\alpha + v_2\beta$ will be used in the rest of the proof. Note that, since $(d^o, k^o) \in \Omega_0$, and $\bar{y}^o$ is non-fake, then v_2, T_0 and $y_2^o - k^o$ are all not zero (see Remark 3.19(1), page 90). Now, substituting $\mathcal{P}(t)$ into the polynomial $s^1(d^o, k^o, \bar{y})$ leads to a power series, whose zero-order term coefficient A_0 must vanish (because $\bar{y}^o \in \mathcal{S}^1_{(d^o,k^o)}$). This term is:

$$A_0 = (v_1^2 + v_2^2)(y_2^o - k^o)^2 - (d^o)^2 v_2^2 = T_0^2 (y_2^o - k^o)^2 - (d^o)^2 v_2^2.$$

Therefore we get that:

$$T_0^2 = \frac{-(d^o v_2)^2}{(y_2^o - k^o)^2}.$$

The coefficient of the first-order term A_1 of $s^1(d^o, k^o, \mathcal{P}(t))$ is:

$$A_1 = 2(-(d^o)^2 v_2 \beta + T_0^2 (y_2^o - k^o)\lambda^o v_1 + (v_1\alpha + v_2\beta)(y_2^o - k^o)^2).$$

Next, using $\mathcal{P}(t)$, we generate a place $\mathcal{Q}(t)$ of $\mathcal{O}_{d^o}(\mathcal{C})$ centered at $\bar{x}^o$. Since $v_1^2 + v_2^2 \neq 0$, the power series

$$f_1^2(\mathcal{P}(t)) + f_2^2(\mathcal{P}(t)) = (v_1^2 + v_2^2) + 2(v_1\alpha + v_2\beta)t + \cdots$$

has order zero (is a unit), and hence

$$\frac{1}{\sqrt{f_1^2(\mathcal{P}(t)) + f_2^2(\mathcal{P}(t))}}$$

can be expressed as the following formal power series.

$$\frac{1}{\sqrt{f_1^2(\mathcal{P}(t)) + f_2^2(\mathcal{P}(t))}} = \frac{1}{\sqrt{v_1^2 + v_2^2}} - \frac{v_1\alpha + v_2\beta}{(v_1^2 + v_2^2)^{3/2}} t + \cdots$$

So:

$$\begin{cases} \dfrac{f_1(\mathcal{P}(t))}{\sqrt{f_1^2(\mathcal{P}(t)) + f_2^2(\mathcal{P}(t))}} = \dfrac{v_1}{T_0} + \left(\dfrac{\alpha}{T_0} - \dfrac{T_1 v_1}{T_0{}^3}\right) t + \cdots \\ \dfrac{f_2(\mathcal{P}(t))}{\sqrt{f_1^2(\mathcal{P}(t)) + f_2^2(\mathcal{P}(t))}} = \dfrac{v_2}{T_0} + \left(\dfrac{\beta}{T_0} - \dfrac{T_1 v_2}{T_0{}^3}\right) t + \cdots \end{cases}$$

Therefore, since $\mathcal{Q}(t)$ is one of the two places:

$$\mathcal{Q}(t) = (x_1(t), x_2(t)) = \mathcal{P}(t) \pm d^o \frac{(f_1(\mathcal{P}(t)), f_2(\mathcal{P}(t)))}{\sqrt{f_1^2(\mathcal{P}(t)) + f_2^2(\mathcal{P}(t))}},$$

one has:

$$\begin{cases} x_1(t) = \left(y_1^o \pm \dfrac{d^o v_1}{T_0}\right) + \left(-\lambda^o\, v_2 \pm \dfrac{d^o \alpha}{T_0} \mp \dfrac{d^o\, T_1\, v_1}{{T_0}^3}\right) t + \cdots \\ x_2(t) = \left(y_2^o \pm \dfrac{d^o v_2}{T_0}\right) + \left(\lambda^o\, v_1 \pm \dfrac{d^o \beta}{T_0} \mp \dfrac{d^o\, T_1\, v_2}{{T_0}^3}\right) t + \cdots \end{cases}$$

Substituting $\mathcal{Q}(t)$ in the line $\mathcal{L}_{k^o}$ one has:

$$x_2(t) - k^o = \left(y_2^o \pm \frac{d^o v_2}{T_0} - k^o\right) + \left(\lambda^o\, v_1 \pm \frac{d^o \beta}{T_0} \mp \frac{d^o\, T_1\, v_2}{{T_0}^3}\right) t + \cdots = B_0 + B_1 t + \cdots$$

Now, since $\mathrm{mult}_{\bar{x}^o}(\mathcal{O}_{d^o}(\mathcal{C}), \mathcal{L}_{k^o}) = 1$, one has that

$$B_0 = \left(y_2^o \pm \frac{d^o v_2}{T_0} - k^o\right) = 0, \text{ and } B_1 = \left(\lambda^o\, v_1 \pm \frac{d^o \beta}{T_0} \mp \frac{d^o\, T_1\, v_2}{{T_0}^3}\right) \neq 0.$$

Therefore

$$\pm T_0 = -\frac{d^o v_2}{y_2^o - k^o}$$

Substituting the above equality in B_1 one gets

$$B_1 = \frac{1}{T_0^3}\left(\mp\lambda^o\, v_1 \left(\frac{d^o v_2}{y_2^o - k^o}\right)^3 \pm d^o \beta \left(\frac{d^o v_2}{y_2^o - k^o}\right)^2 \mp d^o T_1 v_2\right) =$$

$$\frac{\mp d^o v_2}{T_0^3 (y_2^o - k^o)^3}\left((d^o)^2 v_2^2 \lambda^o\, v_1 - (d^o)^2 \beta v_2 (y_2^o - k^o) + T_1 (y_2^o - k^o)^3\right).$$

Note that this result does not depend on the previous choice of sign. And using the same equality in A_1 gives:

$$A_1 = 2\left(-(d^o)^2 \beta v_2 + \left(\frac{d^o v_2}{y_2^o - k^o}\right)^2 (y_2^o - k^o)\lambda^o v_1 + T_1 (y_2^o - k^o)^2\right) =$$

$$\frac{2}{y_2^o - k^o}\left(-(d^o)^2 \beta v_2 (y_2^o - k^o) + (d^o)^2 v_2^2 \lambda^o v_1 + T_1 (y_2^o - k^o)^3\right).$$

We observe that the terms in parenthesis in A_1 and B_1 coincide. Since $B_1 \neq 0$, one has that $A_1 \neq 0$ and $\mathrm{mult}_{\bar{y}^o}(\mathcal{C}, \mathcal{S}^1_{(d^o,k^o)}) = 1$.

(4) Since we have assumed that $f_1^2 + f_2^2$ does not divide f (in particular $f_1^2 + f_2^2 \neq 0$), and that $\mathcal{C}$ is not a horizontal or vertical line (in particular $f_2 \neq 0$), all $\mathcal{S}_\alpha$ are algebraic curves. Now

$$\bigcap_\alpha (\overline{\mathcal{C}} \cap \overline{\mathcal{S}_\alpha}) \subset \overline{\mathcal{C}} \cap \overline{\mathcal{S}_{(2,0)}},$$

and by Proposition 3.17, $\overline{\mathcal{C}} \cap \overline{\mathcal{S}_{(2,0)}} \subset \mathcal{F}_1$.

(5) Let $\bar{y}_h^o = (1 : 0 : 0)$ and $A(d,k) = S^1(d,k,\bar{y}_h^o)$. If either $\bar{y}_h^o \in \mathcal{F}_1$ or $\bar{y}_h^o \notin \mathcal{C}$, then no further restriction on Ω_0 is required. Now, let $\bar{y}_h^o \in \mathcal{C}$ and $\bar{y}_h^o \notin \mathcal{F}_1$. Then by Proposition 3.17, $\bar{y}_h^o$ is not a singularity of $\mathcal{C}$. Now, if $F_2(\bar{y}_h^o) \neq 0$, then A is not constant. Moreover, if $F_2(\bar{y}_h^o) = 0$, then $F_1(\bar{y}_h^o) \neq 0$ and A is not constant either. Let Γ be the curve in $\mathbb{C}^2$ defined by A. Then let $\Omega_1^2 = \Omega_1^1 \setminus \Gamma$. Now, if $(d^o,k^o) \in \Omega_1^2$ and $\bar{y}_h^o = (1:0:0) \in \left(\overline{\mathcal{C}} \cap \overline{\mathcal{S}^1_{(d^o,k^o)}}\right) \setminus \mathcal{F}_1$, then $\bar{y}_h^o \in \mathcal{C}$, $A(d^o,k^o) = 0$ and $\bar{y}_h^o \notin \mathcal{F}_1$. Thus $(d^o,k^o) \in \Gamma$, a contradiction.

The above results show that if we take $\Omega_1 = \Omega_1^2$, then statements (1) to (5) hold for $(d^o,k^o) \in \Omega_1$. $\square$

From this Proposition we can derive another characterization of the invariance of the set of fake points.

Corollary 3.22. *Let Ω_1 be as in Proposition 3.21 (page 91). Then one has:*

$$\mathcal{F}_1 = \bigcap_{(d^o,k^o)\in\Omega_1} \left(\overline{\mathcal{S}^1_{(d^o,k^o)}} \cap \overline{\mathcal{C}}\right).$$

Proof. The "$\subset$" inclusion follows from Proposition 3.17. Now, let $\bar{y}_h^o \in \bigcap_{(d^o,k^o)\in\Omega_1} \left(\overline{\mathcal{S}^1_{(d^o,k^o)}} \cap \overline{\mathcal{C}}\right)$. Then $\bar{y}_h^o \in \mathcal{C}$ and $S^1(d,k,\bar{y}_h^o)$ vanishes on Ω_1. This implies that $S^1(d,k,\bar{y}_h^o)$ is identically zero. Thus $\bar{y}_h^o \in \bigcap_\alpha \left(\overline{\mathcal{C}} \cap \overline{\mathcal{S}_\alpha}\right)$, where $\mathcal{S}_\alpha$ is as in Proposition 3.21(4). Then, by Proposition 3.21(4), one has that $\bar{y}_h^o \in \mathcal{F}_1$. $\square$

3.2 Partial Degree Formulae for the Implicit Case

In this section we will obtain two formulae for the partial degree of the generic offset of a curve $\mathcal{C}$, given by its implicit equation. In Subsection 3.2.1 we briefly describe, in Theorem 3.23, the degree formula that one obtains by using Bezout's Theorem combined with the results in the previous section. However, as we already saw in Chapter 2 in the context of the total degree formulae, the formula in Theorem 3.23 is mainly of theoretical interest, and probably not so useful in practice, because it requires an explicit description of the open set Ω_1. In order to overcome this difficulty, we present a second, deterministic formula, that only requires a univariate resultant and gcd computations. The same comments as we did in the introduction to Section 2.3 of Chapter 2 (page 63) apply here: the univariate resultant is a natural tool to compute the intersection multiplicities between two projective plane curves. In fact, in that section we described a common framework (see Theorem 2.30, page 65), designed to derive a deterministic degree formula from this property of the resultant, combined with the invariance of the set of fake points. The same ingredients appear in the partial degree problem, and we will see that, with the help of that framework, the proof of a resultant-based formula for δ_1 is straightforward. This is the content of Subsection 3.2.2, and the formula is obtained in Theorem 3.24 (page 96). This resultant-based formula gives, as expected, an efficient and easy to implement way of computing δ_1.

3.2.1 Partial degree formula using the auxiliary curve

Using the results in the previous section, we derive the first partial degree formula for offset curves. We observe that, if Ω_1 is as in Proposition 3.21 (page 91), then by Bézout's Theorem one has:

$$\deg(\mathcal{C})\deg(\mathcal{S}^1_{(d^o,k^o)}) = \sum_{\bar{y}_h^o \in \overline{\mathcal{C}} \cap \overline{\mathcal{S}^1_{(d^o,k^o)}}} \mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}^1_{(d^o,k^o)}}) =$$

$$\sum_{\bar{y}_h^o \in \mathcal{F}_1} \mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}^1_{(d^o,k^o)}}) + \sum_{\bar{y}_h^o \in \left(\overline{\mathcal{C}} \cap \overline{\mathcal{S}^1_{(d^o,k^o)}}\right) \setminus \mathcal{F}_1} \mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}^1_{(d^o,k^o)}})$$

Moreover, since there are δ_1 non-fake points (see Remark 3.16), and for each of them the multiplicity of intersection is one, taking also Proposition 3.21(1) into account,the following formula holds.

Theorem 3.23 (First partial degree formula). *Let Ω_1 be as in Proposition 3.21. For every $(d^o, k^o) \in \Omega_1$, it holds that:*

$$\boxed{\delta_1 = \deg_{x_1}(\mathcal{O}_{d^o}(\mathcal{C})) = 2\,(\deg(\mathcal{C}))^2 - \sum_{\bar{y}_h^o \in \mathcal{F}_1} \mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}^1_{(d^o,k^o)}})}$$

Proof. See the commentaries preceding the statement of the Theorem. □

3.2.2 Partial degree formulae using resultants

Combining Proposition 3.21 (page 91) with Theorem 2.30 (page 65), we obtain a deterministic formula for the partial degree, requiring the computation of a univariate resultant and gcds.

Theorem 3.24 (Resultant-Based Partial Degree Formula). *Let $\mathcal{C}$ be an irreducible plane curve, and assume that $\mathcal{C}$ is not a line. Then:*

$$\boxed{\delta_1 = \deg_{x_1}(\mathcal{O}_d(\mathcal{C})) = \deg_{\{\bar{y}\}}\left(\mathrm{PP}_{\{d,k\}}\left(\mathrm{Res}_{y_0}(F(\bar{y}_h), S^1(\bar{y}_h, d, k))\right)\right)}$$

We recall that F is the homogeneous implicit equation of $\overline{\mathcal{C}}$, and S^1 is the homogenization of the polynomial introduced in Definition 3.9 (page 87).

Proof. In order to prove the theorem, we apply Theorem 2.30 to $\mathcal{C}$, and $Z(\bar{\omega}, \bar{y}_h) = S^1(d, k, \bar{y}_h)$, where $\bar{\omega} = (d, k)$, and $\Omega = \Omega_1$, where Ω_1 is as in Proposition 3.21. We check that all the hypothesis are satisfied:

- $\mathcal{C}$ is irreducible and it is not a line by assumption.
- S^1 can be written as
$$S^1(d, k, \bar{y}_h) = \left((F_1^2 + F_2^2)k^2 - F_2^2 d^2\right) y_0^2 - 2k(F_1^2 + F_2^2)y_0 + (F_1^2 + F_2^2)y_2^2$$
Thus, since $F_1^2 + F_2^2$ and F_2^2 are not identically zero, S^1 depends on y_0.
- Hypothesis (1),(2),(3) and (5) in Theorem 2.30 follow respectively from (1), (2), (3) and (5) in Proposition 3.21.
- (4) in Theorem 2.30 follows from Corollary 3.22 (page 95).

Then, Theorem 2.30 implies that there exists a non-empty open $\Omega_\star \subset \Omega_1$ such that for $(d^o, k^o) \in \Omega_\star$
$$\#([\overline{\mathcal{S}^1_{(d^o,k^o)}} \cap \overline{\mathcal{C}}] \setminus \mathcal{F}_1) = \deg_{\{\bar{y}\}}\left(\mathrm{PP}_{\{d,k\}}\left(\mathrm{Res}_{y_0}(F(\bar{y}_h), S^1(d, k, \bar{y}_h))\right)\right)$$
Now the theorem follows from Remark 3.19 (page 90). □

We finish this section illustrating the formula in Theorem 3.24 by means of two examples.

Example 3.25. *Let $\mathcal{C}$ be the hyperbola given by $f(\bar{y}) = y_1y_2 - 1$. See Figure 2.4, where the curve $\mathcal{C}$ is depicted in red, and some of its offset curves in blue. Applying the formula in Theorem 2.31 (page 68) ,one deduces that the total degree of the generic offset curve is 8. Now, we apply Theorem 3.24 to compute the partial degrees δ_1 and δ_2. For δ_1 the polynomial $S^1(d, k, \bar{y}_h)$ is:*
$$S^1(d, k, \bar{y}_h) = (y_2^2 + y_1^2)(y_2 - ky_0)^2 - d^2y_1^2y_0^2$$
and
$\mathrm{Res}_{y_0}(F, S^1) = y_2^2(-4\, y_2^3y_1^3k^2 - 2\, y_2^5y_1\, k^2 - 2\, y_2^3y_1^3d^2x + 2\, y_2^4y_1^2 - 2\, y_2\, y_1^5k^2 - 2\, y_2\, y_1^5d^2 + y_2^4y_1^2k^4 + 2\, y_2^2y_1^4k^4 + y_1^6k^4 + y_1^6d^4 - 2\, y_2^2y_1^4k^2d^2 - 2\, y_1^6k^2d^2 + y_2^2y_1^4 + y_2^6)$.
Thus,
$$\delta_1 = \deg_{\{\bar{y}\}}\left(\mathrm{PP}_{\{d,k\}}\left(\mathrm{Res}_{y_0}(F, S^1)\right)\right) = 6.$$
Similarly, exchanging the variables y_1 and y_2 and repeating the process, one gets that $\delta_2 = 6$. In fact, one may check that for this example the generic equation of the offset is:
$g(d, \bar{x}) = 16 + 22\, {x_1}^3x_2d^2 + 7\, {x_1}^2d^4{x_2}^2 - 5\, {x_1}^4d^2{x_2}^2 - 20\, x_1x_2d^4 - 5\, {x_2}^4d^2{x_1}^2 - 20\, d^2{x_1}^2 + 3\, {x_1}^4d^4 - {x_1}^6d^2 - 20\, {x_1}^3{x_2}^3 - 48\, x_2x_1 + {x_1}^6{x_2}^2 + {x_2}^6{x_1}^2 - 20\, d^2{x_2}^2 + 2\, {x_1}^4{x_2}^4 + 3\, {x_2}^4d^4 - {x_2}^6d^2 - 3\, {x_2}^2d^6 + 50\, {x_1}^2{x_2}^2 - 2\, x_2{x_1}^5 - 2\, {x_2}^5x_1 + {x_1}^4 + {x_2}^4 - 8\, d^4 - 3\, {x_1}^2d^6 + 22\, {x_2}^3x_1d^2 + d^8$.

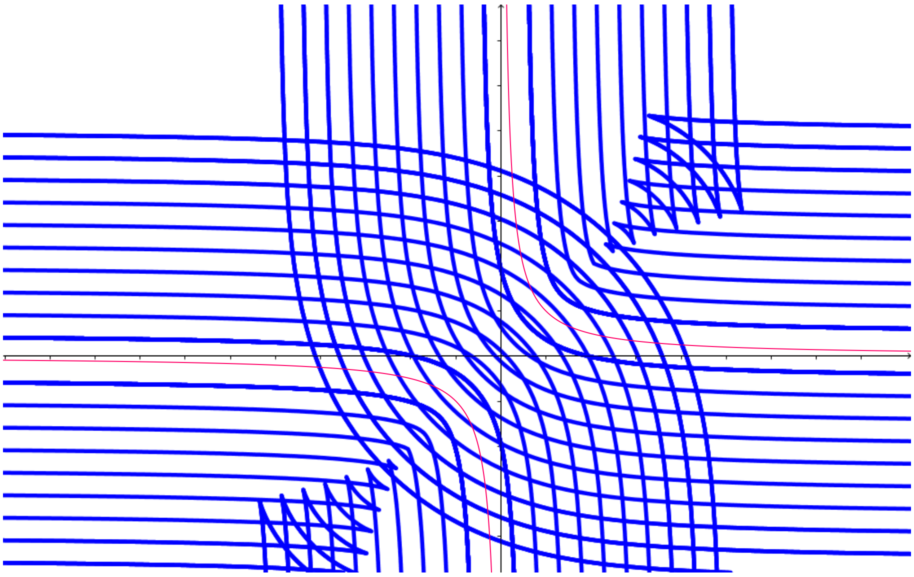

Figure 3.3: A hyperbola and some of its offset curves.

Example 3.26. *Let $\mathcal{C}$ be the cusp given by $f(\bar{y}) = y_1^5 - y_2^3$. In this case the generic offset equation $g(x_1, x_2, d)$ can be computed explicitly using Gröbner basis techniques :*

$g(d, \bar{x}) = -675000\, d^{10} - 11664\, d^6 - 9765625\, d^{14} + 11664\, x_2^6 + 9765625\, x_2^{10} + 11664\, x_1^{10} + 9765625\, x_1^{14} - 10715625\, x_1^4 d^6 + 341796875\, x_1^6 d^8 + 378000\, x_1^6 d^4 - 341796875\, x_1^8 d^6 - 97200\, x_1^8 d^2 + 205078125\, x_1^{10} d^4 - 68359375\, x_1^{12} d^2$
$+68359375\, x_1^2 d^{12} - 11137500\, x_1^2 d^8 - 205078125\, x_1^4 d^{10} - 23328\, x_1^5 x_2^3$
$-19531250\, x_1^5 x_2^7 + 9765625\, x_1^4 x_2^6 + 9765625\, x_1^{10} x_2^4 - 39062500\, x_1^7 x_2^5$
$-19531250\, x_1^9 x_2^3 - 2700000\, x_1^6 x_2^4 + 19531250\, x_1^2 x_2^8 + 1350000\, x_1 x_2^7$
$+19531250\, x_1^{12} x_2^2 + 292968750\, x_1^4 x_2^2 d^8 + 48828125\, x_1^2 x_2^4 d^8$
$-144843750\, x_1^3 x_2 d^8 + 97656250\, x_1^6 x_2^4 d^4 - 238281250\, x_1 x_2^3 d^8 - 1458000\, x_1 x_2 d^6 - 35128125\, x_2^4 d^6 + 9787500\, x_2^2 d^8 + 52396875\, x_1^4 x_2^2 d^4 - 58906250\, x_1^2 x_2^6 d^2 - 4158000\, x_1 x_2^5 d^2 - 97656250\, x_1^4 x_2^4 d^6 + 19531250\, x_2^2 d^{12} - 51496875\, x_1^4 x_2^4 d^2 + 1350000\, x_1^{11} x_2 + 292968750\, x_1^8 x_2^2 d^4 - 34992\, x_2^4 d^2 + 34992\, x_2^2 d^4$
$-7992000\, x_1^6 x_2^2 d^2 - 70143750\, x_1^9 x_2 d^2 + 2531250\, x_1^2 x_2^2 d^6 + 11306250\, x_1^5 x_2 d^6 - 48828125\, x_1^8 x_2^4 d^2 - 421875000\, x_1^5 x_2^5 d^2 - 292656250\, x_1^7 x_2^3 d^2$
$-195312500\, x_1^3 x_2^7 d^2 - 9765625\, x_2^4 d^{10} - 37109375\, x_2^8 d^2 + 144843750\, x_1^3 x_2^3 d^6 + 53359375\, x_2^6 d^4 + 47981250\, x_1^2 x_2^4 d^4 + 265625000\, x_1 x_2^5 d^6 + 4266000\, x_1 x_2^3 d^4 - 390625000\, x_1^6 x_2^2 d^6 - 117187500\, x_1^{10} x_2^2 d^2 - 194400\, x_1^3 x_2^3 d^2 - 97656250\, x_1 x_2^7 d^4 - 117187500\, x_1^2 x_2^2 d^{10} + 70312500\, x_1 x_2 d^{10} + 194400\, x_1^3 x_2 d^4 + 405625000\, x_1^5 x_2^3 d^4 + 195312500\, x_1^3 x_2^5 d^4 - 69984\, x_1^5 x_2 d^2 + 132018750\, x_1^7 x_2 d^4$

It has total degree 14 (and 71 terms), the partial degree in x_1 is $\delta_1 = 14$, and the partial degree in x_2 is $\delta_2 = 10$. In this example $s^1(d, k, \bar{y}) = (25y_1^8 + 9y_2^4)(y_2 - k)^2 - 9d^2 y_2^4$ and

$\mathrm{Res}_{y_0}(F, S^1) = {y_2}^8{y_1}^{16}(450\,{y_1}^{12}k^4{y_2}^2 - 162\,{y_1}^9k^2{y_2}^5 - 900\,{y_1}^7k^2{y_2}^7 + 81\,{y_1}^{14}k^4 + 81\,{y_1}^{14}d^4 + 81\,{y_1}^4{y_2}^{10} + 450\,{y_1}^2{y_2}^{12} + 625\,{y_2}^4k^4{y_1}^{10} - 1250\,{y_2}^9k^2{y_1}^5 - 162\,{y_1}^{14}k^2d^2 - 162\,{y_1}^9d^2{y_2}^5 - 450\,{y_1}^7d^2{y_2}^7 - 450\,{y_1}^{12}d^2{y_2}^2k^2 + 625\,{y_2}^{14})$

And so the formula gives the right result $\delta_1 = \deg_{\{\bar{y}\}}\left(\mathrm{PP}_{\{d,k\}}\left(\mathrm{Res}_{y_0}(F, S^1)\right)\right) = 14$. *Exchanging the roles of* y_1 *and* y_2 *in the above computation gives* 10 *for the other partial degree.*

3.3 Degree in the Distance in the Implicit Case

Since the generic offset equation g also depends on d, it is natural to complete this degree analysis by studying δ_d, the degree of g in d. The strategy that we have used up till now, for the degree problems that we have met (both total and partial), includes the choice of a pencil of lines. This pencil of lines has the property that the cardinal of the intersection of a generic line in the pencil with the offset $\mathcal{O}_{d^o}(\mathcal{C})$ (for a generic choice of d^o) is precisely the degree under study. However, in this case, to study the degree in d, instead of fixing the distance, we need to fix the variable $\bar{x}$. In Subsection 3.3.1 we will describe which is the appropriate notion of auxiliary curve to deal with this problem. See especially Remark 3.27, Theorem 3.29 (page 101) and the comments preceding this theorem. From here we will derive the corresponding notion of fake points (Definition 3.31, page 105), introduced quite naturally as the set of invariant intersection points between $\mathcal{C}$ and the auxiliary curve, where invariant means w.r.t the choice of the parameters that appear in the auxiliary curve. Then, after proving the necessary prerequisites in Proposition 3.35 (page 106), in Subsection 3.3.2 (page 108) we see again the framework introduced in Theorem 2.30 (page 65) coming to fruition. The result, in Theorem 3.36, is a resultant-based formula for δ_d, with the expected advantages of this type of formulae in terms of efficiency.

3.3.1 Auxiliary curve and fake points for the degree in the distance

At first sight, the adequate choice of the auxiliary curve for this degree problem is perhaps not clear. However, as we have said in the introduction of this section, based on the experience of our previous work, it is natural to address this problem using the concept of auxiliary curve, and the framework introduced in Theorem 2.30, page 65.

More precisely: if we consider the generic offset equation as a polynomial in $\mathbb{C}[\bar{x}][d]$, we can write:

$$g(d,\bar{x}) = \sum_{i=0}^{\delta_d} g_{(i)}(\bar{x})d^i$$

If we fix a generic $\bar{x}^o \in \mathbb{C}^2$ (more precisely, we require $g_{(\delta_d)}(\bar{x}^o) \neq 0$), then $g(\bar{x}^o, d) \in \mathbb{C}[d]$ has degree δ_d. Each root d^o of this polynomial corresponds to one of the (finitely many) times that $\mathcal{O}_{d^1}(\mathcal{C})$ passes through $\bar{x}^o$. This implies (since $\bar{x}^o$ is generic) that there is a regular point $\bar{y}^o \in \mathcal{C}$ such that the line defined by $\bar{x}^o$ and $\bar{y}^o$ is the normal line to $\mathcal{C}$ at $\bar{y}^o$ (and of course, the distance between $\bar{x}^o$ and $\bar{y}^o$ equals d^o).

Remark 3.27. *Recall that the equation for the generic normal line to $\mathcal{C}$ is:*

$$\mathrm{nor}(\bar{x},\bar{y}) = f_2(\bar{y})(x_1 - y_1) - f_1(\bar{y})(x_2 - y_2), \tag{3.6}$$

and that

$$\mathrm{Nor}(\bar{x},\bar{y}_h) = F_2(\bar{y}_h)(x_1y_0 - y_1) - F_1(\bar{y}_h)(x_2y_0 - y_2) \tag{3.7}$$

is the homogenization of $\mathrm{nor}(\bar{x},\bar{y})$ *in $\bar{y}$ w.r.t. y_0. For $\bar{x}^o \in \mathbb{C}^2$ we* denote *by $\mathcal{N}_{\bar{x}^o}$ the affine closed set defined by* $\mathrm{nor}(\bar{x}^o,\bar{y}) = 0$ *(observe that there exists an open subset of values of $\bar{x}^o$ such that $\mathcal{N}_{\bar{x}^o}$ is indeed a curve). Let $\overline{\mathcal{N}_{\bar{x}^o}}$* denote *the projective closure of $\mathcal{N}_{\bar{x}^o}$.*

We will show that, using $\bar{x}$ as a parameter, $\mathcal{N}_{\bar{x}^o}$ plays the role of the auxiliary curve $\mathcal{Z}_{\bar{\omega}^o}$ for the δ_d problem. The reader could be wondering where is the elimination step that we have met in previous cases. Note that, since the normal line is in fact part of the Generic Offset System, fixing a value of the parameter (in this case, that means fixing a point $\bar{x}^o$) and eliminating the variable d from the Generic Offset System is bound to lead us to the normal line. Therefore, the answer is that here we need no extra effort to eliminate the d variable. In Figure 3.4 we illustrate the role of the auxiliary curve $\mathcal{N}_{\bar{x}^o}$.

A final observation is needed in order to use the framework that we have developed: because of the symmetry in the offset construction, the offsets $\mathcal{O}_{d^o}(\mathcal{C})$ and $\mathcal{O}_{-d^o}(\mathcal{C})$ are exactly the same (recall Proposition 1.27, page 23). This implies that, for a generic $\bar{x}^o$, there is a $2:1$ correspondence between the roots of $g(\bar{x}^o, d) = 0$ and the points $\bar{y}^o \in \mathcal{C}$ such that $\mathrm{nor}(\bar{x}^o,\bar{y}^o) = 0$. We must take this into account to obtain a correct interpretation of the result of our computations.

Remark 3.28. *In the sequel we* denote *$\delta_d = 2\nu$, where $\nu \in \mathbb{N}$.*

Now we can describe the strategy for the computation of δ_d, via the computation of ν. First, we consider the system

$$\mathfrak{S}_4(\bar{x}) \equiv \begin{cases} f(\bar{y}) = 0 \\ \\ \mathrm{nor}(\bar{x},\bar{y}) = 0 \end{cases}$$

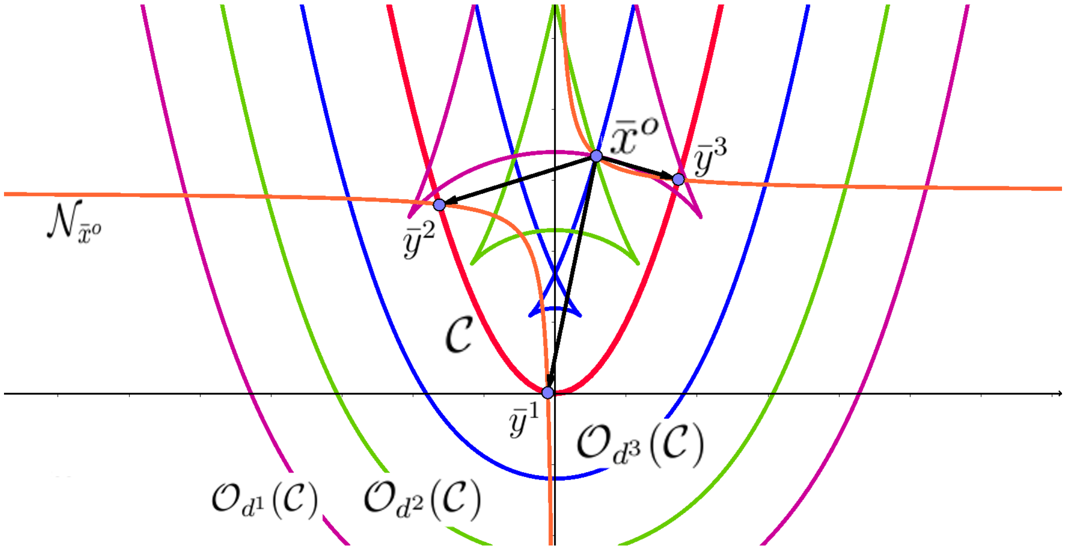

Figure 3.4: *The role of the auxiliary curve. The curve $\mathcal{N}_{\bar{x}^o}$ (in orange) for a parabola $\mathcal{C}$ (in red) is pictured, together with three of the offsets to the parabola at three different values of the distance (in blue, green and magenta). These three offsets meet at point $\bar{x}^o$, which is associated with three points in $\mathcal{C}$, the points $\bar{y}^1$, $\bar{y}^2$ and $\bar{y}^3$. The fundamental property of the auxiliary curve is that $\bar{y}^1$, $\bar{y}^2$ and $\bar{y}^3$ all belong to $\mathcal{C} \cap \mathcal{N}_{\bar{x}^o}$.*

Here we see $\bar{x}$ as parameters. For $\bar{x}^o \in \mathbb{C}^2$ we denote by $\mathfrak{S}_4(\bar{x}^o)$ the specialization of $\mathfrak{S}_4(\bar{x})$, and we denote by $\Psi_4(\bar{x}^o)$ the set of solutions of $\mathfrak{S}_4(\bar{x}^o)$. In Theorem 3.29 we analyze its solutions for a generic choice of $\bar{x}^o$. Based on this analysis, the notions of d-fake and non d-fake points are introduced. Next, the invariance of the set of d-fake points is established in Proposition 3.32. In Proposition 3.35 (page 106), which is similar to Proposition 3.21 (page 91), we check the prerequisites for the use of the framework. Finally, in Theorem 3.36 (page 109) we derive the degree formula applying Theorem 2.30 (page 65).

The first step is the content of the following theorem (compare to Theorem 3.4 and Theorem 3.12).

Theorem 3.29. *There exists a non-empty Zariski open subset Ω_2 of $\mathbb{C}^2$, such that for $\bar{x}^o \in \Omega_2$ the following hold:*

1. *Let $\bar{y}^o$ be an affine regular point of $\mathcal{C}$. If $\bar{y}^o$ is the origin or $\bar{y}^o \in \mathrm{Iso}(\mathcal{C})$, then it is not a solution of $\mathfrak{S}_4(\bar{x}^o)$.*
2. *$\#(\Psi_4(\bar{x}^o)) = \nu$ (see Remark 3.28).*
3. *If $\bar{y}^o \in \Psi_4(\bar{x}^o)$, then*
$$(x_1^o - y_1^o)^2 + (x_2^o - y_2^o)^2 \neq 0.$$

Let then $d_{\bar{y}^o}$ be a particular choice of one of the two solutions in d of this equation:

$$d^2 = (x_1^o - y_1^o)^2 + (x_2^o - y_2^o)^2.$$

Then $\#\{d_{\bar{y}^o}\}_{\bar{y}^o \in \Psi_4(\bar{x}^o)} = \nu$. That is, the correspondence between $\bar{y}^o$ and $d_{\bar{y}^o}$ is a bijection.

4. *For every $\bar{y}^o \in \Psi_4(\bar{x}^o)$, and its corresponding $d_{\bar{y}^o}$ introduced in (3), it holds that $\bar{x}^o$ and $\bar{y}^o$ are associated points in $\mathcal{O}_{d_{\bar{y}^o}}(\mathcal{C})$.*

5. *If $\bar{x}^o \in \mathcal{O}_{d^o}(\mathcal{C})$ for some $d^o \in \mathbb{C}^\times$, then $\bar{x}^o$ is a regular point of $\bar{x}^o \in \mathcal{O}_{d^o}(\mathcal{C})$.*

Proof. The open set Ω_2 is constructed in a finite number of steps, as follows:

(i) Since g is primitive w.r.t d (see Remark 1.23(1), page 21), $g(0, \bar{x})$ cannot be identically zero. Let $\mathcal{D}_1$ be the zero set in $\mathbb{C}^2$ of $g(0, \bar{x})$. And let $\Omega_2^1 = \mathbb{C}^2 \setminus (\mathcal{C} \cup \mathcal{D}_1)$.

(ii) The next open subset ensures that $\deg_{y_0}(\mathrm{Nor}(\bar{x}, \bar{y}_h))$ stays invariant when specializing $\bar{x}$. First, observe that none of F_1, F_2 can be identically zero because $\mathcal{C}$ is irreducible and it is not a horizontal or vertical line. Now, we introduce the polynomial $\Gamma_i(\bar{y})$ as the leading coefficient of F_i w.r.t. y_0 if F_i depends on y_0, and otherwise $\Gamma_i = F_i$. Let $A(\bar{x}, \bar{y})$ be the leading coefficient of $\mathrm{Nor}(\bar{x}, \bar{y}_h)$ w.r.t y_0. Then $A(\bar{x}, \bar{y})$ is either $-\Gamma_2(\bar{y})x_1 + \Gamma_1(\bar{y})x_2$ or $-\Gamma_2(\bar{y})x_1$ or $\Gamma_1(\bar{y})x_2$. In any case, it is clear that there exists an open subset of Ω_2^1, that we denote Ω_2^2, such that for $\bar{x}^o \in \Omega_2^2$, $A(\bar{x}^o, \bar{y})$ does not vanish.

(iii) Let $T(\bar{x}) = \mathrm{Dis}_d(g(d, \bar{x}))$. Note that g is square-free and primitive w.r.t d, and hence T is not identically zero. Let $\mathcal{D}_3$ be the curve defined by T in $\mathbb{C}^2$ if T is not constant and $\mathcal{D}_3 = \emptyset$ otherwise. Then we consider the open subset $\Omega_2^3 = \Omega_2^3 \setminus \mathcal{D}_3$. Now, let $\bar{x}^o \in \Omega_2^2$. Then $g(d, \bar{x}^o)$ has exactly δ_d different roots because of the construction of Ω_2^2 and Ω_2^3. Proposition 1.27 (page 23) implies that these roots can be grouped in pairs, with elements in each pair differing only by multiplication by -1. Let $\Theta(\bar{x}^o) = \{d_1^o, \ldots, d_\nu^o\}$ be a collection of ν roots of $g(d, \bar{x}^o)$, where each d_i^o is only from one of these pairs. Also, observe that because of the construction of Ω_2^1, $d_i^o \neq 0, \forall i = 1 \ldots, \nu$.

(iv) Let Δ be the finite set of distances in Corollary 1.25 (page 21). Let

$$\mathcal{D}_4 = \bigcup_{d^o \in \Delta} \mathcal{O}_{d^o}(\mathcal{C}),$$

and take $\Omega_2^4 = \Omega_2^3 \setminus \mathcal{D}_4$.

(v) Recall (see Remark 1.19, page 18) that,

$$\dim\left(\mathcal{O}_d(\mathcal{C}) \setminus \pi_1(\Psi(\mathcal{C}))\right) < \dim\left(\mathcal{O}_d(\mathcal{C})\right) = 2.$$

The last equality holds because $\dim\left(\mathcal{O}_d(\mathcal{C})\right) = \dim(\mathcal{C}) + 1$. Thus if we let

$$\mathcal{D}_5 = \{\bar{x}^o \in \mathbb{C}^2/(d^o, \bar{x}^o) \in \left(\mathcal{O}_d(\mathcal{C}) \setminus \pi_1(\Psi(\mathcal{C}))\right),\ \text{for some } d^o \in \mathbb{C}^\times\}^*,$$

(the asterisk denotes Zariski closure) then $\dim(\mathcal{D}_5) \leq 1$. Let us take $\Omega_2^5 = \Omega_2^4 \backslash \mathcal{D}_5$.

(vi) Consider the following resultants:

$$R_i(\bar{x}) = \mathrm{Res}_d\left(g(d, \bar{x}), \frac{\partial g}{\partial x_i}(d, \bar{x})\right)$$

for $i = 1, 2$. Note that $\dfrac{\partial g}{\partial x_i}$ cannot be identically zero, because $\mathcal{C}$ is not a line. Also observe that R_i cannot be identically zero, since this would imply that $\dfrac{\partial g}{\partial x_i}(d, \bar{x})$ and $g(d, \bar{x})$ have a common factor of positive degree in d. This factor cannot depend only on d because of the definition of the generic offset equation. Thus, this would imply that for any $d^o \notin \Delta$ (the set in Corollary 1.25, page 21), the offset has infinitely many ramification points, and this is impossible since the offset cannot have multiple components, and it cannot be a line because $\mathcal{C}$ is not a line. Let Φ_i be the zero set of $R_i(\bar{x})$ in $\mathbb{C}^2$. Take $\Omega_2^6 = \Omega_2^5 \setminus (\Phi_1 \cap \Phi_2)$. Now, if $\bar{x}^o \in \Omega_2^6$, and $g(d^o, \bar{x}^o) = 0$, since $d^o \notin \Delta$, it follows that $\bar{x}^o$ is a regular point of $\mathcal{O}_{d^o}(\mathcal{C})$. Otherwise one has

$$g(d^o, \bar{x}^o) = \frac{\partial g}{\partial x_i}(d^o, \bar{x}^o) = 0$$

for $i = 1, 2$. This means that $R_i(\bar{x}^o) = 0$ for $i = 1, 2$, contradicting the construction of Ω_2^6.

(vii) Let $\{\bar{y}_1^o, \ldots, \bar{y}_r^o\}$ be the isotropic affine and regular points of $\mathcal{C}$. This is a finite set because $\mathcal{C}$ is irreducible. For $i = 1, \ldots, r$, let γ_i be the normal line to $\mathcal{C}$ at $\bar{y}_i^o$. Let $\Omega_2^7 = \Omega_2^6 \setminus \bigcup_{i=1}^r \gamma_i$.

(viii) If $\bar{0}$ (the affine origin) belongs to $\mathcal{C}$ and it is regular, let γ_0 be the normal line zero to $\mathcal{C}$ at $\bar{0}$. Define $\Omega_2^8 = \Omega_2^7 \setminus \gamma_0$.

Let us set $\Omega_2 = \Omega_2^8$, and let $\bar{x}^o \in \Omega_2$. We will show that statements (1)-(4) in the Theorem hold. Statement (1) follows from (vii) and (viii). Let $d_i^o \in \Theta(\bar{x}^o)$ (see (iii)), for $i = 1, \ldots, \nu$. Then $g(\pm d_i^o, \bar{x}^o) = 0$. Thus, because of (iv), $(\pm d_i^o, \bar{x}^o) \in \mathcal{O}_d(\mathcal{C})$. Moreover, because of (v), $(\pm d_i^o, \bar{x}^o) \in \pi_1(\Psi(\mathcal{C}))$. Thus, there exist $\bar{y}_i^o \in \mathcal{C}$ and $u_i^o \in \mathbb{C}$

such that $(u_i^o, \bar{x}^o, \bar{y}_i^o)$ is a solution of the Offset System $\mathfrak{S}^1(\pm d_i^o)$ (page 83). In particular, this implies that $\bar{y}_i^o$ is a solution of $\mathfrak{S}_4(\bar{x}^o)$, and that $\bar{y}_i^o$ generates $\bar{x}^o$ in $\mathcal{O}_{\pm d_i^o}(\mathcal{C})$. Let $\mathcal{A} = \{\bar{y}_1^o, \ldots, \bar{y}_\nu^o\}$ be the set of points constructed in this way. Observe that $\bar{y}_i^o \in \mathcal{C}$ and it is affine. Moreover, since $(u_i^o, \bar{x}^o, \bar{y}_i^o)$ is a solution of $\mathfrak{S}^1(\pm d_i^o)$, then $\bar{y}_i^o \in \Psi_4(\bar{x}^o)$. Thus, $\mathcal{A} \subset \Psi_4(\bar{x}^o)$.

Now, since $d_i^o \neq \pm d_j^o$ for $i \neq j$ (see (iii)), and since $\bar{y}_i^o$ belongs to a circle of radius d_i^o, centered at $\bar{x}^o$, one concludes that $\bar{y}_i^o \neq \bar{y}_j^o$. Therefore, $\#\mathcal{A} = \nu$. To prove statement (ii), it remains to show that $\Psi_4(\bar{x}^o) \subset \mathcal{A}$. Let $\bar{y}^o \in \Psi_4(\bar{x}^o)$. Since $\bar{y}^o \notin \mathrm{Iso}(\mathcal{C})$, let $u^o = 1/h(\bar{y}^o)$, and let d^o be any of the solutions in d of this equation:

$$d^2 = (x_1^o - y_1^o)^2 + (x_2^o - y_2^o)^2.$$

Note that, since $\mathrm{nor}(\bar{x}^o, \bar{y}^o) = 0$, and $\bar{y}^o \notin \mathrm{Iso}(\mathcal{C})$, one has $d^o \neq 0$. Thus, $(u^o, \bar{x}^o, \bar{y}^o) \in \Psi_1(d^o)$, and $\bar{y}^o$ generates $\bar{x}^o$ in $\mathcal{O}_{d^o}(\mathcal{C})$. Now observe that, $g(\bar{x}, d^o) = 0$ is the equation of $\mathcal{O}_{d^o}(\mathcal{C})$ (otherwise, $d^o \in \Delta$, contradicting (iv)). It follows that $g(d^o, \bar{x}^o) = 0$. Therefore $\pm d^o \in \Theta(\bar{x}^o)$. That means that there is some $i = 1, \ldots, \nu$ such that $d^o = \pm d_i^o$. We claim that $\bar{y}^o = \bar{y}_i^o \in \mathcal{A}$. In fact, assume that $\bar{y}^o \neq \bar{y}_i^o$. Both these points are regular in $\mathcal{C}$ and generate $\bar{x}^o \in \mathcal{O}_{d_i^o}(\mathcal{C})$. Then, we can take places of $\mathcal{C}$ at both $\bar{y}^o$ and $\bar{y}_i^o$ and lift them to places of $\mathcal{O}_{\pm d_i^o}(\mathcal{C})$ at $\bar{x}^o$. Since $\mathcal{O}_{d_i^o}(\mathcal{C})$ has no special component (by (iv)), these two places cannot lift to the same place of the offset. But if they lift to different places, it follows that $\bar{x}^o$ is not regular in $\mathcal{O}_{d_i^o}(\mathcal{C})$. This contradicts (vi). From this contradiction we conclude that $\bar{y}^o \in \mathcal{A}$, and statement (2) holds.

Statements (3) and (4) follow directly from the construction of $\mathcal{A}$, and from the proof of the identity $\mathcal{A} = \Psi_4(\bar{x}^o)$ that we have just shown. Statement (5) follows from (vi). □

In Section 3.4, and in order to prove the degree formulae for the parametric case, we will need to avoid a certain finite subset $\mathcal{X} \subset \mathcal{C}$. The situation is analogous to Lemma 2.6 of Chapter 2 (page 44), but we can not use directly that lemma: it depends on the pencil of lines through the origin used in Chapter 2. However, we have just seen how to do this, in the proof of Theorem 3.29 (page 101), for the case when $\mathcal{X} = \mathrm{Sing}(\mathcal{C})$ (see the proof of (vii)). And the same argument applies for any finite set. Thus we have proved the following:

Lemma 3.30. *Let Ω_2 be as in Theorem 3.29. If $\mathcal{X} \subset \mathcal{C}$ is a finite set, there exists an open non-empty subset $\Omega^0_{\mathcal{X}} \subset \Omega^0$ such that, if $\bar{x}^o \in \Omega^0_{\mathcal{X}}$, then $\bar{x}^o$ is not associated with any point in $\mathcal{X}$.*

Proof. See (vii) in the proof of Theorem 3.29, page 101. We just need to remove from Ω_2 the (finitely many) normal lines to $\mathcal{C}$ at the points of $\mathcal{X}$. □

In the next definition we extend the terminology of fake and non-fake points to this degree problem.

Definition 3.31. *Let Ω_2 be as in Theorem 3.29. We denote:*

$$\mathcal{F}_d = \bigcap_{\bar{x}^o \in \Omega_2} \left(\overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}}\right)$$

The points of the set $\mathcal{F}_d$ are called d-fake points*. For $\bar{x}^o \in \Omega_2$, the points in $\left(\overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}}\right) \setminus \mathcal{F}_d$ are called* non d-fake points*.*

The next step consists in showing the invariance of the set of d-fake points. This is established in the next proposition (compare to Proposition 3.17). Recall that $\mathrm{Sing}_a(\overline{\mathcal{C}})$ is the affine singular locus of $\overline{\mathcal{C}}$. We also denote by $\mathrm{Iso}_\infty(\overline{\mathcal{C}})$ the set of isotropic points at infinity of $\overline{\mathcal{C}}$; that is, the set of points $\bar{y}_h^o \in \mathcal{C}$ that satisfy $y_0^o = 0$ and $F_1^2(\bar{y}_h^o)+F_2^2(\bar{y}_h^o) = 0$.

Proposition 3.32. *Let Ω_2 be as in Theorem 3.29. The set $\mathcal{F}_d$ is finite. Moreover,*

$$\mathcal{F}_d = \mathrm{Sing}_a(\overline{\mathcal{C}}) \cup \mathrm{Iso}_\infty(\overline{\mathcal{C}})$$

Proof. Let $\bar{y}_h^o \in \mathcal{F}_d$. Then $\bar{y}_h^o \in \overline{\mathcal{C}}$ and $\mathrm{Nor}(\bar{x}^o, \bar{y}_h^o) = 0$ for every $\bar{x}^o \in \Omega_2$. Thus, considering $\mathrm{Nor}(\bar{x}, \bar{y}_h) \in \mathbb{C}[\bar{y}_h][\bar{x}]$, one has that:

$$F_2(\bar{y}_h^o)y_0^o = 0, \quad F_1(\bar{y}_h^o)y_0^o = 0, \quad F_2(\bar{y}_h^o)y_1^o - F_1(\bar{y}_h^o)y_2^o = 0.$$

If $\bar{y}_h^o$ is affine, then $\bar{y}_h^o \in \mathrm{Sing}_a(\overline{\mathcal{C}})$. If $y_0^o = 0$, then using Euler's identity

$$y_1^o F_1(\bar{y}_h^o) + y_2^o F_2(\bar{y}_h^o) b = \deg_{\bar{y}_h}(F) F(\bar{y}_h^o) = 0$$

From this relation and $F_2(\bar{y}_h^o)y_1^o - F_1(\bar{y}_h^o)y_2^o = 0$ one has that $F_1^2(\bar{y}_h^o) + F_2^2(\bar{y}_h^o) = 0$. Thus $\bar{y}_h^o \in \mathrm{Iso}_\infty(\overline{\mathcal{C}})$. Therefore $\mathcal{F}_d \subset \mathrm{Sing}_a(\overline{\mathcal{C}}) \cup \mathrm{Iso}_\infty(\overline{\mathcal{C}})$.

Conversely, let $\bar{y}_h^o \in \mathrm{Sing}_a(\overline{\mathcal{C}}) \cup \mathrm{Iso}_\infty(\overline{\mathcal{C}})$. If $\bar{y}_h^o \in \mathrm{Sing}_a(\overline{\mathcal{C}})$, then $\bar{y}_h^o \in \overline{\mathcal{C}}$ and for every $\bar{x}^o \in \Omega_2$ one has $\mathrm{Nor}(\bar{x}^o, \bar{y}_h^o) = 0$. Thus, $\bar{y}_h^o \in \mathcal{F}_d$. If $\bar{y}_h^o \in \mathrm{Iso}_\infty(\overline{\mathcal{C}})$, then $\bar{y}_h^o \in \mathcal{C}$, $y_0^o = 0$, and $F_1^2(\bar{y}_h^o) + F_2^2(\bar{y}_h^o) = 0$. Using Euler's identity as before, one has

$$F_1(\bar{y}_h^o)y_1^o + F_2(\bar{y}_h^o)y_2^o = 0.$$

From these relations, one gets $\mathrm{Nor}(\bar{x}^o, \bar{y}_h^o) = F_2(\bar{y}_h^o)y_1^o - F_1(\bar{y}_h^o)y_2^o = 0$ for all $\bar{x}^o \in \Omega_2$. Thus, $\bar{y}_h^o \in \mathcal{F}_d$.

The finiteness of $\mathcal{F}_d$ follows from the equality $\mathcal{F}_d = \mathrm{Sing}_a(\overline{\mathcal{C}}) \cup \mathrm{Iso}_\infty(\overline{\mathcal{C}})$. □

Remark 3.33.

1. *The proof of Proposition 3.32 shows that if $\bar{y}_h^o$ is a point at infinity of $\mathcal{C}$, and for some $\bar{x}^o \in \Omega_2$, $\bar{y}_h^o \in \overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}}$, then $\bar{y}_h^o \in \mathrm{Iso}_\infty(\overline{\mathcal{C}})$. In particular, $\bar{y}_h^o \in \mathcal{F}_d$.*

2. *From the definition of $\mathcal{F}_d$ it follows that for every non-empty open subset $\Omega \subset \Omega_2$, one has*
$$\mathcal{F}_d = \bigcap_{\bar{x}^o \in \Omega} \left(\overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}} \right)$$

In the statement of the next proposition we will use the fact that all the points in $\left(\overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}} \right) \setminus \mathcal{F}_d$ are affine (see Remark 3.33(1)). Thus, abusing notation, we think of this set as an affine set.

Proposition 3.34. *Let Ω_2 be as in Theorem 3.29. If $\bar{x}^o \in \Omega_2$, then*
$$\left(\overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}} \right) \setminus \mathcal{F}_d = \Psi_4(\bar{x}^o).$$
In particular,
$$\delta_d = 2\nu = 2\#\left(\left(\overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}} \right) \setminus \mathcal{F}_d \right)$$

Proof. Let $\bar{y}_h^o \in \left(\overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}} \right) \setminus \mathcal{F}_d$. By Remark 3.33 (1), one has $y_0^o \neq 0$, and so $\bar{y}_h^o \in \Psi_4(\bar{x}^o)$.

Conversely, let $\bar{y}^o \in \Psi_4(\bar{x}^o)$, and let $\bar{y}_h^o = (1 : y_1^o : y_2^o)$. Then $\bar{y}_h^o \in \left(\overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}} \right)$. Since $\bar{y}^o \notin \operatorname{Iso}(\mathcal{C})$ (recall Theorem 3.29(1), page 101), by Proposition 3.32 (page 105), $\bar{y}_h^o \notin \mathcal{F}_d$.

The last statement now follows from Theorem 3.29(2). □

The next proposition gathers the information we need when applying the framework introduced in Theorem 2.30 (page 65) to the curves $\overline{\mathcal{C}}$ and $\overline{\mathcal{N}}_{\bar{x}^o}$ (compare to Proposition 3.21, in page 91).

Proposition 3.35. *There exists a non-empty open subset $\Omega_3 \subset \Omega_2$, where Ω_2 is as in Theorem 3.29, such that for every $\bar{x}^o \in \Omega_3$ the following hold:*

1. $\deg(\operatorname{Nor}(\bar{x}^o, \bar{y}_h))$ *does not depend on the choice of $\bar{x}^o$.*

2. *$\mathcal{C}$ and $\mathcal{N}_{\bar{x}^o}$ have no common component.*

3. *If $\bar{y}^o \in \left(\overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}} \right) \setminus \mathcal{F}_d$ (note that in this case the point must be affine), then* $\operatorname{mult}_{\bar{y}^o}(\mathcal{C}, \mathcal{N}_{\bar{x}^o}) = 1$.

4. *Let $\operatorname{Nor}(\bar{x}, \bar{y}_h)$ be considered as an element of $(\mathbb{C}[\bar{y}_h])[\bar{x}]$:*
$$\operatorname{Nor}(\bar{x}, \bar{y}_h) = \operatorname{Nor}_{1,0}(\bar{y}_h)x_1 + \operatorname{Nor}_{0,1}(\bar{y}_h)x_2 + \operatorname{Nor}_{0,0}(\bar{y}_h)$$
where:
$$\begin{cases} \operatorname{Nor}_{1,0}(\bar{y}_h) = -F_2 y_0 \\ \operatorname{Nor}_{0,1}(\bar{y}_h) = F_1 y_0, \\ \operatorname{Nor}_{0,0}(\bar{y}_h) = F_2 y_1 - F_1 y_2, \end{cases}$$

and let $\mathcal{N}_\alpha$ be the zero set in $\mathbb{C}^2$ set of $\mathrm{Nor}_\alpha(\bar{y}_h)$. *Then it holds that:*

$$\bigcap_\alpha (\overline{\mathcal{C}} \cap \overline{\mathcal{N}_\alpha}) \subset \mathcal{F}_d.$$

5. $(1:0:0) \notin \left(\overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}}\right) \setminus \mathcal{F}_d$

Proof.

1. See step (ii) in the proof of Theorem 3.29.
2. Let us consider $\mathrm{nor}(\bar{x}, \bar{y})$ as a polynomial in $\mathbb{C}[\bar{y}][\bar{x}]$. If $\mathrm{nor}(\bar{x}, \bar{y})$ and f have a common factor, one has that $f_1(\bar{y}^o) = f_2(\bar{y}^o) = 0$ for every point $\bar{y}^o \in \mathcal{C}$, which is a contradiction since $\mathcal{C}$ is irreducible.
3. Let $\mathcal{P}(t) = (y_1(t), y_2(t))$, with

$$\begin{cases} y_1(t) = y_1^o + a_1 t + \cdots \\ y_2(t) = y_2^o + b_1 t + \cdots \end{cases}$$

be a place of $\mathcal{C}$ centered at $\bar{y}^o$. Then $\mathrm{mult}_{\bar{y}^o}(\overline{\mathcal{C}}, \overline{\mathcal{N}_{\bar{x}^o}})$ is equal to the order of $\mathrm{nor}(\bar{x}^o, \mathcal{P}(t))$. Let now

$$\begin{cases} f_1(\mathcal{P}(t)) = \alpha_0 + \alpha_1 t + \cdots \\ f_2(\mathcal{P}(t)) = \beta_0 + \beta_1 t + \cdots \end{cases}$$

Note that $\alpha_0^2 + \beta_0^2 \neq 0$ because $\bar{y}^o$ is non d-fake. Besides, the point $\bar{x}^o$ is generated by $\bar{y}^o$ in $\mathcal{O}_{d_{\bar{y}^o}}(\mathcal{C})$, (see Theorem 3.29(4), page 101, and take Proposition 3.34, page 106, into account). To keep a simpler notation we will denote $d^o = d_{\bar{y}^o}$ one has:

$$x_1^o = y_1^o + d^o \frac{\alpha_0}{\sqrt{(\alpha_0^2 + \beta_0^2)}}$$

$$x_2^o = y_2^o + d^o \frac{\beta_0}{\sqrt{(\alpha_0^2 + \beta_0^2)}}$$

Substituting the above expressions in $\mathrm{nor}(\bar{x}^o, \mathcal{P}(t))$ we get:

$$\mathrm{nor}(\bar{x}^o, \mathcal{P}(t)) = \left(-\alpha_0 b_1 + \alpha_1 d^o \frac{\beta_0}{\sqrt{(\alpha_0^2 + \beta_0^2)}} + \beta_0 a_1 - \beta_1 d^o \frac{\alpha_0}{\sqrt{(\alpha_0^2 + \beta_0^2)}} \right) t + \cdots$$

(the order zero term vanishes identically). Now, we will suppose that we have $\mathrm{mult}_{\bar{y}^o}(\mathcal{C}, \mathcal{N}_{\bar{x}^o}) > 1$ and we will arrive at a contradiction. This would imply that

$$-\alpha_0 b_1 + \alpha_1 d^o \frac{\beta_0}{\sqrt{(\alpha_0^2 + \beta_0^2)}} + \beta_0 a_1 - \beta_1 d^o \frac{\alpha_0}{\sqrt{(\alpha_0^2 + \beta_0^2)}} = 0.$$

From this one gets:

$$(-\alpha_1\beta_0 + \beta_1\alpha_0)d^o = -\sqrt{(\alpha_0^2 + \beta_0^2)}(\alpha_0 b_1 - \beta_0 a_1).$$

Now observe that (a_1, b_1) is a tangent vector to $\mathcal{C}$ at $\bar{y}^o$, and (α_0, β_0) is a normal at the same point. Thus $a_1\alpha_0 + b_1\beta_0 = 0$. Thus, if $-\alpha_1\beta_0 + \beta_1\alpha_0 = 0$, since $\alpha_0^2 + \beta_0^2 \neq 0$, one obtains:

$$\begin{cases} \alpha_0 b_1 - \beta_0 a_1 = 0 \\ \beta_0 b_1 + \alpha_0 a_1 = 0 \end{cases}$$

It follows that $a_1 = b_1 = 0$, which is a contradiction, since $\bar{y}^o$ is regular in $\mathcal{C}$. Thus, we have shown that $-\alpha_1\beta_0 + \beta_1\alpha_0 \neq 0$. Therefore

$$d^o = \sqrt{(\alpha_0^2 + \beta_0^2)}\frac{\alpha_0 b_1 - \beta_0 a_1}{\alpha_1\beta_0 - \beta_1\alpha_0}.$$

Now, as in the proof of Proposition 3.21 (page 91), we can offset the place $\mathcal{P}(t)$ to get a place $\mathcal{Q}(t)$ of $\mathcal{O}_{d^o}(\mathcal{C})$ centered at $\bar{x}^o$.

$$\mathcal{Q}(t) = (\mathcal{Q}_1(t), \mathcal{Q}_2(t)) = \left(y_1(t) \pm d^o \frac{f_1(t)}{\sqrt{f_1^2(t) + f_2^2(t)}}, y_2(t) \pm d^o \frac{f_2(t)}{\sqrt{f_1^2(t) + f_2^2(t)}} \right).$$

Substituting the above expressions for $y_1(t), y_2(t), f_1(t), f_2(t)$ and d^o one has, after simplifying the expression:

$$\mathcal{Q}_1(t) = x_1^o + (a_1\alpha_0 + b_1\beta_0)\frac{\alpha_0}{\alpha_0^2 + \beta_0^2}t + \cdots$$

Similarly

$$\mathcal{Q}_2(t) = x_2^o + (a_1\alpha_0 + b_1\beta_0)\frac{\beta_0}{\alpha_0^2 + \beta_0^2}t + \cdots$$

Since $a_1\alpha_0 + b_1\beta_0 = 0$, this would imply that $\bar{x}^o$ is not regular in $\mathcal{O}_{d^o}(\mathcal{C})$, contradicting Theorem 3.29(5) (page 101).

4. If $\bar{y}_h^o \in \bigcap_\alpha(\overline{\mathcal{C}} \cap \overline{\mathcal{N}_\alpha})$, then $y_0^o F_1(\bar{y}_h^o) = 0$ and $y_0^o F_2(\bar{y}_h^o) = 0$. If $y_0^o \neq 0$, it follows that $F_1(\bar{y}_h^o) = F_2(\bar{y}_h^o) = 0$. If $y_0^o = 0$, $F_1^2(\bar{y}_h^o) + F_2^2(\bar{y}_h^o) = 0$ follows by Remark 3.33(1). In either case, by Proposition 3.32, $\bar{y}_h^o \in \mathcal{F}_d$.

5. This follows from statement (1) in Theorem 3.29. □

3.3.2 Resultant-based formula for the degree in the distance

As a consequence of the above results, we can apply Theorem 2.30 (page 65) to derive the following resultant-based formula for computing δ_d.

Theorem 3.36 (Degree formula for the distance, implicit case).

$$\boxed{\delta_d = \deg_d(\mathcal{O}_d(\mathcal{C})) = 2\deg_{\{\bar{y}\}}\left(\mathrm{PP}_{\{\bar{x}\}}\left(\mathrm{Res}_{y_0}(F(\bar{y}_h), \mathrm{Nor}(\bar{x}, \bar{y}_h))\right)\right)}$$

We recall that $F(\bar{y}_h)$ is the homogeneous implicit equation of the curve, and $N(\bar{x}, \bar{y}_h)$ is the polynomial introduced in Remark 3.27 (page 100).

Proof. In order to prove the theorem, we apply Theorem 2.30 to $\mathcal{C}$, and $Z(\bar{\omega}, \bar{y}_h) = \mathrm{Nor}(\bar{x}, \bar{y}_h)$, with $\Xi = \Omega_3$, where Ω_3 is as in Proposition 3.35 (page 106). We check that all the hypothesis are satisfied:

- $\mathcal{C}$ is irreducible and it is not a line by assumption.
- Nor can be written as
$$\mathrm{Nor}(\bar{x}, \bar{y}_h) = (-F_2(\bar{y}_h)x_1 + F_1(\bar{y}_h)x_2)y_0 + (y_1F_2(\bar{y}_h) - y_2F_1(\bar{y}_h))$$
Thus, since $F_1(\bar{y}_h)$ and $F_2(\bar{y}_h)$ are not identically zero, $\mathrm{Nor}(\bar{x}, \bar{y}_h)$ depends on y_0.
- (1) and (2) in Theorem 2.30 follow from (1) and (2) in Proposition 3.35.
- The equality
$$\mathcal{F}_d = \bigcap_{\bar{x}^o \in \Omega_3} (\overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}})$$
follows from Remark 3.33(2) and $\Omega_3 \subset \Omega_2$.
- In this situation, hypothesis (3), (4) and (5) in Theorem 2.30 follow from Proposition 3.35 (3), (4) and (5) (page 106).

Then, Theorem 2.30 implies that there exists a non-empty open $\Omega_* \subset \Omega_3$ such that for $\bar{x}^o \in \Omega_*$

$$\#([\overline{\mathcal{N}(\bar{x}^o)} \cap \overline{\mathcal{C}}] \setminus d\mathcal{F}) = \deg_{\{\bar{y}\}}\left(\mathrm{PP}_{\bar{x}}\left(\mathrm{Res}_{y_0}(F(\bar{y}_h), \mathrm{Nor}(\bar{x}, \bar{y}_h))\right)\right)$$

Now the theorem follows from Proposition 3.34, page 106 (note the factor 2). □

We finish this section illustrating the above formula by means of an example.

Example 3.37. *Let $\mathcal{C}$ be the Three Petal Rose, given by the implicit equation*

$$f(\bar{y}) := (y_1^2 + y_2^2)^2 + y_1(3y_2^2 - y_1^2) = 0.$$

See Figure 3.5, where the curve $\mathcal{C}$ is depicted in red, and some of its offset curves in blue. Applying the formula in Theorem 2.31 (page 68), one deduces that the total *degree of the generic offset curve is* 14*. Now, we apply Theorem 3.36 to compute* δ_d*. The polynomial* $\mathrm{Nor}(\bar{x}, \bar{y}_h)$ *is:*

$$\mathrm{Nor}(\bar{x}, \bar{y}_h) = (4\,y_1^3+4\,y_2^2y_2+3\,y_0\,y_2^2-3\,y_1^2y_0)(x_2\,y_0-y_2)-(4\,y_1^2y_2+4\,y_2^3+6\,y_2\,y_0\,y_2)(x_1\,y_0-y_2)$$

and
$\mathrm{Res}_{y_0}(F, \mathrm{Nor}) = (y_1^2+y_2^2)^2(-11\,y_1^4y_2^2x_2+y_1^6x_2-10\,y_1^5y_2\,x_1-9\,y_2^4y_1^2x_2-4\,y_2^3y_1^3x_1+3\,y_2^6x_2+6\,y_2^5y_1\,x_1+9\,y_1\,y_2^5-30\,y_2^3y_1^3+9\,y_1^5y_2)$.
Thus, we conclude that

$$\delta_d = \deg_d(\mathcal{O}_d(\mathcal{C})) = 2\deg_{\{\bar{y}\}}\left(\mathrm{PP}_{\{\bar{x}\}}\left(\mathrm{Res}_{y_0}(F(\bar{y}_h), \mathrm{Nor}(\bar{x}, \bar{y}_h))\right)\right) = 12.$$

In fact, the generic offset polynomial for this curve, obtained by elimination methods is (the terms have been collected w.r.t. to their degree in d):
$g(d,\bar{x}) = 11664\,d^{12} + (6912\,x_1^3 - 20736\,x_1x_2^2 - 54432\,x_1^2 - 54432\,x_2^2 - 34992)d^{10} + (1024\,x_1^6 + 3072\,x_1^4x_2^2 + 3072\,x_1^2x_2^4 + 1024\,x_2^6 - 29952\,x_1^5 + 59904\,x_1^3x_2^2 + 89856\,x_1x_2^4 + 101520\,x_1^4 + 203040\,x_1^2x_2^2 + 101520\,x_2^4 - 21600\,x_1^3 + 64800\,x_1x_2^2 + 91368\,x_1^2 + 91368\,x_2^2 + 34992)d^8 + (-4096\,x_1^8 - 16384\,x_1^6x_2^2 - 24576\,x_1^4x_2^4 - 16384\,x_1^2x_2^6 - 4096\,x_2^8 + 50432\,x_1^7 - 50432\,x_1^5x_2^2 - 252160\,x_1^3x_2^4 - 151296\,x_1x_2^6 - 98944\,x_1^6 - 287616\,x_1^4x_2^2 - 302976\,x_1^2x_2^4 - 97920\,x_2^6 + 49824\,x_1^5 - 99648\,x_1^3x_2^2 - 149472\,x_1x_2^4 - 78192\,x_1^4 - 156384\,x_1^2x_2^2 - 78192\,x_2^4 + 43848\,x_1^3 - 131544\,x_1x_2^2 - 54432\,x_1^2 - 54432\,x_2^2 - 11664)d^6 + (6144\,x_1^{10} + 30720\,x_1^8x_2^2 + 61440\,x_1^6x_2^4+61440\,x_1^4x_2^6+30720\,x_1^2x_2^8+6144\,x_2^{10}-40704\,x_1^9+244224\,x_1^5x_2^4+325632\,x_1^3x_2^6+122112\,x_1x_2^8+53760\,x_1^8+187392\,x_1^6x_2^2+313344\,x_1^4x_2^4+230400\,x_1^2x_2^6+50688\,x_2^8-33984\,x_1^7+33984\,x_1^5x_2^2 + 169920\,x_1^3x_2^4 + 101952\,x_1x_2^6 + 39960\,x_1^6 - 16200\,x_1^4x_2^2 + 210600\,x_1^2x_2^4 + 24840\,x_2^6 - 62208\,x_1^5 + 124416\,x_1^3x_2^2 + 186624\,x_1x_2^4 + 22113\,x_1^4 + 44226\,x_1^2x_2^2 + 22113\,x_2^4 - 5832\,x_1^3+17496\,x_1x_2^2+17496\,x_1^2+17496\,x_2^2)d^4+(-4096\,x_1^{12}-24576\,x_1^{10}x_2^2-61440\,x_1^8x_2^4-81920\,x_1^6x_2^6-61440\,x_1^4x_2^8-24576\,x_1^2x_2^{10}-4096\,x_2^{12}+15360\,x_1^{11}+15360\,x_1^9x_2^2-92160\,x_1^7x_2^4-215040\,x_1^5x_2^6 - 168960\,x_1^3x_2^8 - 46080\,x_1x_2^{10} - 14592\,x_1^{10} - 45312\,x_1^8x_2^2 - 109056\,x_1^6x_2^4 - 152064\,x_1^4x_2^6-85248\,x_1^2x_2^8-11520\,x_2^{10}+7488\,x_1^9-44928\,x_1^5x_2^4-59904\,x_1^3x_2^6-22464\,x_1x_2^8-21600\,x_1^8 + 84672\,x_1^6x_2^2 - 72576\,x_1^4x_2^4 - 181440\,x_1^2x_2^6 - 2592\,x_2^8 + 20088\,x_1^7 - 20088\,x_1^5x_2^2 - 100440\,x_1^3x_2^4 - 60264\,x_1x_2^6 - 3402\,x_1^6 + 7290\,x_1^4x_2^2 - 21870\,x_1^2x_2^4 - 1458\,x_2^6 + 7290\,x_1^5 - 14580\,x_1^3x_2^2 - 21870\,x_1x_2^4 - 6561\,x_1^4 - 13122\,x_1^2x_2^2 - 6561\,x_2^4)d^2 + 1024\,x_1^{14} + 7168\,x_1^{12}x_2^2 + 21504\,x_1^{10}x_2^4+35840\,x_1^8x_2^6+35840\,x_1^6x_2^8+21504\,x_1^4x_2^{10}+7168\,x_1^2x_2^{12}+1024\,x_2^{14}-2048\,x_1^{13}-4096\,x_1^{11}x_2^2+10240\,x_1^9x_2^4+40960\,x_1^7x_2^6+51200\,x_1^5x_2^8+28672\,x_1^3x_2^{10}+6144\,x_1x_2^{12}+1024\,x_1^{12}-3072\,x_1^{10}x_2^2-6144\,x_1^8x_2^4+10240\,x_1^6x_2^6+21504\,x_1^4x_2^8+9216\,x_1^2x_2^{10}-1728\,x_1^{11}-1728\,x_1^9x_2^2+10368\,x_1^7x_2^4+24192\,x_1^5x_2^6+19008\,x_1^3x_2^8+5184\,x_1x_2^{10}+3456\,x_1^{10}-13824\,x_1^8x_2^2-6912\,x_1^6x_2^4+41472\,x_1^4x_2^6 + 31104\,x_1^2x_2^8 - 1728\,x_1^9 + 15552\,x_1^7x_2^2 - 46656\,x_1^5x_2^4 + 46656\,x_1^3x_2^6 + 729\,x_1^8 + 2916\,x_1^6x_2^2+4374\,x_1^4x_2^4+2916\,x_1^2x_2^6+729\,x_2^8-1458\,x_1^7+1458\,x_1^5x_2^2+7290\,x_1^3x_2^4+4374\,x_1x_2^6+729\,x_1^6 - 4374\,x_1^4x_2^2 + 6561\,x_1^2x_2^4$,
and so the distance degree value agrees with the one predicted by our formula.

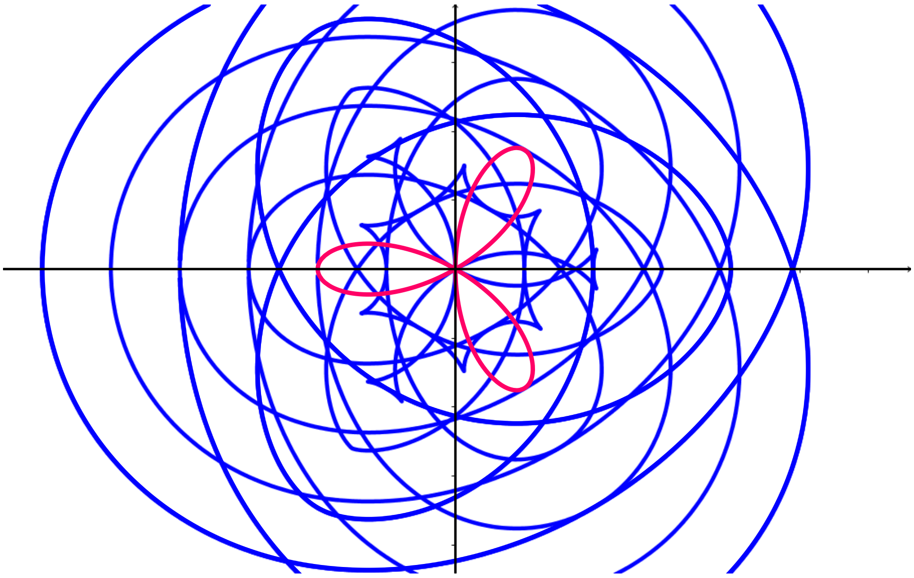

Figure 3.5: The Three Petal Rose and some of its offset curves.

3.4 Extension of the Formulae to the Parametric Case

To complete the degree analysis, in this final section of the chapter we will present degree formulae for δ_1, δ_2 and δ_d, when the curve $\mathcal{C}$ is given parametrically. The situation is similar to Section 2.4 (page 71) of Chapter 2. As we did there, we aim to translate the information contained in the auxiliary curve associated with each of the degree problems (partial and w.r.t. the distance), into the parameter space. The result of this approach is, for each degree problem, a univariate auxiliary polynomial. The invariant solutions are reflected in the content w.r.t. the corresponding parameters of these curves, and taking this into account we are able to obtain the degree formulae. In Subsection 3.4.1 this work is done for the partial degree problem, with the auxiliary polynomial in Definition 3.38 (page 112), and the degree formula appearing in Theorem 3.42 (page 115). Then, in Subsection 3.4.2 (page 117) we do a similar work for the degree w.r.t. d. The auxiliary polynomial is described in Definition 3.44 (page 117), and the degree formula appears in Theorem 3.48 (page 118).

The formulae obtained in this section only require the computation of the degree of univariate gcds of polynomials directly related to the parametrization. Thus, together with the results in Section 2.4 of Chapter 2 they provide a complete and efficient solution of the generic offset degree problem for the specially important class of rational curves, given parametrically.

We will use the notation for parametric curves introduced in Section 2.4. Thus, let

$$P(t) = \left(\frac{X(t)}{W(t)}, \frac{Y(t)}{W(t)}\right)$$

be a proper rational parametrization of a plane curve $\mathcal{C}$. Recall that $X, Y, Z, W \in \mathbb{C}[t]$, with $\gcd(X, Y, W) = 1$. And let $N(t)$ and $H_P(t)$ be the normal vector and parametric hodograph, introduced in Definitions 2.33 and 2.34 (page 71), respectively. We will also need the relation (see Remark 2.35, page 72):

$$f_i(P(t)) = \frac{Q(t)}{(W(t))^{\mu}} N_i(t) \text{ for } i = 1, 2. \tag{3.8}$$

for some $\mu \in \mathbb{N}$ and $Q(t) \in \mathbb{C}[t]$, with $\gcd(Q(t), W(t)) = 1$.

3.4.1 Partial degree formula in the parametric case

The auxiliary curve for the partial degree problem (more precisely, for the computation of δ_1) is defined by the polynomial:

$$s^1(d, k, \bar{y}) = h(\bar{y})(y_2 - k)^2 - d^2 f_2^2(\bar{y}),$$

see Definition 3.9 (page 87). Now, substituting $\bar{y} = P(t)$ in $s^1(d, k, \bar{y})$, and taking Equation 3.8 into account, one has:

$$s^1(d, k, P(t)) = \frac{Q^2(t)}{(W(t))^{2\mu+1}} \left(H_P(t)(Y(t) - kW(t))^2 - d^2 N_2^2(t) W^2(t)\right). \tag{3.9}$$

Thus we are led to consider the following definition.

Definition 3.38. *The polynomial:*

$$s_P^{(1)}(d, k, t) = H_P(t)(Y(t) - kW(t))^2 - d^2 N_2^2(t) W^2(t)$$

is called the parametric δ_1-auxiliary polynomial.

In Section 3.2 we have seen that, for a generic choice of (d^o, k^o), the number of non-invariant solutions of $f(\bar{y}) = s^1(d^o, k^o, \bar{y}) = 0$, that is, the cardinal of $\left(\overline{\mathcal{S}^1_{(d^o,k^o)} \cap \mathcal{C}}\right) \setminus \mathcal{F}_1$, equals the partial degree δ_1 (see in particular Remark 3.19, page 90). Thus, now it seems natural to look at those non-invariant values t^o (that is, depending on the choice of (d^o, k^o)) such that $s_P^{(1)}(d^o, k^o, t^o) = 0$. The strategy for the degree formula now follows closely the structure of Section 2.4. The first step is the analysis of the content of $s_P^{(1)}(d, k, t)$, in order to determine the invariant solutions. Let:

$$G(t) = \gcd(W(t), Y(t))$$

and

$$W_0(t) = \frac{W(t)}{G(t)}, \quad Y_0(t) = \frac{Y(t)}{G(t)}.$$

Thus, $\gcd(W_0, Y_0) = 1$.

Lemma 3.39. $\mathrm{Con}_{(d,k)}(s_P^{(1)}(d,k,t)) = G^2(t) \gcd(H_P(t), W_0^2(t))$.

Proof. Considering $s_P^{(1)}(d,k,t)$ as a polynomial in $\mathbb{C}[d,k][t]$, one has:

$\mathrm{Con}_{(d,k)}(s_P^{(1)}(d,k,t)) = \gcd(H_P Y^2, H_P Y W, H_P W^2, N_2^2 W^2) = \gcd(H_P \gcd(Y,W)^2, N_2^2 W^2)$

Therefore, taking $\gcd(H_P, N_2) = 1$ (this follows from $\gcd(N_1, N_2) = 1$) into account:

$$\mathrm{Con}_{(d,k)}(s_P^{(1)}(d,k,t)) = G^2 \gcd(H_P, N_2^2 W_0^2) = G^2 \gcd(H_P, W_0^2),$$

and the proof is finished. □

Remark 3.40. *We* denote:

$$U(t) = G^2(t) \gcd\left(H_P(t), W_0^2(t)\right)$$

Note that $W(t^o) \neq 0$ implies $U(t^o) \neq 0$.

Now we will establish the relation between the non-invariant solutions of $s_P^{(1)}(d,k,t)$ and the points in $\left(\overline{\mathcal{S}^1_{(d^o,k^o)}} \cap \overline{\mathcal{C}}\right) \setminus \mathcal{F}_1$. This is analogous to Proposition 2.38, page 73, and the proof will be very similar.

Proposition 3.41. *Let Ω_1 be as in Proposition 3.21 (page 91). There exists an open non-empty subset $\Omega_4 \subset \Omega_1$ such that, for $(d^o, k^o) \in \Omega_4$:*

1. $\left(\overline{\mathcal{S}^1_{(d^o,k^o)}} \cap \overline{\mathcal{C}}\right) \setminus \mathcal{F}_1 \subset P(\mathbb{C})$.

2. *There is a one-to-one correspondence between the points of $\left(\overline{\mathcal{S}^1_{(d^o,k^o)}} \cap \overline{\mathcal{C}}\right) \setminus \mathcal{F}_1$ and the values $t^o \in \mathbb{C}$ that verify $s_P^{(1)}(d^o, k^o, t^o) = 0$ and $U(t^o) \neq 0$.*

Proof.
We proceed as in the proof of Proposition 2.38, constructing first the set Ω_4. Let $A \subset \mathbb{C}$ be the finite set of roots of $Q(t)$ (see Remark 2.35, page 72). Since $\gcd(W, Q) = 1$, if $t^o \in A$, then $W(t^o) \neq 0$. Thus, the set $\mathcal{A}_1 = P(A)$ is a well-defined finite subset of $\mathcal{C}$. Besides, the set $\mathcal{A}_2 = \mathcal{C} \setminus P(\mathbb{C})$ is also a finite subset of $\mathcal{C}$. Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and let $\Omega^0_{\mathcal{A}}$ be the set provided by Lemma 3.6, page 86, when one takes $\mathcal{X} = \mathcal{A}$. We consider $\Omega^0_4 = \Omega^0_{\mathcal{A}} \cap \Omega_1$, where Ω_1 is as in Proposition 3.21 (not to be confused with the set Ω_1 in Proposition 2.38 of Chapter 2). Note that $W(t^o) \neq 0$ implies $U_d(t^o) \neq 0$.

Next, since $\gcd(W, Q) = 1$, if $Q(t^o) = 0$, then $W(t^o) \neq 0$. Thus, since the set of roots of Q is finite and $\gcd(H_P, N_2) = 1$ (note the difference with 2.38), there exists a non-empty open $\Omega_4^1 \subset \Omega_4^0$ such that, for $(d^o, k^o) \in \Omega_4^1$, $Q(t^o) = 0$ implies $s_P^{(1)}(d^o, k^o, t^o) \neq 0$.

The next step also differs slightly from the proof of 2.38: if $W(t^o) = 0$ but $U(t^o) \neq 0$, then $G(t^o) \neq 0$ and $\gcd(H, W_0^2)(t^o) \neq 0$. From $W_0(t^o) = 0$, it follows that $Y_0(t^o) \neq 0$. Thus, $Y(t^o) \neq 0$. And from $\gcd(H, W_0^2)(t^o) \neq 0$ it follows that $H(t^o) \neq 0$. In this case we have:

$$s_P^{(1)}(d, k, t^o) = H_P(t^o)Y^2(t^o) \neq 0.$$

Therefore, we need not impose more restrictions on the open set. Then, just as in the proof of 2.38, we use that P is a proper parametrization, to choose $\Omega_4^2 \subset \Omega_4^1$ such that for $(d^o, k^o) \in \Omega_4^2$, if t_1^o, t_2^o are roots of $s_P(d^o, k^o, t)$ with $U(t_1^o)U(t_2^o) \neq 0$ and $P(t_1^o) = P(t_2^o)$, then $t_1^o = t_2^o$.

And finally, set $\Omega^4 = \Omega_4^2$. Let us show that, for $(d^o, k^o) \in \Omega^4$, statements (1) and (2) hold. The proof of this claim follows almost verbatim the proof of Proposition 2.38 from this point, using in this case Equation 3.9 (page 112) instead of 2.9, that was used there.

(1) Let us assume that $\bar{y}^o \in \left(\mathcal{S}^1_{(d^o,k^o)} \cap \mathcal{C}\right) \setminus \mathcal{F}_1$ (recall that all the points in $\left(\mathcal{S}^1_{(d^o,k^o)} \cap \mathcal{C}\right) \setminus \mathcal{F}_1$ are affine, see Remark 3.16(1), page 90). This implies (see Theorem 3.12(b), page 2.17 and Remark 3.19(1), page 90) that $\bar{y}^o$ is associated with $\bar{x}^o \in \mathcal{O}_{d^o}(\mathcal{C}) \cap \mathcal{L}_{k^o}$. Because of the construction of Ω_4^0, this implies that there exists $t^o \in \mathbb{C}$ with $P(t^o) = \bar{y}^o$. Thus, (1) holds.

(2) Let us suppose first that $\bar{y}^o \in \left(\mathcal{S}^1_{(d^o,k^o)} \cap \mathcal{C}\right) \setminus \mathcal{F}_1$. We have proved in the preceding paragraph that there is $t^o \in \mathbb{C}$ such that $P(t^o) = \bar{y}^o$. In particular $W(t^o) \neq 0$, and so $U(t^o) \neq 0$. Since $\bar{y}^o \notin \mathrm{Sing}_a(\mathcal{C})$, one concludes from Remark 2.35 (page 72) that $Q(\bar{t}^o) \neq 0$. Then, from Equation 3.9 (page 112), one has $s_P^{(1)}(d^o, k^o, t^o) = 0$. Conversely, let us suppose that $t^o \in \mathbb{C}$ and $s_P^{(1)}(d^o, k^o, t^o) = 0$, with $U(t^o) \neq 0$. The construction of Ω_4^1 guarantees that in this case $Q(t^o) \neq 0$ and $W(t^o) \neq 0$. Therefore, $\bar{y}^o = P(t^o)$ is a well defined affine point of $\mathcal{C}$, and $s^1(d^o, k^o, \bar{y}^o) = 0$ follows again from the Equation 3.9. Besides, since

$$f_i(\bar{y}^o) = \frac{Q(t^o)}{(W(t^o))^{\mu}} N_i(t^o) \text{ for } i = 1, 2,$$

and $Q(t^o) \neq 0$, then $\bar{y}^o \notin \mathrm{Sing}_a(\mathcal{C})$. Therefore, $P(t^o) \in \left(\mathcal{S}^1_{(d^o,k^o)} \cap \mathcal{C}\right) \setminus \mathcal{F}_1$. Finally, the construction of Ω_4^2 implies that the correspondence is one-to-one. Thus, (2) holds. □

Now we are ready for the partial degree formula for parametrically given curves.

Theorem 3.42 (Partial degree formula in the parametric case).

$$\boxed{\delta_1 = \deg_{x_1}(\mathcal{O}_d(\mathcal{C})) = \deg_t\left(\mathrm{PP}_{\{d,k\}}\left(s_P^{(1)}(d,k,t)\right)\right) = \deg_t\left(s_P^{(1)}(d,k,t)\right) - \deg_t\left(U(t)\right)}$$

Proof. There is an open set of values (d^o, k^o) for which

$$\deg_t(s_P^{(1)}(d,k,t)) = \deg_t(s_P^{(1)}(d^o,k^o,t))$$

Thus, it suffices to prove that for $(d^o, k^o) \in \Omega_4$, (with Ω_4 as in Proposition 3.41, page 113)

$$\delta_1 = \deg_{x_1}(\mathcal{O}_d(\mathcal{C})) = \deg_t\left(s_P^{(1)}(d^o,k^o,t)\right) - \deg_t\left(U(t)\right)$$

Now, since $\Omega_4 \subset \Omega_0$, one has that for $(d^o, k^o) \in \Omega_4$, $\#\left(\left(\overline{\mathcal{S}^1_{(d^o,k^o)}} \cap \overline{\mathcal{C}}\right) \setminus \mathcal{F}_1\right) = \delta_1$ (see Remark 3.19, page 90). By Proposition 3.41, this implies that there are precisely δ_1 different roots of $s_P(k^o, d^o, t)$ which are not roots of $U(t)$. Besides, by Proposition 3.21(3) (page 91), and Lemma 2.39 (page 74), these roots are simple. □

We finish this section illustrating the above formula by means of one example.

Example 3.43. *Let $\mathcal{C}$ be the trisectrix given by $f(\bar{y}) = y_1(y_1^2 + y_2^2) - (y_2^2 - 3y_1^2)$ (see Figure 3.6). We consider the proper parametrization given by*

$$X(t) = t^2 - 3, \quad Y(t) = t(t^2 - 3), \quad W(t) = 1 + t^2.$$

From this, one has

$$\begin{cases} N_1(t) = -(1+t^2)(3-3t^2) + 2t^2(3-t^2) \\ N_2(t) = -2(1+t^2)t - 2t(3-t^2) \end{cases}$$

We apply the formula in Theorem 3.42, in order to compute the partial degrees δ_1 and δ_2. The parametric auxiliary curve in this case is the following polynomial:

$$s_P^{(1)}(d,k,t) = \left(9+t^2\right)\left(1+t^2\right)^3\left(-t\left(3-t^2\right) - k\left(1+t^2\right)\right)^2 - 64\,d^2t^2\left(1+t^2\right)^2,$$

with $\deg_t(s_P^{(1)}(d,k,t)) = 14$, *and its content w.r.t.* $\{d,k\}$ *is* $(1+t^2)^2$. *Thus,*

$$\delta_1 = \deg_t\left(\mathrm{PP}_{\{d,k\}}\left(s_P^{(1)}(d,k,t)\right)\right) = 14 - 4 = 10.$$

Similarly,

$$s_P^{(2)}(d,k,t) = \left(9+t^2\right)\left(1+t^2\right)^3\left(-3+t^2-k\left(1+t^2\right)\right)^2 - d^2\left(3-t^4-6\,t^2\right)^2\left(1+t^2\right)^2$$

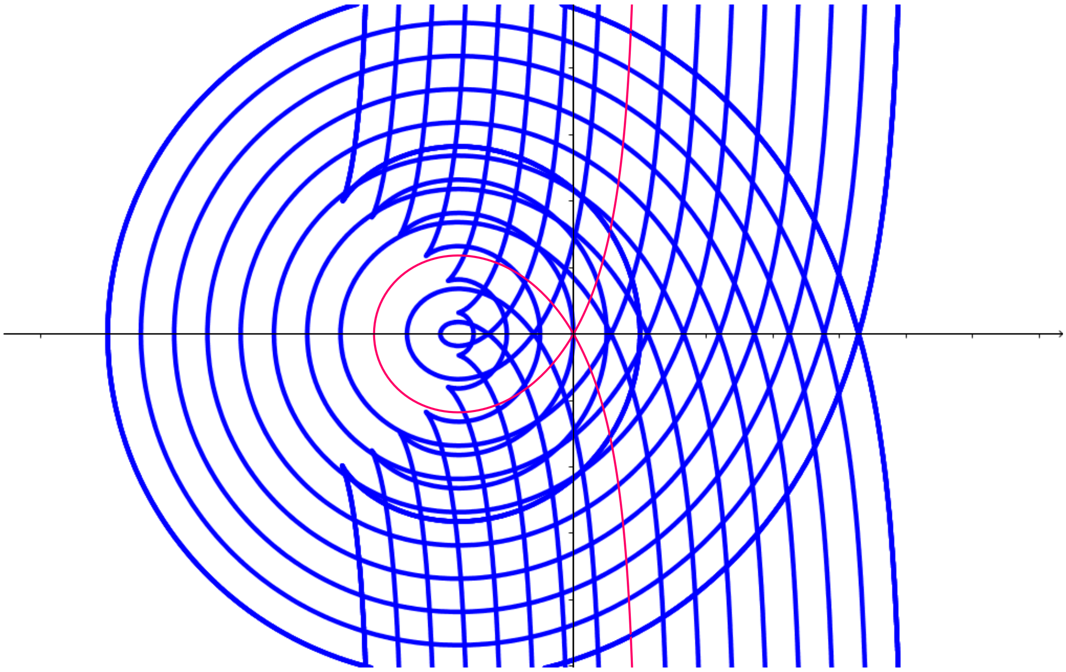

Figure 3.6: The Trisectrix and some of its offset curves.

with $\deg_t(s_P^{(1)}(d,k,t)) = 12$*, and its content w.r.t.* $\{d,k\}$ *is again* $(1+t^2)^2$*. Thus,*

$$\delta_2 = \deg_t\left(\mathrm{PP}_{\{d,k\}}\left(s_P^{(2)}(d,k,t)\right)\right) = 12 - 4 = 8.$$

The generic offset polynomial for this curve can be computed using elimination techniques:

$g(\bar{x},d) = -18\,x_2^4x_1^4d^2 + 8\,x_2^2xd^6 - x_2^8d^2 + x_2^8{x_1}^2 + 6\,x_2^4x_1^6 - 5\,d^2x_1^8 - 6\,x_2^4d^6 + 4\,x_2^2d^8 + 4\,x_2^6d^4 - 10\,d^6x_1^4 + 10\,d^4x_1^6 + 5\,d^8x_1^2 - 296\,x_2^4x_1^2 - 4320\,x_1^3d^2 - 88\,x_2^6x - 3456\,x_1^2d^2 + 2688\,xd^4 + 760\,x_1^2d^4 + 1080\,x_2^2x_1^4 - 192\,x_2^4x - 1440\,x_1^4d^2 + 184\,x_1^5d^2 - 296\,x_1^3d^4 - 864\,x_2^2x_1^2 + 136\,xd^6 - 8\,x_2^6x_1^3 - 8\,x_2^2x_1^7 - 12\,x_2^4x_1^5 - 2\,x_2^8x + 312\,x_2^4x_1^3 + 376\,x_2^2x_1^5 - 1152\,d^2x_2^2 - 480\,d^2x_2^4 + 840\,d^4x_2^2 + 116\,x_1^2d^6 - 210\,x_1^4d^4 + 164\,x_1^6d^2 + 54\,x_1^4x_2^4 - 44\,x_1^6x_2^2 - 52\,x_2^6d^2 + 78\,x_2^4d^4 + 52\,x_2^6x_1^2 - 4\,d^6x_2^2 - 2\,xd^8 - 12\,x_1^5d^4 + 8\,x_1^7d^2 + 8\,x_1^3d^6 + 144\,x_2^4 + 40\,x_2^6 + 1296\,x_1^4 + 1728\,x_1^5 - d^{10} + x_1^{10} - 24\,x_1^7 - 47\,x_1^8 - 23\,d^8 + 648\,x_1^6 + 32\,d^6 + 2304\,d^4 + x_2^8 - 2\,x_1^9 + 4\,x_2^2{x_1}^8 + 4\,x_2^6x_1^4 - 12\,x_2^4xd^4 - 1632\,x_1d^2x_2^2 + 216\,x_1d^4x_2^2 - 592\,x_1^3d^2x_2^2 - 2048\,x_2^2x_1^2d^2 + 24\,x_2^4x_1^3d^2 - 264\,x_1d^2x_2^4 - 132\,x_1^2d^2y^4 - 36\,x_1^2d^4x_2^2 - 16\,x_2^2x_1^2d^6 - 16\,x_2^2x_1^6d^2 + 24\,x_2^2x_1^4d^4 + 18\,x_2^4x_1^2d^4 - 8\,x_2^6x_1^2d^2 + 84\,x_1^4d^2x_2^2 + 8\,x_2^6x_1d^2 - 24\,x_2^2x_1^3d^4 + 24\,x_2^2x_1^5d^2.$

and this agrees with the result predicted by our formulae.

3.4.2 Degree in the distance in the parametric case

With the experience of the previous subsections, the strategy for the proof of a formula for δ_d is clear. The necessary notions and results are shown below. Recall that the auxiliary curve for the δ_d problem is defined by the polynomial:

$$\operatorname{nor}(\bar{x}, \bar{y}) = -f_2(\bar{y})(x_1 - y_1) + f_1(\bar{y})(x_2 - y_2),$$

see Remark 3.27 (page 100). Substituting $\bar{y} = P(t)$ in $\operatorname{nor}(\bar{x}, \bar{y})$, and taking Equation 3.8 into account, one has:

$$\operatorname{nor}(\bar{x}, P(t)) = \frac{Q(t)}{(W(t))^{\mu+1}} \big(N_2(t)\big(W(t)x_1 - X(t)\big) - N_1(t)\big(W(t)x_2 - Y(t)\big)\big). \quad (3.10)$$

This leads to consider the following.

Definition 3.44. *The polynomial:*

$$\operatorname{nor}_P(\bar{x}, t) = N_2(t)\big(W(t)x_1 - X(t)\big) - N_1(t)\big(W(t)x_2 - Y(t)\big)$$

is called the parametric δ_d-auxiliary polynomial.

Lemma 3.45. $\operatorname{Con}_{\bar{x}}(\operatorname{nor}_P(\bar{x}, t)) = \gcd\left(W(t), X(t)N_2(t) - Y(t)N_1(t)\right)$.

Proof. This is straightforward, considering $\operatorname{nor}_P(\bar{x}, t)$ as a polynomial in $\mathbb{C}[\bar{x}][t]$, and taking into account that $\gcd(N_1, N_2) = 1$. □

Remark 3.46. *We* denote:

$$U_d(t) = \gcd\left(W(t), X(t)N_2(t) - Y(t)N_1(t)\right)$$

Proposition 3.47. *Let Ω_3 be as in Proposition 3.35 (page 106). There exists an open non-empty subset $\Omega_5 \subset \Omega_3$ such that, for $\bar{x}^o \in \Omega_5$:*

1. $\left(\overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}}\right) \setminus \mathcal{F}_d \subset P(\mathbb{C})$.
2. *There is a one-to-one correspondence between the points of $\left(\overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}}\right) \setminus \mathcal{F}_d$ and the values $t^o \in \mathbb{C}$ that verify $\operatorname{nor}_P(\bar{x}^o, t^o) = 0$ and $U_d(t^o) \neq 0$.*

Proof. This is closely related to the proof of Proposition 3.41 (page 113). We start by constructing the set Ω_5. Let $A \subset \mathbb{C}$ be the finite set of roots of $Q(t)$ (see Remark 2.35, page 72). Since $\gcd(W, Q) = 1$, if $t^o \in A$, then $W(t^o) \neq 0$. Thus, the set $\mathcal{A}_1 = P(A)$ is a well-defined finite subset of $\mathcal{C}$. Besides, the set $\mathcal{A}_2 = \mathcal{C} \setminus P(\mathbb{C})$ is also a finite subset of $\mathcal{C}$. Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and let $\Omega^0_{\mathcal{A}}$ be the set provided by Lemma 3.30, page 104, when one takes $\mathcal{X} = \mathcal{A}$ We consider $\Omega^0_5 = \Omega^0_{\mathcal{A}} \cap \Omega_3$. Note that $W(t^o) \neq 0$ implies $U_d(t^o) \neq 0$.

Next, since $\gcd(W, Q) = 1$, if $Q(t^o) = 0$, then $W(t^o) \neq 0$. Thus, since the set of roots of Q is finite and $\gcd(N_1, N_2) = 1$ (see Remark 2.35, page 72), there exists a non-empty open $\Omega^1_5 \subset \Omega^0_5$ such that, for $\bar{x}^o \in \Omega^1_5$, $Q(t^o) = 0$ implies $\operatorname{nor}_P(\bar{x}^o, t^o) \neq 0$. Similarly, if $W(t^o) = 0$ but $U(t^o) \neq 0$, then

$$\operatorname{nor}_P(\bar{x}^o, t^o) = -X(t)N_2(t^o) + Y(t^o)N_1(t^o) \neq 0,$$

independently of $\bar{x}^o$.

Using that P is a proper parametrization, we can choose $\Omega^2_5 \subset \Omega^1_5$ such that for $\bar{x}^o \in \Omega^2_5$, if t^o_1, t^o_2 are roots of $\operatorname{nor}_P(\bar{x}^o, t)$ with $U(t^o_1)U(t^o_2) \neq 0$ and $P(t^o_1) = P(t^o_2)$, then $t^o_1 = t^o_2$.

Finally, set $\Omega_5 = \Omega^2_5$. We will prove that for $\bar{x}^o \in \Omega_5$, statements (1) and (2) hold.

(1) Let us assume that $\bar{y}^o \in \left(\overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}}\right) \setminus \mathcal{F}_d$ (recall that all the points in $\left(\overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}}\right) \setminus \mathcal{F}_d$ are affine, see Remark 3.33(1), page 105). This implies (see Proposition 3.34, page 106 and Theorem 3.29, page 101) that $\bar{y}^o$ is associated with $\bar{x}^o$ in $\mathcal{O}_{d_{\bar{y}^o}}(\mathcal{C})$. Because of the construction of Ω^0_5, this implies that there exists $t^o \in \mathbb{C}$ with $P(t^o) = \bar{y}^o$. Thus, (1) holds.

(2) Let us suppose first that $\bar{y}^o \in \left(\overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}}\right) \setminus \mathcal{F}_d$. We have proved in the preceding paragraph that there is $t^o \in \mathbb{C}$ such that $P(t^o) = \bar{y}^o$. In particular $W(t^o) \neq 0$, and so $U_d(t^o) \neq 0$. Since $\bar{y}^o \notin \operatorname{Sing}_a(\mathcal{C})$ (see Theorem 3.32, page 105), one concludes from Remark 2.35 (page 72) that $Q(\bar{t}^o) \neq 0$. Then, from Equation 3.10, one has $\operatorname{nor}_P(\bar{x}^o, t^o) = 0$. Conversely, let us suppose that $t^o \in \mathbb{C}$ and $\operatorname{nor}_P(\bar{x}^o, t^o) = 0$, with $U(t^o) \neq 0$. The construction of Ω^1_5, guarantees that in this case $Q(t^o) \neq 0$ and $W(t^o) \neq 0$. Therefore, $\bar{y}^o = P(t^o)$ is a well defined affine point of $\mathcal{C}$, and $\operatorname{nor}(\bar{x}^o, \bar{y}^o) = 0$ follows again from the above equality 3.10. Besides, since

$$f_i(\bar{y}^o) = \frac{Q(t^o)}{(W(t^o))^\mu} N_i(t^o) \text{ for } i = 1, 2,$$

and $Q(t^o) \neq 0$, then $\bar{y}^o \notin \operatorname{Sing}_a(\mathcal{C})$. Therefore, $P(t^o) \in \left(\overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}}\right) \setminus \mathcal{F}_d$. Finally, the construction of Ω^2_5 implies that the correspondence is one-to-one. Thus, (2) holds. □

Then, the formula for the degree in the distance, for parametrically given curves, is as follows.

Theorem 3.48 (Degree formula for the distance, parametric case).

$$\boxed{\delta_d = \deg_d(\mathcal{O}_d(\mathcal{C})) = 2\deg_t\left(\operatorname{PP}_{\{\bar{x}\}}\left(\operatorname{nor}_P(\bar{x}, t)\right)\right) = 2\deg_t\left(\operatorname{nor}_P(\bar{x}, t)\right) - 2\deg_t\left(U(t)\right)}$$

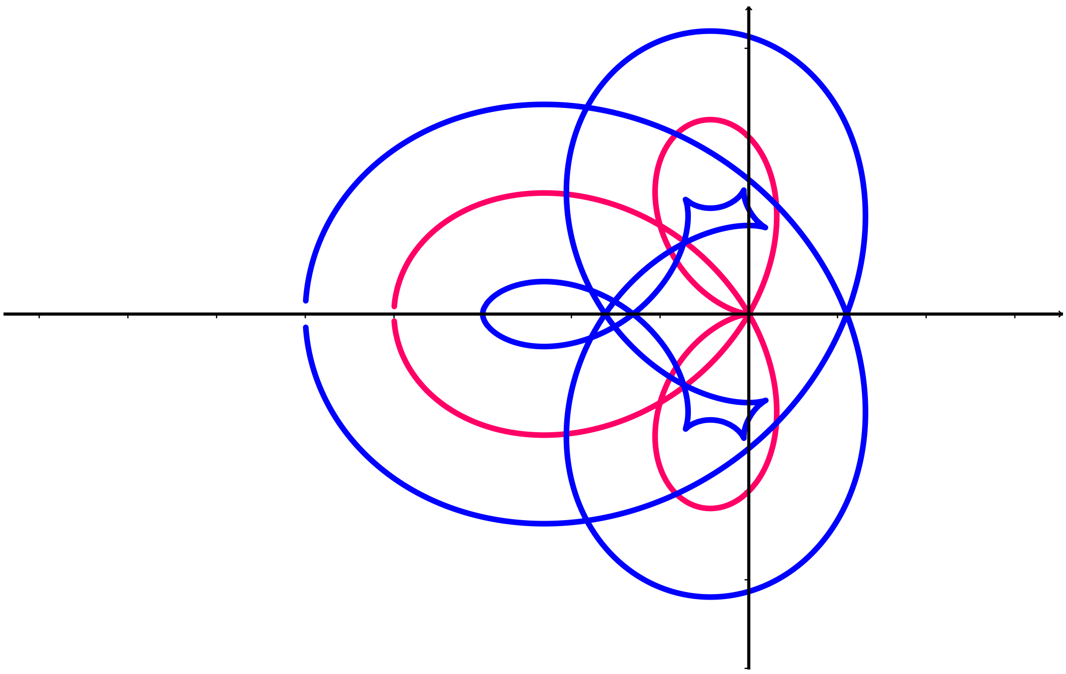

Figure 3.7: The Scarabeus and one of its offset curves.

Proof. There is an open set of values (d^o, k^o) for which

$$\deg_t(\mathrm{nor}_P(\bar{x}, t)) = \deg_t(\mathrm{nor}_P(\bar{x}^o, t))$$

Thus, it suffices to prove that for $\bar{x}^o \in \Omega_5$, (with Ω_5 as in Proposition 3.47, page 117)

$$\delta_d = \deg_d(\mathcal{O}_d(\mathcal{C})) = 2\left(\deg_t\left(\mathrm{nor}_P(\bar{x}^o, t)\right) - \deg_t\left(U_d(t)\right)\right)$$

Now, since $\Omega_5 \subset \Omega_3 \subset \Omega_2$, one has that for $\bar{x}^o \in \Omega_5$, $\delta_d = 2\nu = 2\#\big((\overline{\mathcal{N}_{\bar{x}^o}} \cap \overline{\mathcal{C}}) \setminus \mathcal{F}_d\big)$ (see Proposition 3.34, page 106). By Proposition 3.47, this implies that there are precisely $\delta_d/2$ different roots of $\mathrm{nor}_P(\bar{x}^o, t)$ which are not roots of $U_d(t)$. Besides, by Proposition 3.35(3) (page 106), and Lemma 2.39 (page 74), these roots are simple. $\square$

We finish this section with an example, illustrating the use of the formula in Theorem 3.48.

Example 3.49. *Let $\mathcal{C}$ be the Scarabeus curve given by $f(\bar{y}) = (y_1^2+y_2^2)(y_1^2+y_2^2+y_1)^2 - (y_1^2-y_2^2)^2$ (see Figure 3.7). In this case the generic offset equation $g(\bar{x}, d)$ can again be computed explicitly using elimination techniques. It turns out to be a polynomial of total degree* 18 *with* 321 *terms and with most coefficients being* 16*-digits integers (see Appendix B, page 205). The partial degrees in x_1 and x_2 are both equal to* 18*, and the degree w.r.t. d is* 14*. A proper parametrization is given by*

$$X(t) = -2t^2(-3+t^2)(-1+t^2), \quad Y(t) = 4t^3(-3+t^2), \quad W(t) = (1+t^2)^3$$

From this, one has

$$\begin{cases} N_1(t) = t(t^4 - 14t^2 + 9) \\ N_2(t) = -7t^4 + 14t^2 - 3 \end{cases}$$

Now, we apply the formula in Theorem 3.48. One has:
$\mathrm{nor}_P(\bar{x}, t) = -(7\,t^4 - 14\,t^2 + 3)((1 + t^2)^3 x_1 - t(t^4 - 14\,t^2 + 9)((1 + t^2)^3 x_2 - 4\,t^3(-3 + t^2)) + 2\,t^2(-3 + t^2)(t - 1)(t + 1))$.
This is a polynomial of degree and its content w.r.t. $\{\bar{x}\}$ *is* $(1 + t^2)^2$*. Thus,*

$$\delta_d = \deg_d(\mathcal{O}_d(\mathcal{C})) = 2\deg_t \left(\mathrm{PP}_{\{\bar{x}\}}\left(\mathrm{nor}_P(\bar{x}, t)\right)\right) = 2(11 - 4) = 14,$$

as expected.

Chapter 4

Degree Formulae for Rational Surfaces

In contrast with the case of curves, even in the case of a generating parametric surface, there are, up to our knowledge, no available results for the offset degree problem in the scientific literature. In this chapter we will provide (in Theorem 4.45, page 172) a formula for the total offset degree computation in the case of rational surfaces, given in parametric form. The parametrization of the surface is not assumed to be proper, and the formula in fact provides the product of the total offset degree times the tracing index of the parametrization. However, since there are available efficient algorithms for computing the tracing index of a surface parametrization (see [34]) this does not limit the applicability of the formula.

The strategy for this offset degree problem is, as in the previous chapters, based in the analysis of the intersection between the generic offset and a pencil of lines through the origin. The restriction to the rational case, combined with this strategy, results in a reduction in the dimension of the space needed to study of the intersection problem. Thus, we are led to consider again an intersection problem of plane curves. The auxiliary curves involved in this case are obtained by eliminating the variables corresponding to a point in the generating surface from the offset-line intersection system. The main technical differences between this chapter and the previous ones are that:

- Here we need to consider more than two intersection curves. Thus, the total degree formula is expressed as a generalized resultant of the equations of these auxiliary curves.

- Furthermore, all the curves involved in the intersection problem depend on parameters. Thus, the notion of fake point and their characterization is technically more demanding.

Generally speaking, the dimensional advantage gained by working with a parametric representation is partially compensated by the fact that we are not dealing directly with the points of the surface but with their parametric representation, and thus we are losing some geometric intuition. In the general situation of an implicitly given generating surface, if one were to apply a similar strategy to the offset degree problem, we believe that one is bound to consider a surface intersection problem, instead of the simpler curve intersection problem used here. However, in this thesis we do not address the offset degree problem in that general situation.

As those skilled in the art know, going from the curve to the surface case usually implies a huge step in the difficulty of the proofs. For us, this has indeed been the case. As a result, some of the proofs in this chapter are rather technical. And in one particular case, we have not been able to extend to the surface case the proof of a result that we obtained for plane curves. Specifically, in Proposition 2.3 (page 40) of Chapter 2 we proved that there are only finitely many distance values d^o for which the origin belongs to $\mathcal{O}_{d^o}(\mathcal{C})$. Our conjecture is that a similar property in holds for all algebraic surfaces. However, as we said, we have not been able to provide a proof (recall Remark 2.4 in page 40, that shows that this property does not hold if we consider the analytic case).

Besides, because of its own nature, our strategy fails in the case of some simple surfaces. We have met similar situations in previous chapters, when we needed to exclude circles centered at the origin and lines through the origin from our considerations. Correspondingly, in this chapter we need to exclude the case in which the generating surface is a sphere centered at the origin. In this case, however, the generic offset degree (in fact the generic offset equation) is known beforehand. Therefore, excluding it does not really affect the generality of the degree formula that we present here. The above observations are the reason for the following assumptions:

Assumptions 4.1. *Let Σ denote the generating surface. In this chapter, we assume that:*

(1) There exists a finite subset Δ^1 of $\mathbb{C}$ such that, for $d^o \notin \Delta^1$ the origin does not belong to $\mathcal{O}_{d^o}(\Sigma)$.

(2) Σ is not a sphere centered at the origin.

In the case of a parametric surface, as we have said, the dimensional gain provided by the parametrization helps to turn the offset degree problem into a curve intersection problem. A similar situation arises when one considers some special types of surfaces which are derived from a curve by some sort of geometrical construction. For example, this happens for the surface of revolution, obtained from a plane curve $\mathcal{C}$. In this case, using the geometric properties of the revolution construction, we are able to relate the offset of the surface of revolution generated by $\mathcal{C}$ with the surface of revolution of the offset to $\mathcal{C}$. In particular, this allows us to apply the formulae derived in Chapters

2 and 3 to provide a complete and efficient solution for the offset degree problem for surfaces of revolution. This approach gives an alternative –and more efficient– method for surfaces of revolution, when compared with the general degree formula in Theorem 4.45. As a byproduct, we derive an efficient method to obtain the implicit equation of the surface of revolution generated by a planar curve, given either implicitly or parametrically.

The results in this chapter, concerning surfaces of revolution, appear in [45] and [44].

The structure of the chapter is the following:

- In Section 4.1 (page 125) the theoretical foundation of the strategy is established. In Subsection 4.1.1, we recall some basic notions on parametric algebraic surfaces, and some technical lemmas about them. We also introduce the notion of associated normal vector, and we review some of its properties. In Subsection 4.1.2 (page 128) we construct a parametric analogous of the Generic Offset System; this analogous system is System 4.4. The final Subsection 4.1.3 (page 130) contains the analysis of the intersection between the generic offset and a pencil of lines through the origin. The main result in this section is Theorem 4.13, (page 133).

- In Section 4.2 we will see that, when elimination techniques are brought into our strategy, the dimension of the space in which we count the points in $\mathcal{O}_d(\Sigma)\cap\mathcal{L}_{\bar{k}}$ is reduced, and we arrive again at an intersection problem between projective plane curves. Then we begin the analysis of that problem. Specifically, in Subsection 4.2.1 we describe the auxiliary polynomials obtained by using elimination techniques in the Parametric Offset-Line System, and we introduce the Auxiliary System 4.7 (page 137), denoted by $\mathfrak{S}_3^P(d,\bar{k})$. Some geometric properties of the solutions of $\mathfrak{S}_3^P(d,\bar{k})$ (see Proposition 4.16, page 139 and Lemma 4.18, page 142) will be used in the sequel to study the relation between the solution sets of Systems $\mathfrak{S}_2^P(d,\bar{k})$ and $\mathfrak{S}_3^P(d,\bar{k})$. In Subsection 4.2.2 (page 143) we define the corresponding notion of fake points and invariant points for the Affine Auxiliary System $\mathfrak{S}_3^P(d,\bar{k})$. The relation between these two notions is then shown in Proposition 4.23 (page 145).

- The statement and proof of the degree formula appear in Section 4.3 (page 146). This rather long section is structured into four subsections as follows. In Subsection 4.3.1 we study the projective version of the auxiliary curves introduced in the preceding section, and we introduce the Projective Auxiliary System 4.25 (page 150). The polynomials that define this system are the basic ingredients of the degree formula. Subsection 4.3.2. (page 150) deals with the invariant solutions of the Projective Auxiliary System. In Subsection 4.3.3 (page 160) we will prove that the value of the multiplicity of intersection of the auxiliary curves at their non-invariant points of intersection equals one (in Proposition 4.43, page

160). Subsection 4.3.4 (page 170) contains the statement and proof of the degree formula, in Theorem 4.45 (page 172).

- Section 4.4 (page 180) is independent of the preceding results in this chapter. It is dedicated to the offset degree problem for a special type of surface, namely the surface of revolution obtained from a plane curve $\mathcal{C}$. Therefore, even though we are working with surfaces, it is connected with the results about curves in Chapters 2 and 3. In Subsection 4.4.1, we formally define the surface of revolution by means of incidence diagrams, and we obtain some of its basic properties. In Theorem 4.53 (page 184) we prove that the implicit equation of the revolution surface can be obtained from the implicit equation of the initial curve by a straightforward method. This can be used to solve effectively the implicitization problem when the generating curve is given parametrically. Subsection 4.4.2 (page 186) turns to the offsetting process for revolution surfaces. The main result is Theorem 4.58, which shows that the offset of a revolution surface generated by a curve is the surface of revolution generated by the offset of that curve. Then we use this result to derive degree formulae for the offset of a surface of revolution, both when the generating curve is given implicitly or parametrically.

Notation and terminology for this chapter

In this chapter we will adapt some of the notational conventions introduced in page 1 to the case of surfaces.

- Since $n = 3$, then $\bar{x} = (x_1, x_2, x_3), \bar{y} = (y_1, y_2, y_3)$, while their homogeneous counterparts are $\bar{x}_h = (x_0 : x_1 : x_2 : x_3), \bar{y}_h = (y_0 : y_1 : y_2 : y_3)$
- The symbol Σ denotes a rational algebraic surface defined over $\mathbb{C}$ by the irreducible polynomial $f(\bar{y}) \in \mathbb{C}[\bar{y}]$.
- We assume that we are given a non-necessarily proper rational parametrization of Σ:
$$P(\bar{t}) = \left(\frac{P_1(\bar{t})}{P_0(\bar{t})}, \frac{P_2(\bar{t})}{P_0(\bar{t})}, \frac{P_3(\bar{t})}{P_0(\bar{t})}\right).$$
Here $\bar{t} = (t_1, t_2)$, and $P_0, \ldots, P_3 \in \mathbb{C}[\bar{t}]$ with $\gcd(P_0, \ldots, P_3) = 1$.
- The projectivization P_h of P is obtained by homogenizing the components of P w.r.t. a new variable t_0, multiplying both the numerators and denominators if necessary by a suitable power of t_0. It will be denoted by
$$P_h(\bar{t}_h) = \left(\frac{X(\bar{t}_h)}{W(\bar{t}_h)}, \frac{Y(\bar{t}_h)}{W(\bar{t}_h)}, \frac{Z(\bar{t}_h)}{W(\bar{t}_h)}\right)$$
where $\bar{t}_h = (t_0 : t_1 : t_2)$, and $X, Y, Z, W \in \mathbb{C}[\bar{t}_h]$ are homogeneous polynomials of the same degree d_P, for which $\gcd(X, Y, Z, W) = 1$ holds.

- Given two vectors $(a_1, b_1, c_1), (a_2, b_2, c_2) \in \mathbb{K}^3$, their cross product is defined as

$$(a_1, b_1, c_1) \wedge (a_2, b_2, c_2) = (b_1c_2 - b_2c_1, a_2c_1 - a_1c_2, b_1c_2 - b_2c_1)$$

4.1 Offset-Line Intersection for Rational Surfaces

In this section we build the theoretical foundation of our strategy. The section is structured as follows: in Subsection 4.1.1, we first recall some basic notions on parametric algebraic surfaces, as well as some technical lemmas that will be used through the chapter. Besides, we introduce the notion of associated normal vector, and its properties. The use of the parametric representation of the generating surface requires the construction of a parametric analogous of the Generic Offset System of Chapter 1 (recall System 1.3, page 17). This analogous system is System 4.4, introduced in Subsection 4.1.2 (page 128). Finally, Subsection 4.1.3 (page 130) is devoted to the analysis of the intersection between the generic offset and a pencil of lines through the origin. The results in this subsection (see Theorem 4.13, page 133) constitute the theoretical foundation of the degree formula to be derived in Section 4.3 of this chapter.

4.1.1 Surface parametrizations and their associated normal vector

An algebraic set Σ over $\mathbb{K}$ (in our case, $\mathbb{K} = \mathbb{C}$) is called a surface if all of its irreducible components have dimension 2 over $\mathbb{K}$. The surface Σ is unirational (or parametric) if there exists a rational map $P : \mathbb{K}^2 \mapsto \Sigma$ such that the image of P is dense in Σ w.r.t. the Zariski topology. The map P is called an (affine) parametrization of Σ. If P is a birational map, then Σ is called a rational surface, and P is called a proper parametrization of Σ. In this chapter we will not assume that P is proper (see Lemma 4.2 in page 126, and the observations preceding it). It is well known that a rational surface is always irreducible.

Thus, a parametrization P of Σ is given through a non-constant triplet of rational functions in two parameters. We will use $\bar{t} = (t_1, t_2)$ for the parameters of P and, as usual, $\bar{t}^o = (t_1^o, t_2^o)$ stands for a particular value in $\mathbb{K}^2$ of the pair of parameters. By a simple algebraic manipulation, we can assume that the three components of P have a common denominator. Thus, we can write:

$$P(\bar{t}) = \left(\frac{P_1(\bar{t})}{P_0(\bar{t})}, \frac{P_2(\bar{t})}{P_0(\bar{t})}, \frac{P_3(\bar{t})}{P_0(\bar{t})}\right) \tag{4.1}$$

where $P_0, \ldots, P_3 \in \mathbb{C}[\bar{t}]$ and $\gcd(P_0, \ldots, P_3) = 1$. The number

$$d_P = \max_{i=0,\ldots,3}(\{\deg_{\bar{t}}(P_i)\})$$

is then called the degree of P.

Over the algebraically closed field $\mathbb{K}$, the notions of rational and parametric surface are equivalent (see the Castelnuovo Theorem [11]). Furthermore, there exists an algorithm by Schicho (see [48]) to obtain a proper parametrization of a rational surface given by its implicit equation. Thus, in principle, given a non-proper parametrization of a surface, it is possible (though computationally very expensive) to implicitize, and then apply Schicho's algorithm to obtain a proper parametrization. In addition, [32] shows how to properly reparametrize certain special families of rational surfaces. However, in this chapter we will not assume that P is proper, and the degree formulas below take this fact into account.

The parametrization P has two associated tangent vectors, denoted by

$$\frac{\partial P(\bar{t})}{\partial t_1} \text{ and } \frac{\partial P(\bar{t})}{\partial t_2}. \tag{4.2}$$

That is:

$$\frac{\partial P}{\partial t_i} = \left(\frac{P_{1,i}P_0 - P_1 P_{0,i}}{(P_0)^2}, \frac{P_{2,i}P_0 - P_2 P_{0,i}}{(P_0)^2}, \frac{P_{3,i}P_0 - P_3 P_{0,i}}{(P_0)^2} \right)$$

where $P_{j,i}$ denotes the partial derivative of P_j w.r.t. t_i, for $j = 0, \ldots, 2$ and $i = 1, 2$.

The following Lemma states those properties of the surface parametrization P that we will need in the sequel.

Lemma 4.2. *There are non-empty Zariski open subsets $\Upsilon_1 \subset \mathbb{C}^2$ and $\Upsilon_2 \subset \Sigma$ such that:*

$$P : \Upsilon_1 \mapsto \Upsilon_2$$

is a surjective regular application of degree m. In particular, this means that P defines a $m : 1$ correspondence between Υ_1 and Υ_2. Thus, given $\bar{y}^o \in \Upsilon_2$, there are precisely m different values $\bar{t}_1^o, \ldots, \bar{t}_m^o$ of the parameter $\bar{t}$ such that $P(\bar{t}_i^o) = \bar{y}^o$ for $i = 1, \ldots, m$. Furthermore, if $\bar{t}^o \in \Upsilon_1$, the rank of the Jacobian matrix $\left(\dfrac{\partial P}{\partial \bar{t}}\right)$ evaluated at $\bar{t}^o$ is two.

Proof. See e.g. [33]. □

Remark 4.3.

1. *The number m is also called, as in the case of curves, the* tracing index *of P. See [52] for an algorithm to compute m. In the sequel, we will* denote *by m the tracing index of P.*

2. *As a consequence of this lemma, the part of the surface Σ not covered by the image of P is a proper closed subset (i.e. a finite collection of curves and points).*

Starting with the parametrization P of Σ as in (4.1) above, we will construct a polynomial normal vector to Σ, that will be used in the statements of the degree formulas for rational surfaces. This particular choice of normal vector will be called in the sequel the associated normal vector of P, and it will be denoted by $\bar{n}(\bar{t})$.

To construct $\bar{n}(\bar{t})$, we first take the cross product of the associated tangent vectors introduced in 4.2, page 126. Let us denote:

$$V(\bar{t}) = \frac{\partial P(\bar{t})}{\partial t_1} \wedge \frac{\partial P(\bar{t})}{\partial t_2}$$

This vector $V(\bar{t})$ has the following form:

$$V(\bar{t}) = \left(\frac{A_1(\bar{t})}{A_0(\bar{t})}, \frac{A_2(\bar{t})}{A_0(\bar{t})}, \frac{A_3(\bar{t})}{A_0(\bar{t})}\right)$$

where $A_i \in \mathbb{K}[\bar{t}]$. Let $G(\bar{t}) = \gcd(A_1, A_2, A_3)$.

Definition 4.4. *With the above notation, the* associated normal vector $\bar{n} = (n_1, n_2, n_3)$ *to P is the vector whose components are the polynomials:*

$$n_i(\bar{t}) = \frac{A_i(\bar{t})}{G(\bar{t})} \text{ for } i = 1, 2, 3.$$

Remark 4.5.

1. *Note that $\bar{n}$ is a normal vector to Σ at $P(\bar{t})$, vanishing at most at a finite set of points in the $\bar{t}$ plane. To see this observe that, because of their construction, n_1, n_2, n_3 have no common factors. Besides, at most one of the polynomials n_i is constant (otherwise the surface is a plane). Thus, the non constant components of $\bar{n}$ define a system of at least two plane curves without common components.*

2. *In particular, by a similar argument as the one used in the case of rational curves (see Remark 2.35 in page 72), there are some $\mu \in \mathbb{N}$ and $\beta(\bar{t}) \in \mathbb{C}[\bar{t}]$, with $\gcd(\beta, P_0) = 1$, such that*

$$f_i(P(\bar{t})) = \frac{\beta(\bar{t})}{P_0(\bar{t})^\mu} n_i(\bar{t}) \text{ for } i = 1, 2, 3. \tag{4.3}$$

   That is:

$$\nabla f(P(\bar{t})) = \frac{\beta(\bar{t})}{P_0(\bar{t})^\mu} \cdot \bar{n}(\bar{t})$$

3. *Note that the polynomial $\beta(\bar{t})$ introduced above is not identically zero. Otherwise, one has $f_i(P(\bar{t})) = 0$ for $i = 1, 2, 3$, and this implies that $f(\bar{y})$ is a constant polynomial, which is a contradiction.*

Definition 4.6. *The polynomial $h \in \mathbb{C}[\bar{t}]$ defined as*

$$h(\bar{t}) = n_1(\bar{t})^2 + n_2(\bar{t})^2 + n_3(\bar{t})^2$$

is called the parametric (affine) normal-hodograph *of the parametrization P.*

Remark 4.7. *In this chapter, if we need to refer to the implicit normal-hodograph introduced in Chapter 1 (see page 9), we will denote it by H_{imp} in the projective case, resp. h_{imp} in the affine case.*

The following lemma will be used below to exclude from our discussion certain pathological cases, associated to some particular parameter values.

Lemma 4.8. *The sets Υ_1 and Υ_2 in Lemma 4.2 (page 126) can be chosen so that if $\bar{t}^o \in \Upsilon_1$, then*

$$P_0(\bar{t}^o)h(\bar{t}^o)\beta(\bar{t}^o) \neq 0.$$

In particular, $\bar{n}(\bar{t}^o) \neq 0$.

Proof. Note that P_0, h and β are non-zero polynomials. Thus, the equation:

$$P_0(\bar{t})h(\bar{t})\beta(\bar{t}) = 0$$

defines an algebraic curve. Let us call it $\mathcal{C}$. Then it suffices to replace Υ_1 (resp. Υ_2) in Lemma 4.2 with $\Upsilon_1 \setminus \mathcal{C}$ (resp. $\Upsilon_2 \setminus P(\mathcal{C})$). ◻

4.1.2 Parametric system for the generic offset

Let Σ and P be as above. In order to describe $\mathcal{O}_d(\Sigma)$ from a parametric point of view, we introduce the following system, to be called the parametric system for the generic offset:

$$\mathfrak{S}_1^P(d) \equiv \begin{cases} b^P(d, \bar{t}, \bar{x}) : (P_0x_1 - P_1)^2 + (P_0x_2 - P_2)^2 + (P_0x_3 - P_3)^2 - d^2{P_0}^2 = 0 \\ \text{nor}^P_{(1,2)}(\bar{t}, \bar{x}) : \ n_1 \cdot (P_0x_2 - P_2) - n_2 \cdot (P_0x_1 - P_1) = 0 \\ \text{nor}^P_{(1,3)}(\bar{t}, \bar{x}) : \ n_1 \cdot (P_0x_3 - P_3) - n_3 \cdot (P_0x_1 - P_1) = 0 \\ \text{nor}^P_{(2,3)}(\bar{t}, \bar{x}) : \ n_2 \cdot (P_0x_3 - P_3) - n_3 \cdot (P_0x_2 - P_2) = 0 \\ w^P(r, \bar{t}) : \ r \cdot P_0 \cdot h \cdot \beta - 1 = 0 \end{cases} \tag{4.4}$$

Our first result will show that this system provides an alternative description for the generic offset. To state this, we will introduce some additional notation. Let

$$\Psi^P \subset \mathbb{C} \times \mathbb{C} \times \mathbb{C}^2 \times \mathbb{C}^3$$

be the set of solutions, in the variables $(d, r, \bar{t}, \bar{x})$, of the system $\mathfrak{S}_1^P(d)$. We also consider the projection maps

$$\begin{cases} \pi_1^P : \mathbb{C} \times \mathbb{C} \times \mathbb{C}^2 \times \mathbb{C}^3 \mapsto \mathbb{C} \times \mathbb{C}^3 \\ \pi_1^P(d, r, \bar{t}, \bar{x}) = (d, \bar{x}) \end{cases} \quad \text{and} \quad \begin{cases} \pi_2^P : \mathbb{C} \times \mathbb{C} \times \mathbb{C}^2 \times \mathbb{C}^3 \mapsto \mathbb{C} \times \mathbb{C}^2 \\ \pi_2^P(d, r, \bar{t}, \bar{x}) = (r, \bar{t}) \end{cases}$$

and we define $\mathcal{A}^P = \pi_1^P(\Psi^P)$. Recall that $\left(\mathcal{A}^P\right)^*$ denotes the Zariski closure of $\mathcal{A}^P$.

Proposition 4.9.

$$\mathcal{O}_d(\Sigma) = \left(\mathcal{A}^P\right)^*.$$

Proof. With the notation introduced in Definition 1.18, page 17, recall that

$$\mathcal{O}_d(\Sigma) = \mathcal{A}(\Sigma)^* = \pi_1(\Psi(\Sigma))^*.$$

Note that in this proof we use π_1, π_2 as in page 17, to be distinguished from π^P, π_2^P introduced above. Let Υ_1, Υ_2 be as in Lemma 4.8, page 128, and let us denote

$$\mathcal{B}_\Sigma^P = \pi_2^{-1}(\mathbb{C} \times \Upsilon_2).$$

$\mathcal{B}_\Sigma^P$ is a non-empty dense subset of $\Psi(\Sigma)$, because $\mathbb{C} \times \Upsilon_2$ is dense in $\mathbb{C} \times \Sigma$. It follows that $\mathcal{O}_d(\Sigma) = \pi_1(\mathcal{B}_\Sigma^P)^*$. We will show that $\pi_1(\mathcal{B}_\Sigma^P) = \mathcal{A}^P$, thus completing the proof.

If $(d^o, \bar{x}^o) \in \pi_1(\mathcal{B}_\Sigma^P)$, there are $\bar{y}^o, u^o$ and $\bar{t}^o \in \Upsilon_1$ such that $(d^o, \bar{x}^o, \bar{y}^o, u^o) \in \Psi(\Sigma)$, with $\bar{y}^o = P(\bar{t}^o)$. Since $u^o \neq 0$ and also $P_0(\bar{t}^o)h(\bar{t}^o)\beta(\bar{t}^o) \neq 0$, we can define:

$$r^o = \frac{u^o\,\beta(\bar{t}^o)}{P_0(\bar{t}^o)^{2\mu+1}}.$$

where μ is as in Equation 4.3, page 127. Then, substituting $P(\bar{t}^o)$ by $\bar{y}^o$ in System 4.4, and using also Equation 4.3, one has that:

$$\begin{cases} b^P(d^o, \bar{t}^o, \bar{x}^o) = P_0(\bar{t}^o)^2 b(d^o, \bar{x}^o, \bar{y}^o) = 0 \\ \mathrm{nor}^P_{(i,j)}(\bar{t}^o, \bar{x}^o) = \dfrac{P_0(\bar{t}^o)^{\mu+1}}{\beta(\bar{t}^o)} \mathrm{nor}_{(i,j)}(\bar{x}^o, \bar{y}^o) = 0 \\ w^P(r^o, \bar{t}^o) = w(u^o, \bar{y}^o) = 0, \end{cases} \tag{4.5}$$

because $(d^o, \bar{x}^o, \bar{y}^o, u^o) \in \Psi(\Sigma)$. Therefore, one concludes that $(d^o, r^o, \bar{t}^o, \bar{x}^o) \in \Psi^P$, and so $(d^o, \bar{x}^o) \in \mathcal{A}^P$. This proves that $\pi_1(\mathcal{B}_\Sigma^P) \subset \mathcal{A}^P$.

Conversely, let $(d^o, \bar{x}^o) \in \mathcal{A}^P$. Then, there are $\bar{t}^o, r^o$ such that $(d^o, r^o, \bar{t}^o, \bar{x}^o) \in \Psi^P$. Since $P_0(\bar{t}^o)h(\bar{t}^o)\beta(\bar{t}^o) \neq 0$,

$$\bar{y}^o = P(\bar{t}^o) \quad \text{and} \quad u^o = \frac{r^o\,P_0(\bar{t}^o)^{2\mu+1}}{\beta(\bar{t}^o)}$$

are well defined. The equations (4.5) still hold, and in this case, they imply that $(d^o, \bar{x}^o, \bar{y}^o, u^o) \in \Psi(\Sigma)$. Besides, $\pi_2(d^o, \bar{x}^o, \bar{y}^o, u^o) = (d^o, \bar{y}^o) \in \mathbb{C} \times \Upsilon_2$, and so $(d^o, \bar{x}^o) \in \pi_1(\mathcal{B}_\Sigma^P)$. This proves that $\mathcal{A}^P \subset \pi_1(\mathcal{B}_\Sigma^P)$, thus finishing the proof. $\square$

4.1.3 Intersection with lines

As in the case of plane curves, we will address the degree problem for surfaces by counting the number of intersection points between $\mathcal{O}_d(\Sigma)$ and a generic line through the origin. More precisely, let us consider a family of lines through the origin, denoted by $\mathcal{L}_{\bar{k}}$, whose direction is determined (see page 2) by the values of the variable $\bar{k} = (k_1, k_2, k_3)$. The family $\mathcal{L}_{\bar{k}}$ is described by the following set of parametric equations:

$$\mathcal{L}_{\bar{k}} \equiv \begin{cases} \ell_1(\bar{k}, l, \bar{x}) : \; x_1 - k_1\, l = 0 \\ \ell_2(\bar{k}, l, \bar{x}) : \; x_2 - k_2\, l = 0 \\ \ell_3(\bar{k}, l, \bar{x}) : \; x_3 - k_3\, l = 0 \end{cases}$$

A particular line of the family, corresponding to the value $\bar{k}^o$, will be denoted by $\mathcal{L}_{\bar{k}^o}$. We add the equations ℓ_1, ℓ_2, ℓ_3 of $\mathcal{L}_{\bar{k}}$ to the equations of the parametric system for the generic offset (System 4.4 in page 128), and we arrive at the following system:

$$\mathfrak{S}_2^P(d, \bar{k}) \equiv \begin{cases} b^P(d, \bar{t}, \bar{x}) : (P_0x_1 - P_1)^2 + (P_0x_2 - P_2)^2 + (P_0x_3 - P_3)^2 - d^2 {P_0}^2 = 0 \\ \mathrm{nor}^P_{(1,2)}(\bar{t}, \bar{x}) : \; n_1 \cdot (P_0x_2 - P_2) - n_2 \cdot (P_0x_1 - P_1) = 0 \\ \mathrm{nor}^P_{(1,3)}(\bar{t}, \bar{x}) : \; n_1 \cdot (P_0x_3 - P_3) - n_3 \cdot (P_0x_1 - P_1) = 0 \\ \mathrm{nor}^P_{(2,3)}(\bar{t}, \bar{x}) : \; n_2 \cdot (P_0x_3 - P_3) - n_3 \cdot (P_0x_2 - P_2) = 0 \\ w^P(r, \bar{t}) : \; r \cdot P_0 \cdot \beta \cdot h - 1 = 0 \\ \ell_1(\bar{k}, l, \bar{x}) : \; x_1 - k_1\, l = 0 \\ \ell_2(\bar{k}, l, \bar{x}) : \; x_2 - k_2\, l = 0 \\ \ell_3(\bar{k}, l, \bar{x}) : \; x_3 - k_3\, l = 0 \end{cases} \tag{4.6}$$

We will refer to this as the Parametric Offset-Line System. The next step is the study of the generic solutions of this system. We need to exclude certain degenerate situations that arise for a set of values of $(d, \bar{k})$. The following lemma is the basic tool: for a given proper closed subset $\mathfrak{F} \subset \Sigma$, it shows

1. how to avoid the set of values $(d^o, \bar{k}^o)$ such that $\mathcal{L}_{\bar{k}^o} \setminus \{\bar{0}\}$ meets Σ in a point $\bar{y}^o \in \mathfrak{F}$,
2. and how to avoid the set of values $(d^o, \bar{k}^o)$ such that $\mathcal{L}_{\bar{k}^o} \setminus \{\bar{0}\}$ meets $\mathcal{O}_{d^o}(\Sigma)$ in a point $\bar{x}^o$ associated to $\bar{y}^o \in \mathfrak{F}$.

This lemma generalizes Lemma 2.6 (page 44). The proof of that lemma used the finiteness of the set of singularities of a plane irreducible curve. But now, since we are dealing with a closed, possibly one-dimensional subset of Σ, the proof must be different.

In the proof of the Lemma we will use the polynomials f, h, b and $\mathrm{nor}_{(i,j)}$ (for $i, j = 1, \ldots, 3; i < j$), introduced with System $\mathfrak{S}_1(d)$ in page 17. For the convenience of the reader we repeat that system here, recalling that in this chapter $n = 3$:

$$\left.\begin{array}{lr} & f(\bar{y}) = 0 \\ \mathrm{nor}_{(i,j)}(\bar{x}, \bar{y}): & f_i(\bar{y})(x_j - y_j) - f_j(\bar{y})(x_i - y_i) = 0 \\ (\text{for } i, j = 1, \ldots, 3; i < j) & \\ b(d, \bar{x}, \bar{y}): & (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 - d^2 = 0 \\ w(\bar{y}, u): & u \cdot (\|\nabla f(\bar{y})\|^2) - 1 = 0 \end{array}\right\} \equiv \mathfrak{S}_1(d).$$

and that $h(\bar{t}) = n_1(\bar{t})^2 + n_2(\bar{t})^2 + n_3(\bar{t})^2$, while $h_{\mathrm{imp}}(\bar{t}) = \|\nabla f(\bar{y})\|^2$.

Lemma 4.10. *Let $\mathfrak{F} \subsetneq \Sigma$ be closed. There exists an open $\Omega_{\mathfrak{F}} \subset \mathbb{C}^4$, such that if $(d^o, \bar{k}^o) \in \Omega_{\mathfrak{F}}$, the following hold:*

(1) $\mathcal{L}_{\bar{k}^o} \cap (\mathfrak{F} \setminus \{\bar{0}\}) = \emptyset$.

(2) If $\bar{x}^o \in (\mathcal{L}_{\bar{k}^o} \cap \mathcal{O}_{d^o}(\Sigma)) \setminus \{\bar{0}\}$, there is no solution $(d^o, \bar{x}^o, \bar{y}^o, u^o)$ of System $\mathfrak{S}_1(d)$ (System 1.3 in page 17) with $\bar{y}^o \in \mathfrak{F}$.

Proof. If $\mathfrak{F}$ is empty, the result is trivial. Thus, let us assume that $\mathfrak{F} \neq \emptyset$, and let the defining polynomials of $\mathfrak{F}$ be $\{\phi_1(\bar{y}), \ldots, \phi_p(\bar{y})\} \subset \mathbb{C}[\bar{y}]$. We will show that one may take $\Omega_{\mathfrak{F}} = \Omega^1_{\mathfrak{F}} \cap \Omega^2_{\mathfrak{F}}$, where $\Omega^1_{\mathfrak{F}}, \Omega^2_{\mathfrak{F}}$ are two open sets constructed as follows:

(a) Let us consider the following ideal in $\mathbb{C}[\bar{k}, \rho, v, \bar{y}]$:

$$\mathcal{I} =< f(\bar{y}), \phi_1(\bar{y}), \ldots, \phi_p(\bar{y}), \bar{y} - \rho \cdot \bar{k}, v \cdot \rho - 1 >,$$

and the projection maps defined in its solution set $\mathbf{V}(\mathcal{I})$ as follows:

$$\pi_{(1,1)}(\bar{k}, \rho, v, \bar{y}) = \bar{y}, \quad \pi_{(1,2)}(\bar{k}, \rho, v, \bar{y}) = \bar{k}$$

We show first that $\pi_{(1,1)}(\mathbf{V}(\mathcal{I})) = \mathfrak{F}$. The inclusion $\pi_{(1,1)}(\mathbf{V}(\mathcal{I})) \subset \mathfrak{F}$ is trivial; and if $\bar{y}^o \in \mathfrak{F}$, then since $\mathcal{F} \subset \Sigma$, $(\bar{y}^o, 1, 1, \bar{y}^o) \in \mathbf{V}(\mathcal{I})$ proves the reversed inclusion. Therefore, since $\mathcal{F} \subsetneq \Sigma$, $\dim(\pi_{(1,1)}(\mathbf{V}(\mathcal{I}))) = \dim(\mathfrak{F}) < 2$. Besides, for every $\bar{y}^o \in \pi_{(1,1)}(\mathbf{V}(\mathcal{I}))$ one has:

$$\pi^{-1}_{(1,1)}(\bar{y}^o) = \{(v^o \bar{y}^o, \frac{1}{v^o}, v^o, \bar{y}^o) \mid v^o \in \mathbb{C}^\times\}$$

from where one has that $\dim(\pi^{-1}_{(1,1)}(\bar{y}^o)) = 1$. Since the dimension of the fiber does not depend on $\bar{y}^o$, applying Lemma 1.5 (page 12), we obtain $\dim(\mathbf{V}(\mathcal{I})) < 3$. Thus, $\dim(\pi_{(1,2)}(\mathbf{V}(\mathcal{I}))) < 3$. It follows that $(\pi_{(1,2)}(\mathbf{V}(\mathcal{I})))^*$ is a proper closed subset of $\mathbb{C}^3$. Let $\Theta^1 = \mathbb{C}^3 \setminus (\pi_{(1,2)}(\mathbf{V}(\mathcal{I})))^*$, and let $\Omega^1_{\mathfrak{F}} = \mathbb{C} \times \Theta^1$.

(b) Let us consider the following ideal in $\mathbb{C}[d, \bar{k}, \rho, v, \bar{x}, \bar{y}]$:

$$\begin{aligned}\mathcal{J} = &< f(\bar{y}), b(d, \bar{x}, \bar{y}), \mathrm{nor}_{(1,2)}(\bar{x}, \bar{y}), \mathrm{nor}_{(1,3)}(\bar{x}, \bar{y}), \mathrm{nor}_{(2,3)}(\bar{x}, \bar{y}),\\ &\bar{x} - \rho \cdot \bar{k}, v \cdot \rho \cdot d \cdot h_{\mathrm{imp}}(\bar{y}) - 1, \phi_1(\bar{y}), \ldots, \phi_p(\bar{y}) >\end{aligned}$$

and the projection maps defined in its solution set $\mathbf{V}(\mathcal{J}) \subset \mathbb{C}^{12}$ as follows:

$$\pi_{(2,1)}(d, \bar{k}, \rho, v, \bar{x}, \bar{y}) = \bar{y}, \quad \pi_{(2,2)}(d, \bar{k}, \rho, v, \bar{x}, \bar{y}) = (d, \bar{k})$$

Then $\pi_{(2,1)}(\mathbf{V}(\mathcal{J})) \subset \mathfrak{F}$. Therefore $\dim(\pi_{(2,1)}(\mathbf{V}(\mathcal{J}))) \leq 1$. Let $\bar{y}^o \in \pi_{(2,1)}(\mathbf{V}(\mathcal{J}))$. Note that then $h_{\mathrm{imp}}(\bar{y}^o) \neq 0$. We denote $\sigma^o = \sqrt{h_{\mathrm{imp}}(\bar{y}^o)}$ (a particular choice of the square root); clearly $\sigma^o \neq 0$. Then, it holds that:

$$\pi_{(2,1)}^{-1}(\bar{y}^o) = \left\{ \left(d^o, \frac{1}{\rho^o} \left(\bar{y}^o \pm \frac{d^o}{\sigma^o} \nabla(\bar{y}^o) \right), \rho^o, \frac{1}{(\sigma^o)^2 \rho^o d^o}, \bar{y}^o \pm \frac{d^o}{\sigma^o} \nabla(\bar{y}^o), \bar{y}^o \right) \middle| d^o, \rho^o \in \mathbb{C}^\times \right\}$$

Therefore $\dim(\pi_{(2,1)}^{-1}(\bar{y}^o)) = 2$. Applying Lemma 1.5 again, one has

$$\dim(\mathbf{V}(\mathcal{J})) = 2 + \dim(\pi_{(2,1)}(\mathbf{V}(\mathcal{J})) \leq 3.$$

It follows that $\dim(\pi_{(2,2)}(\mathbf{V}(\mathcal{J}))) \leq 3$. Let us take $\Omega^2_{\mathfrak{F}} = \mathbb{C}^4 \setminus \pi_{(2,2)}(\mathcal{V})^*$.

Let $\Omega_{\mathfrak{F}} = \Omega^1_{\mathfrak{F}} \cap \Omega^2_{\mathfrak{F}}$ and let $(d^o, \bar{k}^o) \in \Omega_{\mathfrak{F}}$.

1. If $\bar{y}^o \in \mathcal{L}_{\bar{k}^o} \cap (\mathfrak{F} \setminus \{\bar{0}\})$, then there is some $\rho^o \in \mathbb{C}^\times$ such that $\bar{y}^o = \rho^o \bar{k}^o$. It follows that $(\bar{k}^o, \rho^o, \frac{1}{\rho^o}, \bar{y}^o) \in \mathbf{V}(\mathcal{I})$, and so $\bar{k}^o \in \pi_{(1,2)}(\mathbf{V}(\mathcal{I}))$, contradicting the construction of $\Omega^1_{\mathfrak{F}}$. This proves statement (1).
2. If $\bar{x}^o \in (\mathcal{L}_{\bar{k}^o} \cap \mathcal{O}_{d^o}(\Sigma)) \setminus \{\bar{0}\}$, and there is a solution $(d^o, \bar{x}^o, \bar{y}^o, u^o)$ of System $\mathfrak{S}_1(d)$ with $\bar{y}^o \in \mathfrak{F}$, then there is some $\rho^o \in \mathbb{C}^\times$ such that $\bar{x}^o = \rho^o \bar{k}^o$. It follows that $(d^o, \bar{k}^o, \rho^o, \frac{1}{\rho^o \cdot d^o \cdot h_{\mathrm{imp}}(\bar{y}^o)}, \bar{x}^o, \bar{y}^o) \in \mathbf{V}(\mathcal{J})$. Therefore $(d^o, \bar{k}^o) \in \pi_{(2,2)}(\mathbf{V}(\mathcal{J}))$, contradicting the construction of $\Omega^2_{\mathfrak{F}}$. This proves statement (2). □

Remark 4.11. *Note that the origin may belong to $\mathfrak{F}$. In that case, Lemma 4.10(1) guarantees that the origin is the only point in $\mathcal{L}_{\bar{k}^o} \cap \mathcal{F}$. Correspondingly, part (2) of the lemma guarantees that the remaining points in $\mathcal{L}_{\bar{k}^o} \cap \mathcal{O}_{d^o}(\Sigma)$ cannot be extended to a solution $(d^o, \bar{x}^o, \bar{y}^o, u^o)$ of System $\mathfrak{S}_1(d)$ with $\bar{y}^o \in \mathfrak{F}$.*

Our next goal is to prove a theorem (Theorem 4.13 below), that gives the theoretical foundation for our approach to the degree problem. Theorem 4.13 is the analogous of Theorem 2.5 (page 41) in Chapter 2. That theorem is preceded by Proposition 2.3, that states that for a curve $\mathcal{C}$, $\bar{0} \in \mathcal{O}_{d^o}(\mathcal{C})$ for at most finitely many values $d^o \in \mathbb{C}$. However, the proof of Proposition 2.3 does not extend directly to the case of surfaces: the main difficulty is that a surface can have infinitely many singular points. Even if we restrict ourselves to the case of rational surfaces, we still have to take into account the possible existence of a singular curve contained in Σ, and not contained in the image of the parametrization. Besides, in the proof of the theorem we will use Lemma 1.14 (page 14), that does not apply when Σ is a sphere centered at the origin. This is the reason for the Assumptions 4.1 (see page 122), that we recall here. In the sequel, we assume that:

(1) There exists a finite subset Δ^1 of $\mathbb{C}$ such that, for $d^o \notin \Delta^1$ the origin does not belong to $\mathcal{O}_{d^o}(\Sigma)$.

(2) Σ is not a sphere centered at the origin.

Before stating the theorem we have to introduce some terminology.

Remark 4.12. *For $(d^o, \bar{k}^o) \in \mathbb{C}^4$ we will* denote *by $\Psi_2^P(d^o, \bar{k}^o)$ the set of solutions of System $\mathfrak{S}_2^P(d^o, \bar{k}^o)$ in the variables $(l, r, \bar{t}, \bar{x})$ (see (4.6) in page 130).*

Theorem 4.13. *Let Σ satisfy the hypothesis in Remark 4.1. There exists a non-empty Zariski-open subset $\Omega_0 \subset \mathbb{C}^4$, such that if $(d^o, \bar{k}^o) \in \Omega_0$, then*

(a) if $\bar{y}^o \in \mathcal{L}_{\bar{k}_0} \cap (\Sigma \setminus \{\bar{0}\})$, then no normal vector to Σ at $\bar{y}^o$ is parallel to $\bar{y}^o$.

(b) $\Psi_2^P(d^o, \bar{k}^o)$ has precisely $m\delta$ elements (recall that m is the tracing index of P and δ the total degree of the generic offset). Besides, the set $\Psi_2^P(d^o, \bar{k}^o)$ can be partitioned as a disjoint union:

$$\Psi_2^P(d^o, \bar{k}^o) = \Psi_2^1(d^o, \bar{k}^o) \cup \cdots \cup \Psi_2^\delta(d^o, \bar{k}^o),$$

such that:

(b1) $\#\Psi_2^i(d^o, \bar{k}^o) = m$ for $i = 1, \ldots, \delta$.

(b2) The m elements of $\#\Psi_2^i(d^o, \bar{k}^o)$ have the same values of the variables $(l, r, \bar{x})$, and differ only in the value of $\bar{t}$. Besides, for $(l^o, r^o, \bar{t}^o, \bar{x}^o) \in \Psi_2^i(d^o, \bar{k}^o)$, the point $P(\bar{t}^o) \in \Sigma$ does not depend on the choice of $\bar{t}^o$.

Let us denote by $(l_i^o, r_i^o, \bar{t}_{h,i}^o, \bar{x}_i^o)$ an element of $\Psi_2^i(d^o, \bar{k}^o)$. Then

(b3) The points $\bar{x}_1^o, \ldots, \bar{x}_\delta^o$ are all different (and different from $\bar{0}$), and

$$\mathcal{L}_{\bar{k}^o} \cap \mathcal{O}_{d^o}(\Sigma) = \{\bar{x}_1^o, \ldots, \bar{x}_\delta^o\}.$$

Furthermore, $\bar{x}_i^o$ is non normal-isotropic in $\mathcal{O}_{d^o}(\Sigma)$, for $i = 1, \ldots, \delta$.

(b4) The δ *points*

$$\bar{y}_1^o = P(\bar{t}_{h,1}^o), \cdots, \bar{y}_\delta^o = P(\bar{t}_{h,\delta}^o)$$

are affine, distinct and non normal-isotropic points of Σ.

(c) $k_i^o \neq 0$ *for* $i = 1, 2, 3$.

Proof. Let $\Delta_0^1 = \{d^o \in \mathbb{C} \mid g(d^o, \bar{0}) \neq 0\}$. The assumption in Remark 4.1 (page 122) implies that Δ_0^1 is an open non-empty subset of $\mathbb{C}$. Let Δ be as in Corollary 1.25, (Chapter 1, page 21), and let $\Omega_0^0 = (\Delta_0^1 \cap (\mathbb{C} \setminus \Delta)) \times (\mathbb{C}^3 \setminus \big(\{\bar{k}^o / k_i^o = 0 \text{ for some } i = 1, 2, 3\}\big))$. Next, let us consider $g(d, \bar{x})$ expressed as follows:

$$g(d, \bar{x}) = \sum_{i=0}^{\delta} g_i(d, \bar{x})$$

where g_i is a degree i form in $\bar{x}$. We consider:

$$\tilde{g}(d, \bar{k}, \rho) = g(d, \rho\bar{k}) = \sum_{i=0}^{\delta} g_i(d, \bar{k})\rho^i.$$

This polynomial is not identically zero, is primitive w.r.t. $\bar{x}$ (see Lemma 1.22, page 20), and it is squarefree; note that $g(d, \bar{x})$ is square-free by Remark 1.23 (page 21), and therefore $\tilde{g}$ is square-free too. Thus, the discriminant

$$Q(d, \bar{k}) = \mathrm{Dis}_\rho \left(\tilde{g}(d, \bar{k}, \rho)\right)$$

is not identically zero either.

In this situation, let us take

$$\Omega_1^0 = \Omega_0^0 \setminus \{(d^o, \bar{k}^o) \in \mathbb{C}^4 / Q(d^o, \bar{k}^o) \cdot g_0(d^o, \bar{k}^o) \cdot g_\delta(d^o, \bar{k}^o) = 0\}.$$

For $(d^o, \bar{k}^o) \in \Omega_1^0$, $g(d^o, \rho\bar{k}_0)$ has δ different and non-zero roots; say, $\rho_1, \ldots, \rho_\delta$. Therefore, $\mathcal{L}_{\bar{k}^o}$ intersects $\mathcal{O}_{d^o}(\Sigma)$ in δ different points:

$$\bar{x}_1^o = \rho_1\bar{k}^o, \ldots, \bar{x}_\delta^o = \rho_\delta\bar{k}^o,$$

and none of these points is the origin.

We will now construct an open subset $\Omega_2^0 \subset \Omega_1^0$ such that for $(d^o, \bar{k}^o) \in \Omega_2^0$, the points $\bar{x}_1^o, \ldots, \bar{x}_\delta^o$ are non-normal isotropic points in $\mathcal{O}_{d^o}(\Sigma)$, and each one of them is associated with a unique non-normal isotropic point of Σ. To do this, recall that $\mathrm{Iso}(\Sigma)$ is the closed set of normal-isotropic points of Σ (see page 9), and let $\Omega_{\mathrm{Iso}(\Sigma)}$ be the set obtained when applying Lemma 4.10 (page 131) to the closed subset $\mathfrak{F} = \mathrm{Iso}(\Sigma)$. Note that if

$(d^o, \bar{k}^o) \in \Omega_1^0 \cap \Omega_{\mathrm{Iso}(\Sigma)}$, then the points $\bar{x}_1^o, \ldots, \bar{x}_\delta^o$ are not associated with normal-isotropic points of Σ. Let us consider the polynomial

$$\Gamma(d, \bar{x}) = \left(\frac{\partial g}{\partial x_1}(d, \bar{x})\right)^2 + \left(\frac{\partial g}{\partial x_2}(d, \bar{x})\right)^2 + \left(\frac{\partial g}{\partial x_3}(d, \bar{x})\right)^2.$$

This polynomial is not identically zero, because in that case for every $d^o \notin \Delta$ all the points in $\mathcal{O}_{d^o}(\Sigma)$ would be normal-isotropic, contradicting Proposition 1.11(3) (page 13). Let then $\tilde{\Gamma}(d, \bar{k}, r) = \Gamma(d, r\bar{k})$, and consider the resultant:

$$\Phi(d, \bar{k}) = \mathrm{Res}_r(\tilde{g}(d, \bar{k}, r), \tilde{\Gamma}(d, \bar{k}, r))$$

If $\Phi(d, \bar{k}) \equiv 0$, then $\tilde{g}(d, \bar{k}, r)$ y $\tilde{\Gamma}(d, \bar{k}, r)$ have a common factor of positive degree in r. Let us show that this leads to a contradiction. Suppose that

$$\begin{cases} \tilde{g}(d, \bar{k}, r) = M(d, \bar{k}, r)G(d, \bar{k}, r), \\ \tilde{\Gamma}(d, \bar{k}, r) = M(d, \bar{k}, r)\Gamma^*(d, \bar{k}, r). \end{cases}$$

Then M depends on $\bar{k}$ (because $\tilde{g}$ cannot have a non constant factor in $\mathbb{C}[d, r]$). Take therefore $r^o \in \mathbb{C}^\times$ such that $M(d, \frac{\bar{k}}{r^o}, r^o)$ depends on $\bar{k}$. Then:

$$\begin{cases} g(d, \bar{x}) = g(d, r^o \frac{\bar{x}}{r^o}) = \tilde{g}(d, \frac{\bar{x}}{r^o}, r^o) = M(d, \frac{\bar{x}}{r^o}, r^o)G(d, \frac{\bar{x}}{r^o}, r^o) \\ \Gamma(d, \bar{x}) = \Gamma(d, r^o \frac{\bar{x}}{r^o}) = \tilde{\Gamma}(d, \frac{\bar{x}}{r^o}, r^o) = M(d, \frac{\bar{x}}{r^o}, r^o)\Gamma^*(d, \frac{\bar{x}}{r^o}, r^o) \end{cases}$$

But since g has at most two irreducible components, this would imply that for $d^o \notin \Delta$, $\mathcal{O}_{d^o}(\Sigma)$ would have at least a normal-isotropic component, contradicting Proposition 1.11(3) (page 13). Therefore, the equation $\Phi(d, \bar{k}) = 0$ defines a proper closed subset of $\mathbb{C}^4$. This shows that we can take:

$$\Omega_2^0 = \left(\Omega_1^0 \cap \Omega_{\mathrm{Iso}(\Sigma)}\right) \setminus \{(d^o, \bar{k}^o) : \Phi(d^o, \bar{k}^o) = 0\}$$

Then, for $(d^o, \bar{k}^o) \in \Omega_2^0$, each of the points $\bar{x}_i^o$, for $i = 1, \ldots, \delta$, is associated with a unique non-normal isotropic point $\bar{y}_i^o$ of Σ (recall that $d^o \in \Delta$, and so the irreducible components of $\mathcal{O}_{d^o}(\Sigma)$ are simple).

Let $\Omega_\perp$ be the open subset of $\mathbb{C} \times \mathbb{C}^3$ obtained by applying Lemma 4.10 (page 131) to the closed subset $\Sigma_\perp$ whose existence is guaranteed by Lemma 1.14 (page 14). Recall that, by assumption (see Remark 4.1(2), page 122), Σ is not a sphere centered at the origin. Besides, let $\Theta = \mathbb{C}^3 \setminus \mathcal{L}_0$, where

$$\mathcal{L}_0 = \begin{cases} \emptyset \text{ if } \bar{0} \notin \Sigma \text{ or if } \bar{0} \in \mathrm{Sing}(\Sigma) \\ \text{the normal line to } \Sigma \text{ at } \bar{0} \text{ otherwise.} \end{cases}$$

and set

$$\Omega_3^0 = \Omega_2^0 \cap \Omega_\perp \cap (\mathbb{C} \times \Theta).$$

Then for $(d^o, \bar{k}^o) \in \Omega_3^o$, the points $\bar{y}_i^o, i = 1, \ldots, \delta$, are different. To prove this, note that if $\bar{y}_i = \bar{y}_j$, with $i \neq j$, then $\bar{y}_i^o$ generates $\bar{x}_i^o$ and $\bar{x}_j^o$. Thus, since $\bar{y}_i^o, \bar{x}_i^o, \bar{x}_j^o$ are all in the normal line to Σ at $\bar{y}_i^o$ and in $\mathcal{L}_{\bar{k}^o}$, it follows that these two lines coincide. This means that $\bar{y}_i^o \in \Sigma_\perp$. Since $(d^o, \bar{k}^o) \in \Omega_\perp$, then (by Lemma 4.10) $\bar{y}_i^o \in \mathcal{L}_{\bar{k}^o} \cap \Omega_\perp$ implies that $\bar{y}_i^o = \bar{0}$, in contradiction with $\bar{k}^o \in \mathbb{C} \times \Theta$.

We will now show that it is possible to restrict the values of $(d, \bar{k})$ so that the points $\bar{y}_i^o$ belong to the image of the parametrization P. Let Υ_2 be as in Lemma 4.2 (page 126), and let $\Omega_{\Upsilon_2} \subset \mathbb{C} \times \mathbb{C}^3$ be the open subset obtained applying Lemma 4.10 to $\Sigma \setminus \Upsilon_2$. Then take $\Omega_4^0 = \Omega_3^0 \cap \Omega_{\Upsilon_2}$.

Let us show that we can take $\Omega_0 = \Omega_4^0$. If $(d^o, \bar{k}^o) \in \Omega_4^0$, then for each of the points $\bar{y}_i^o$ there are μ values $\bar{t}_{(i,j)}^o$ (with $i = 1, \ldots, \delta$, $j = 1, \ldots, m$) such that $P(\bar{t}_{(i,j)}^o) = \bar{y}_i^o$. Setting $\Psi_2^i(d^o, \bar{k}^o) = \{\bar{t}_{(i,j)}^o\}_{j=1,\ldots,m}$, one has that

$$\Psi_2^P(d^o, \bar{k}^o) = \Psi_2^1(d^o, \bar{k}^o) \cup \cdots \cup \Psi_2^\delta(d^o, \bar{k}^o)$$

and so the first part of claim (2) is proved. Furthermore:

- claim (a) holds because of the construction of Ω_3^0.
- the structure of $\Psi_2(d^o, \bar{k}^o)$ in claims (b1) and (b2) holds because of the construction of Ω_4^0.
- Claims (b3) and (b4) hold because of the construction of Ω_0^0, Ω_1^0 and Ω_2^0.
- Claim (c) follows the construction of Ω_0^0.

□

Remark 4.14. *Note that, by the construction of Ω_0^0 in the proof of Theorem 4.13 (page 133), if $(d^o, \bar{k}^o) \in \Omega_0$, then $g(d^o, \bar{x}) = 0$ is the equation of $\mathcal{O}_{d^o}(\Sigma)$.*

4.2 Auxiliary Curves for Rational Surfaces

As we said in the introduction to this chapter, the rational character of Σ results in a reduction of the dimension of the space in which we count the points in $\mathcal{O}_d(\Sigma) \cap \mathcal{L}_{\bar{k}}$. This is so because, instead of counting directly those points, we count the values of the $\bar{t}$ parameters that generate them. In this section we will show how, with this approach, we are led again to an intersection problem between projective plane curves, and we will analyze that problem. More precisely, in Subsection 4.2.1 we describe the auxiliary polynomials obtained by using elimination techniques in the Parametric Offset-Line System, and we introduce a new auxiliary system, see System 4.7. Also, we obtain some geometric properties of the solutions of this new system $\mathfrak{S}_3^P(d,\bar{k})$ in Proposition 4.16 (page 139) and the subsequent Lemma 4.18 (page 142). These results will be used in the sequel to elucidate the relation between the solution sets of Systems $\mathfrak{S}_2^P(d,\bar{k})$ and $\mathfrak{S}_3^P(d,\bar{k})$. Then, in Subsection 4.2.2 (page 143) we define the corresponding notion of fake points and invariant points for the Affine Auxiliary System $\mathfrak{S}_3^P(d,\bar{k})$. The main result of this subsection is Proposition 4.23 (page 145), that shows the relation between these two notions.

4.2.1 Elimination and auxiliary polynomials

To continue with our strategy, we proceed to eliminate the variables $(l,r,\bar{x})$ in the Parametric Offset-Line System $\mathfrak{S}_2^P(d,\bar{k})$ (page 130). This elimination process leads us to consider the following system of equations:

$$\mathfrak{S}_3^P(d,\bar{k}) \equiv \begin{cases} s_1(d,\bar{k},\bar{t}) := h(\bar{t})(k_2P_3 - k_3P_2)^2 - d^2P_0(\bar{t})^2(k_2n_3 - k_3n_2)^2 = 0, \\ s_2(d,\bar{k},\bar{t}) := h(\bar{t})(k_1P_3 - k_3P_1)^2 - d^2P_0(\bar{t})^2(k_1n_3 - k_3n_1)^2 = 0, \\ s_3(d,\bar{k},\bar{t}) := h(\bar{t})(k_1P_2 - k_2P_1)^2 - d^2P_0(\bar{t})^2(k_1n_2 - k_2n_1)^2 = 0. \end{cases} \tag{4.7}$$

We will refer to this as the Affine Auxiliary System.

We recall that $P = \left(\dfrac{P_1}{P_0}, \dfrac{P_2}{P_0}, \dfrac{P_3}{P_0}\right)$, $\bar{k} = (k_1,k_2,k_3)$, $\bar{n} = (n_1,n_2,n_3)$ and $h(\bar{t}) = n_1(t)^2 + n_2(t)^2 + n_3(t)^2$. Along with the polynomials s_1, s_2, s_3 introduced in the above system, we will also need to consider the following polynomial:

$$s_0(\bar{k},\bar{t}) = k_1(P_2n_3 - P_3n_2) - k_2(P_1n_3 - P_3n_1) + k_3(P_1n_2 - P_2n_1)$$

The geometrical meaning of s_0 is clear when one expresses it as a determinant, as follows:

$$s_0(\bar{k},\bar{t}) = \det\begin{pmatrix} k_1 & k_2 & k_3 \\ P_1 & P_2 & P_3 \\ n_1 & n_2 & n_3 \end{pmatrix}. \tag{4.8}$$

We will introduce some additional notation to simplify the expression of the polynomials s_i for $i = 1,2,3$. More precisely, we denote:

$$\begin{cases} M_1(\bar{k},\bar{t}) = k_2P_3 - k_3P_2, & G_1(\bar{k},\bar{t}) = k_2n_3 - k_3n_2, \\ M_2(\bar{k},\bar{t}) = k_3P_1 - k_1P_3, & G_2(\bar{k},\bar{t}) = k_3n_1 - k_1n_3, \\ M_3(\bar{k},\bar{t}) = k_1P_2 - k_2P_1, & G_3(\bar{k},\bar{t}) = k_1n_2 - k_2n_1. \end{cases} \tag{4.9}$$

With this notation one has

$$s_i(d,\bar{k},\bar{t}) = h(\bar{t})M_i^2(\bar{k},\bar{t}) - d^2P_0(\bar{t})^2G_i^2(\bar{k},\bar{t}) \text{ for } i = 1,2,3.$$

Note also that

$$\begin{cases} (M_1, M_2, M_3)(\bar{k}, \bar{t}) = \bar{k} \wedge \big(P_1(\bar{t}), P_2(\bar{t}), P_3(\bar{t})\big) \\ (G_1, G_2, G_3)(\bar{k}, \bar{t}) = \bar{k} \wedge \bar{n}(\bar{t}). \end{cases} \tag{4.10}$$

Let

$$I_2^P(d) = < b^P, \mathrm{nor}^P_{(1,2)}, \mathrm{nor}^P_{(1,3)}\mathrm{nor}^P_{(2,3)}, w^P, \ell_1, \ell_2, \ell_3 > \subset \mathbb{C}[d, \bar{k}, l, r, \bar{t}, \bar{x}]$$

be the ideal generated by the polynomials that define the Parametric Offset-Line System $\mathfrak{S}_2^P(d, \bar{k})$. We consider the projection associated with the elimination:

$$\pi_{(2,1)} : \mathbb{C} \times \mathbb{C}^3 \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}^2 \times \mathbb{C}^3 \mapsto \mathbb{C} \times \mathbb{C}^3 \times \mathbb{C}^2$$

given by

$$\pi_{(2,1)}(d, \bar{k}, l, r, \bar{t}, \bar{x}) = (d, \bar{k}, \bar{t})$$

The next lemma relates the polynomials $s_0, \ldots, s_3 \in \mathbb{C}[d, \bar{k}, \bar{t}]$ in System $\mathfrak{S}_3(d, \bar{k})$ with the elimination process. We denote by $\tilde{I}_2^P(d)$ the elimination ideal $I_2^P(d) \cap \mathbb{C}[d, \bar{k}, \bar{t}]$. For $(d^o, \bar{k}^o) \in \mathbb{C} \times \mathbb{C}^3$, the set of solutions of the Parametric Offset-Line system is denoted by $\Psi_2^P(d^o, \bar{k}^o)$, and the set of solutions of $\mathfrak{S}_3^P(d^o, \bar{k}^o)$ is denoted by $\Psi_3^P(d^o, \bar{k}^o)$. Note that $\Psi_2^P(d^o, \bar{k}^o) = \mathbf{V}(I_2^P(d))$.

Lemma 4.15. *$s_i \in \tilde{I}_2^P(d)$ for $i = 0, \ldots, 3$. In particular, if $(l^o, r^o, \bar{t}^o, \bar{x}^o) \in \Psi_2^P(d^o, \bar{k}^o)$, then $\bar{t}^o \in \Psi_3^P(d^o, \bar{k}^o)$.*

Proof. The polynomials s_i can be expressed as follows:

$$s_i = c_1^{(i)}\, b^P + c_2^{(i)}\, \mathrm{nor}^P_{(1,2)} + c_3^{(i)}\, \mathrm{nor}^P_{(1,3)} + c_4^{(i)}\, \mathrm{nor}^P_{(2,3)} + c_5^{(i)}\, w^P + c_6^{(i)}\, \ell_1 + c_7^{(i)}\, \ell_2 + c_8^{(i)}\, \ell_3$$

where $c_j^{(i)} \in \mathbb{C}[d, \bar{k}, l, r, \bar{t}, \bar{x}]$ for $i = 0, \ldots, 3$, $j = 1, \ldots, 8$. This polynomials (obtained with the CAS Singular [21]) can be found in Appendix B (page 201). □

The next step appears naturally to be the converse analysis: which are the $\bar{t}^o \in \mathfrak{S}_3^P(d^o, \bar{k}^o)$ that can be extended to a solution $(l^o, r^o, \bar{t}^o, \bar{x}^o) \in \Psi_2^P(d^o, \bar{k}^o)$? In order to describe them, we need some notation and a lemma. Let $\mathcal{A}$ denote the set of values $\bar{t}^o \in \mathbb{C}^2$ such that:

$$\begin{cases} P_0(\bar{t}^o)h(\bar{t}^o)\big(P_2(\bar{t}^o)n_3(\bar{t}^o) - P_3(\bar{t}^o)n_2(\bar{t}^o)\big) \neq 0 \\ \text{or} \\ P_0(\bar{t}^o)h(\bar{t}^o)\big(P_1(\bar{t}^o)n_3(\bar{t}^o) - P_3(\bar{t}^o)n_1(\bar{t}^o)\big) \neq 0 \\ \text{or} \\ P_0(\bar{t}^o)h(\bar{t}^o)\big(P_1(\bar{t}^o)n_2(\bar{t}^o) - P_2(\bar{t}^o)n_1(\bar{t}^o)\big) \neq 0 \end{cases} \tag{4.11}$$

Now we can describe which solutions of $\bar{t}^o \in \mathfrak{S}_3^P(d^o, \bar{k}^o)$ can be extended.

Proposition 4.16. *Let Ω_0 be as in Theorem 4.13 (page 133), $(d^o, \bar{k}^o) \in \Omega_0$ and $\bar{t}^o \in \Psi_3^P(d^o, \bar{k}^o)$. Then the following holds:*

(a) There exists $\lambda^o \in \mathbb{C}^\times$ such that:

$$\bar{k}^o \wedge \left(P_1(\bar{t}^o), P_2(\bar{t}^o), P_3(\bar{t}^o)\right) = \lambda^o \left(\bar{k}^o \wedge \bar{n}(\bar{t}^o)\right).$$

That is,

$$M_i(\bar{k}^o, \bar{t}^o) = \lambda^o G_i(\bar{k}^o, \bar{t}^o) \text{ for } i = 1, 2, 3.$$

(b) If $\bar{t}^o \in \mathcal{A}$, then $s_0(d^o, k^o, \bar{t}^o) = 0$.

(c) $(d^o, \bar{k}^o, \bar{t}^o) \in \pi_{(2,1)}(\Psi_2^P(d^o, \bar{k}^o))$ if and only if $\bar{t}^o \in \mathcal{A}$.

In particular,

$$m\delta = \#\left(\mathcal{A} \cap \Psi_3^P(d^o, \bar{k}^o)\right).$$

Recall that m is the tracing index of P, and δ is the total degree w.r.t $\bar{x}$ of the generic offset equation.

Proof.

(a) To prove the existence of λ^o, notice that $\bar{t}^o \in \Psi_3^P(d^o, \bar{k}^o)$ implies:

$$h(\bar{t}^o) M_i^2(\bar{k}^o, \bar{t}^o) = (d^o)^2 P_0(\bar{t}^o)^2 G_i^2(\bar{k}^o, \bar{t}^o) \text{ for } i = 1, 2, 3.$$

Since $\bar{t}^o \in \mathcal{A}$, $h(\bar{t}^o) \neq 0$. Therefore one concludes that there exist ϵ_i, with $\epsilon_i^2 = 1$, such that

$$M_i(\bar{k}^o, \bar{t}^o) = \epsilon_i \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} G_i(\bar{k}^o, \bar{t}^o) \text{ for } i = 1, 2, 3.$$

Since there are three of them, two of the ϵ_i must coincide. We will show that the third one must coincide as well. That is, we will show that either $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, or $\epsilon_1 = \epsilon_2 = \epsilon_3 = -1$ holds. We will study one particular case, the other possible combinations can be treated similarly. Let us suppose, e.g., that $\epsilon_1 = \epsilon_2 = 1$. Then:

$$\begin{cases} k_2^o P_3(\bar{t}^o) - k_3^o P_2(\bar{t}^o) = \dfrac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} \left(k_2^o n_3(\bar{t}^o) - k_3^o n_2(\bar{t}^o)\right) \\ k_3^o P_1(\bar{t}^o) - k_1^o P_3(\bar{t}^o) = \dfrac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} \left(k_3^o n_1(\bar{t}^o) - k_1^o n_3(\bar{t}^o)\right) \end{cases}$$

Multiplying the first equation by k_1^o and the second by k_2^o, and subtracting one has:

$$k_3^o \left(k_1^o P_2(\bar{t}^o) - k_2^o P_1(\bar{t}^o)\right) = k_3^o \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} \left(k_1^o n_2(\bar{t}^o) - k_2^o n_1(\bar{t}^o)\right)$$

Since $(d^o, \bar{k}^o) \in \Omega_0$, we have $k_3^o \neq 0$ (see Theorem 4.13(c), page 133). Thus, we have shown that

$$k_1^o P_2(\bar{t}^o) - k_2^o P_1(\bar{t}^o) = \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} \left(k_1^o n_2(\bar{t}^o) - k_2^o n_1(\bar{t}^o)\right)$$

and so $\epsilon_3 = 1$. Therefore, $\lambda^o = \dfrac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}}$, and it is non-zero because $\bar{t}^o \in \mathcal{A}$.

(b) From the identity in (a) it follows immediately that $\bar{k}^o$, $\big(P_1(\bar{t}^o), P_2(\bar{t}^o), P_3(\bar{t}^o)\big)$ and $\bar{n}(\bar{t}^o)$ are coplanar vectors. Thus, $s_0(d^o, \bar{k}^o, \bar{t}^o) = 0$ (recall the geometric interpretation of s_0 in equation 4.8, page 137).

(c) If $(d^o, \bar{k}^o, \bar{t}^o) \in \pi_{(2,1)}(\Psi_2^P(d^o, \bar{k}^o))$, then $P_0(\bar{t}^o)\beta(\bar{t}^o)h(\bar{t}^o) \neq 0$ follows from equation w^P in the Parametric Offset-Line System 4.6 (page 130). Besides,

$$\big(P_2 n_3 - P_3 n_2\big)(\bar{t}^o) = \big(P_1 n_3 - P_3 n_1\big)(\bar{t}^o) = \big(P_1 n_2 - P_2 n_1\big)(\bar{t}^o) = 0$$

is impossible because of Theorem 4.13(a) (page 133). Thus $\bar{t}^o \in \mathcal{A}$.
Conversely, let us suppose that $\bar{t}^o \in \mathcal{A}$. More precisely, let us suppose w.l.o.g. that

$$P_0(\bar{t}^o)h(\bar{t}^o)\big(P_2(\bar{t}^o)n_3(\bar{t}^o) - P_3(\bar{t}^o)n_2(\bar{t}^o)\big) \neq 0.$$

The other cases can be proved in a similar way. First we note that

$$G_1(\bar{k}^o, \bar{t}^o) = k_3^o n_2(\bar{t}^o) - k_2^o n_3(\bar{t}^o) \neq 0.$$

since, using that $s_1(d^o, \bar{k}^o, \bar{t}^o) = 0$ and $h(\bar{t}^o) \neq 0$, one has that

$$k_2^o P_3(\bar{t}^o) - k_3^o P_2(\bar{t}^o) = 0.$$

Then, from the system:

$$\begin{cases} k_2^o n_3(\bar{t}^o) - k_3^o n_2(\bar{t}^o) = 0 \\ k_2^o P_3(\bar{t}^o) - k_3^o P_2(\bar{t}^o) = 0 \end{cases}$$

and the fact that $k_2^o k_3^o \neq 0$ (again, this is Theorem 4.13(c)), one deduces that

$$P_2(\bar{t}^o)n_3(\bar{t}^o) - P_3(\bar{t}^o)n_2(\bar{t}^o) = 0,$$

that is a contradiction. Thus, we can define

$$r^o = \frac{1}{P_0(\bar{t}^o)\beta(\bar{t}^o)h(\bar{t}^o)}, \text{ and } l^o = \frac{P_3(\bar{t}^o)n_2(\bar{t}^o) - P_2(\bar{t}^o)n_3(\bar{t}^o)}{-P_0(\bar{t}^o)G_1(\bar{k}^o, \bar{t}^o)}$$

We also define $\bar{x}^o = l^o \bar{k}^o$. We claim that $(l^o, r^o, \bar{t}^o, \bar{x}^o) \in \Psi_2^P(d^o, \bar{k}^o)$, and therefore $(d^o, \bar{k}^o, \bar{t}^o) \in \pi_{(2,1)}(\Psi_2^P(d^o, \bar{k}^o))$. To prove our claim we substitute $(l^o, r^o, \bar{t}^o, \bar{x}^o)$

in the equations of the Parametric Offset-Line System 4.6 (page 130), and we check that all of them vanish. The vanishing of $w^P(r^o,\bar{t}^o)$ and $\ell_i(\bar{k}^o,l^o,\bar{x}^o)$ for $i=1,2,3$ is a trivial consequence of the definitions. Substitution in $\mathrm{nor}^P_{(2,3)}$ leads to a polynomial whose numerator vanishes immediately. Substituting in $\mathrm{nor}^P_{(1,2)}$ (resp. in $\mathrm{nor}^P_{(1,3)}$) one obtains:

$$\mathrm{nor}^P_{(1,2)}(\bar{t}^o,\bar{x}^o)=\frac{n_2(\bar{t}^o)s_0(\bar{k}^o,\bar{t}^o)}{n_2\bar{t}^o k^o_3-n_3(\bar{t}^o)k^o_2}=0,$$

(respectively

$$\mathrm{nor}^P_{(1,3)}(\bar{t}^o,\bar{x}^o)=\frac{n_3(\bar{t}^o)s_0(\bar{k}^o,\bar{t}^o)}{n_2\bar{t}^o k^o_3-n_3(\bar{t}^o)k^o_2}=0),$$

where both equations hold because of part (a). Finally, substituting in $b^P(d^o,\bar{t}^o,\bar{x}^o)$ one has:

$$b^P(d^o,\bar{t}^o,\bar{x}^o)=\frac{s_2(d^o,\bar{k}^o,\bar{t}^o)+\phi_1(\bar{k}^o,\bar{t}^o)s_0(\bar{k}^o,\bar{t}^o)}{(n_2\bar{t}^o k^o_3-n_3(\bar{t}^o)k^o_2)^2}=0 \tag{4.12}$$

with $\phi_1(\bar{k},\bar{t})=k_2n_1P_3+k_2n_3P_1-k_3n_1P_2-k_3n_2P_1-k_1n_3P_2+k_1n_2P_3$. Equation 4.12 holds because of part (a) and because $s_2(d^o,\bar{k}^o,\bar{t}^o)=0$.

The claim that

$$m\delta=\#\left(\mathcal{A}\cap\Psi^P_3(d^o,\bar{k}^o)\right)$$

follows easily from Theorem 4.13(b) (page 133) and the above result (c). This shows that, for $(d^o,\bar{k}^o)\in\Omega_0$, there is a bijection (under $\pi_{(2,1)}$) between the points of $\Psi^P_2(d^o,\bar{k}^o)$ and the points in $\mathcal{A}\cap\Psi^P_3(d^o,\bar{k}^o)$. This finishes the proof of the proposition. □

Remark 4.17. *In the proof of Proposition 4.16 (page 4.16) we have seen that there is a vector equality:*

$$\bar{M}(\bar{k}^o,\bar{t}^o)=\epsilon\frac{d^oP_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}}\bar{G}(\bar{k}^o,\bar{t}^o)\ \textit{for}\ i=1,2,3.$$

where $\bar{M}=(M_1,M_2,M_3)$, $\bar{G}=(G_1,G_2,G_3)$ and $\epsilon=\pm1$. In the next lemma we will see that the value of $\epsilon=1$ determines the sign that appears in the offsetting construction. More precisely, in the proof of Proposition 4.16 we have seen that if $\bar{y}^o=P(\bar{t}^o)$, and

$$P_0(\bar{t}^o)h(\bar{t}^o)\big(P_2(\bar{t}^o)n_3(\bar{t}^o)-P_3(\bar{t}^o)n_2(\bar{t}^o)\big)\neq0.$$

then it holds that

$$k^o_2n_3(\bar{t}^o)-k^o_3n_2(\bar{t}^o)\neq0\ \textit{and}\ k^o_2P_3(\bar{t}^o)-k^o_3P_2(\bar{t}^o)\neq0.$$

Furthermore, the point $\bar{x}^o$, constructed as follows

$$\bar{x}^o = \frac{P_3(\bar{t}^o)n_2(\bar{t}^o) - P_2(\bar{t}^o)n_3(\bar{t}^o)}{-P_0(\bar{t}^o)G_1(\bar{k}^o, \bar{t}^o)}\bar{k}^o, \tag{4.13}$$

is the point in $\mathcal{O}_{d^o}(\Sigma) \cap \mathcal{L}_{\bar{k}^o}$ associated with $\bar{y}^o$. Thus, one has:

$$\bar{x}^o = \bar{y}^o + \epsilon' \frac{d^o \nabla f(\bar{y}^o)}{\sqrt{h_{\mathrm{imp}(\bar{y}^o)}}}.$$

where $\epsilon' = \pm 1$.

Lemma 4.18. *With the notation of Remark 4.17, it holds that $\epsilon = \epsilon'$.*

Proof. From the Equations

$$M_2(\bar{k}^o, \bar{t}^o) = \epsilon \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} G_2(\bar{k}^o, \bar{t}^o) \text{ and } M_3(\bar{k}^o, \bar{t}^o) = \epsilon \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} G_3(\bar{k}^o, \bar{t}^o),$$

multiplying the first equation by $n_2(\bar{t}^o)$, the second by $n_3(\bar{t}^o)$ and adding the results, one has:

$$-G_1(\bar{k}^o, \bar{t}^o)P_1(\bar{t}^o) - k_1^o(P_3 n_2 - P_2 n_3)(\bar{t}^o) = \epsilon\, n_1(\bar{t}^o) \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} G_1(\bar{k}^o, \bar{t}^o)$$

Using Equation 4.13 in Remark 4.17, this is:

$$-G_1(\bar{k}^o, \bar{t}^o)P_1(\bar{t}^o) + x_1^o G_1(\bar{k}^o, \bar{t}^o)P_0(\bar{t}^o) = \epsilon\, n_1(\bar{t}^o) \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} G_1(\bar{k}^o, \bar{t}^o).$$

Dividing by $G_1(\bar{k}^o, \bar{t}^o)P_0(\bar{t}^o)$:

$$-\frac{P_1(\bar{t}^o)}{P_0(\bar{t}^o)} + x_1^o = \epsilon \frac{d^o n_1(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}},$$

and finally

$$x_1^o = \frac{P_1(\bar{t}^o)}{P_0(\bar{t}^o)} + \epsilon \frac{d^o n_1(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}}.$$

Similar results are obtained for x_2^o and x_3^o. Thus we have proved that $\epsilon' = \epsilon$. $\square$

4.2.2 Fake points

Using Proposition 4.16 (page 139) we can now define the set of fake points associated with this problem.

Definition 4.19. *A point $\bar{t}^o \in \mathbb{C}^2$ is a* fake point *if*

$$\begin{cases} P_0(\bar{t}^o)h(\bar{t}^o)\big(P_2(\bar{t}^o)n_3(\bar{t}^o) - P_3(\bar{t}^o)n_2(\bar{t}^o)\big) = 0 \\ \text{and} \\ P_0(\bar{t}^o)h(\bar{t}^o)\big(P_1(\bar{t}^o)n_3(\bar{t}^o) - P_3(\bar{t}^o)n_1(\bar{t}^o)\big) = 0 \\ \text{and} \\ P_0(\bar{t}^o)h(\bar{t}^o)\big(P_1(\bar{t}^o)n_2(\bar{t}^o) - P_2(\bar{t}^o)n_1(\bar{t}^o)\big) = 0 \end{cases}$$

Equivalently,

$$P_0(\bar{t}^o)h(\bar{t}^o) = 0 \text{ or } (P_1(\bar{t}^o), P_2(\bar{t}^o), P_3(\bar{t}^o)) \wedge \bar{n}(\bar{t}^o) = \bar{0} \tag{4.14}$$

The set of fake points will be denoted *by $\mathcal{F}$.*

Definition 4.20. *Let Ω_0 be as in Theorem 4.13 (page 133) and let Ω be any open subset of Ω_0.* The set of invariant solutions of $\mathfrak{S}_3^P(d, \bar{k})$ w.r.t Ω. *is defined as the set:*

$$\mathcal{I}_3^P(\Omega) = \bigcap_{(d^o,\bar{k}^o)\in\Omega} \Psi_3^P(d^o, \bar{k}^o)$$

Remark 4.21. *Note that if $\bar{t}^o \in \mathcal{F}$, we do not assume that $\bar{t}^o \in \Psi_3^P(d^o, \bar{k}^o)$ for some $(d^o, \bar{k}^o) \in \mathbb{C} \times \mathbb{C}^3$.*

We have introduced the fake points starting from the notion non-extendable solutions of $\mathfrak{S}_3^P(d^o, \bar{k}^o)$. Another point of view is to define fake points as the *invariant solutions* of $\mathfrak{S}_3^P(d, \bar{k})$. First we will define what we mean by invariant in this context, and then we will show that, in a certain open subset of values $(d, \bar{k})$, both notions actually coincide.

To prove the equivalence between the notions of fake points and invariant points we need to further restrict the set of values of $(d, \bar{k})$. The following lemma gives the required restrictions.

Lemma 4.22. *Let Ω_0 be the open set in Theorem 4.13. There exists an open non-empty $\Omega_1 \subset \Omega_0$ such that if $(d^o, \bar{k}^o) \in \Omega_1$, then*

(1) $\bar{k}^o$ is not isotropic.

(2) d^o is not a critical distance of Σ (see Corollary 1.16 in page 15).

(3) The system

$$\{P_0(\bar{t}) = M_1(\bar{k}^o, 1, \bar{t}) = M_2(\bar{k}^o, 1, \bar{t}) = M_3(\bar{k}^o, 1, \bar{t}) = 0\} \tag{4.15}$$

has no solutions unless $P_0(\bar{t}^o) = P_1(\bar{t}^o) = P_2(\bar{t}^o) = P_3(\bar{t}^o) = 0$.

Proof.

(1) Set $\Omega_1^1 = \Omega_0 \cap (\mathbb{C} \times \mathfrak{Q})$, where $\mathfrak{Q} = \{\bar{k}^o/(k_1^o)^2 + (k_2^o)^2 + (k_3^o)^2 = 0\}$ is the cone of isotropy in $\bar{k}$.

(2) Let $\Upsilon(\Sigma^\perp)$ is the set of critical distances of Σ (defined in page 15), and set $\Omega_1^2 = \Omega_1^1 \cap (\Upsilon(\Sigma^\perp) \times \mathbb{C}^3)$.

(3) First we will show that the set of values $\bar{k}^o \neq \bar{0}$ for which the System 4.15 has a solution is contained in an at most two-dimensional closed subset $\mathfrak{R} \subset \mathbb{C}^3$. If P_0 is constant the result is trivial. Assuming that P_0 is not constant, let $\mathcal{C}_0$ be the affine curve defined by P_0, and let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ be the varieties defined by P_1, P_2, P_3 respectively. Let $\mathcal{J}_{P_1} \subset \mathbb{C}[\bar{k}, \bar{t}, v]$ be the ideal defined as follows:

$$\mathcal{J}_{P_1} =< P_0, k_2P_3 - k_3P_2, k_1P_3 - k_3P_1, k_1P_2 - k_2P_1, vP_1 - 1 >,$$

and let $\mathbf{V}(\mathcal{J}_{P_1}) \subset \mathbb{C}^3 \times \mathbb{C}^2 \times \mathbb{C}$ be the solution set of this ideal. Consider the projections defined by:

$$\begin{cases} \pi_1(\bar{k}, \bar{t}, v) = \bar{k} \\ \pi_2(\bar{k}, \bar{t}, v) = \bar{t} \end{cases}$$

Let A_0 be an irreducible component of $\mathbf{V}(\mathcal{J}_{P_1})$, and let $(\bar{k}^o, \bar{t}^o, v^o) \in A_0$. Then the points in $\pi_2^{-1}(\pi_2(\bar{k}^o, \bar{t}^o, v^o))$ are the solutions of the following system:

$$\begin{cases} \bar{t} = \bar{t}^o, \\ M_1(\bar{k}, 1, \bar{t}^o) = M_2(\bar{k}, 1, \bar{t}^o) = M_3(\bar{k}, 1, \bar{t}^o) = 0, \\ vP_1(\bar{t}^o) - 1 = 0 \end{cases}$$

The dimension of the set of solutions is 1. On the other hand, $\pi_2(A_0) \subset \mathcal{C}_0$ implies that $\dim(\pi_2(A_0)) \leq 1$. Thus, using Lemma 1.5 (page 12), one has that $\dim(\mathbf{V}(\mathcal{J}_{P_1})) \leq 2$. Thus, $\dim(\pi_1 (\mathbf{V}(\mathcal{J}_{P_1}))^*) \leq 2$. Now, defining $\mathcal{J}_{P_2}$ and $\mathcal{J}_{P_3}$ in a similar way (that is, replacing the equation $vP_1(\bar{t}^o) - 1 = 0$ by $vP_2(\bar{t}^o) - 1 = 0$ and $vP_3(\bar{t}^o) - 1 = 0$ respectively), we set:

$$\mathfrak{R} = \bigcup_{i=1,2,3} \pi_1 (\mathbf{V}(\mathcal{J}_{P_i}))^*$$

Now let $\Omega_1^3 = \Omega_1^2 \cap (\mathbb{C} \times \mathfrak{R})$.

The above construction shows that $\Omega_1 = \Omega_1^3$ satisfies the required properties. □

Now we can prove the announced equivalence between the notions of fake points and invariant points.

Proposition 4.23. *Let Ω_1 be as in Lemma 4.22 (page 143). If Ω is a non-empty open subset of Ω_1, then it holds that:*

$$\mathcal{I}_3^P(\Omega) = \mathcal{F} \cap \left(\bigcup_{(d^o, \bar{k}^o) \in \Omega} \Psi_3^P(d^o, \bar{k}^o) \right).$$

Proof. Let $\bar{t}^o \in \mathcal{I}_3^P(\Omega)$. Then $s_i(d^o, \bar{k}^o, \bar{t}^o) = 0$ for $i = 1, 2, 3$ and any $(d^o, \bar{k}^o) \in \Omega$. Thus $\bar{t}^o \in \cup_{(d^o, \bar{k}^o) \in \Omega} \Psi_3^P(d^o, \bar{k}^o)$. Furthermore, considering s_i as polynomials in $\mathbb{C}[\bar{t}][d, \bar{k}]$, it follows that $\bar{t}^o$ must be a solution of:

$$\begin{cases} h(\bar{t})P_1(\bar{t}) = h(\bar{t})P_2(\bar{t}) = h(\bar{t})P_3(\bar{t}) = 0 \\ P_0(\bar{t})n_1(\bar{t}) = P_0(\bar{t})n_2(\bar{t}) = P_0(\bar{t})n_3(\bar{t}) = 0 \end{cases}$$

It follows that $h(\bar{t}^o)P_0(\bar{t}^o) = 0$, and so $\bar{t}^o \in \mathcal{F}$. In fact, if we suppose $h(\bar{t}^o)P_0(\bar{t}^o) \neq 0$, then from $P_0(\bar{t}^o) \neq 0$ one gets $\bar{n}(\bar{t}^o) = 0$, and so $h(\bar{t}^o) = 0$, a contradiction.
Conversely, let $\bar{t}^o \in \mathcal{F} \cap \left(\bigcup_{(d^o, \bar{k}^o) \in \Omega} \Psi_3^P(d^o, \bar{k}^o) \right)$. Then:

1. If $P_0(\bar{t}^o) = h(\bar{t}^o) = 0$, then $s_i(d, \bar{k}, \bar{t}^o) = 0$ identically in $(d, \bar{k})$ for $i = 1, 2, 3$, and so $\bar{t}^o \in \mathcal{I}_3^P(\Omega)$.
2. If $P_0(\bar{t}^o) \neq 0$ and $h(\bar{t}^o) = 0$, then since $\bar{t}^o \in \Psi_3^P(d^o, \bar{k}^o)$ for some $(d^o, \bar{k}^o) \in \Omega$, one has the following two possibilities:
   (a) $\bar{n}(\bar{t}^o)$ is isotropic and parallel to $\bar{k}^o$. This is impossible because of the construction of Ω_1 (see Lemma 4.22(1), page 143).
   (b) $\bar{n}(\bar{t}^o) = \bar{0}$. In this case, again $s_i(d, \bar{k}, \bar{t}^o) = 0$ identically in $(d, \bar{k})$ for $i = 1, 2, 3$, and so $\bar{t}^o \in \mathcal{I}_3^P(\Omega)$.
3. Let us suppose that $P_0(\bar{t}^o) = 0$ and $h(\bar{t}^o) \neq 0$. Then, since $\bar{t}^o \in \Psi_3^P(d^o, \bar{k}^o)$ for some $(d^o, \bar{k}^o) \in \Omega$, one has that $\bar{t}^o$ is a solution of:

   $$P_0(\bar{t}) = 0, \quad M_1(\bar{t}, \bar{k}^o) = M_2(\bar{t}, \bar{k}^o) = M_3(\bar{t}, \bar{k}^o) = 0,$$

   Thus, two cases are possible:
   (a) $(P_1, P_2, P_3)(\bar{t}^o) = \bar{0}$. In this case, $s_i(d, \bar{k}, \bar{t}^o) = 0$ identically in $(d, \bar{k})$ for $i = 1, 2, 3$, and so $\bar{t}^o \in \mathcal{I}_3^P(\Omega)$.
   (b) $(P_1, P_2, P_3)(\bar{t}^o)$ is non-zero. This contradicts the construction of Ω_1 in Lemma 4.22(3).

4. Finally, let us suppose that $P_0(\bar{t}^o)h(\bar{t}^o) \neq 0$. Then it follows that the point $P(\bar{t}^o)$ is well defined, and it belongs to $\Sigma_\perp^*$ (recall that $\Sigma_\perp$ was introduced in Lemma 1.14, page 14). Thus d^o would be one of the critical distances, and this contradicts the construction of Ω_0 in Lemma 4.22(2). □

4.3 Total Degree Formula for Rational Surfaces

According to Proposition 4.16 (page 139), if $(d^o, \bar{k}^o) \in \Omega_0$ it holds that

$$m\delta = \#\left(\mathcal{A} \cap \Psi_3^P(d^o, \bar{k}^o)\right).$$

Recall that m is the tracing index of P, and δ is the total degree w.r.t $\bar{x}$ of the generic offset equation. Moreover, $\mathcal{A}$ was introduced in Equation 4.11 (page 138), and $\Psi_3^P(d^o, \bar{k}^o)$ was also introduced in page 138, as the solution set of System 4.7 (page 137). In this section, we will derive a formula for the total degree δ, using the tools in Section 1.3 (page 27) to analyze the intersection $\mathcal{A} \cap \Psi_3^P(d^o, \bar{k}^o)$.

In order to do this, in Subsection 4.3.1 we will consider the projective closure of the auxiliary curves introduced in the preceding section. This in turn, requires as a first step the projectivization of the parametrization P. At the end of the subsection we introduce the Projective Auxiliary System 4.25 (page 150), which will play a key rôle in the degree formula. Subsection 4.3.2. (page 150) is devoted to the study of the invariant solutions of the Projective Auxiliary System, connecting them with the corresponding affine notions in Section 4.2. As we have seen in previous chapters, a crucial step in our strategy concerns the multiplicity of intersection of the auxiliary curves at their non-invariant points of intersection. In Subsection 4.3.3 (page 160) we will prove that the value of that multiplicity of intersection is one (in Proposition 4.43, page 160). After this is done, everything is ready for the proof of the degree formula, which is the topic of Subsection 4.3.4 (page 170). The formula appears in Theorem 4.45 (page 172).

4.3.1 Projectivization of the parametrization and auxiliary curves

Let P be the parametrization of Σ, introduced in Equation (4.1). If we homogenize the components of P w.r.t. a new variable t_0, multiplying both the numerators and denominators if necessary by a suitable power of t_0 we arrive at an expression of the form:

$$P_h(\bar{t}_h) = \left(\frac{X(\bar{t}_h)}{W(\bar{t}_h)}, \frac{Y(\bar{t}_h)}{W(\bar{t}_h)}, \frac{Z(\bar{t}_h)}{W(\bar{t}_h)}\right) \tag{4.16}$$

where $\bar{t}_h = (t_0 : t_1 : t_2)$, and $X, Y, Z, W \in \mathbb{C}[\bar{t}_h]$ are homogeneous polynomials of the same degree d_P, for which $\gcd(X, Y, Z, W) = 1$ holds. This P_h will be called the projectivization of P.

Remark 4.24. *Note that those projective values of $\bar{t}_h$ of the form $(0 : a : b)$ correspond to points at infinity in the parameter plane.*

In Section 4.1.1 (page 125) we defined $\bar{n} = (n_1, n_2, n_3)$, the associated normal vector to P. A similar construction, applied to P_h, leads to a normal vector $N = (N_1, N_2, N_3)$, where N_i are homogeneous polynomials in $\bar{t}_h$ of the same degree, such that $\gcd(N_1, N_2, N_3) = 1$. This vector N will be called the associated homogeneous normal vector to P_h. The homogeneous polynomial H defined by

$$H(\bar{t}_h) = (N_1(\bar{t}))^2 + (N_2(\bar{t}))^2 + (N_3(\bar{t}))^2$$

is the parametric projective normal-hodograph of the parametrization P_h.

Remark 4.25. *The polynomials N_i are, up to multiplication by a power of t_0, the homogenization of the components of $\bar{n}$ w.r.t. t_0. However, since $\gcd(N_1, N_2, N_3) = 1$, at least one of the components N_i $(i = 1, 2, 3)$ is not divisible by t_0. Besides, note that if two components N_i, N_j, with $i \neq j$, are divisible by t_0, then H is not.*

Lemma 4.26.

1. *If W does not depend on t_0, then at least one of the polynomials X, Y, Z must depend on t_0.*
2. *If W does not depend on t_0, and there is exactly one of the polynomials X, Y, Z depending on t_0, then the surface is a cylinder with its axis parallel to the direction of the component with numerator depending on t_0.*

Proof

1. Otherwise, the rank of the jacobian matrix of P would be less than two. To see this, let us suppose that X, Y, Z, W depend only on t_1, t_2. Let $\partial_i P_h$ be the vector obtained as the partial derivative of P_h w.r.t. t_i, that is;
$$\partial_i P_h = \left(\frac{X_i W - X W_i}{W^2}, \frac{Y_i W - Y W_i}{W^2}, \frac{Z_i W - Z W_i}{W^2} \right)$$
where X_i, Y_i, Z_i, W_i denotes the partial derivative of X, Y, Z, W w.r.t. t_i. Using Euler's formula, and taking into account that the polynomials X, Y, Z, W have the same degree n, one has that $t_1 \partial_1 P_h = -t_2 \partial_2 P_h$. Substituting $t_0 = 1$, we see that the rank of the jacobian of P would be less than 2.
2. Assume w.l.o.g. that X, Y do not depend on t_0, but Z does. The rational map
$$\phi(\bar{t}) = \left(\frac{X(\bar{t})}{W(\bar{t})}, \frac{Y(\bar{t})}{W(\bar{t})} \right)$$

has rank one, because X, Y, W are homogeneous polynomials in $\bar{t}$ of the same degree. Thus, ϕ parametrizes a curve $\mathcal{C}$ in the (y_1, y_2)-plane. Let $\mathrm{Cyl}(\mathcal{C})$ be the cylinder over $\mathcal{C}$ with axis parallel to the y_3-axis. The points of the form $(\phi(\bar{t}^o), y_3^o)$, with $W(\bar{t}^o) \neq 0$, are dense in $\mathrm{Cyl}(\mathcal{C})$. Given one of these points, let t_0^o be any solution of the equation (in t_0):

$$Z(\bar{t}^o, t_0) = y_3^o W(\bar{t}^o)$$

Then we have

$$P_h(\bar{t}^o, t_0^o) = (\phi(\bar{t}^o), y_3^o)$$

and so $P_h(\mathbb{P}^2)$ is dense in $\mathrm{Cyl}(\mathcal{C})$. □

Now we are ready to introduce the projective auxiliary polynomials. We consider the following system:

$$\mathfrak{S}_4^{P_h}(d, \bar{k}) \equiv \begin{cases} S_0(\bar{k}, \bar{t}_h) := k_1(YN_3 - ZN_2) - k_2(XN_3 - ZN_1) + k_3(XN_2 - YN_1) \\ S_1(d, \bar{k}, \bar{t}_h) := H(\bar{t}_h)(k_2Z - k_3Y)^2 - d^2W(\bar{t}_h)^2(k_2N_3 - k_3N_2)^2 \\ S_2(d, \bar{k}, \bar{t}_h) := H(\bar{t}_h)(k_1Z - k_3X)^2 - d^2W(\bar{t}_h)^2(k_1N_3 - k_3N_1)^2 \\ S_3(d, \bar{k}, \bar{t}_h) := H(\bar{t}_h)(k_1Y - k_2X)^2 - d^2W(\bar{t}_h)^2(k_1N_2 - k_2N_1)^2 \end{cases} \tag{4.17}$$

As usual, for $(d^o, \bar{k}^o) \in \mathbb{C}^4$, we denote by $\Psi_4^{P_h}(d^o, \bar{k}^o)$ the set of projective solutions of $\mathfrak{S}_4^{P_h}(d^o, \bar{k}^o)$. Our next goal is the analysis of the relation between $\Psi_4^{P_h}(d^o, \bar{k}^o)$ and $\Psi_3^P(d^o, \bar{k}^o)$ (the set of solutions of $\mathfrak{S}_3^P(d^o, \bar{k}^o)$, see Subsection 4.2.1, page 137). In particular, an in order to obtain the degree formula, we will characterize those points in $\Psi_4^{P_h}(d^o, \bar{k}^o)$ that correspond to the points in $\mathcal{A} \cap \Psi_3^P(d^o, \bar{k}^o)$. In Proposition 4.23 (page 145) we have seen that the invariant solutions of $\Psi_3^P(d, \bar{k})$ correspond to fake points. Thus, as a first step, we will characterize certain invariant solutions of $\mathfrak{S}_4^{P_h}(d, \bar{k})$.

Lemma 4.27. *Let $S = c_1S_1 + c_2S_2 + c_3S_3$. Then:*

$$\mathrm{Con}_{(d,\bar{k})}(S(\bar{c}, d, \bar{k}, \bar{t}_h)) = \gcd(H, W^2).$$

Proof. Since $S = c_1S_1 + c_2S_2 + c_3S_3$, one has:

$$\mathrm{Con}_{(d,\bar{k})}(S) = \gcd\left(\mathrm{Con}_{(d,\bar{k})}(S_1), \mathrm{Con}_{(d,\bar{k})}(S_2), \mathrm{Con}_{(d,\bar{k})}(S_3)\right).$$

Now, considering S_i for $i = 1, 2, 3$ as polynomials in $\mathbb{C}[\bar{t}_h][d, \bar{k}]$ one has:

$$\mathrm{Con}_{(d,\bar{k})}(S_1) = \gcd(HZ^2, HZY, HY^2, W^2N_2^2, W^2N_2N_3, W^2N_3^2).$$

That is,

$$\mathrm{Con}_{(d,\bar{k})}(S_1) = \gcd(H\gcd(Y, Z)^2, W^2\gcd(N_2, N_3)).$$

Similarly,

$$\mathrm{Con}_{(d,\bar{k})}(S_2) = \gcd(H\gcd(X,Z)^2, W^2\gcd(N_1,N_3)).$$

and

$$\mathrm{Con}_{(d,\bar{k})}(S_3) = \gcd(H\gcd(X,Y)^2, W^2\gcd(N_1,N_2))$$

Taking into account that $\gcd(N_1,N_2,N_3)=1$ and $\gcd(X,Y,Z,W)=1$, one has

$$\mathrm{Con}_{(d,\bar{k})}(S) = \gcd(H\gcd(X,Y,Z)^2, W^2) = \gcd(H,W^2).$$

□

In order to use the above results, and to state the degree formula, we need to introduce some additional notation. We denote by:

$$Q_0(\bar{t}_h) = \mathrm{Con}_{\bar{k}}(S_0(\bar{k},\bar{t}_h)) \quad \text{and} \quad Q(\bar{t}_h) = \mathrm{Con}_{(d,\bar{k})}(S(\bar{c},d,\bar{k},\bar{t}_h)). \tag{4.18}$$

Observe that, by Lemma 4.27, Q does not depend on $\bar{c}$, a fact that is reflected in our notation. Furthermore, note that:

$$Q_0(\bar{t}_h) = \gcd(YN_3 - ZN_2, XN_3 - ZN_1, XN_2 - YN_1)$$

and

$$Q(\bar{t}_h) = \gcd(H, W^2).$$

We also denote by:

$$\tilde{H}(\bar{t}_h) = \frac{H(\bar{t}_h)}{Q(\bar{t}_h)}, \qquad \tilde{W}(\bar{t}_h) = \frac{W^2(\bar{t}_h)}{Q(\bar{t}_h)}, \tag{4.19}$$

and

$$\begin{cases} U_1(\bar{t}_h) = \dfrac{(YN_3 - ZN_2)(\bar{t}_h)}{Q_0(\bar{t}_h)}, \\ U_2(\bar{t}_h) = \dfrac{(ZN_1 - XN_3)(\bar{t}_h)}{Q_0(\bar{t}_h)}, \\ U_3(\bar{t}_h) = \dfrac{(XN_2 - YN_1)(\bar{t}_h)}{Q_0(\bar{t}_h)}. \end{cases} \tag{4.20}$$

Thus, one has:

$$S_0(\bar{k},\bar{t}_h) = Q_0(\bar{t}_h)(k_1U_1(\bar{t}_h) + k_2U_2(\bar{t}_h) + k_3U_3(\bar{t}_h)).$$

We denote as well:

$$\begin{cases} M_{h,1}(\bar{k},\bar{t}_h) = k_2Z(\bar{t}_h) - k_3Y(\bar{t}_h), \quad G_{h,1}(\bar{k},\bar{t}_h) = k_2N_3(\bar{t}_h) - k_3N_2(\bar{t}_h), \\ M_{h,2}(\bar{k},\bar{t}_h) = k_3X(\bar{t}_h) - k_1Z(\bar{t}_h), \quad G_{h,2}(\bar{k},\bar{t}_h) = k_3N_1(\bar{t}_h) - k_1N_3(\bar{t}_h), \\ M_{h,3}(\bar{k},\bar{t}_h) = k_1Y(\bar{t}_h) - k_2X(\bar{t}_h), \quad G_{h,3}(\bar{k},\bar{t}_h) = k_1N_2(\bar{t}_h) - k_2N_1(\bar{t}_h). \end{cases} \tag{4.21}$$

and so, for $i = 1, 2, 3$,

$$S_i(d, \bar{k}, \bar{t}_h) = Q(\bar{t}_h)\left(\tilde{H}(\bar{t}_h)M^2_{h,i}(\bar{k}, \bar{t}_h) - d^2\tilde{W}(\bar{t}_h)G^2_{h,i}(\bar{k}, \bar{t}_h)\right)$$

We denote:

$$T_0(\bar{k}, \bar{t}_h) = \frac{S_0(\bar{k}, \bar{t}_h)}{Q_0(\bar{t}_h)} \tag{4.22}$$

and, for $i = 1, 2, 3$,

$$T_i(d, \bar{k}, \bar{t}_h) = \frac{S_i(d, \bar{k}, \bar{t}_h)}{Q(\bar{t}_h)}, \tag{4.23}$$

Finally, we denote:

$$T(\bar{c}, d, \bar{k}, \bar{t}_h) = \frac{S(\bar{c}, d, \bar{k}, \bar{t}_h)}{Q(\bar{t}_h)}. \tag{4.24}$$

Note that:

$$T(\bar{c}, d, \bar{k}, \bar{t}_h) = c_1T_1(d, \bar{k}, \bar{t}_h) + c_2T_2(d, \bar{k}, \bar{t}_h) + c_3T_3(d, \bar{k}, \bar{t}_h).$$

With this notation we can introduce the system of equations that will play the central role in the degree formula:

$$\mathfrak{S}_5^{P_h}(d, \bar{k}) \equiv \begin{cases} T_0(\bar{k}, \bar{t}_h) = k_1U_1(\bar{t}_h) + k_2U_2(\bar{t}_h) + k_3U_3(\bar{t}_h) = 0 \\ T_i(d, \bar{k}, \bar{t}_h) = \tilde{H}(\bar{t}_h)M^2_{h,i}(\bar{k}, \bar{t}_h) - d^2\tilde{W}(\bar{t}_h)G^2_{h,i}(\bar{k}, \bar{t}_h), \\ \text{for } i = 1, 2, 3. \end{cases} \tag{4.25}$$

We will refer to this as the Projective Auxiliary System.

4.3.2 Invariant solutions of the projective auxiliary system

In passing from $\mathfrak{S}_3^{P}(d, \bar{k})$ to $\mathfrak{S}_4^{P_h}(d, \bar{k})$, and then to $\mathfrak{S}_5^{P_h}(d, \bar{k})$, we have introduced additional solutions at infinity, in the space of parameters (that is, with $t_0 = 0$). The following results will show that, in a certain open subset of values of $(d, \bar{k})$, these solutions at infinity are invariant w.r.t. $(d, \bar{k})$. We start with some technical lemmas.

Lemma 4.28. *There is always $i^o \in \{1, 2, 3\}$ such that $U_{i^o}(0, t_1, t_2)$ and $T_{i^o}(d, \bar{k}, 0, t_1, t_2)$ are both not identically zero.*

Proof. First, let us prove that there are always $i, j \in \{1, 2, 3\}$ such that t_0 does not divide T_i and T_j. Suppose, on the contrary that, for example $T_1(d, \bar{k}, 0, t_1, t_2) \equiv 0$ and

$T_2(d, \bar{k}, 0, t_1, t_2) \equiv 0$. Considering T_1 and T_2 as polynomials in $\mathbb{C}[\bar{t}][d, \bar{k}]$, if t_0 divides T_1 and T_2 one concludes that t_0 must divide

$$\tilde{H}X, \tilde{H}Y, \tilde{H}Z, \tilde{W}N_1, \tilde{W}N_2 \text{ and } \tilde{W}N_3.$$

If one assumes that t_0 divides $\tilde{W}$, then it does not divide $\tilde{H}$, because $\gcd(\tilde{H}, \tilde{W}) = 1$. Thus it divides X, Y and Z. But this is again a contradiction, since $\gcd(X, Y, Z, W) = 1$, and $\tilde{W}$ divides W. Thus, t_0 does not divide $\tilde{W}$. Then it must divide N_1, N_2, N_3. This is also a contradiction, since $\gcd(N_1, N_2, N_3) = 1$. Therefore we can assume w.l.o.g. that e.g. t_0 does not divide T_1 and T_2. To finish the proof in this case we need to show that, if t_0 divides T_3, then it does not divide at least one of U_1 and U_2. The hypothesis that t_0 divides T_3 implies that it divides

$$\tilde{H}X, \tilde{H}Y, \tilde{W}N_1 \text{ and } \tilde{W}N_2.$$

If t_0 divides $\tilde{W}$, again, it must divide X and Y. Thus it does not divide Z. Now, observe that $XU_1 + YU_2 + ZU_3 = 0$. Therefore, one concludes that t_0 divides U_3. Thus, t_0 does not divide at least one of U_1 and U_2, since $\gcd(U_1, U_2, U_3) = 1$. If t_0 does not divide $\tilde{W}$, then it divides N_1 and N_2. Observing that $N_1U_1 + N_2U_2 + N_3U_3 = 0$, we again conclude that t_0 does not divide at least one of U_1 and U_2, since $\gcd(U_1, U_2, U_3) = 1$. □

Lemma 4.29. *Let $i^o \in \{1, 2, 3\}$ be such that $T_{i^o}(d, \bar{k}, 0, t_1, t_2)$ and $U_{i^o}(0, t_1, t_2)$ are both not identically zero (see Lemma 4.28). Then*

$$\gcd(T_0(\bar{k}, 0, t_1, t_2), T_{i^o}(d, \bar{k}, 0, t_1, t_2))$$

does not depend on $\bar{k}$ (it certainly does not depend on d).

Proof. The claim follows observing that $T_0(\bar{k}, 0, t_1, t_2)$ depends linearly on k_{i^o}, and $T_{i^o}(d, \bar{k}, 0, t_1, t_2)$ does not. □

In order to describe what we mean when we say that a solution is invariant w.r.t. $(d, \bar{k})$, we make the following definition (recall Definition 4.20, page 143):

Definition 4.30. *Let Ω_1 be as in Lemma 4.22 (page 143), and let Ω be a non-empty open subset of Ω_1.* The set of invariant solutions of $\mathfrak{S}_5^{P_h}(d, \bar{k})$ w.r.t Ω. *is defined as the set:*

$$\mathcal{I}_5^{P_h}(\Omega) = \bigcap_{(d^o, \bar{k}^o) \in \Omega} \Psi_5^{P_h}(d^o, \bar{k}^o)$$

Remark 4.31.

1. *Considering T_i (for $i = 0, \ldots, 3$) as polynomials in $\mathbb{C}[\bar{t}_h][d, \bar{k}]$, it is easy to see that $\mathcal{I}_5^{P_h}(\Omega)$ is the set of solutions of:*

$$\begin{cases} U_1(\bar{t}_h) = U_2(\bar{t}_h) = U_3(\bar{t}_h) = 0, \\ (\tilde{H} \cdot X)(\bar{t}_h) = (\tilde{H} \cdot Y)(\bar{t}_h) = (\tilde{H} \cdot Z)(\bar{t}_h) = 0, \\ (\tilde{W} \cdot N_1)(\bar{t}_h) = (\tilde{W} \cdot N_2)(\bar{t}_h) = (\tilde{W} \cdot N_3)(\bar{t}_h) = 0. \end{cases} \tag{4.26}$$

2. *In particular, since* $\gcd(U_1, U_2, U_3) = 1$, *the set $\mathcal{I}_5^{P_h}(\Omega)$ is always a finite set.*

The following proposition shows that, restricting the values of $(d, \bar{k})$ to a certain open set, we can ensure that all the solutions at infinity of $\mathfrak{S}_5^{P_h}(d, \bar{k})$ are invariant w.r.t. the particular choice of $(d, \bar{k})$ in that open set.

Proposition 4.32. *There exists an open non-empty subset $\Omega_2 \subset \Omega_1$ (with Ω_1 as in Lemma 4.22, page 143), such that if $(d^o, \bar{k}^o) \in \Omega_2$, and $\bar{t}_h^o = (0 : t_1^o : t_2^o) \in \Psi_5^{P_h}(d^o, \bar{k}^o)$, then $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega_2)$.*

Proof. We know that $T_0(\bar{k}, 0, \bar{t}) \not\equiv 0$. Suppose, in the first place, that $T_0(\bar{k}, 0, \bar{t})$ depends only in $\bar{k}$ and one of the variables t_1, t_2. Say, e.g., $T_0(\bar{k}, 0, \bar{t}) = T_0^*(\bar{k}) t_1^p$ for some $p \in \mathbb{N}$. This implies that, for any given $(d^o, \bar{k}^o)$ such that $T_0^*(\bar{k}^o) \neq 0$, $(0 : 0 : 1)$ is the only possible point of $\Psi_5^{P_h}(d^o, \bar{k}^o)$ with $t_0 = 0$. Obviously, if

$$T_i(d, \bar{k}, 0, 0, 1) \equiv 0 \text{ for } i = 1, 2, 3,$$

then one may take $\Omega_2 = \Omega_1 \cap \{(d^o, \bar{k}^o) \in \mathbb{C}^4 / T_0^*(\bar{k}^o) \neq 0\}$, and the result is proved. On the other hand, if not all $T_i(d, \bar{k}, 0, 0, 1) \equiv 0$, say w.l.o.g that

$$T_1(d, \bar{k}, 0, 0, 1) \not\equiv 0$$

then we may take $\Omega_2 = \Omega_1 \cap \{(d^o, \bar{k}^o) / T_0^*(\bar{k}^o) T_1(d^o, \bar{k}^o, 0, 0, 1) \neq 0\}$, and the result is proved.

Thus, w.l.o.g. we can assume that $T_0(\bar{k}, 0, \bar{t})$ depends on both t_1 and t_2. Let $i^o \in \{1, 2, 3\}$ be such that $U_i(0, \bar{t}) \not\equiv 0$ and $T_i(d, \bar{k}, 0, \bar{t}) \not\equiv 0$ (see Lemma 4.29, 151). By Lemma 4.28 (page 150) we know that this is the case at least for one value of i^o. Let us consider (see Lemma 4.29):

$$T_{i^o}^*(\bar{t}) = \gcd(T_0(\bar{k}, 0, \bar{t}), T_{i^o}(d, \bar{k}, 0, \bar{t})) \in \mathbb{C}[\bar{t}].$$

Note that $T_{i^o}^*$ is homogeneous in $\bar{t}$, and so, if it is not constant, it factors as:

$$T_{i^o}^*(\bar{t}) = \gamma \prod_{j=1}^{p} (\beta_j t_1 - \alpha_j t_2)$$

for some $\gamma \in \mathbb{C}^{\times}$, and $(\alpha_j, \beta_j) \in \mathbb{C}^2, j = 1, \ldots, p$. For each point $(0 : \alpha_j : \beta_j)$ we can repeat the construction that we did for $(0 : 0 : 1)$. Thus, one obtains a non-empty open set $\Omega_2^1 \subset \Omega_1$ such that, if $(d^o, \bar{k}^o) \in \Omega_2^1$, and $T^*_{i^o}(\alpha_j, \beta_j) = 0$, then either $(0 : \alpha_j : \beta_j) \notin \Psi_5^{P_h}(d^o, \bar{k}^o)$, or $(0 : \alpha_j : \beta_j) \in \mathcal{I}_5^{P_h}(\Omega_2^1)$.

Let
$$T'_0(\bar{k}, \bar{t}) = \frac{T_0(\bar{k}, 0, \bar{t})}{T^*_{i^o}(\bar{t})}, \quad \text{and} \quad T'_{i^o}(d, \bar{k}, \bar{t})) = \frac{T_{i^o}(d, \bar{k}, 0, \bar{t})}{T^*_{i^o}(\bar{t})}.$$
Note that both $T'_0(\bar{k}, \bar{t})$ and $T'_{i^o}(d, \bar{k}, \bar{t})$ are homogeneous in $\bar{t}$, and by construction they have a trivial gcd. If we define:

$$\Gamma(d, \bar{k}, t_2) = \begin{cases} \mathrm{Res}_{t_1}(T'_0(\bar{k}, \bar{t}), T'_{i^o}(d, \bar{k}, \bar{t})) & \text{if } \deg_{t_1}(T'_0(\bar{k}, \bar{t})) > 0, \\ \\ T'_0(\bar{k}, \bar{t}) & \text{in other case.} \end{cases}$$

Then Γ is not identically zero, and since T'_0 and T'_{i^o} are both homogeneous in $\bar{t}$, we have a factorization:
$$\Gamma(d, \bar{k}, t_2) = t_2^q \Gamma^*(d, \bar{k})$$
for some $q \in \mathbb{N}$. Note also that, by construction, since $\gcd(T'_0, T'_{i^o}) = 1$, t_2 cannot divide both T'_0 and T'_{i^o}. In particular, since these polynomials are homogeneous in $\bar{t}$, one concludes that $\bar{t}^o = (1, 0)$ is not a solution of

$$T'_0(\bar{k}, \bar{t}) = T'_{i^o}(d, \bar{k}, \bar{t}) = 0.$$

We define
$$\Omega_2 = \Omega_2^1 \cap \{(d^o, \bar{k}^o)/\Gamma^*(d^o, \bar{k}^o) \neq 0\}$$
If $(d^o, \bar{k}^o) \in \Omega_2$, and $\bar{t}^o_h = (0 : t^o_1 : t^o_2) \in \Psi_5^{P_h}(d^o, \bar{k}^o)$, then either $T'_0(\bar{k}^o, \bar{t}^o) \neq 0$ or $T'_{i^o}(d^o, \bar{k}^o, \bar{t}^o) \neq 0$. In any case, one has $T^*_{i^o}(\bar{t}^o) = 0$ (that is, $(0 : t^o_1 : t^o_2) = (0 : \alpha_j : \beta_j)$ for some $j = 1, \ldots, p$). The construction of Ω_2^1 implies that $(0 : t^o_1 : t^o_2) \in \mathcal{I}_5^{P_h}(\Omega_2)$. □

Let $\mathcal{A}_h$ denote the set of values $\bar{t}^o_h \in \mathbb{P}^2$ such that (compare to the Definition of the set $\mathcal{A}$ in Equation 4.11, page 138):

$$\begin{cases} t^o_0\, W(\bar{t}^o_h)H(\bar{t}^o_h)\big(Y(\bar{t}^o_h)N_3(\bar{t}^o_h) - Z(\bar{t}^o_h)N_2(\bar{t}^o_h)\big) \neq 0 \\ \text{or} \\ t^o_0\, W(\bar{t}^o_h)H(\bar{t}^o_h)\big(Z(\bar{t}^o_h)N_1(\bar{t}^o_h) - X(\bar{t}^o_h)N_3(\bar{t}^o_h)\big) \neq 0 \\ \text{or} \\ t^o_0\, W(\bar{t}^o_h)H(\bar{t}^o_h)\big(X(\bar{t}^o_h)N_1(\bar{t}^o_h) - Y(\bar{t}^o_h)N_1(\bar{t}^o_h)\big) \neq 0 \end{cases} \tag{4.27}$$

Equivalently, $\mathcal{A}_h$ consists in those points $\bar{t}^o_h \in \mathbb{P}^2$ such that:

$$t^o_0 W(\bar{t}^o_h)H(\bar{t}^o_h) \neq 0 \text{ and } (X(\bar{t}^o_h), Y(\bar{t}^o_h), Z(\bar{t}^o_h)) \wedge \bar{N}(\bar{t}^o_h) \neq \bar{0}. \tag{4.28}$$

We will see that the non-invariant solutions of $\Psi_5^{P_h}(d,\bar{k})$ are points in $\mathcal{A}_h$. Note that we are explicitly asking these points to be affine (recall Proposition 4.32, page 152)

With this notation we are ready to state the main theorem about System $\mathfrak{S}_5^{P_h}(d,\bar{k})$.

Theorem 4.33. *Let Ω_2 be as in Proposition 4.32. If $(d^o,\bar{k}^o)\in\Omega_2$,*

$$\bar{t}^o=(t_1^o,t_2^o)\in\mathcal{A}\cap\Psi_3^P(d^o,\bar{k}^o)\Leftrightarrow \bar{t}_h^o=(1:t_1^o:t_2^o)\in\mathcal{A}_h\cap\Psi_5^{P_h}(d^o,\bar{k}^o)$$

Recall that every point in $\mathcal{A}_h$ is affine, see Equation 4.11 (page 138), for the Definition of $\mathcal{A}$, and page 138 for the definition of $\Psi_3^P(d^o,\bar{k}^o)$.

Proof. Let us prove that $\Rightarrow$ holds. If $\bar{t}^o\in\mathcal{A}\cap\Psi_3^P(d^o,\bar{k}^o)$, then, e.g.,

$$P_0(\bar{t}^o)h(\bar{t}^o)(P_2n_3-P_3n_2)(\bar{t}^o)\neq 0.$$

Therefore,

$$W(\bar{t}_h^o)H(\bar{t}_h^o)(YN_3-ZN_2)(\bar{t}_h^o)\neq 0,$$

and so $\bar{t}_h^o\in\mathcal{A}_h$. Besides, this last inequality implies that $Q_0(\bar{t}_h^o)Q(\bar{t}_h^o)\neq 0$. Since $\bar{t}^o\in\mathcal{A}\cap\Psi_3^P(d^o,\bar{k}^o)$, by Proposition 4.16(b) (page 139), one has that $(1:t_1^o:t_2^o)\in\Psi_4^{P_h}(d^o,\bar{k}^o)$. From Equations 4.22 and 4.23 (page 150), and from $Q_0(\bar{t}_h^o)Q(\bar{t}_h^o)\neq 0$, one concludes that $\bar{t}_h^o=(1:t_1^o:t_2^o)\in\mathcal{A}_h\cap\Psi_5^{P_h}(d^o,\bar{k}^o)$. Thus, $\Rightarrow$ is proved.
The proof of $\Leftarrow$ is similar, simply reversing the implications. $\square$

Remark 4.34. *Let Ω_2 be as in Proposition 4.32, and let $(d^o,\bar{k}^o)\in\Omega_2$. Then, Theorem 4.33 implies that:*

$$m\delta=\#\left(\mathcal{A}\cap\Psi_3^P(d^o,\bar{k}^o)\right)=\#\left(\mathcal{A}_h\cap\Psi_5^{P_h}(d^o,\bar{k}^o)\right).$$

Theorem 4.33 establishes the link between $\mathcal{A}\cap\Psi_3^P(d^o,\bar{k}^o)$ and $\mathcal{A}_h\cap\Psi_5^{P_h}(d^o,\bar{k}^o)$ for a fixed $(d^o,\bar{k}^o)\in\Omega_2$. As we said before, the non-invariant solutions of $\Psi_5^{P_h}(d,\bar{k})$ should be the points in $\mathcal{A}_h$. As a first step, we have this result.

Proposition 4.35. *Let Ω_2 be as in Proposition 4.32 (page 152). If $\bar{t}_h^o\in\mathcal{A}_h$, then $\bar{t}_h^o\notin\mathcal{I}_5^{P_h}(\Omega_2)$.*

Proof. Let us suppose that for every $(d^o,\bar{k}^o)\in\Omega_2$, one has $\bar{t}_h^o\in\left(\mathcal{A}_h\cap\Psi_5^{P_h}(d^o,\bar{k}^o)\right)$. Then (recall Equation 4.11, page 138), $\bar{t}_h^o$ is of the form $(1:t_1^o:t_2^o)$ and, by Theorem 4.33,

$$\bar{t}^o=(t_1^o,t_2^o)\in\bigcap_{(d^o,\bar{k}^o)\in\Omega_2}\Psi_3^P(d^o,\bar{k}^o)=\mathcal{I}_3^P(\Omega_2).$$

Since $\Omega_2 \subset \Omega_1$, Proposition 4.23 (page 145) applies, and one concludes that $(t_1^o, t_2^o) \in \mathcal{F}$. But then,

$$P_0(\bar{t}^o)h(\bar{t}^o) = 0 \text{ or } (P_1(\bar{t}^o), P_2(\bar{t}^o), P_3(\bar{t}^o)) \wedge \bar{n}(\bar{t}^o) = \bar{0}$$

This implies that:

$$W(\bar{t}_h^o)H(\bar{t}_h^o) = 0 \text{ or } (X(\bar{t}_h^o), Y(\bar{t}_h^o), Z(\bar{t}_h^o)) \wedge \bar{N}(\bar{t}_h^o) = \bar{0},$$

contradicting $\bar{t}_h^o \in \mathcal{A}_h$. □

As we said before, we will prove the converse of this proposition. That is, if $\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)$ for some $(d^o, \bar{k}^o) \in \Omega_2$, but $\bar{t}_h^o \notin \mathcal{I}_5^{P_h}(\Omega_2)$, then we would like to conclude that $\bar{t}_h^o \in \mathcal{A}_h$. However, the open set Ω_2 can be too large for this to hold. More precisely, the problem is caused by the solutions of $Q(\bar{t}_h)Q_0(\bar{t}_h) = 0$ (when Q and Q_0 are not equal 1). These points do not belong to $\mathcal{A}_h$. However, since $\mathcal{I}_5^{P_h}(\Omega_2)$ is finite (see Remark 4.31, page 152), most of the solutions $Q(\bar{t}_h)Q_0(\bar{t}_h) = 0$ are not invariant. Therefore, we need to impose some more restrictions in the values of $(d, \bar{k})$. Note, however, that we have already dealt with the points at infinity; thus, we need only consider the affine solutions of $Q(\bar{t}_h)Q_0(\bar{t}_h) = 0$. We do the necessary technical work in the following lemma. First, we introduce some notation for the affine versions of some polynomials. We denote:

$$\begin{cases} q(\bar{t}) = Q(1, t_1, t_2), q_0(\bar{t}) = Q_0(1, t_1, t_2), \tilde{w}(\bar{t}) = \tilde{W}(1, t_1, t_2), \tilde{h}(\bar{t}) = \tilde{H}(1, t_1, t_2), \\ u_i(\bar{t}) = U_i(1, t_1, t_2) \text{ for } i = 1, 2, 3. \end{cases}$$

and we consider a new auxiliary set of variables $\bar{\rho} = (\rho_1, \rho_2, \rho_3)$, in order to do the necessary Rabinowitsch's tricks.

Let $\mathcal{G} \subset \mathbb{C} \times \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^2$ be the set of solutions (in the variables $(d, \bar{k}, \bar{\rho}, \bar{t})$) of this system of equations:

$$\begin{cases} q(\bar{t})q_0(\bar{t}) = 0 \\ \tilde{h}(\bar{t})M_i^2(\bar{k}, \bar{t}) - d^2\tilde{w}(\bar{t})G_i^2(\bar{k}, \bar{t}) = 0 \quad \text{for } i = 1, 2, 3. \\ k_1u_1(\bar{t}) + k_2u_2(\bar{t}) + k_3u_3(\bar{t}) = 0 \\ \rho_1\tilde{w}(\bar{t})\tilde{h}(\bar{t}) - 1 = 0 \\ \prod_{i=1}^3(\rho_2u_i(\bar{t}) - 1) = 0 \\ \prod_{i=1}^3(\rho_3n_i(\bar{t}) - 1) = 0 \end{cases} \tag{4.29}$$

and consider the projection $\pi_1(d, \bar{k}, \lambda, \bar{\rho}, \bar{t}) = (d, \bar{k})$.

Lemma 4.36. *$\mathcal{G}$ is empty or* $\dim(\pi_1(\mathcal{G})) \leq 3$.

Proof. Let $\mathcal{G} \neq \emptyset$. Then $q(\bar{t})q_0(\bar{t})$ is not constant. We will use Lemma 1.5 (page 12), to prove that $\dim(\mathcal{G}) \leq 3$. From this the result follows immediately. Consider the projection $\pi_2(d, \bar{k}, \bar{\rho}, \bar{t}) = \bar{t}$. Clearly, $\pi_2(\mathcal{G})$ is contained in the affine curve defined by $q(\bar{t})q_0(\bar{t}) = 0$. Thus, $\dim(\pi_2(\mathcal{G})) \leq 1$. Let $\bar{t}^o \in \pi_2(\mathcal{G})$. First, let us suppose that for all $(d^o, \bar{k}^o, \bar{\rho}^o, \bar{t}^o) \in \pi_2^{-1}(\bar{t}^o)$, one has $\bar{k}^o = \bar{0}$. Then, if $(d^o, \bar{0}, \bar{\rho}^o, \bar{t}^o) \in \pi_2^{-1}(\bar{t}^o)$, ρ_1^o, ρ_2^o, and ρ_3^o must be one of the finitely many solutions of the polynomial equations:

$$\rho_1\tilde{w}(\bar{t}^o)\tilde{h}(\bar{t}^o) - 1 = 0, \quad \prod_{i=1}^{3}(\rho_2 u_i(\bar{t}^o) - 1) = 0, \text{ and } \prod_{i=1}^{3}(\rho_3 n_i(\bar{t}^o) - 1) = 0.$$

The condition $\bar{t}^o \in \pi_2(\mathcal{G})$ implies that these equations can be solved. Note that, in this case, $(d^o, \bar{0}, \bar{\rho}^o, \bar{t}^o) \in \pi_2^{-1}(\bar{t}^o)$ does not impose any condition on d^o. It follows that, in this case, one has $\mu = \dim(\pi_2^{-1}(\bar{t}^o)) = 1$.

Now, let us suppose that $(d^o, \bar{k}^o, \bar{\rho}^o, \bar{t}^o) \in \pi_2^{-1}(\bar{t}^o)$, with $\bar{k}^o \neq \bar{0}$. Then, by a similar argument to the proof of Proposition 4.16(a) (page 139), and taking $\tilde{w}(\bar{t}^o)\tilde{h}(\bar{t}^o) \neq 0$ into account, there exists $\lambda^o \in \mathbb{C}^\times$ such that

$$M_i(\bar{k}^o, \bar{t}^o) = \lambda^o G_i(\bar{k}^o, \bar{t}^o) \text{ for } i = 1, 2, 3.$$

Thus, in this case d^o must be a solution of:

$$\tilde{h}(\bar{t}^o)(\lambda^o)^2 - (d^o)^2\tilde{w}(\bar{t}^o) = 0.$$

Besides, there exists also $j^o \in \{1, 2, 3\}$ with $u_{j^o}(\bar{t}^o) \neq 0$. Then, $\bar{k}^o$ must belong to the two-dimensional space defined by

$$k_1u_1(\bar{t}^o) + k_2u_2(\bar{t}^o) + k_3u_3(\bar{t}^o) = 0.$$

Finally, ρ_1^o, ρ_2^o, and ρ_3^o must be one of the finitely many solutions of the polynomial equations:

$$\rho_1\tilde{w}(\bar{t}^o)\tilde{h}(\bar{t}^o) - 1 = 0, \quad \prod_{i=1}^{3}(\rho_2 u_i(\bar{t}^o) - 1) = 0, \text{ and } \prod_{i=1}^{3}(\rho_3 n_i(\bar{t}^o) - 1) = 0.$$

The condition $\bar{t}^o \in \pi_2(\mathcal{G})$ implies that these equations can be solved. These remarks show that for every $\bar{t}^o \in \pi_2(\mathcal{G})$, one has $\mu = \dim(\pi_2^{-1}(\bar{t}^o)) \leq 2$. Thus, using Lemma 1.5:

$$\dim(\mathcal{G}) = \dim(\pi_2(\mathcal{G})) + \mu \leq 1 + 2 = 3,$$

and the lemma is proved. □

If $\bar{t}_h^o$ is such that

$$T_0(\bar{k}, \bar{t}_h^o) \equiv 0 \text{ and } T_i(d, \bar{k}, \bar{t}_h^o) \equiv 0, \text{ for } i = 1, 2, 3,$$

then $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega)$ for any choice of Ω. However, if this is not the case, then sometimes we need to remove from Ω precisely those values $(d^o, \bar{k}^o)$ such that $\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)$. In the proof of the following lemma we will need to do this several times. Thus we introduce the necessary notation.

Definition 4.37. *Let $\Omega \subset \mathbb{C} \times \mathbb{C}^3$ be non-empty and open. For $\bar{t}_h^o \in \mathbb{P}^2$ we define:*

$$\Omega^{inv}(\bar{t}_h^o) = \begin{cases} \Omega, & \text{if } T_0(\bar{k}, \bar{t}_h^o) \equiv 0 \text{ and } T_i(d, \bar{k}, \bar{t}_h^o) \equiv 0, \text{ for } i = 1, 2, 3. \\ \Omega \setminus \left\{(d^o, \bar{k}^o)/\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)\right\}, \text{in other case.} \end{cases}$$

Remark 4.38.

(1) *Note that if $\Omega \subset \Omega^{inv}(\bar{t}_h^o)$, then $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega)$.*

(2) *Observe that $\Omega^{inv}(\bar{t}_h^o) \neq \emptyset$.*

Lemma 4.39. *Let Ω_2 be as in Proposition 4.32. There exists an open non-empty set $\Omega_3 \subset \Omega_2$ such that the following hold:*

(a) *If $\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)$ for some $(d^o, \bar{k}^o) \in \Omega_3$, and $Q_0(\bar{t}_h^o)Q(\bar{t}_h^o) = 0$, then $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega_3)$.*

(b) *If $\bar{t}_h^o$ satisfies*

$$T_1(d, k, \bar{t}_h^o) = T_2(d, k, \bar{t}_h^o) = T_3(d, k, \bar{t}_h^o) = 0 \text{ identically in } (d, \bar{k}),$$

then $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega_3)$.

Proof. Let

$$A_0 = \{\bar{t}_h^o \mid X(\bar{t}_h^o) = Y(\bar{t}_h^o) = Z(\bar{t}_h^o) = W(\bar{t}_h^o) = 0\}.$$

Since $\gcd(X, Y, Z, W) = 1$, one sees that A_0 is (empty or) a finite set. Thus, if we define (see Definition 4.37):

$$\Omega_3^0 = \Omega_2 \cap \left(\bigcap_{\bar{t}_h^o \in A_0} \Omega^{inv}(\bar{t}_h^o) \right).$$

By Remark 4.38, Ω_3^0 is an open non-empty set. Let

$$A_1 = \{\bar{t}_h^o \mid N(\bar{t}_h^o) = \bar{0}\},$$

where $N = (N_1, N_2, N_3)$. Recalling that $\gcd(N_1, N_2, N_3) = 1$, A_1 is (empty or) a finite set. We define:

$$\Omega_3^1 = \Omega_3^0 \cap \left(\bigcap_{\bar{t}_h^o \in A_1} \Omega^{inv}(\bar{t}_h^o) \right).$$

By Remark 4.38, Ω_3^1 is an open non-empty set.
Similarly, since $\gcd(\tilde{H}, \tilde{W}) = 1$, the set

$$A_2 = \{\bar{t}_h^o \mid \tilde{H}(\bar{t}_h^o) = \tilde{W}(\bar{t}_h^o) = 0\}$$

is (empty or) finite. We define

$$\Omega_3^2 = \Omega_3^1 \cap \left(\bigcap_{\bar{t}_h^o \in A_2} \Omega^{inv}(\bar{t}_h^o) \right),$$

and Ω_3^2 is an open non-empty set. Moreover, since $\gcd(U_1, U_2, U_3) = 1$, the set

$$A_3 = \{\bar{t}_h^o \,|\, U_1(\bar{t}_h^o) = U_2(\bar{t}_h^o) = U_3(\bar{t}_h^o) = 0\}$$

is (empty or) finite. We define

$$\Omega_3^3 = \Omega_3^2 \cap \left(\bigcap_{\bar{t}_h^o \in A_3} \Omega^{inv}(\bar{t}_h^o) \right),$$

and Ω_3^3 is an open non-empty set. We define

$$\Omega_3^4 = \Omega_3^3 \setminus (\pi_1(\mathcal{G})^*),$$

where $\pi_1(\mathcal{G})$ is as in Lemma 4.36 (page 155), and, as usual, the asterisk denotes Zariski closure.

Finally, since $T(\bar{c}, d, \bar{k}, \bar{t})$ is primitive w.r.t. $(d, \bar{k})$ (recall Equations 4.18, page 149, and 4.24, page 150), it follows that the set

$$A_4 = \left\{\bar{t}_h^o \,|\, T_1(d, k, \bar{t}_h^o) = T_2(d, k, \bar{t}_h^o) = T_3(d, k, \bar{t}_h^o) = 0 \text{ identically in } (d, \bar{k})\right\}$$

is (empty or) finite. We define

$$\Omega_3 = \Omega_3^4 \cap \left(\bigcap_{\bar{t}_h^o \in A_4} \Omega^{inv}(\bar{t}_h^o) \right).$$

Let us now suppose that $(d^o, \bar{k}^o) \in \Omega_3$ and $\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)$, with $Q_0(\bar{t}_h^o)Q(\bar{t}_h^o) = 0$. We will show that $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega_3)$. This will prove that statement (a) holds. If $\bar{t}_h^o$ is of the form $(0 : t_1^o : t_2^o)$, by Proposition 4.32, $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega_3)$ holds trivially. Thus, in the rest of the proof we can assume w.l.o.g. that $\bar{t}_h^o$ is of the form $(1 : t_1^o : t_2^o)$.

If $\bar{t}_h^o \in \cup_{i=0,\ldots,3} A_i$, then we have $\Omega_3 \subset \Omega_3^4 \subset \Omega^{inv}(\bar{t}_h^o)$, and by Remark 4.38, $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega_3)$. So, let $\bar{t}_h^o \notin \cup_{i=0,\ldots,3} A_i$. Then the following hold:

(0) Since $\bar{t}_h^o \notin A_0$, $P_i(\bar{t}^o) \neq 0$ for some $i = 0, \ldots, 3$ (recall that $\bar{t}_h^o = (1 : \bar{t}^o)$).

(1) Since $\bar{t}_h^o \notin A_1$, $N(\bar{t}_h^o) \neq \bar{0}$.

(2) Since $\bar{t}_h^o \notin A_2$, $\tilde{H}(\bar{t}_h^o) \neq 0$ or $\tilde{W}(\bar{t}_h^o) \neq 0$.

(3) Since $\bar{t}_h^o \notin A_3$, $U_i(\bar{t}_h^o) \neq 0$ for some $i = 1, 2, 3$.

Let us show that (0) and (2) imply the following:

(4) $\tilde{H}(\bar{t}_h^o)\tilde{W}(\bar{t}_h^o) \neq 0$.

Indeed, if we suppose that $\tilde{H}(\bar{t}_h^o) = 0$ but $\tilde{W}(\bar{t}_h^o) \neq 0$, then from $\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)$ one concludes that $d^o G_i(\bar{k}^o, \bar{t}_h^o) = 0$ for $i = 1, 2, 3$. Since $d^o \neq 0$ and $\bar{k}^o \neq \bar{0}$ in Ω_3, one has that $N(\bar{t}_h^o)$ is isotropic and parallel to $\bar{k}^o$, contradicting Lemma 4.22(1) (page 143). On the other hand, if we suppose $\tilde{H}(\bar{t}_h^o) \neq 0$ but $\tilde{W}(\bar{t}_h^o) = 0$, then from $\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)$ one concludes that $d^o G_i(\bar{k}^o, \bar{t}_h^o) = 0$ for $i = 1, 2, 3$. Since $d^o \neq 0$, we conclude that $G_i(\bar{k}^o, \bar{t}_h^o) = 0$ for $i = 1, 2, 3$. Thus, $\bar{t}^o$ is a solution of:

$$P_0(\bar{t}) = M_1(\bar{k}^o, 1, \bar{t}) = M_2(\bar{k}^o, 1, \bar{t}) = M_3(\bar{k}^o, 1, \bar{t}) = 0.$$

However, by (0), there exists $j^o \in \{0, 1, 2, 3\}$ such that $P_j(\bar{t}^o) \neq 0$. Therefore, we get a contradiction with Lemma 4.22(3) (page 143).

From (1), (3), (4), and since $\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)$ and $Q_0(\bar{t}_h^o)Q(\bar{t}_h^o) = 0$, it follows that $(d^o, \bar{k}^o, \bar{t}_h^o)$ can be extended to $(d^o, \bar{k}^o, \bar{\rho}^o, \bar{t}^o) \in \mathcal{G}$. Thus, one has $(d^o, \bar{k}^o) \in \pi_1(\mathcal{G})$, contradicting the construction of Ω_3^3. This finishes the proof of statement (a).

The proof of statement (b) is a consequence of the construction of Ω_3 (in particular, see the construction of A_4); indeed, if $\bar{t}_h^o$ satisfies

$$T_1(d, k, \bar{t}_h^o) = T_2(d, k, \bar{t}_h^o) = T_3(d, k, \bar{t}_h^o) = 0 \text{ identically in } (d, \bar{k}),$$

then $\bar{t}_h^o \in A_4$. It follows that $\Omega_3 \subset \Omega^{inv}(\bar{t}_h^o)$ and so $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega)$ (see Remark 4.38(1), page 157). ◻

Now, restricting the values of $(d, \bar{k})$ to a new open set, we are ready to prove the announced converse of Proposition 4.35 (page 154).

Proposition 4.40. *Let Ω_3 be as in Lemma 4.39 (page 157). If $\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)$ for some $(d^o, \bar{k}^o) \in \Omega_3$, but $\bar{t}_h^o \notin \mathcal{I}_5^{P_h}(\Omega_3)$, then $\bar{t}_h^o \in \mathcal{A}_h$.*

Proof. If $\bar{t}_h^o \notin \mathcal{I}_5^{P_h}(\Omega_3)$, then $t_0^o \neq 0$ (by Proposition 4.32, page 152). Let us write $\bar{t}_h^o = (1 : t_1^o : t_2^o)$. Then, since $\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)$, one has $\bar{t}^o \in \Psi_3^P(d^o, \bar{k}^o)$. Note also that, since $\bar{t}_h^o \notin \mathcal{I}_5^{P_h}(\Omega_3)$, by Lemma 4.39, we must have $Q_0(\bar{t}_h^o)Q(\bar{t}_h^o) \neq 0$. If we suppose $\bar{t}_h^o \notin \mathcal{A}_h$, then $\bar{t}^o \notin \mathcal{A}$. Thus $\bar{t}^o \in \mathcal{F}$, and by Proposition 4.23 (page 145), $\bar{t}^o \in \mathcal{I}_3^P(\Omega_3)$. Taking Equation 4.23 (page 150) into account, and using $Q(\bar{t}_h^o) \neq 0$, we conclude that

$$T_1(d, k, \bar{t}_h^o) = T_2(d, k, \bar{t}_h^o) = T_3(d, k, \bar{t}_h^o) = 0 \text{ identically in } (d, \bar{k}).$$

Then. by Lemma 4.39(b) (page 157), one has that $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega_3)$. This is a contradiction, and so we obtain that $t_h^o \in \mathcal{A}_h$. ◻

4.3.3 Multiplicity of intersection at non-fake points

The auxiliary polynomials S_i (for $i = 0, \ldots, 3$) were introduced in Section 4.2 (page 136), in order to reduce the offset degree problem to a problem of intersection between planar curves. More precisely, the preceding results in this chapter indicate that the offset degree problem can be reduced to an intersection problem between the planar curves defined by the auxiliary polynomials T_i. A crucial step in this reduction concerns the multiplicity of intersection of these curves at their non-invariant points of intersection. In this subsection we will prove that the value of that multiplicity of intersection is one (in Proposition 4.43, page 160). We first introduce some notation for the curves involved in this problem.

Definition 4.41. *Let Ω_0 be as in Theorem 4.13 (page 133). If $(d^o, \bar{k}^o) \in \Omega_0$, and T_i (for $i = 0, \ldots, 3$) are the polynomials introduced in Equations 4.22 and 4.23 (page 150), we denote by $\mathcal{T}_0^a(\bar{k}^o)$ (resp. $\mathcal{T}_i^a(d^o, \bar{k}^o)$ for $i = 1, 2, 3$) the affine algebraic set defined by the polynomial $T_0(\bar{k}^o, 1, \bar{t})$ (resp. $T_i(d^o, \bar{k}^o, 1, \bar{t})$ for $i = 1, 2, 3$). Similarly, we denote by $\mathcal{T}_0^h(\bar{k}^o)$ (resp. $\mathcal{T}_i^h(d^o, \bar{k}^o)$ for $i = 1, 2, 3$) the projective algebraic set defined by the polynomial $T_0(\bar{k}^o, \bar{t}_h)$ (resp. $T_i(d^o, \bar{k}^o, \bar{t}_h)$ for $i = 1, 2, 3$).*

Remark 4.42. *Note that the homogenization of the polynomials $T_0(\bar{k}^o, 1, \bar{t})$ and $T_i(d^o, \bar{k}^o, 1, \bar{t})$ w.r.t. t_0 does not necessarily coincide with $T_0(\bar{k}^o, \bar{t}_h)$ and $T_i(d^o, \bar{k}^o, \bar{t}_h)$. They may differ in a power of t_0. In particular, it is not necessarily true that $\overline{\mathcal{T}_0^a(\bar{k}^o)} = \mathcal{T}_0^h(\bar{k}^o)$ and $\overline{\mathcal{T}_i^a(d^o, \bar{k}^o)} = \mathcal{T}_i^h(d^o, \bar{k}^o)$ (the overline denotes projective closure, as usual). However, it holds that $\mathcal{T}_i^h(d^o, \bar{k}^o) \cap \mathbb{C}^n = \mathcal{T}_i^a(d^o, \bar{k}^o)$ and $\mathcal{T}_0^h(d^o, \bar{k}^o) \cap \mathbb{C}^n = \mathcal{T}_0^a(\bar{k}^o)$.*

Proposition 4.43. *Let Ω_3 be as in Lemma 4.39 (page 157). There exists a non-empty open $\Omega_4 \subset \Omega_3$, such that if $(d^o, \bar{k}^o) \in \Omega_4$, and $\bar{t}_h^o \in \mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)$, then:*

$$\min_{i=1,2,3} \left(\mathrm{mult}_{\bar{t}^o}(\mathcal{T}_o(\bar{k}^o), \mathcal{T}_i(d^o, \bar{k}^o)) \right) = 1.$$

Proof. Since $\bar{t}_h^o \in \mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)$, we can write $\bar{t}_h^o = (1 : t_1^o : t_2^o)$. Let $\bar{t}^o = (t_1^o, t_2^o)$. By Theorem 4.33 (page 154) we know that $\bar{t}^o = (t_1^o, t_2^o) \in \mathcal{A} \cap \Psi_3^P(d^o, \bar{k}^o)$. W.l.o.g. we will suppose that

$$P_0(\bar{t}^o) h(\bar{t}^o) \big(P_2(\bar{t}^o) n_3(\bar{t}^o) - P_3(\bar{t}^o) n_2(\bar{t}^o) \big) \neq 0$$

(see the definition of the set $\mathcal{A}$ in Equation 4.11, page 138). In this case, it holds (see Remark 4.17, page 141) that

$$k_2^o n_3(\bar{t}^o) - k_3^o n_2(\bar{t}^o) \neq 0 \text{ and } k_2^o P_3(\bar{t}^o) - k_3^o P_2(\bar{t}^o) \neq 0.$$

Furthermore, by Remark 4.5 (page 127), one has

$$f_j(P(\bar{t})) = \frac{\beta(\bar{t})}{P_0^{\mu}(\bar{t})} n_j(\bar{t}) \text{ for } j = 1, 2, 3, \tag{4.30}$$

and therefore

$$\sqrt{h_{\mathrm{imp}}(P(\bar{t}))} = \frac{\beta(\bar{t})}{P_0^{\mu}(\bar{t})}\sqrt{h(\bar{t})}. \tag{4.31}$$

with $\beta(\bar{t}^o) \neq 0$ (see Lemma 4.8, page 128).

For this case we will construct a non-empty open set $\Omega_{4,1} \subset \Omega_3$ such that if $(d^o, \bar{k}^o) \in \Omega_{4,1}$, and $\bar{t}_h^o \in \mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)$, then:

$$\mathrm{mult}_{\bar{t}^o}\left(\mathcal{T}_o(\bar{k}^o), \mathcal{T}_1(d^o, \bar{k}^o)\right) = 1.$$

If the second, respectively third, defining equation of $\mathcal{A}$ is used, then analogous open subsets $\Omega_{4,2}$, respectively $\Omega_{4,3}$ can be constructed, and the corresponding result for $\mathcal{T}_2(d^o, \bar{k}^o)$, respectively $\mathcal{T}_3(d^o, \bar{k}^o)$, is obtained. Finally, it suffices to take

$$\Omega_4 = \Omega_{4,1} \cap \Omega_{4,2} \cap \Omega_{4,3}.$$

The construction of $\Omega_{4,1}$ will proceed in several steps:

(1) By Proposition 4.16 (page 139), $(d^o, \bar{k}^o, \bar{t}^o) \in \pi_{(2,1)}(\Psi_2^P(d^o, \bar{k}^o))$. Thus, by Theorem 4.13 (page 133), the point $\bar{y}^o = P(\bar{t}^o)$ is an affine, non normal-isotropic point of Σ, and it is associated with $\bar{x}^o \in \mathcal{L}_{\bar{k}^o} \cap \mathcal{O}_{d^o}(\Sigma)$, where $\bar{x}^o$ is a non normal-isotropic point of $\mathcal{O}_{d^o}(\Sigma)$. Besides, since $(d^o, \bar{k}^o) \in \Omega_3 \subset \Omega_0$, (see Remark 4.14, page 136), $g(d^o, \bar{x})$ is the defining polynomial of $\mathcal{O}_{d^o}(\Sigma)$.

It follows that there is an open neighborhood U^0 of $(d^o, \bar{y}^o)$ (in the usual unitary topology of $\mathbb{C} \times \mathbb{C}^3$) such that the equation

$$\left(\frac{\partial f}{\partial y_1}(\bar{y})\right)^2 + \left(\frac{\partial f}{\partial y_2}(\bar{y})\right)^2 + \left(\frac{\partial f}{\partial y_3}(\bar{y})\right)^2 = 0$$

has no solutions in U^0. Similarly, there is an open neighborhood V^0 of $(d^o, \bar{x}^o)$ (in the usual unitary topology of $\mathbb{C} \times \mathbb{C}^3$) such that the equation

$$\left(\frac{\partial g}{\partial x_1}(d, \bar{x})\right)^2 + \left(\frac{\partial g}{\partial x_2}(d, \bar{x})\right)^2 + \left(\frac{\partial g}{\partial x_3}(d, \bar{x})\right)^2 = 0$$

has no solutions in V^0. Let us consider the map:

$$\varphi : U^0 \to \mathbb{C}^3$$

defined by

$$\bar{\varphi}(d, \bar{y}) = (\varphi_1(d, \bar{y}), \varphi_2(d, \bar{y}), \varphi_3(d, \bar{y})) = \bar{y} \pm d\frac{\nabla f(\bar{y})}{\sqrt{h_{\mathrm{imp}}(\bar{y})}} \tag{4.32}$$

We assume w.l.o.g. that the + sign in this expression is chosen so that $\bar{\varphi}(d^o, \bar{y}^o) = \bar{x}^o$; our discussion does not depend on this choice of sign in this expression, as will be shown below. According to Remark 4.17 and Lemma 4.18 (page 142), this implies that:

$$\bar{M}(\bar{k}^o, \bar{t}^o) = \epsilon \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} \bar{G}(\bar{k}^o, \bar{t}^o) \text{ for } i = 1, 2, 3. \tag{4.33}$$

We will use Equation 4.33 later in the proof. Since $\bar{y}^o$ is not normal-isotropic in Σ, it follows that $\bar{\varphi}$ is analytic in U^0. Furthermore, we consider the map

$$\bar{\eta} : V^0 \to \mathbb{C}^3$$

defined by:

$$\bar{\eta}(d, \bar{x}) = \bar{x} + d\frac{\nabla_{\bar{x}} g(d, \bar{x})}{\|\nabla_{\bar{x}} g(d, \bar{x})\|}.$$

Here $\nabla_{\bar{x}}$ refers to the gradient computed w.r.t. $\bar{x}$; that is:

$$\nabla_{\bar{x}} g(d, \bar{x}) = \left(\frac{\partial g}{\partial x_1}(d, \bar{x}), \frac{\partial g}{\partial x_2}(d, \bar{x}), \frac{\partial g}{\partial x_3}(d, \bar{x}) \right).$$

In the definition of $\bar{\eta}$, w.l.o.g. the sign + is chosen so that $\bar{\eta}(d^o, \bar{x}^o) = \bar{y}^o$. Then, since $\bar{x}^o$ is non normal-isotropic in $\mathcal{O}_{d^o}(\Sigma)$, it follows that $\bar{\eta}$ is analytic in V^o. Thus, there are open neighborhoods U^1 of $(d^o, \bar{y}^o)$ and V^1 of $(d^o, \bar{x}^o)$ (in the unitary topology of $\mathbb{C} \times \mathbb{C}^3$), such that $\bar{\varphi}$ is an analytic isomorphism between U^1 and V^1, with inverse given by $\bar{\eta}$. We can assume w.l.o.g. that $\|\nabla f(\bar{y})\| \neq 0$ holds in U^1, and $\|\nabla_{d,\bar{x}} g(\bar{x})\| \neq 0$ holds in V^1. Note also that if $(d^o, \bar{y}^1) \in U^1$, with $\bar{y}^1 \in \Sigma$, then $\bar{\varphi}(d^o, \bar{y}^1) \in \mathcal{O}_{d^o}(\Sigma)$. It follows that the map $\bar{\varphi}_{d^o}$, obtained by restricting $\bar{\varphi}$ to $d = d^o$, induces an isomorphism:

$$d\bar{\varphi}_{d^o} : T_{\bar{y}^o}(\Sigma) \to T_{\bar{x}^o}(\mathcal{O}_{d^o}(\Sigma))$$

where $T_{\bar{y}^o}(\Sigma)$ is the tangent plane to Σ at $\bar{y}^o$, and $T_{\bar{x}^o}(\mathcal{O}_{d^o}(\Sigma))$ is the tangent plane to $\mathcal{O}_{d^o}((\Sigma)$ at $\bar{x}^o$.

Since $(d^o, \bar{k}^o) \in \Omega_3 \subset \Omega_0$, we have $\bar{t}^o \in \Upsilon_1$, with Υ_1 as in Lemma 4.2, page 126 (see the construction of Ω_0^4 in the proof of Theorem 4.13, 133). Thus, the jacobian $\dfrac{\partial P}{\partial \bar{t}}(\bar{t}^o)$ has rank two. It follows that P induces an isomorphism:

$$dP : T_{\bar{t}^o}(\mathbb{C}^2) \to T_{\bar{y}^o}(\Sigma)$$

where $T_{\bar{t}^o}(\mathbb{C}^2)$ is the tangent plane to $\mathbb{C}^2$ at $\bar{t}^o$. Therefore, the map defined by

$$\bar{\nu}_{d^o}(\bar{t}) = \bar{\varphi}_{d^o}(P(\bar{t})) = \bar{\varphi}(d^o, P(\bar{t})) \tag{4.34}$$

induces an isomorphism $d\bar{\nu}_{d^o}$ between $T_{\bar{t}^o}(\mathbb{C}^2)$ and $T_{\bar{x}^o}(\mathcal{O}_{d^o}(\Sigma))$.

(2) Consider the following polynomials in $\mathbb{C}[d, \bar{k}, \rho]$:

$$\begin{cases} K(d, \bar{k}, \rho) = k_1 \dfrac{\partial g}{\partial x_1}(d, \rho\bar{k}) + k_2 \dfrac{\partial g}{\partial x_2}(d, \rho\bar{k}) + k_3 \dfrac{\partial g}{\partial x_3}(d, \rho\bar{k}) \\ \tilde{g}(d, \bar{k}, \rho) = g(d, \rho\bar{k}) \end{cases}$$

and let

$$\Theta(d, \bar{k}) = \mathrm{Res}_\rho(K(d, \bar{k}, \rho), \tilde{g}(d, \bar{k}, \rho)).$$

Let us show that this resultant does not vanish identically. If it does, then there are $A, B_1, B_2 \in \mathbb{C}[d, \bar{k}, \rho]$, with $\deg_\rho(A(d, \bar{k}, \rho)) > 0$, such that

$$\begin{cases} K(d, \bar{k}, \rho) = A(d, \bar{k}, \rho) B_1(d, \bar{k}, \rho), \\ \tilde{g}(d, \bar{k}, \rho) = A(d, \bar{k}, \rho) B_2(d, \bar{k}, \rho). \end{cases}$$

Then $g(d, \rho\bar{k}) = A(d, \bar{k}, \rho)B_2(d, \bar{k}, \rho)$, and $\deg_{\bar{k}}(A(d, \bar{k}, \rho)) > 0$ (because $\tilde{g}$ cannot have a non constant factor in $\mathbb{C}[d, \rho]$). Thus, setting $\rho = 1$ and $\bar{k} = \bar{x}$, one has $g(d, \bar{x}) = A(d, \bar{x}, 1)B_2(d, \bar{x}, 1)$. It follows (see Remark 1.23(1), page 21) that if $\tilde{A}(d, \bar{x})$ is any irreducible factor of $A(d, \bar{x}, 1)$, then $\tilde{A}(d, \bar{x})$ defines an irreducible component $\mathcal{M}$ of the generic offset, such that

$$x_1 \frac{\partial g}{\partial x_1}(d, \bar{x}) + x_2 \frac{\partial g}{\partial x_2}(d, \bar{x}) + x_3 \frac{\partial g}{\partial x_3}(d, \bar{x}) = 0$$

holds identically on $\mathcal{M}$. Besides, for an open set of points $\bar{x}^o \in \mathcal{M}$, one has $\nabla_{\bar{x}} g(d, \bar{x}^o) = \nabla_{\bar{x}} \tilde{A}(d, \bar{x}^o)$. Thus the above equation implies that

$$x_1 \frac{\partial \tilde{A}}{\partial x_1}(d, \bar{x}) + x_2 \frac{\partial \tilde{A}}{\partial x_2}(d, \bar{x}) + x_3 \frac{\partial \tilde{A}}{\partial x_3}(d, \bar{x}) = 0$$

holds identically in $\mathcal{M}$. Therefore, since $\tilde{A}$ is irreducible, we get

$$x_1 \frac{\partial \tilde{A}}{\partial x_1}(d, \bar{x}) + x_2 \frac{\partial \tilde{A}}{\partial x_2}(d, \bar{x}) + x_3 \frac{\partial \tilde{A}}{\partial x_3}(d, \bar{x}) = \kappa^o \tilde{A}(d, \bar{x})$$

for some constant κ^o. This implies that the polynomial $\tilde{A}(d, \bar{x})$ is homogeneous w.r.t. $\bar{x}$, and it follows that, for any value $d^o \notin \Delta$ (with Δ as in Corollary 1.25, page 21), $\mathcal{O}_{d^o}(\Sigma)$, has a homogeneous component. This implies that $\bar{0} \in \mathcal{O}_{d^o}(\Sigma)$ for $d^o \notin \Delta$, which is a contradiction with our hypothesis (see Remark 4.1(1), page 122). Thus, $\Theta(d, \bar{k})$ is not constant. Let us define $\Omega^1_{4,1} = \Omega_3 \backslash \{(d^o, \bar{k}^o) / \Theta(d^o, \bar{k}^o) = 0\}$.

(3) Let us consider the following polynomials in $\mathbb{C}[d, \bar{k}, \bar{x}]$

$$\begin{cases} \sigma_0(d, \bar{k}, \bar{x}) = \det(\bar{k}, \bar{x}, \nabla_{\bar{x}} g(d, \bar{x})) \\ \sigma_1(d, \bar{k}, \bar{x}) = k_2 x_3 - k_3 x_2 \end{cases} \tag{4.35}$$

Let $\Omega^2_{4,1} \subset \Omega^1_{4,1}$ be such that, for $(d^o, \bar{k}^o) \in \Omega^2_{4,1}$, these polynomials are non identically zero (note that σ_0 and σ_1 are both homogeneous w.r.t. $\bar{k}$). Therefore, for $(d^o, \bar{k}^o) \in \Omega^2_{4,1}$, and for $i = 0, 1$, $\sigma_i(d^o, \bar{k}^o, \bar{x})$ defines a surface $\Sigma_i(d^o, \bar{k}^o)$. From

$$\nabla_{\bar{x}}\sigma_1(\bar{k}, \bar{x}) = (0, -k_3, k_2)$$

one has

$$\nabla_{\bar{x}}g \wedge \nabla_{\bar{x}}\sigma_1 = \left(k_2\frac{\partial g}{\partial x_2} + k_3\frac{\partial g}{\partial x_3}, -k_2\frac{\partial g}{\partial x_1}, -k_3\frac{\partial g}{\partial x_1}\right).$$

Let $(d^o, \bar{k}^o) \in \Omega^2_{4,1}$ and $\bar{x}^o \in \mathcal{O}_{d^o}(\Sigma) \cap \mathcal{L}_{\bar{k}^o}$. We will show that

$$\nabla_{\bar{x}}g(d^o, \bar{x}^o) \wedge \nabla_{\bar{x}}\sigma_1(\bar{k}^o, \bar{x}^o) \neq \bar{0}. \tag{4.36}$$

First, note that there is $\rho^o \in \mathbb{C}$ such that $\bar{x}^o = \rho^o\bar{k}^o$. If $\dfrac{\partial g}{\partial x_1}(d^o, \bar{x}^o) \neq 0$, then since $k_i \neq 0$ for $i = 1, 2, 3$, the result follows. Thus, let $\dfrac{\partial g}{\partial x_1}(d^o, \bar{x}^o) = 0$. If we suppose that $\nabla_{\bar{x}}g(d^o, \bar{x}^o) \wedge \nabla_{\bar{x}}\sigma_1(\bar{k}^o, \bar{x}^o) = \bar{0}$, then

$$k_2^o\frac{\partial g}{\partial x_2}(d^o, \bar{x}^o) + k_3^o\frac{\partial g}{\partial x_3}(d^o, \bar{x}^o) = 0.$$

Thus, one obtains

$$\begin{cases} g(d^o, \rho^o\bar{k}^o) = 0 \\ k_1^o\dfrac{\partial g}{\partial x_1}(d^o, \rho^o\bar{k}^o) + k_2^o\dfrac{\partial g}{\partial x_2}(d^o, \rho^o\bar{k}^o) + k_3^o\dfrac{\partial g}{\partial x_3}(d^o, \rho^o\bar{k}^o) = 0 \end{cases}$$

and it follows that $\Theta(d^o, \bar{k}^o) = 0$ (with Θ as in step (2) of the proof), contradicting the construction of $\Omega^1_{4,1}$. Thus, Equation 4.36 is proved.

We will prove the analogous result for σ_0. That is, for $(d^o, \bar{k}^o) \in \Omega^2_{4,1}$ and $\bar{x}^o \in \mathcal{O}_{d^o}(\Sigma) \cap \mathcal{L}_{\bar{k}^o}$, we will show that:

$$\nabla_{\bar{x}}g(d^o, \bar{x}^o) \wedge \nabla_{\bar{x}}\sigma_0(d^o, \bar{k}^o, \bar{x}^o) \neq \bar{0}. \tag{4.37}$$

From

$$\sigma_0(d, \bar{k}, \bar{x}) = \det(\bar{k}, \bar{x}, \nabla_{\bar{x}}g(d, \bar{x}))$$

and applying the derivation properties of determinants, one has, e.g.

$$\frac{\partial \sigma_0}{\partial x_1}(d, \bar{k}, \bar{x}) = \begin{vmatrix} k_1 & k_2 & k_3 \\ 1 & 0 & 0 \\ \partial_1 g & \partial_2 g & \partial_3 g \end{vmatrix} + \begin{vmatrix} k_1 & k_2 & k_3 \\ x_1 & x_2 & x_3 \\ \partial_{1,1} g & \partial_{2,1} g & \partial_{3,1} g \end{vmatrix}$$

where $\partial_i g = \dfrac{\partial g}{\partial x_i}$ and $\partial_{i,j} g = \dfrac{\partial^2 g}{\partial x_i \partial x_j}$ for $i, j \in \{1, 2, 3\}$. Let as before, $\bar{x}^o = \rho^o \bar{k}^o$ for some $\rho^o \in \mathbb{C}$. Then:

$$\frac{\partial \sigma_0}{\partial x_1}(d^o, \bar{k}^o, \bar{x}^o) = \begin{vmatrix} k_1^o & k_2^o & k_3^o \\ 1 & 0 & 0 \\ \partial_1 g & \partial_2 g & \partial_3 g \end{vmatrix} + \begin{vmatrix} k_1^o & k_2^o & k_3^o \\ \rho^o k_1^o & \rho^o k_2^o & \rho^o k_3^o \\ \partial_{1,1} g & \partial_{2,1} g & \partial_{3,1} g \end{vmatrix}$$

with all the partial derivatives evaluated at $(d^o, \bar{x}^o)$. Since the second determinant in the above equation vanishes, one concludes that

$$\frac{\partial \sigma_0}{\partial x_1}(d^o, \bar{k}^o, \bar{x}^o) = k_3^o \frac{\partial g}{\partial x_2}(d^o, \bar{x}^o) - k_2^o \frac{\partial g}{\partial x_3}(d^o, \bar{x}^o).$$

Similar results are obtained for the other two partial derivatives, leading to:

$$\nabla_{\bar{x}} \sigma_0(d^o, \bar{k}^o, \bar{x}^o) = \nabla_{\bar{x}} g(d^o, \bar{x}^o) \wedge \bar{k}^o.$$

Therefore,

$$\nabla_{\bar{x}} g(d^o, \bar{x}^o) \wedge \nabla_{\bar{x}} \sigma_0(d^o, \bar{k}^o, \bar{x}^o) = \nabla_{\bar{x}} g(d^o, \bar{x}^o) \wedge (\nabla_{\bar{x}} g(d^o, \bar{x}^o) \wedge \bar{k}^o).$$

Note that $\nabla_{\bar{x}} g(d^o, \bar{x}^o) \wedge \bar{k}^o \neq \bar{0}$ because, by construction, $\mathcal{L}_{\bar{k}^o}$ is not normal to $\mathcal{O}_{d^o}(\Sigma)$ at $\bar{x}^o$. If we suppose that

$$\nabla_{\bar{x}} g(d^o, \bar{x}^o) \wedge \nabla_{\bar{x}} \sigma_0(d^o, \bar{k}^o, \bar{x}^o) = \bar{0},$$

then the vectors $\nabla_{\bar{x}} g(d^o, \bar{x}^o)$ and $\nabla_{\bar{x}} g(d^o, \bar{x}^o) \wedge \bar{k}^o$ are parallel and perpendicular to each other. However, if two vectors are parallel and perpendicular, and one of them is not zero, then the other one must be isotropic. One concludes that $\|\nabla_{\bar{x}} g(d^o, \bar{x}^o)\| = 0$. This is a contradiction (see step (1) of the proof); therefore, Equation 4.37 is proved.

From Equations 4.36 and 4.37, and using Theorem 9 in [14] (page 480), we conclude that $\bar{x}^o$ is a regular point in $\Sigma_i(d^o, \bar{k}^o) \cap \mathcal{O}_{d^o}(\Sigma)$ (for $i = 0, 1$). Besides, $\bar{x}^o$ belongs to a unique one-dimensional component of $\Sigma_i(d^o, \bar{k}^o) \cap \mathcal{O}_{d^o}(\Sigma)$. For $i = 0, 1$, let $\mathcal{C}_i(d^o, \bar{k}^o)$ be the one-dimensional component of $\Sigma_i(d^o, \bar{k}^o) \cap \mathcal{O}_{d^o}(\Sigma)$ containing $\bar{x}^o$.

(4) The non-zero vector

$$\bar{v}_i(d^o, \bar{k}^o, \bar{x}^o) = \nabla_{\bar{x}} g(d^o, \bar{x}^o) \wedge \nabla_{\bar{x}} \sigma_i(\bar{k}^o, \bar{x}^o), \quad (i = 0, 1)$$

obtained in step (3) of the proof, is a tangent vector to $\mathcal{C}_i(d^o, \bar{k}^o)$ at $\bar{x}^o$. We will show that

$$\bar{v}_0(d^o, \bar{k}^o, \bar{x}^o) \wedge \bar{v}_1(d^o, \bar{k}^o, \bar{x}^o) \neq \bar{0}. \tag{4.38}$$

It holds that

$$\bar{v}_0(d^o,\bar{k}^o,\bar{x}^o)\wedge\bar{v}_1(d^o,\bar{k}^o,\bar{x}^o)=$$
$$-(k_3^o\frac{\partial g}{\partial x_2}-k_2^o\frac{\partial g}{\partial x_3})\cdot\left(k_1^o\frac{\partial g}{\partial x_1}+k_2^o\frac{\partial g}{\partial x_2}+k_3^o\frac{\partial g}{\partial x_3}\right)\cdot\nabla_{\bar{x}}g(d^o,\bar{x}^o),$$

with all the partial derivatives evaluated at $(d^o,\bar{x}^o)$. Since

$$\|\nabla f(\bar{y}^o)\|\cdot\|\nabla_{\bar{x}}g(d^o,\bar{x}^o)\|\neq 0,$$

by the fundamental property of the offset (Proposition 1.4, page 11), there is some $\kappa^o\in\mathbb{C}^\times$ such that

$$\nabla_{\bar{x}}g(d^o,\bar{x}^o)=\kappa^o\nabla f(\bar{y}^o).$$

Then, using Equation 4.30 (page 160) one has (see Remark 4.17, page 141) that

$$k_3^o\frac{\partial g}{\partial x_2}-k_2^o\frac{\partial g}{\partial x_3}=\kappa^o(k_3^of_2(\bar{y}^o)-k_2^of_3(\bar{y}^o))=\kappa^o\frac{\beta(\bar{t}^o)}{P_0^\mu(\bar{t}^o)}(k_3^on_2(\bar{t}^o)-k_2^on_3(\bar{t}^o))\neq 0.$$

Besides, in step (2) of the proof we have already seen that

$$\left(k_1^o\frac{\partial g}{\partial x_1}+k_2^o\frac{\partial g}{\partial x_2}+k_3^o\frac{\partial g}{\partial x_3}\right)\neq 0.$$

Thus, the proof of Equation 4.38 is finished.

(5) From Equation 4.33 (page 162) one has

$$M_1\bar{k}^o,\bar{t}^o)=\epsilon\frac{d^oP_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}}G_1(\bar{k}^o,\bar{t}^o).$$

Therefore:

$$\sqrt{h(\bar{t}^o)}(k_2^oP_3(\bar{t}^o)-k_3^oP_2(\bar{t}^o))=d^oP_0(\bar{t}^o)(k_2(\bar{t}^o)n_3(\bar{t}^o)-k_3(\bar{t}^o)n_2(\bar{t}^o)).\tag{4.39}$$

Multiplying by $\dfrac{\beta(\bar{t}^o)}{P_0^{\mu+1}(\bar{t}^o)}$, it holds that

$$\frac{\beta(\bar{t}^o)\sqrt{h(\bar{t}^o)}}{P_0^\mu(\bar{t}^o)}\left(k_2^o\frac{P_3(\bar{t}^o)}{P_0(\bar{t}^o)}-k_3^o\frac{P_2(\bar{t}^o)}{P_0(\bar{t}^o)}\right)=d^o\left(k_2(\bar{t}^o)\frac{\beta(\bar{t}^o)n_3(\bar{t}^o)}{P_0^\mu(\bar{t}^o)}-k_3(\bar{t}^o)\frac{\beta(\bar{t}^o)n_2(\bar{t}^o)}{P_0^\mu(\bar{t}^o)}\right).$$

Using Equation 4.30 (page 160), one obtains (recall that $\bar{y}^o=P(\bar{t}^o)$):

$$\sqrt{h_{\mathrm{imp}}(\bar{y}^o)}(k_2^oy_3^o-k_3^oy_2^o)-d^o(k_2^of_3^o(\bar{y}^o)-k_3^of_2(\bar{y}^o))=0.\tag{4.40}$$

Note also that, since $\sqrt{h(\bar{t}^o)}(k_2^o P_3(\bar{t}^o) - k_3^o P_2(\bar{t}^o)) \neq 0$ we also have

$$\sqrt{h_{\mathrm{imp}}(\bar{y}^o)}(k_2^o y_3^o - k_3^o y_2^o) + d^o(k_2^o f_3^o(\bar{y}^o) - k_3^o f_2(\bar{y}^o)) \neq 0. \tag{4.41}$$

Observe that, if the sign $\epsilon = -1$ is used in the offsetting construction (see step (1) of the proof), the results in Equations 4.40 and 4.41 are reversed.

Recall (see Equation 4.7, page 137) that the auxiliary polynomial s_1 is given by:

$$s_1(d, \bar{k}, \bar{t}) = h(\bar{t})(k_2 P_3 - k_3 P_2)^2 - d^2 P_0(\bar{t})^2 (k_2 n_3 - k_3 n_2)^2.$$

Thus, one has:

$$\begin{aligned}&\frac{\beta^2(\bar{t})}{P_0^{2\mu+2}(\bar{t})} s_1(d, \bar{k}, \bar{t}) = \\ &\frac{\beta^2(\bar{t})}{P_0^{2\mu}(\bar{t})}\left(k_2 \frac{P_3(\bar{t})}{P_0(\bar{t})} - k_3 \frac{P_2(\bar{t})}{P_0(\bar{t})}\right)^2 - d^2\left(k_2 \frac{\beta(\bar{t}) n_3(\bar{t})}{P_0^{\mu}(\bar{t})} - k_3 \frac{\beta(\bar{t}) n_2(\bar{t})}{P_0^{\mu}(\bar{t})}\right)^2.\end{aligned}$$

And substituting $\bar{y} = P(\bar{t})$ in $\dfrac{\beta^2(\bar{t})}{P_0^{2\mu+2}(\bar{t})} s_1(d, \bar{k}, \bar{t})$, one obtains:

$$\frac{\beta^2(\bar{t})}{P_0^{2\mu+2}(\bar{t})} s_1(d, \bar{k}, \bar{t}) = h_{\mathrm{imp}}(\bar{y})(k_2 y_3 - k_3 y_2)^2 - d^2(k_2 f_3(\bar{y}) - k_3 f_2(\bar{y}))^2. \tag{4.42}$$

Let us consider the polynomial $\sigma_1' \in \mathbb{C}[d, \bar{k}, \bar{y}]$ defined by

$$\sigma_1'(d, \bar{k}, \bar{y}) = h_{\mathrm{imp}}(\bar{y})(k_2 y_3 - k_3 y_2)^2 - d^2(k_2 f_3(\bar{y}) - k_3 f_2(\bar{y}))^2,$$

and let $\Sigma_1'(d^o, \bar{k}^o) \subset \mathbb{C}^3$ be the algebraic closed set defined by the equation $\sigma_1'(d^o, \bar{k}^o, \bar{y}) = 0$. Let $\bar{\tau} = (\tau^1, \tau^2)$, and let $\mathcal{Q}_1(\bar{\tau})$ be a place of $\mathcal{T}_1^a(d^o, \bar{k}^o)$ centered at $\bar{t}^o$. We assume that $\mathcal{Q}_1(\bar{0}) = \bar{t}^o$. Since $T_1(d^o, \bar{k}^o, 1, \mathcal{Q}_1(\bar{\tau})) = 0$ identically in $\bar{\tau}$, from Equation 4.23 (page 150) it follows that

$$s_1(d^o, \bar{k}^o, \mathcal{Q}_1(\bar{\tau})) = S_1(d^o, \bar{k}^o, 1, \mathcal{Q}_1(\bar{\tau})) = 0$$

identically in $\bar{\tau}$. Thus, from Equation 4.42 (recall that $\bar{y} = P(\bar{t})$ in the lhs of Equation 4.42) one has:

$$\sigma_1'(d^o, \bar{k}^o, P(\mathcal{Q}_1(\bar{\tau}))) = 0$$

identically in $\bar{\tau}$. Note that:

$$\sigma_1'(d, \bar{k}^o, \bar{y}) = \sigma_{1,+}'(d, \bar{k}^o, \bar{y})\sigma_{1,-}'(d, \bar{k}^o, \bar{y})$$

with

$$\begin{cases} \sigma'_{1,+}(d, \bar{k}^o, \bar{y}) = \sqrt{h_{\text{imp}}(\bar{y})}(k_2^o y_3 - k_3^o y_2) + d(k_2^o f_3(\bar{y}) - k_3^o f_2(\bar{y})), \\ \sigma'_{1,-}(d, \bar{k}^o, \bar{y}) = \sqrt{h_{\text{imp}}(\bar{y})}(k_2^o y_3 - k_3^o y_2) - d(k_2^o f_3(\bar{y}) - k_3^o f_2(\bar{y})). \end{cases}$$

The functions $\sigma'_{1,+}(d, \bar{k}^o, \bar{y})$ and $\sigma'_{1,-}(d, \bar{k}^o, \bar{y})$ are analytic in the neighborhood U^1 of $(d^o, \bar{x}^o)$ introduced in step (1) of the proof. Therefore:

$$\sigma'_{1,+}(d^o, \bar{k}^o, \mathcal{Q}_1(\bar{\tau}))\sigma'_{1,-}(d^o, \bar{k}^o, \mathcal{Q}_1(\bar{\tau})) = 0,$$

identically in $\bar{\tau}$. However, evaluating at $\bar{\tau} = \bar{0}$, and taking Equations 4.40 and 4.41 into account, one sees that

$$\sigma'_{1,+}(d, \bar{k}^o, \bar{y}^o) \neq 0, \text{ while } \sigma'_{1,-}(d, \bar{k}^o, \bar{y}^o) = 0.$$

By the analytic character of the functions, one concludes that

$$\sigma'_{1,-}(d^o, \bar{k}^o, \mathcal{Q}_1(\bar{\tau})) = 0,$$

identically in $\bar{\tau}$. Dividing by $\sqrt{h_{\text{imp}}(\bar{y})}$, this relation implies that:

$$k_2^o \left(\frac{P_3(\mathcal{Q}_1(\bar{\tau}))}{P_0(\mathcal{Q}_1(\bar{\tau}))} + d^o \frac{f_3(P(\mathcal{Q}_1(\bar{\tau})))}{\sqrt{h_{\text{imp}}(P_3(\mathcal{Q}_1(\bar{\tau})))}} \right) - k_3^o \left(\frac{P_2(\mathcal{Q}_1(\bar{\tau}))}{P_0(\mathcal{Q}_1(\bar{\tau}))} + d^o \frac{f_2(P(\mathcal{Q}_1(\bar{\tau})))}{\sqrt{h_{\text{imp}}(P_2(\mathcal{Q}_1(\bar{\tau})))}} \right) = 0.$$

That is,

$$k_3^o \varphi_2(d^o, P(\mathcal{Q}_1(\bar{\tau}))) - k_2^o \varphi_3(d^o, P(\mathcal{Q}_1(\bar{\tau}))) = 0$$

identically in $\bar{\tau}$, where $\bar{\varphi} = (\varphi_2, \varphi_2, \varphi_3)$ was defined in step (1) of the proof (see Equation 4.32, page 161). With the notation introduced in step (3) of the proof (see Equation 4.35, page 163), this is

$$\sigma_1(d^o, \bar{\varphi}(d^o, P(\mathcal{Q}_1(\bar{\tau})))) = 0,$$

identically in $\bar{\tau}$. This implies that if $\mathcal{B}_1$ is the branch of $\mathcal{T}_1^a(d^o, \bar{k}^o)$ at $\bar{t}^o$ determined by $\mathcal{Q}_1(\bar{\tau})$, then

$$\bar{\nu}_{d^o}(\mathcal{B}_1) \subset \mathcal{C}_1(d^o, \bar{k}^o),$$

where $\bar{\nu}_{d^o}$ was defined in Equation 4.34 (page 162), and $\mathcal{C}_1(d^o, \bar{k}^o)$ was introduced at the end of step (3) of the proof.

(6) Let $\mathcal{Q}_0(\bar{\tau})$ be a place of $\mathcal{T}_0^a(\bar{k}^o)$ centered at $\bar{t}^o$. We assume that $\mathcal{Q}_0(\bar{0}) = \bar{t}^o$. Since $T_0(\bar{k}^o, 1, \mathcal{Q}_0(\bar{\tau})) = 0$ identically in $\bar{\tau}$, from Equation 4.22 (page 150) it follows that

$$s_0(\bar{k}^o, \mathcal{Q}_0(\bar{\tau})) = S_0(\bar{k}^o, 1, \mathcal{Q}_0(\bar{\tau})) = 0,$$

identically in $\bar{\tau}$. That is,

$$s_0(\bar{k}^o, \mathcal{Q}_0(\bar{\tau})) = \det \begin{pmatrix} k_1^o & k_2^o & k_3^o \\ P_1(\mathcal{Q}_0(\bar{\tau})) & P_2(\mathcal{Q}_0(\bar{\tau})) & P_3(\mathcal{Q}_0(\bar{\tau})) \\ n_1(\mathcal{Q}_0(\bar{\tau})) & n_2(\mathcal{Q}_0(\bar{\tau})) & n_3(\mathcal{Q}_0(\bar{\tau})) \end{pmatrix}, \tag{4.43}$$

identically in $\bar{\tau}$. Multiplying this by $\dfrac{\beta(\mathcal{Q}_0(\bar{\tau}))}{P_0^{\mu+1}(\mathcal{Q}_0(\bar{\tau}))}$, one has:

$$\det \begin{pmatrix} k_1^o & k_2^o & k_3^o \\ \dfrac{P_1(\mathcal{Q}_0(\bar{\tau}))}{P_0(\bar{\tau})} & \dfrac{P_2(\mathcal{Q}_0(\bar{\tau}))}{P_0(\mathcal{Q}_0(\bar{\tau}))} & \dfrac{P_3(\mathcal{Q}_0(\bar{\tau}))}{P_0(\mathcal{Q}_0(\bar{\tau}))} \\ \dfrac{\beta(\mathcal{Q}_0(\bar{\tau}))n_1(\mathcal{Q}_0(\bar{\tau}))}{P_0^{\mu}(\mathcal{Q}_0(\bar{\tau}))} & \dfrac{\beta(\bar{\tau})n_2(\mathcal{Q}_0(\bar{\tau}))}{P_0^{\mu}(\mathcal{Q}_0(\bar{\tau}))} & \dfrac{\beta(\bar{\tau})n_3(\mathcal{Q}_0(\bar{\tau}))}{P_0^{\mu}(\mathcal{Q}_0(\bar{\tau}))} \end{pmatrix} = 0,$$

identically in $\bar{\tau}$. Using Equation 4.30 (page 160), this implies that:

$$\det(\bar{k}^o, P(\mathcal{Q}_0(\bar{\tau})), \nabla f(P(\mathcal{Q}_0(\bar{\tau})))) = 0.$$

Since

$$\bar{\varphi}(d^o, P(\mathcal{Q}_0(\bar{\tau}))) = P(\mathcal{Q}_0(\bar{\tau})) \pm d^o \frac{\nabla f(P(\mathcal{Q}_0(\bar{\tau})))}{\sqrt{h_{\text{imp}}(\bar{P}(\mathcal{Q}_0(\bar{\tau})))}}$$

and the second term in the sum is parallel to $\nabla f(P(\mathcal{Q}_0(\bar{\tau})))$, we have

$$\det(\bar{k}^o, \bar{\varphi}(d^o, P(\mathcal{Q}_0(\bar{\tau}))), \nabla f(P(\mathcal{Q}_0(\bar{\tau})))) = 0.$$

Besides, by the fundamental property of the offset (Proposition 1.4, page 11), and the construction in step (1) of the proof, the vectors

$$\nabla f(y) \text{ and } \nabla_{\bar{x}} g(d^o, \bar{\varphi}(d^o, \bar{y}))$$

are parallel for every value of $(d^o, \bar{y})$ in V^1. It follows that

$$\det(\bar{k}^o, \bar{\varphi}(d^o, P(\mathcal{Q}_0(\bar{\tau}))), \nabla_{\bar{x}} g(d^o, \bar{\varphi}(d^o, P(\mathcal{Q}_0(\bar{\tau}))))) = 0,$$

identically in $\bar{\tau}$. Recalling the definition of σ_0 in Equation 4.35 (page 163), this implies that

$$\sigma_0(d^o, \bar{k}^o, \bar{\varphi}(d^o, P(\mathcal{Q}_0(\bar{\tau})))) = 0,$$

identically in $\bar{\tau}$. It follows that, if $\mathcal{B}_0$ is the branch of $\mathcal{T}_0^a(\bar{k}^o)$ at $\bar{t}^o$ determined by $\mathcal{Q}_0(\bar{\tau})$, then

$$\bar{\nu}_{d^o}(\mathcal{B}_0) \subset \mathcal{C}_0(d^o, \bar{k}^o),$$

where $\bar{\nu}_{d^o}$ was defined in Equation 4.34 (page 162), and $\mathcal{C}_0(d^o, \bar{k}^o)$ was introduced at the end of step (3) of the proof.

Now we can finish the proof of the proposition. In steps (5) and (6) of the proof we have shown that any branch at $\bar{t}^o$ of the curves $\mathcal{T}_0^a(\bar{k}^o)$ or $\mathcal{T}_1^a(d^o, \bar{k}^o)$ is mapped by $\bar{\nu}_{d^o}$ respectively into the curve $\mathcal{C}_1(d^o, \bar{k}^o)$ or $\mathcal{C}_0(d^o, \bar{k}^o)$ (these curves are constructed in step (3)). Since $d\bar{\nu}_{d^o}$ is an isomorphism of vector spaces (see step (1)), it follows that:

- By the results in step (3), there is only one branch at $\bar{t}^o$ of each of the curves $\mathcal{T}_0^a(\bar{k}^o)$ and $\mathcal{T}_1^a(d^o, \bar{k}^o)$. Besides, since the rank of the Jacobian matrix (and therefore, the condition in [14], Theorem 9, page 480) is preserved under analytic isomorphisms, the unique branch of each the curves $\mathcal{T}_0^a(\bar{k}^o)$ and $\mathcal{T}_1^a(d^o, \bar{k}^o)$ passing through $\bar{t}^o$ is regular at that point.
- By the results in step (4), if ℓ_1 and ℓ_0 are the two tangent lines of these two branches, then ℓ_1 and ℓ_0 are different.

Then

$$\mathrm{mult}_{\bar{t}^o}(\mathcal{T}_o, \mathcal{T}_1) = 1$$

follows from Theorem 5.10 in [56] (page 114). □

4.3.4 The degree formula

Before the statement of the degree formula we need to introduce some more notation and a technical lemma. Let

$$R(\bar{c}, d, \bar{k}, \bar{t}) = \mathrm{Res}_{t_0}\left(T_0(\bar{k}, \bar{t}_h), T(\bar{c}, d, \bar{k}, \bar{t}_h)\right)$$

(for the definition of T_0 and T see Equations 4.22 and 4.24, in page 150). Then R factors as follows:

$$R(\bar{c}, d, \bar{k}, \bar{t}) = N_1(d, \bar{k}, \bar{t}) M_3(\bar{c}, d, \bar{k}, \bar{t})$$

where $N_1(d, \bar{k}, \bar{t}) = \mathrm{Con}_{\bar{c}}(R)$ and $M_3(\bar{c}, d, \bar{k}, \bar{t}) = \mathrm{PP}_{\bar{c}}(R)$.

Besides, N_1 factors as follows:

$$N_1(d, \bar{k}, \bar{t}) = M_1(\bar{t}) M_2(d, \bar{k}, \bar{t})$$

where $M_1(\bar{t}) = \mathrm{Con}_{(d,\bar{k})}(N_1)$ and $M_2(d, \bar{k}, \bar{t}) = \mathrm{PP}_{(d,\bar{k})}(N_1)$. Thus, one has

$$R(\bar{c}, d, \bar{k}, \bar{t}) = M_1(\bar{t}) M_2(d, \bar{k}, \bar{t}) M_3(\bar{c}, d, \bar{k}, \bar{t})$$

and

$$M_2(d, \bar{k}, \bar{t}) = \mathrm{PP}_{(d,\bar{k})}(\mathrm{Con}_{\bar{c}}(R)).$$

Note that M_1, M_2 and M_3 are homogeneous polynomials in $\bar{t} = (t_1, t_2)$.

The following lemma deals with the specialization of the resultant $R(\bar{c}, d, \bar{k}, \bar{t})$. More precisely, for $(d^o, \bar{k}^o) \in \mathbb{C} \times \mathbb{C}^3$ we denote:

$$T_0^{\bar{k}^o}(\bar{t}_h) = T_0(d^o, \bar{k}^o, \bar{t}_h), \quad T^{(d^o,\bar{k}^o)}(\bar{c}, \bar{t}_h) = T(\bar{c}, d^o, \bar{k}^o, \bar{t}_h).$$

and

$$R^{(d^o,\bar{k}^o)}(\bar{c}, \bar{t}) = \mathrm{Res}_{t_0}\left(T_0^{\bar{k}^o}(\bar{t}_h), T^{(d^o,\bar{k}^o)}(\bar{c}, \bar{t}_h)\right)$$

Lemma 4.44. *Let Ω_4 be as in Proposition 4.43 (page 160). There exists a non-empty open Ω_5, such that for $(d^o, \bar{k}^o) \in \Omega_5$ the following hold:*

(a) The resultant $R(\bar{c}, d, \bar{k}, \bar{t})$ specializes properly:

$$R^{(d^o,\bar{k}^o)}(\bar{c}, \bar{t}) = R(\bar{c}, d^o, \bar{k}^o, \bar{t}) = M_1(\bar{t}) M_2(d^o, \bar{k}^o, \bar{t}) M_3(\bar{c}, d^o, \bar{k}^o, \bar{t}).$$

(b) The content w.r.t $\bar{c}$ also specializes properly:

$$\mathrm{Con}_{\bar{c}}(R^{(d^o,\bar{k}^o)})(\bar{t}) = M_1(\bar{t}) M_2(d^o, \bar{k}^o, \bar{t}).$$

(c) the coprimality of M_1 and M_2 is invariant under specialization:

$$\gcd\left(M_1(\bar{t}), M_2(d^o, \bar{k}^o, \bar{t})\right) = 1$$

Proof. For (a), consider T_0 and T as polynomials in $\mathbb{C}[\bar{c}, d, \bar{k}, \bar{t}][t_0]$. Let $\mathrm{lc}(T_0)(\bar{k}, \bar{t})$, (resp. $\mathrm{lc}(T)(\bar{c}, d, \bar{k}, \bar{t})$) be a leading coefficient w.r.t. t_0 of T_0 (resp. T). Take $A_1(\bar{k})$ (resp. $B_1(d, \bar{k})$) to be the coefficient of a term of $\mathrm{lc}(T_0)(\bar{k}, \bar{t})$ (resp. $\mathrm{lc}(T)(d, \bar{k}, \bar{t})$) of degree equal to $\deg_{\bar{t}}(\mathrm{lc}(T_0)(\bar{k}, \bar{t}))$ (resp. $\deg_{\{\bar{c},\bar{t}\}}(\mathrm{lc}(T)(\bar{k}, \bar{t}))$). Now, if $A_1(\bar{k}^o) B_1(d^o, \bar{k}^o) \neq 0$, then (a) holds. Thus, set

$$\Omega_5^1 = \Omega_4 \cap \{(d^o, \bar{k}^o) / A_1(\bar{k}^o) B_1(d^o, \bar{k}^o) \neq 0\}.$$

For (b), we know that $M_3(\bar{c}, d, \bar{k}, \bar{t})$ is primitive w.r.t. $\bar{c}$. If $M_3(\bar{c}, d, \bar{k}, \bar{t})$ (considered as a polynomial in $\mathbb{C}[d, \bar{k}, \bar{t}][\bar{c}]$) has only one term, then its coefficient w.r.t. $\bar{c}$ must be constant, and so M_3 remains primitive under specialization of $(d, \bar{k})$. Suppose, on the other hand, that $M_3(\bar{c}, d, \bar{k}, \bar{t})$ has more than one term, and let:

$$M_{3,1}(d, \bar{k}, \bar{t}), \ldots, M_{3,\rho}(d, \bar{k}, \bar{t})$$

be an (arbitrary) ordering of its non-zero coefficients w.r.t. $\bar{c}$. Let $\Gamma_1(d, \bar{k}, \bar{t}) = M_{3,1}(d, \bar{k}, \bar{t})$, and for $j = 2, \ldots, \rho$ let

$$\Gamma_j(d, \bar{k}, \bar{t}) = \gcd\left(M_{3,j}(d, \bar{k}, \bar{t}), \Gamma_{j-1}(d, \bar{k}, \bar{t})\right)$$

Note that, for $j = 1, \ldots, \rho$, the $M_{3,j}$ are homogeneous in $\bar{t}$ of the same degree. Thus, the Γ_j are either homogeneous in $\bar{t}$, or they only depend on $(d, \bar{k})$.

Since M_3 is primitive w.r.t. $\bar{c}$, let j^o be the first index value in $1, \ldots, \rho$ for which $\Gamma_{j^o}(d, \bar{k}, \bar{t}) = 1$. If $j^o = 1$, then $M_{3,1}(d, \bar{k}, \bar{t})$ is a constant, and in this case it is obvious that M_3 remains primitive under specialization of $(d, \bar{k})$. If $j^o > 1$, we consider:

$$\mathrm{Res}_{t_1}(M_{3,j^o}(d, \bar{k}, \bar{t}), \Gamma_{j^o-1}(d, \bar{k}, \bar{t}))$$

This resultant is not identically zero, because we have assumed that $\Gamma_{j^o-1}(d, \bar{k}, \bar{t}) = 1$. Since the involved polynomials are homogeneous in $\bar{t}$, this resultant is of the form $t_2^p \Phi(d, \bar{k})$ for some $p \in \mathbb{N}$ and some $\Phi \in \mathbb{C}[d, \bar{k}]$. Now, because of the construction, if $\Phi(d^o, \bar{k}^o) \neq 0$, the specialization $M_3(\bar{c}, d^o, \bar{k}^o, \bar{t})$ is primitive w.r.t. $\bar{c}$. Thus, set:

$$\Omega_5^2 = \Omega_5^1 \cap \{(d^o, \bar{k}^o)/\Phi(d^o, \bar{k}^o) \neq 0\}.$$

For (c) we use a similar construction. If either M_1 or M_2 do not depend on $\bar{t}_h$, the result is trivial. Otherwise, M_1 and M_2 are both homogeneous polynomials in $\bar{t}$, so the resultant

$$\mathrm{Res}_{t_1}(M_1(\bar{t}), M_2(d, \bar{k}, \bar{t}))$$

is of the form $t_2^{\tilde{p}} \tilde{\Phi}(d, \bar{k})$ for some $\tilde{p} \in \mathbb{N}$ and some $\tilde{\Phi}_1 \in \mathbb{C}[d, \bar{k}]$. Thus, if $\tilde{\Phi}_1(d^o, \bar{k}^o) \neq 0$, then $M_1(\bar{t})$ and $M_2(d^o, \bar{k}^o, \bar{t})$ do not have common factors of positive degree in t_1. A similar construction can be carried out w.r.t. t_2, obtaining a certain $\tilde{\Phi}_2$. Thus, set:

$$\Omega_5^3 = \Omega_5^2 \cap \{(d^o, \bar{k}^o)/\tilde{\Phi}_1(d^o, \bar{k}^o)\tilde{\Phi}_2(d^o, \bar{k}^o) \neq 0\}.$$

The construction shows that the lemma holds for $\Omega_5 = \Omega_5^3$. □

Finally, we are ready to state and prove the degree formula.

Theorem 4.45 (Total Degree Formula for the Offset of a Rational Surface). *Let T_0 and T be as in Equations 4.22 and 4.24 (page 150). Then:*

$$\boxed{m \cdot \delta = \deg_{\{\bar{t}\}}\left(\mathrm{PP}_{(d,\bar{k})}\left(\mathrm{Con}_{\bar{c}}\left(\mathrm{Res}_{t_0}\left(T_0(\bar{k}, \bar{t}_h), T(\bar{c}, d, \bar{k}, \bar{t}_h)\right)\right)\right)\right) = \deg_{\bar{t}}(M_2(d, \bar{k}, \bar{t}))}$$

where m is the tracing index of P (see Remark 4.3, page 126), and if $g(d, \bar{x})$ is the generic offset polynomial of Σ, then $\delta = \deg_{\bar{x}}(g(d, \bar{x}))$.

Proof. Recall that (see Remark 4.34, page 154), since $\Omega_3 \subset \Omega_2$, if $(d^o, \bar{k}^o) \in \Omega_3$ it holds that

$$m\delta = \#\left(\mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)\right).$$

Thus, to prove the theorem it suffices to show that for any of these $(d^o, \bar{k}^o)$, it holds that

$$\deg_{\bar{t}}\left(M_2(d, \bar{k}, \bar{t})\right) = \#\left(\mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)\right).$$

In order to do this, we will specialize at $(d^o, \bar{k}^o)$. More specifically, we will show that there is an open non-empty subset $\Omega_6 \subset \Omega_5$, such that, if $(d^o, \bar{k}^o) \in \Omega_6$, then $\deg_{\bar{t}}\left(M_2(d^o, \bar{k}^o, \bar{t})\right)$ equals $\#\left(\mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)\right)$.

Let Ω_5 be as in Lemma 4.44 (page 171), and let $(d^o, \bar{k}^o) \in \Omega_5$. Note that $M_1(\bar{t})$ and $M_2(d^o, \bar{k}^o, \bar{t})$ both factor as product of linear factors. there exists $\gamma \in \mathbb{C}$ such that $(\gamma : \alpha : \beta) \in \Psi_5^P(d^o, \bar{k}^o)$. Let us see that if $M_1(\bar{t}^o) = 0$, with $\bar{t}^o = (t_1^o, t_2^o)$, and there is t_0^o such that $(t_0^o : t_1^o : t_2^o) \in \Psi_5^P(d^o, \bar{k}^o)$, then $\bar{t}_h^o \notin \mathcal{A}_h$. In fact, if $t_0^o = 0$, the result follows from Proposition 4.32 (page 152) and Proposition 4.35 (page 154). Thus, w.l.o.g we suppose that $\bar{t}_h^o = (1 : t_1^o : t_2^o)$, with $\bar{t}_h^o \in \mathcal{A}_h$. Then using Proposition 4.35 (page 154), we get $\bar{t}_h^o \notin \mathcal{I}_5^{P_h}(\Omega_2)$. This is a contradiction, since $M_1(\bar{t})$ does not depend on $(d, \bar{k})$.

We will now show that there is an open set $\Omega_6 \subset \Omega_5$ such that if $(d^o, \bar{k}^o) \in \Omega_6$ and $M_2(d^o, \bar{k}^o, \bar{t}^o) = 0$, then there is t_0^o such that $\bar{t}_h^o = (t_0^o, \bar{t}^o) \in \Psi_5^P(d^o, \bar{k}^o)$. This follows from Lemma 1.35, page 32. Let us define:

$$\begin{cases} \Omega_6^1 = \Omega_5 \cap \{(d^o, \bar{k}^o) \mid \deg_{\bar{t}_0}(T_i(d^o, \bar{k}^o, \bar{t}_h)) = \deg_{\bar{t}_0}(T_i(d, \bar{k}, \bar{t}_h)) \text{ for } i = 1, 2, 3\} \\ \Omega_6^2 = \Omega_6^1 \cap \{(d^o, \bar{k}^o) \mid \deg_{\bar{t}_h}(T_i(d^o, \bar{k}^o, \bar{t}_h)) = \deg_{\bar{t}_h}(T_i(d, \bar{k}, \bar{t}_h)) \text{ for } i = 1, 2, 3\} \\ \Omega_6^3 = \Omega_6^2 \cap \{(d^o, \bar{k}^o) \mid \gcd(T_1(d^o, \bar{k}^o, \bar{t}_h), T_2(d^o, \bar{k}^o, \bar{t}_h), T_3(d^o, \bar{k}^o, \bar{t}_h)) = 1\} \end{cases}$$

The sets Ω_6^1 and Ω_6^2 are open and non-empty because they are defined by the non-vanishing of the corresponding leading coefficients. The fact that Ω_6^3 is open and non-empty follows from a similar argument to the proof of Lemma 4.44(c) (page 171). Finally, take $\Omega_6 = \Omega_6^3$. Then, (i), (ii) and (iii) in Lemma 1.35 hold because of the construction of Ω_6^i for $i = 1, 2, 3$, respectively. And also

$$\mathrm{lc}_{t_0}(T_0)(\bar{t}^o) \cdot \mathrm{lc}_{t_0}(T)(\bar{c}, \bar{t}^o) \neq 0$$

holds because of the construction of Ω_5^1 in Lemma 4.44 (page 171), and because $\Omega_6 \subset \Omega_5$.

Let $(d^o, \bar{k}^o) \in \Omega_6$. If $\bar{t}_h^o \in \mathcal{A} \cap \Psi_5^{P_h}(d^o, \bar{k}^o)$, then $M_1(\bar{t}^o) M_2(d^o, \bar{k}^o, \bar{t}^o) = 0$. Since we have seen that $M_1(\bar{t}^o) \neq 0$, one concludes that $M_2(d^o, \bar{k}^o, \bar{t}^o) = 0$. Conversely, let $\bar{t}^o$ be such that $M_2(d^o, \bar{k}^o, \bar{t}^o) = 0$. Then, by the construction of Ω_6, there is t_0^o such that $(t_0^o : \bar{t}^o) \in \Psi_5^P(d^o, \bar{k}^o)$. Let us see that $\bar{t}_h^o \in \mathcal{A}_h$. If $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega_2)$, then because of the invariance, $M_1(\bar{t}^o) = 0$, and this contradicts Lemma 4.44(c) (page 171). Thus, $\bar{t}_h^o \notin \mathcal{I}_5^{P_h}(\Omega_2)$, and by Proposition 4.40 (page 159), one has $\bar{t}_h^o \in \mathcal{A}_h$.

Thus, we have shown that for each of the factors of $M_2(d^o, \bar{k}^o, \bar{t})$ there is a point $\bar{t}_h^o \in \mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)$ such that $M_2(d^o, \bar{k}^o, \bar{t}^o) = 0$, and conversely. Let $L^{(\alpha,\beta)}(\bar{t}) = \beta t_1 - \alpha t_2$ be one of these factors of $M_2(d^o, \bar{k}^o, \bar{t})$, and let $\mathcal{L}^{(\alpha,\beta)}$ the line defined by the equation $L^{(\alpha,\beta)}(\bar{t}) = 0$. By Lemma 4.44(c) (page 171), one has

$$\#\left(\mathcal{L}^{(\alpha,\beta)} \cap \mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)\right) = \#\left(\mathcal{L}^{(\alpha,\beta)} \cap \Psi_5^{P_h}(d^o, \bar{k}^o)\right). \tag{4.44}$$

If we define

$$p(\alpha,\beta) = \#\big(\mathcal{L}^{(\alpha,\beta)} \cap \Psi_5^{P_h}(d^o,\bar{k}^o)\big),$$

then we will show that $L^{(\alpha,\beta)}(\bar{t})$ appears in $M_2(d^o,\bar{k}^o,\bar{t})$ with exponent equal to $p(\alpha,\beta)$. From this it will follow that:

$$\#\left(\mathcal{A}_h \cap \Psi_5^{P_h}(d^o,\bar{k}^o)\right) = \sum_{(\alpha,\beta)} \#\big(\mathcal{L}^{(\alpha,\beta)} \cap \Psi_5^{P_h}(d^o,\bar{k}^o)\big) = \sum_{(\alpha,\beta)} p(\alpha,\beta) = \deg_{\bar{t}}\left(M_2(d,\bar{k},\bar{t})\right),$$

and this will conclude the proof of the theorem.

To prove our claim, note that $\mathcal{A}_h \cap \Psi_5^{P_h}(d^o,\bar{k}^o)$ is a finite set, and by Proposition 4.43 (page 160), if $\bar{t}_h^o \in \mathcal{A}_h \cap \Psi_5^{P_h}(d^o,\bar{k}^o)$, then:

$$\min_{i=1,2,3}\left(\mathrm{mult}_{\bar{t}^o}(\mathcal{T}_o(\bar{k}^o),\mathcal{T}_i(d^o,\bar{k}^o))\right) = 1. \tag{4.45}$$

Recall that

$$T_0^{\bar{k}^o}(\bar{t}_h) = T_0(d^o,\bar{k}^o,\bar{t}_h)$$

and

$$T^{(d^o,\bar{k}^o)}(\bar{c},\bar{t}_h) = T(\bar{c},d^o,\bar{k}^o,\bar{t}_h) = c_1T_1(d^o,\bar{k}^o,\bar{t}_h) + c_2T_2(d^o,\bar{k}^o,\bar{t}_h) + c_3T_3(d^o,\bar{k}^o,\bar{t}_h).$$

For $\bar{c}^o \in \mathbb{K}^3$, let $\mathcal{T}^{(\bar{c}^o,d^o,\bar{k}^o)}$ be the algebraic closed subset of $\mathbb{P}^2$ defined by the equation $T^{(d^o,\bar{k}^o)}(\bar{c}^o,\bar{t}_h) = 0$. Note that there is an open set of values $\bar{c}^o$ for which $\mathcal{T}^{(\bar{c}^o,d^o,\bar{k}^o)}$ is indeed a curve. Let us see that there is an open subset $A(\bar{t}_h^o) \subset \mathbb{K}^3$, such that if $c^o \in A(\bar{t}_h^o)$, then

$$\mathrm{mult}_{\bar{t}_h^o}(\mathcal{T}_0^{\bar{k}^o},\mathcal{T}^{(\bar{c}^o,d^o,\bar{k}^o)}) = 1.$$

To prove this, let $\mathcal{P}(\bar{\tau})$ be a place of $\mathcal{T}_0^{\bar{k}^o}$ at $\bar{t}_h^o$. Then, by Equation 4.45 the order of the power series $T^{(d^o,\bar{k}^o)}(\bar{c},\mathcal{P}(\bar{\tau}))$ is one. From this, one sees that it suffices to take $A(\bar{t}_h^o)$ to be the open set of values $\bar{c}^o$ for which the order of $T^{(d^o,\bar{k}^o)}(\bar{c},\mathcal{P}(\bar{\tau}))$ does not increase.

Let now

$$\bar{c}^o \in \bigcap_{\bar{t}_h^o \in \mathcal{A}_h \cap \Psi_5^{P_h}(d^o,\bar{k}^o)} A(\bar{t}^o).$$

Applying Lemma 1.34 (page 30) to the curves $\mathcal{T}_0^{\bar{k}^o}$ and $\mathcal{T}^{(\bar{c}^o,d^o,\bar{k}^o)}$, and the line $\mathcal{L}^{(\alpha,\beta)}$, one concludes that the factor $\beta t_1 - \alpha_j t_2$ appears in $\mathrm{Res}_{t_0}\left(T_0^{\bar{k}^o}(\bar{t}_h), T^{(d^o,\bar{k}^o)}(\bar{c}^o,\bar{t}_h)\right)$ with exponent equal to:

$$\sum_{\bar{t}_h^o \in \mathcal{L}^{(\alpha,\beta)} \cap \mathcal{A}_h \cap \Psi_5^{P_h}(d^o,\bar{k}^o)} \mathrm{mult}_{\bar{t}_h^o}(\mathcal{T}_0^{\bar{k}^o},\mathcal{T}^{(\bar{c}^o,d^o,\bar{k}^o)}) = \#\big(\mathcal{L}^{(\alpha,\beta)} \cap \mathcal{A}_h \cap \Psi_5^{P_h}(d^o,\bar{k}^o)\big).$$

Taking Equation 4.44 into account, this finishes the proof of our claim, and of the theorem. $\square$

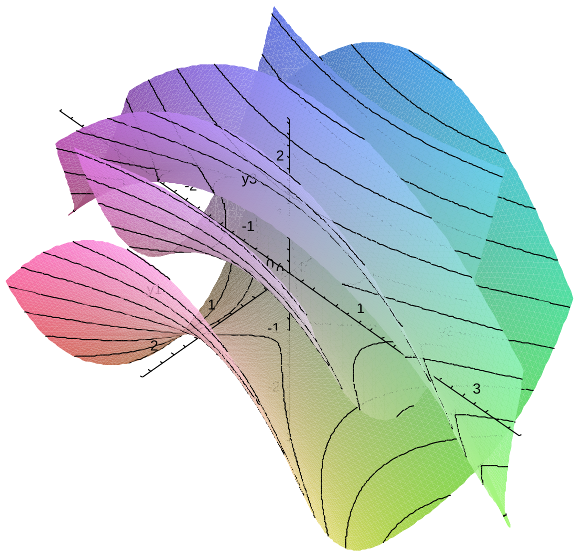

Figure 4.1: Hyperbolic paraboloid and one of its offsets

We will finish this section with some examples, illustrating the use of the degree formula in Theorem 4.45 (page 172). The implicit equations in these examples have been obtained with the Computer Algebra System CoCoA (see [13]).

Example 4.46. *Let Σ be the surface (a hyperbolic paraboloid) with implicit equation*

$$y_3 - y_1^2 + \frac{y_2^2}{4} = 0.$$

A rational –in fact polynomial– parametrization of Σ is given by:

$$P(t_1, t_2) = (t_1, 2t_2, t_1^2 - t_2^2).$$

From the form of its two first components, it is clear that this is a proper parametrization. This surface and its offset at $d^o = 1$ are illustrated in Figure 4.1. The homogeneous associated normal vector is

$$N(\bar{t}_h) = (-2t_1, t_2, t_0).$$

Then the auxiliary curves are:

$T_0(\bar{t}_h) = 2\,k_1\,t_2\,t_0^2 - k_1\,t_2\,t_1^2 + k_1\,t_2^3 - t_1\,t_0^2 k_2 + 5\,t_1\,t_0\,k_3\,t_2 - 2\,k_2\,t_1^3 + 2\,t_1\,k_2\,t_2^2,$

$T_1(\bar{t}_h) = 4\,t_1^6 k_2^2 - 7\,t_1^4 k_2^2 t_2^2 - 16\,t_1^4 k_2\,k_3\,t_2\,t_0 + 2\,t_1^2 k_2^2 t_2^4 + 12\,t_1^2 k_2\,t_2^3 k_3\,t_0 + 16\,t_1^2 k_3^2 t_2^2 t_0^2 + t_2^6 k_2^2 + 4\,t_2^5 k_2\,k_3\,t_0 + 4\,t_2^4 k_3^2 t_0^2 + t_0^2 k_2^2 t_1^4 - 2\,t_0^2 k_2^2 t_1^2 t_2^2 - 4\,t_0^3 k_2\,t_1^2 k_3\,t_2 + t_0^2 k_2^2 t_2^4 + 4\,t_0^3 k_2\,t_2^3 k_3 + 4\,t_0^4 k_3^2 t_2^2 - d^2 t_0^6 k_2^2 + 2\,d^2 t_0^5 k_2\,k_3\,t_2 - d^2 t_0^4 k_3^2 t_2^2,$

$T_2(\bar{t}_h) = 4\,t_1^4t_0^2k_3^2 - 8\,t_1^5t_0\,k_3\,k_1 + 6\,t_1^3t_0\,k_3\,k_1\,t_2^2 + 4\,t_1^6k_1^2 - 7\,t_1^4k_1^2t_2^2 + 2\,t_1^2k_1^2t_2^4 + t_1^2k_3^2t_2^2t_0^2 + 2\,t_2^4t_1\,t_0\,k_3\,k_1 + t_2^6k_1^2 + t_0^4t_1^2k_3^2 - 2\,t_0^3t_1^3k_3\,k_1 + 2\,t_0^3t_1\,k_3\,k_1\,t_2^2 + t_0^2k_1^2t_1^4 - 2\,t_0^2k_1^2t_1^2t_2^2 + t_0^2k_1^2t_2^4 - 4\,d^2t_0^4t_1^2k_3^2 - 4\,d^2t_0^5k_3\,t_1\,k_1 - d^2t_0^6k_1^2,$

$T_3(\bar{t}_h) = t_0^2(4\,k_2^2t_1^4 - 16\,t_1^3k_2\,k_1\,t_2 + 16\,k_1^2t_1^2t_2^2 + k_2^2t_1^2t_2^2 - 4\,t_2^3k_2\,t_1\,k_1 + 4\,k_1^2t_2^4 + t_0^2k_2^2t_1^2 - 4\,k_2\,t_1\,t_0^2k_1\,t_2 + 4\,k_1^2t_2^2t_0^2 - 4\,t_0^2d^2k_2^2t_1^2 - 4\,t_0^2d^2t_1\,k_2\,k_1\,t_2 - t_0^2d^2k_1^2t_2^2).$

Denoting, as in the degree formula,

$$R(\bar{c}, d, \bar{k}, \bar{t}) = \mathrm{Res}_{t_0}\left(T_0(\bar{k}, \bar{t}_h), T(\bar{c}, d, \bar{k}, \bar{t}_h)\right),$$

one has:

$R(\bar{c}, d, \bar{k}, \bar{t}) = (t_1-t_2)^2(t_1+t_2)^2(4\,t_2^4c_3^2k_1^4 + t_2^4c_2^2k_1^4 + t_2^4c_1^2k_2^4 + t_1^2t_2^2c_3^2k_1^2k_2^2 + 4\,t_1^2t_2^2c_1\,c_3\,k_1^2k_2^2 + 4\,t_1^2t_2^2c_2\,c_3\,k_1^2k_2^2 + 4\,t_1^2t_2^2c_2\,c_3\,k_1^4 + 4\,t_1^2t_2^2c_1\,c_3\,k_2^4 + 4\,t_1^2t_2^2c_1\,c_2\,k_2^2k_3^2 + 4\,t_1^2t_2^2c_1\,c_2\,k_1^2k_3^2 + 17\,t_1^2t_2^2c_1\,c_3\,k_2^2k_3^2 + 17\,t_1^2t_2^2c_2\,c_3\,k_1^2k_3^2 - 4\,t_1^2t_2^2c_1^2k_2^2k_3^2 + 17\,t_1^2c_2\,c_1\,k_3^4t_2^2 - 4\,t_1^2t_2^2c_2^2k_1^2k_3^2 - 6\,t_1\,t_2^3c_2\,c_3\,k_1^3k_2 - 6\,t_1\,t_2^3c_1\,c_3\,k_1\,k_2^3 + 12\,t_1\,t_2^3c_1\,c_3\,k_1\,k_2\,k_3^2 - 12\,t_1\,t_2^3c_1\,c_2\,k_1\,k_2\,k_3^2 + 12\,t_1\,t_2^3c_3^2k_1^3k_2 + 2\,t_2^4c_1\,c_2\,k_1^2k_2^2 - 4\,t_2^4c_1\,c_3\,k_1^2k_2^2 - 4\,t_2^4c_2\,c_3\,k_1^4 - 4\,t_1^4c_1\,c_3\,k_2^4 - 2\,t_1^2t_2^2c_2^2k_1^4 - 2\,t_1^2t_2^2c_1^2k_2^4 + 4\,t_1^4c_2^2k_3^4 - 4\,t_1^4c_1\,c_2\,k_2^2k_3^2 + 4\,t_1^4c_2^2k_1^2k_3^2 + 2\,t_1^4k_1^2k_2^2c_2\,c_1 - 4\,t_1^4c_2\,c_3\,k_1^2k_2^2 + 8\,t_1^4c_2\,c_3\,k_2^2k_3^2 - 12\,t_1^3t_2\,c_3^2k_1\,k_2^3 + 6\,t_1^3t_2\,c_1\,c_3\,k_1\,k_2^3 + 12\,t_1^3t_2\,c_1\,c_2\,k_1\,k_2\,k_3^2 - 12\,t_1^3t_2\,c_2\,c_3\,k_1\,k_2\,k_3^2 + 6\,t_1^3t_2\,c_2\,c_3\,k_1^3k_2 - 4\,t_1^2t_2^2c_1\,c_2\,k_1^2k_2^2 - 4\,t_2^4c_1\,c_2\,k_1^2k_3^2 + 8\,t_2^4c_1\,c_3\,k_1^2k_3^2 + 4\,c_1^2t_2^4k_3^4 + 4\,t_2^4c_1^2k_2^2k_3^2 + t_1^4k_2^4c_1^2 + 4\,t_1^4c_3^2k_2^4 + t_1^4k_1^4c_2^2) \cdot (-48\,d^2t_1^6t_2^4k_2^6 + 64\,d^2t_1^2t_2^8k_1^6 - 72\,d^2t_1^4t_2^6k_1^6 - 128\,d^4t_1^8t_2^2k_2^6 + 64\,d^4t_1^6t_2^4k_2^6 + d^4t_1^4t_2^6k_1^6 - 2\,d^4t_1^2t_2^8k_1^6 + 25\,t_1^6k_2^4t_2^4k_3^2 + 4\,k_2^6t_1^{10} - 78\,t_1^5t_2^5k_1\,k_2^5 - 154\,t_1^7t_2^3k_1\,k_2^5 + d^4k_1^6t_2^{10} + 64\,d^4k_2^6t_1^{10} + 8\,d^2k_1^6t_2^{10} + 32\,d^2k_2^6t_1^{10} + 136\,d^2t_1^5t_2^5k_1\,k_2^5 + 88\,d^2t_1^7t_2^3k_1\,k_2^5 + 320\,d^2t_1^8t_2^2k_1^2k_2^4 + 440\,d^2t_1^4t_2^6k_1^4k_2^2 - 100\,d^2t_1^6k_2^2k_3^2t_2^4k_1^2 - 170\,d^2t_1^3t_2^7k_1^3k_2^3 - 110\,d^2t_1^5t_2^5k_1^3k_2^3 + 280\,d^2t_1^7t_2^3k_1^3k_2^3 + 1200\,d^2t_1^7k_2^3k_3^2t_2^3k_1 - 400\,d^2t_1^8k_2^4k_3^2t_2^2 + 300\,d^2t_1^5k_2^3k_3^2t_2^5k_1 - 1200\,d^2t_1^5k_2\,k_3^2t_2^5k_1^3 - 25\,d^2t_1^4k_2^2k_3^2t_2^6k_1^2 - 400\,d^2t_1^4k_3^2t_2^6k_1^4 - 100\,d^2t_1^2k_3^2t_2^8k_1^4 - 370\,d^2t_1^6t_2^4k_1^4k_2^2 - 344\,d^2t_1^5t_2^5k_1^5k_2 + 16\,d^2t_1\,t_2^9k_1^5k_2 + 328\,d^2t_1^3t_2^7k_1^5k_2 - 300\,d^2t_1^3k_2\,k_3^2t_2^7k_1^3 - 70\,d^2t_1^2t_2^8k_1^4k_2^2 + 465\,t_1^8t_2^2k_1^2k_2^4 - 120\,d^4t_1^4t_2^6k_1^4k_2^2 + 160\,d^4t_1^3t_2^7k_1^3k_2^3 - 24\,d^4t_1^3t_2^7k_1^5k_2 + 60\,d^4t_1^2t_2^8k_1^4k_2^2 + 12\,d^4t_1\,t_2^9k_1^5k_2 + 2480\,t_1^4t_2^6k_1^4k_2^2 + 2400\,t_1^6k_2^2k_3^2t_2^4k_1^2 - 440\,t_1^3t_2^7k_1^3k_2^3 - 1920\,t_1^5t_2^5k_1^3k_2^3 - 1640\,t_1^7t_2^3k_1^3k_2^3 - 800\,t_1^7k_2^3k_3^2t_2^3k_1 + 100\,t_1^8k_2^4k_3^2t_2^2 + 16\,d^2t_1^8t_2^2k_2^6 - 200\,t_1^5k_2^3k_3^2t_2^5k_1 - 3200\,t_1^5k_2\,k_3^2t_2^5k_1^3 + 600\,t_1^4k_2^2k_3^2t_2^6k_1^2 + 1600\,t_1^4k_3^2t_2^6k_1^4 + 400\,t_1^2k_3^2t_2^8k_1^4 + 3160\,t_1^6t_2^4k_1^4k_2^2 - 3168\,t_1^5t_2^5k_1^5k_2 - 128\,t_1\,t_2^9k_1^5k_2 - 1504\,t_1^3t_2^7k_1^5k_2 - 800\,t_1^3k_2\,k_3^2t_2^7k_1^3 + 12\,t_1^8t_2^2k_2^6 + 9\,t_1^6t_2^4k_2^6 + 360\,t_1^2t_2^8k_1^4k_2^2 - 224\,d^2t_1^9t_2\,k_1\,k_2^5 + 20\,d^2t_1^4t_2^6k_1^2k_2^4 - 340\,d^2t_1^6t_2^4k_1^2k_2^4 + 192\,d^4t_1^9t_2\,k_1\,k_2^5 + 240\,d^4t_1^8t_2^2k_1^2k_2^4 - 384\,d^4t_1^7t_2^3k_1\,k_2^5 + 160\,d^4t_1^7t_2^3k_1^3k_2^3 - 480\,d^4t_1^6t_2^4k_1^2k_2^4 + 60\,d^4t_1^6t_2^4k_1^4k_2^2 + 192\,d^4t_1^5t_2^5k_1\,k_2^5 - 320\,d^4t_1^5t_2^5k_1^3k_2^3 + 12\,d^4t_1^5t_2^5k_1^5k_2 + 240\,d^4t_1^4t_2^6k_1^2k_2^4 + 16\,k_1^6t_2^{10} + 288\,t_1^2t_2^8k_1^6 + 1296\,t_1^4t_2^6k_1^6 - 68\,t_1^9t_2\,k_1\,k_2^5 - 100\,d^2t_1^6k_2^4t_2^4k_3^2 + 265\,t_1^4t_2^6k_1^2k_2^4 + 770\,t_1^6t_2^4k_1^2k_2^4).$

From this expression it is easy to check that $\mathrm{PP}_{(d,\bar{k})}\left(\mathrm{Con}_{\bar{c}}\left(\mathrm{Res}_{t_0}\left(T_0(\bar{k}, \bar{t}_h), T(\bar{c}, d, \bar{k}, \bar{t}_h)\right)\right)\right)$ *is the last factor in the above expression, and so:*

$$\deg_{\bar{t}}\left(\mathrm{PP}_{(d,\bar{k})}\left(\mathrm{Con}_{\bar{c}}\left(\mathrm{Res}_{t_0}\left(T_0(\bar{k}, \bar{t}_h), T(\bar{c}, d, \bar{k}, \bar{t}_h)\right)\right)\right)\right) = 10.$$

Using Theorem 4.45 one has that the total offset degree is $\delta = 10$. *In fact, in this case, using elimination techniques, it is possible to check this result, computing the generic* *offset polynomial:*

$g(d,\bar{x}) = -256x_1^{10} - 640x_1^8x_2^2 - 256x_1^8x_3^2 + 1408x_1^8d^2 - 400x_1^6x_2^4 - 384x_1^6x_2^2x_3^2 + 3232x_1^6x_2^2d^2 + 1152x_1^6x_3^2d^2 - 3088x_1^6d^4 + 80x_1^4x_2^6 - 16x_1^4x_2^4x_3^2 + 2448x_1^4x_2^4d^2 + 1696x_1^4x_2^2x_3^2d^2 - 5904x_1^4x_2^2d^4 - 1936x_1^4x_3^2d^4 + 3376x_1^4d^6 + 80x_1^2x_2^8 + 96x_1^2x_2^6x_3^2 + 832x_1^2x_2^6d^2 + 736x_1^2x_2^4x_3^2d^2 - 3744x_1^2x_2^4d^4 - 2272x_1^2x_2^2x_3^2d^4 + 4672x_1^2x_2^2d^6 + 1440x_1^2x_3^2d^6 - 1840x_1^2d^8 - 16x_2^{10} - 16x_2^8x_3^2 + 208x_2^8d^2 + 192x_2^6x_3^2d^2 - 928x_2^6d^4 - 736x_2^4x_3^2d^4 + 1696x_2^4d^6 + 960x_2^2x_3^2d^6 - 1360x_2^2d^8 - 400x_3^2d^8 + 400d^{10} + 3200x_1^8x_3 - 320x_1^6x_2^2x_3 + 3072x_1^6x_3^3 - 12608x_1^6x_3d^2 - 1560x_1^4x_2^4x_3 - 2944x_1^4x_2^2x_3^3 + 496x_1^4x_2^2x_3d^2 - 9088x_1^4x_3^3d^2 + 18088x_1^4x_3d^4 + 1640x_1^2x_2^6x_3 + 1696x_1^2x_2^4x_3^3 + 7016x_1^2x_2^4x_3d^2 + 5184x_1^2x_2^2x_3^3d^2 - 2184x_1^2x_2^2x_3d^4 + 8480x_1^2x_3^3d^4 - 11080x_1^2x_3d^6 - 320x_2^8x_3 - 288x_2^6x_3^3 + 2912x_2^6x_3d^2 + 2272x_2^4x_3^3d^2 - 7072x_2^4x_3d^4 - 3680x_2^2x_3^3d^4 + 2080x_2^2x_3d^6 - 2400x_3^3d^6 + 2400x_3d^8 + 2544x_1^8 - 9144x_1^6x_2^2 - 10752x_1^6x_3^2 - 10520x_1^6d^2 + 4479x_1^4x_2^4 - 6976x_1^4x_2^2x_3^2 + 25770x_1^4x_2^2d^2 - 11776x_1^4x_3^4 + 21568x_1^4x_3^2d^2 + 16583x_1^4d^4 - 684x_1^2x_2^6 + 10304x_1^2x_2^4x_3^2 + 2700x_1^2x_2^4d^2 + 9088x_1^2x_2^2x_3^4 + 25056x_1^2x_2^2x_3^2d^2 - 23444x_1^2x_2^2d^4 + 14720x_1^2x_3^4d^2 - 7840x_1^2x_3^2d^4 - 11980x_1^2d^6 + 24x_2^8 - 2472x_2^6x_3^2 + 160x_2^6d^2 - 1936x_2^4x_3^4 + 14488x_2^4x_3^2d^2 - 2752x_2^4d^4 + 8480x_2^2x_3^4d^2 - 15160x_2^2x_3^2d^4 + 6080x_2^2d^6 - 400x_3^4d^4 - 3000x_3^2d^6 + 3400d^8 - 19008x_1^6x_3 + 25896x_1^4x_2^2x_3 + 3328x_1^4x_3^3 + 44072x_1^4x_3d^2 - 6534x_1^2x_2^4x_3 + 23616x_1^2x_2^2x_3^3 + 2484x_1^2x_2^2x_3d^2 + 15360x_1^2x_3^5 + 15680x_1^2x_3^3d^2 - 31790x_1^2x_3d^4 + 312x_2^6x_3 - 9112x_2^4x_3^3 + 3112x_2^4x_3d^2 - 5760x_2^2x_3^5 + 31120x_2^2x_3^3d^2 - 22360x_2^2x_3d^4 + 9600x_3^5d^2 - 17400x_3^3d^4 + 7800x_3d^6 - 6240x_1^6 + 3360x_1^4x_2^2 + 29984x_1^4x_3^2 + 15816x_1^4d^2 - 510x_1^2x_2^4 - 18472x_1^2x_2^2x_3^2 + 11022x_1^2x_2^2d^2 + 14080x_1^2x_3^4 + 8440x_1^2x_3^2d^2 - 16520x_1^2d^4 + 15x_2^6 + 1319x_2^4x_3^2 - 669x_2^4d^2 - 15680x_2^2x_3^4 + 15010x_2^2x_3^2d^2 + 1045x_2^2d^4 - 6400x_3^6 + 27200x_3^4d^2 - 28825x_3^2d^4 + 8025d^6 + 14880x_1^4x_3 - 4800x_1^2x_2^2x_3 - 13120x_1^2x_3^3 + 14320x_1^2x_3d^2 + 270x_2^4x_3 + 1720x_2^2x_3^3 - 4270x_2^2x_3d^2 - 9600x_3^5 + 17400x_3^3d^2 - 7800x_3d^4 - 400x_1^4 + 200x_1^2x_2^2 - 11040x_1^2x_3^2 + 7840x_1^2d^2 - 25x_2^4 + 1440x_2^2x_3^2 - 140x_2^2d^2 - 400x_3^4 - 3000x_3^2d^2 + 3400d^4 + 800x_1^2x_3 - 200x_2^2x_3 + 2400x_3^3 - 2400x_3d^2 - 400x_3^2 + 400d^2$

This is, as predicted by our formula, a polynomial of degree 10 *in* $\bar{x}$.

Example 4.47. *To illustrate the behavior of the degree formula in the case of non-proper parametrizations, let us consider the surface* Σ *defined by the parametrization:*

$$P(t_1,t_2) = (t_1^3, t_2, t_1^6 + t_2^2).$$

This is a parametrization of the circular paraboloid with implicit equation $y_3 = y_1^2 + y_2^2$*; the parametrization* P *has been obtained by replacing* t_1 *with* t_1^3 *in the usual proper parametrization* $\tilde{P}$ *of* Σ*, which is given by:*

$$\tilde{P}(t_1,t_2) = (t_1, t_2, t_1^2 + t_2^2).$$

Thus, the tracing index of the parametrization P *in this example is* $\mu = 3$*. In Example 4.63 (page 190) of Section 4.4, applying the formulae for surfaces of revolution we will see that in this example one has* $\deg_{\bar{x}}(\mathcal{O}_d(\Sigma)) = 6$*. Computing with* P *we obtain the following associated normal vector:*

$$N(\bar{t}_h) = (-2t_1^3, -2t_0^2t_2, t_0^3).$$

Then the auxiliary curves are:

$T_0(\bar{t}_h) = -k_2\, t_1^3 + k_1\, t_0^2 t_2$

$T_1(\bar{t}_h) = 4\, t_1^{18} {k_2}^2 + 12\, t_1^{12} {k_2}^2 t_2^2 t_0^4 - 8\, t_1^{12} k_2\, k_3\, \mathit{t2}\, t_0^5 + 12\, t_1^6 {k_2}^2 t_2^4 t_0^8 - 16\, t_1^6 k_2\, t_2^3 t_0^9 k_3 + 4\, t_1^6 {k_3}^2 t_2^2 t_0^{10} + 4\, t_2^6 t_0^{12} {k_2}^2 - 8\, t_2^5 t_0^{13} k_2\, k_3 + 4\, t_2^4 t_0^{14} {k_3}^2 + t_0^6 {k_2}^2 t_1^{12} + 2\, t_0^{10} {k_2}^2 t_1^6 t_2^2 - 2\, t_0^{11} k_2\, t_1^6 k_3\, t_2 + t_0^{14} {k_2}^2 t_2^4 - 2\, t_0^{15} k_2\, t_2^3 k_3 + t_0^{16} {k_3}^2 t_2^2 - d^2 t_0^{18} {k_2}^2 - 4\, d^2 t_0^{17} k_2\, k_3\, \mathit{t2} - 4\, d^2 t_0^{16} {k_3}^2 t_2^2$

$T_2(\bar{t}_h) = 4\, t_1^{12} {k_3}^2 t_0^6 - 8\, t_1^{15} k_3\, t_0^3 k_1 - 16\, t_1^9 k_3\, t_0^7 k_1\, t_2^2 + 4\, t_1^{18} {k_1}^2 + 12\, t_1^{12} {k_1}^2 t_2^2 t_0^4 + 12\, t_1^6 {k_1}^2 t_2^4 t_0^8 + 4\, t_1^6 {k_3}^2 t_2^2 t_0^{10} - 8\, t_2^4 t_0^{11} k_3\, t_1^3 k_1 + 4\, t_2^6 t_0^{12} {k_1}^2 + t_0^{12} {k_3}^2 t_1^6 - 2\, t_0^9 k_3\, t_1^9 k_1 - 2\, t_0^{13} k_3\, t_1^3 k_1\, t_2^2 + t_0^6 {k_1}^2 t_1^{12} + 2\, t_0^{10} {k_1}^2 t_1^6 t_2^2 + t_0^{14} {k_1}^2 t_2^4 - 4\, d^2 t_0^{12} t_1^6 {k_3}^2 - 4\, d^2 t_0^{15} k_3\, \mathit{t1}^3 k_1 - d^2 t_0^{18} {k_1}^2$

$T_3(\bar{t}_h) = t_0^6 (k_2\, t_1^3 - k_1\, t_0^2 t_2)^2 (4\, t_1^6 + t_0^6 + 4\, t_2^2 t_0^4 - 4\, d^2 t_0^6)$

Denoting, as in the degree formula,

$$R(\bar{c}, d, \bar{k}, \bar{t}) = \mathrm{Res}_{t_0}\left(T_0(\bar{k}, \bar{t}_h), T(\bar{c}, d, \bar{k}, \bar{t}_h)\right),$$

one has:
$R(\bar{c}, d, \bar{k}, \bar{t}) = (k_1^2 a_2 + k_2^2 a_1)^2 t_1^{36} (-8\, k_2^{12} t_1^{12} k_1^2 t_2^6 k_3^4 d^2 + 16\, k_1^{18} t_2^{18} + 16\, k_2^{12} k_1^6 t_2^{18} + 96\, k_2^{10} k_1^8 t_2^{18} + 240\, k_2^8 k_1^{10} t_2^{18} + 320\, k_2^6 k_1^{12} t_2^{18} + 240\, k_1^{14} t_2^{18} k_2^4 + 96\, k_1^{16} t_2^{18} k_2^2 + t_1^{18} d^4 k_2^{18} + t_1^6 k_1^4 t_2^{12} k_2^{14} + 8\, t_1^3 k_1^5 t_2^{15} k_2^{13} + 4\, t_1^6 k_1^6 t_2^{12} k_2^{12} + 40\, t_1^3 k_1^7 \mathit{t2}^{15} k_2^{11} + 6\, t_1^6 k_1^8 t_2^{12} k_2^{10} + 80\, t_1^3 k_1^9 t_2^{15} k_2^9 + 80\, t_1^3 k_1^{11} t_2^{15} k_2^7 + 4\, k_2^8 t_1^6 k_1^{10} t_2^{12} + 40\, k_2^5 t_1^3 k_1^{13} t_2^{15} + k_2^6 t_1^6 k_1^{12} t_2^{12} + 8\, k_2^3 t_1^3 k_1^{15} t_2^{15} - 32\, k_2^6 t_1^6 k_1^{10} t_2^{12} d^2 k_3^2 + 16\, k_2^{10} t_1^6 k_1^4 t_2^{12} k_3^4 + 32\, k_2^8 t_1^6 k_1^6 t_2^{12} k_3^4 - 192\, k_2^7 t_1^3 k_1^9 t_2^{15} k_3^2 - 128\, k_2^5 t_1^3 k_1^{11} t_2^{15} k_3^2 + 16\, k_2^6 t_1^6 k_1^8 t_2^{12} k_3^4 - 32\, k_2^3 t_1^3 k_1^{13} t_2^{15} k_3^2 - 4\, k_2^{11} t_1^9 k_1^5 t_2^9 k_3^2 + 8\, k_2^{11} t_1^9 k_1^3 t_2^9 k_3^4 - 2\, k_2^9 t_1^9 k_1^7 t_2^9 k_3^2 + 8\, k_2^9 t_1^9 k_1^5 t_2^9 k_3^4 - 48\, k_2^8 t_1^6 k_1^8 t_2^{12} k_3^2 - 16\, k_2^6 t_1^6 k_1^{10} t_2^{12} k_3^2 - 32\, k_2^{12} k_1^4 t_2^{12} t_1^6 d^2 \mathit{k3}^2 - 32\, k_2^{11} k_1^5 t_2^{15} t_1^3 k_3^2 - 128\, k_2^9 k_1^7 t_2^{15} t_1^3 k_3^2 + 16\, k_2^{12} t_1^{12} k_1^2 t_2^6 d^4 k_3^4 - 144\, k_2^{11} t_1^9 k_1^5 t_2^9 d^2 k_3^2 - 32\, k_2^{11} t_1^9 k_1^3 t_2^9 d^2 k_3^4 - 96\, k_2^{10} t_1^6 k_1^6 t_2^{12} d^2 k_3^2 - 2\, t_1^{12} d^2 k_2^{16} k_1^2 t_2^6 - 2\, t_1^{15} d^2 k_2^{15} k_1\, t_2^3 k_3^2 - 8\, t_1^9 d^2 k_2^{15} k_1^3 t_2^9 - 8\, t_1^{15} d^4 k_2^{15} k_1\, t_2^3 k_3^2 - 4\, t_1^{12} d^2 k_2^{14} k_1^4 t_2^6 - 24\, t_1^{12} d^2 k_2^{14} \mathit{k1}^2 t_2^6 k_3^2 - 24\, t_1^9 d^2 k_2^{13} k_1^5 t_2^9 - 2\, t_1^{12} d^2 k_2^{12} k_1^6 t_2^6 - 24\, t_1^{12} d^2 k_2^{12} k_1^4 t_2^6 k_3^2 - 24\, t_1^9 d^2 k_2^{11} k_1^7 t_2^9 - 8\, t_1^9 d^2 k_2^9 k_1^9 t_2^9 - 2\, t_1^9 k_1^3 t_2^9 k_2^{13} k_3^2 - 72\, t_1^9 k_1^3 t_2^9 k_2^{13} d^2 k_3^2 - 16\, t_1^6 k_1^4 t_2^{12} k_2^{12} k_3^2 - 48\, t_1^6 k_1^6 t_2^{12} k_2^{10} k_3^2 + k_2^{12} t_1^{12} k_1^2 t_2^6 k_3^4 - 72\, k_2^9 t_1^9 k_1^7 t_2^9 d^2 k_3^2 - 32\, k_2^9 t_1^9 k_1^5 t_2^9 d^2 k_3^4 - 96\, k_2^8 t_1^6 k_1^8 t_2^{12} d^2 k_3^2)$.

From this expression it is easy to check that

$$\deg_{\bar{t}}\left(\mathrm{PP}_{(d,\bar{k})}\left(\mathrm{Con}_{\bar{c}}\left(\mathrm{Res}_{t_0}\left(T_0(\bar{k}, \bar{t}_h), T(\bar{c}, d, \bar{k}, \bar{t}_h)\right)\right)\right)\right) = 18.$$

This agrees with the expected result $\mu \cdot \delta = 3 \cdot 6 = 18$.

Example 4.48. *Let Σ be the surface (Whitney Umbrella) with implicit equation $y_1^2 - y_2^2 y_3 = 0$. A proper rational parametrization of Σ is given by:*

$$P(t_1, t_2) = (t_1 t_2, t_2, t_1).$$

This surface is illustrated in Figure 4.2. The homogeneous associated normal vector is

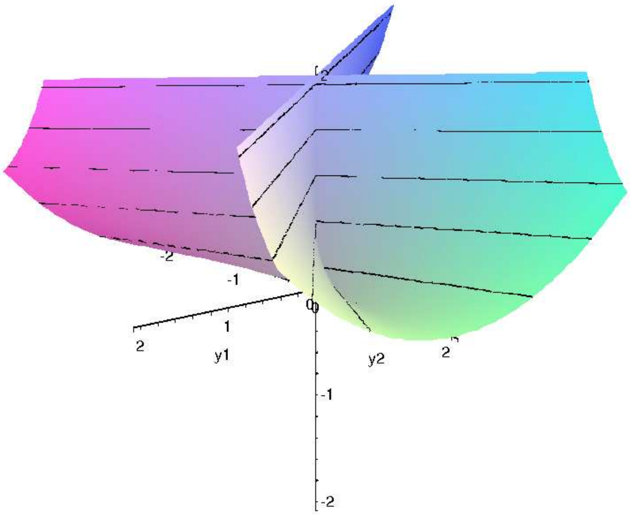

Figure 4.2: The Whitney Umbrella

$$N(\bar{t}_h) = (2t_1t_2, -2t_1^2, -t_0t_2).$$

Then the auxiliary curves are:

$T_0(\bar{t}_h) = -k_1\, t_0^2t_2^2 + 2\, k_1\, t_1^4 + t_1\, t_0^2k_2\, t_2 - 2\, t_1^3t_0\, k_3 + 2\, t_1^3t_2\, k_2 - 2\, t_1\, t_2^2k_3\, t_0$

$T_1(\bar{t}_h) = 4\, t_1^6t_2^2k_2^2 - 8\, t_1^4t_2^3k_2\, k_3\, t_0 + 4\, t_1^2t_2^4k_3^2t_0^2 + 4\, t_1^8k_2^2 - 8\, t_1^6k_2\, k_3\, t_0\, t_2 + 4\, t_1^4k_3^2t_0^2t_2^2 + t_0^2t_2^2k_2^2t_1^4 - 2\, t_0^3t_2^3k_2\, t_1^2k_3 + t_0^4t_2^4k_3^2 - d^2t_2^6k_2^2t_0^2 + 4\, d^2t_2^5k_2\, t_1^2k_3\, t_0 - 4\, d^2t_2^4k_3^2t_1^4$

$T_2(\bar{t}_h) = 4\, t_1^4k_3^2t_0^2t_2^2 - 8\, t_1^5t_2^2k_3\, t_0\, k_1 + 4\, t_1^6t_2^2k_1^2 + 4\, t_1^6k_3^2t_0^2 - 8\, t_1^7k_3\, t_0\, k_1 + 4\, t_1^8k_1^2 + t_0^4t_2^2k_3^2t_1^2 - 2\, t_0^3t_2^2k_3\, t_1^3k_1 + t_0^2t_2^2k_1^2t_1^4 - 4\, d^2t_2^6t_1^2k_3^2 - 4\, d^2t_2^6t_1\, k_3\, k_1\, t_0 - d^2t_2^6k_1^2t_0^2$

$T_3(\bar{t}_h) = 4\, t_0^2t_2^2k_2^2t_1^4 - 8\, t_1^3t_2^3k_2\, t_0^2k_1 + 4\, t_1^2t_2^4k_1^2t_0^2 + 4\, t_1^6k_2^2t_0^2 - 8\, t_1^5k_2\, t_0^2k_1\, t_2 + 4\, t_0^2t_2^2k_1^2t_1^4 + t_0^4t_2^2k_2^2t_1^2 - 2\, t_0^4t_2^3k_2\, t_1\, k_1 + t_0^4t_2^4k_1^2 - 4\, d^2t_2^6k_2^2t_1^2 - 8\, d^2t_2^5t_1^3k_2\, k_1 - 4\, d^2t_2^4k_1^2t_1^4$

Denoting, as in the degree formula,

$$R(\bar{c}, d, \bar{k}, \bar{t}) = \mathrm{Res}_{t_0}\left(T_0(\bar{k}, \bar{t}_h), T(\bar{c}, d, \bar{k}, \bar{t}_h)\right),$$

one has:

$R(\bar{c}, d, \bar{k}, \bar{t}) = 4\, t_2^2\, t_1^4\, (-28\, t_1^6t_2^8k_3^2d^2k_1^2 + 4\, k_2^4t_1^8t_2^6d^2 + k_1^4d^4t_2^{12}t_1^2 - 6\, d^2k_1^4t_2^8t_1^6 + k_2^2k_1^2d^4t_2^{14} - 4\, k_1^4d^2t_2^{10}t_1^4 + 12\, k_1^4t_2^6t_1^8 - 20\, t_1^4t_2^{10}d^2k_1^2k_3^2 - 4\, k_2^2k_1^2d^4t_2^{12}t_1^2 - 4\, k_2^2t_1^8t_2^6k_3^2d^2 - 24\, k_2\, t_1^{11}t_2^3k_1\, k_3^2 - 2\, k_2\, d^4k_1^3t_2^{11}t_1^3 + 4\, k_2\, k_1^3d^2t_2^9t_1^5 + 2\, k_2\, k_1^3d^4t_2^{13}t_1 + 16\, k_2\, t_1^5t_2^9d^2k_3^2k_1 - 8\, k_2\, t_1^{13}t_2\, k_3^2k_1 - 8\, k_2\, t_1^7t_2^7k_3^2k_1 + 32\, k_2\, t_1^7t_2^7k_3^2d^2k_1 + 16\, k_2\, d^2k_1^3t_2^7t_1^7 + 16\, k_2\, k_3^2k_1\, d^2t_2^5t_1^9 - 24\, k_2\, k_3^2k_1\, t_2^5t_1^9 - 12\, t_1^8t_2^6k_3^2d^2k_1^2 - 4\, t_1^{10}t_2^4k_3^4d^2 + 4\, t_1^{12}t_2^2k_1^2k_3^2 + 4\, t_1^6t_2^8k_3^2k_1^2 - 16\, t_1^8t_2^6k_3^4d^2 - 4\, t_1^2t_2^{12}d^2k_3^4 + 12\, t_1^8t_2^6k_3^2k_1^2 + 12\, t_1^{10}t_2^4k_3^2k_1^2 -$

$6\,k_2^3k_1\,t_2^5t_1^9+12\,t_1^{10}t_2^4k_3^2k_2^2+4\,t_1^8t_2^6k_3^2k_2^2-16\,t_1^4t_2^{10}k_3^4d^2+4\,t_1^4t_2^{10}k_3^2d^2k_2^2-24\,t_1^6t_2^8k_3^4d^2+t_1^2t_2^{12}d^4k_2^4+$
$2\,t_1^6t_2^8d^2k_2^4+4\,t_1^6t_2^8k_3^2d^2k_2^2-20\,k_2^3k_1\,t_2\,t_1^{13}+4\,k_2^4t_1^{14}+9\,k_1^4t_2^4t_1^{10}-22\,k_2^3k_1\,t_2^3t_1^{11}+37\,k_2^2k_1^2t_2^2t_1^{12}+$
$44\,k_2^2k_1^2t_2^4t_1^{10}+10\,k_2^2k_1^2d^2t_2^{10}t_1^4+13\,k_2^2k_1^2t_2^6t_1^8-12\,k_2\,k_1^3t_2^7t_1^7-30\,k_2\,k_1^3t_2^3t_1^{11}-38\,k_2\,k_1^3t_2^5t_1^9-$
$4\,k_2\,d^2k_1^3t_2^{11}t_1^3+t_1^{10}t_2^4k_2^4-4\,t_1^2t_2^{12}d^2k_1^2k_3^2+4\,k_1^4t_2^8t_1^6+4\,k_2^2t_1^{14}k_3^2+4\,k_2^4t_1^{12}t_2^2+12\,t_1^{12}t_2^2k_3^2k_2^2-$
$8\,k_2^3k_1\,d^2t_2^9t_1^5-12\,k_2^3k_1\,d^2t_2^7t_1^7+2\,k_2^3k_1\,d^4t_2^{11}t_1^3+4\,k_2^3k_1\,d^2t_2^5t_1^9-2\,k_2^3k_1\,d^4t_2^{13}t_1+8\,k_2^2k_1^2d^2t_2^8t_1^6-$
$14\,k_2^2k_1^2d^2t_2^6t_1^8-4\,k_2^2t_1^{10}t_2^4d^2k_3^2+k_2^2k_1^2d^4t_2^{10}t_1^4)\cdot(8\,c_3\,c_1\,k_1\,k_3^2k_2\,t_1\,t_2^3-4\,c_3\,c_1\,k_1\,t_1^3k_2^3t_2-$
$8\,c_3\,c_1\,k_1\,t_1^3k_3^2k_2\,t_2+4\,c_3\,c_1\,k_3^2k_2^2t_2^4-4\,c_3\,c_1\,k_2^4t_1^2t_2^2+4\,c_3\,c_1\,t_1^4k_2^2k_3^2-4\,c_1^2k_3^2k_2^2t_1^2t_2^2+8\,c_3^2k_1^3t_1\,k_2\,t_2^3-$
$8\,c_3^2k_1^3t_1^3t_2\,k_2+4\,c_3^2k_2^2k_1^2t_2^4+4\,c_3^2k_1^2t_1^4k_2^2-16\,c_3^2k_1^2t_2^2k_2^2t_1^2-8\,c_3^2k_1\,k_2^3t_1\,t_2^3+8\,c_3^2k_1\,t_1^3k_2^3t_2+$
$4\,{c_3}^2k_2^4t_1^2t_2^2+c_2^2t_1^2t_2^2k_1^4+4\,c_2^2t_2^2t_1^2k_3^4+4\,c_2\,c_1\,k_3^4t_2^4+4\,c_2\,c_1\,k_3^4t_1^4+4\,c_3\,c_2\,t_1^2t_2^2k_1^4+4\,c_3\,c_2\,k_1^3t_1\,k_2\,t_2^3-$
$4\,c_3\,c_2\,k_1^3t_1^3t_2\,k_2+4\,c_3\,c_2\,k_1^2t_2^4k_3^2-4\,c_3\,c_2\,k_1^2t_2^2k_2^2t_1^2+4\,c_3\,c_2\,k_1^2t_1^4k_3^2-8\,c_3\,c_2\,k_1\,k_3^2k_2\,t_1\,t_2^3+$
$8\,c_3\,c_2\,k_1\,t_1^3k_3^2k_2\,t_2+8\,c_3\,c_2\,k_3^2k_2^2t_1^2t_2^2+4\,c_2^2k_1^2t_1^2k_3^2t_2^2+4\,c_2\,c_1\,k_1^2t_1^2k_3^2t_2^2+2\,c_2\,c_1\,k_1^2t_2^2k_2^2t_1^2+$
$8\,c_2\,c_1\,k_1\,k_3^2k_2\,t_1\,t_2^3-8\,c_2\,c_1\,k_1\,t_1^3k_3^2k_2\,t_2-4\,c_2\,c_1\,k_3^2k_2^2t_1^2t_2^2+c_1^2k_2^4t_1^2t_2^2+4\,c_1^2t_2^2t_1^2k_3^4+4\,{c_3}^2t_1^2t_2^2k_1^4+$
$8\,c_3\,c_1\,k_1^2t_1^2k_3^2t_2^2+4\,c_3\,c_1\,k_1^2t_2^2k_2^2t_1^2+4\,c_3\,c_1\,k_1\,k_2^3t_1\,t_2^3).$

From this expression it is easy to check that

$$\deg_{\bar{t}}\left(\mathrm{PP}_{(d,\bar{k})}\left(\mathrm{Con}_{\bar{c}}\left(\mathrm{Res}_{t_0}\left(T_0(\bar{k},\bar{t}_h),T(\bar{c},d,\bar{k},\bar{t}_h)\right)\right)\right)\right)=14,$$

and then, using Theorem 4.45 one concludes that the total offset degree in $\bar{x}$ is $\delta = 14$.

In fact, in this case, using elimination techniques, it is possible to check this result, computing the generic offset polynomial (see Appendix B, page 208). This is indeed a polynomial of degree 14 in $\bar{x}$.

4.4 Alternative Approach for Surfaces of Revolution

This section is independent of the preceding results in this chapter. We consider another case in which a dimensional gain, in this case due to the symmetry inherent in the construction of the surface, helps to solve the offset degree problem. We will see that this is in fact the case for the surface of revolution obtained from a planer curve $\mathcal{C}$. Thus, using the geometric properties of the revolution construction, we are able to relate the offset of the surface of revolution generated by $\mathcal{C}$ with the surface of revolution of the offset to $\mathcal{C}$. .

Revolution surfaces are very common objects in Computer Aided Geometric Design, and offsetting a surface is also a frequently used process in the applications. Thus, it is natural to study the offsetting process for these special surfaces. In the Geometric Modeling literature, revolution surfaces are often introduced informally, and under the assumption that they are generated by a rational plane curve (see e.g. [1], [16], [31]).

Here we address the more general algebraic situation, in which the generating curve is an algebraic plane curve $\mathcal{C}$, given by its implicit equation.

In order to do this, in Subsection 4.4.1, we introduce the formal definition of surface of revolution by means of incidence diagrams, and from there we state some preliminary properties. Then we show, in Theorem 4.53 (page 184) that the implicit equation of the revolution surface can be obtained from the implicit equation of the initial curve by a straightforward method. As a by-product, we show that even when the generating curve is a rational curve given parametrically, a very efficient way to obtain the implicit equation of a revolution surface is to apply the most suitable curve implicitization method, and then use the result in Theorem 4.53.

Then, in Subsection 4.4.2 (page 186), we apply the above ideas and results to the offsetting process for the case of revolution surfaces. The main result in this context is Theorem 4.58, where we prove that the offset of a revolution surface is the surface of revolution of the offset curve. From this result, many properties of the offset to a surface of revolution may be traced back to the properties of the generating curve. Here, in the spirit of the present work, we focus on the degree problem for offset surfaces. Thus, we show how the formulae in Chapters 2 and 3 generalize to surfaces of revolution. Therefore, they provide a complete and efficient solution to the offset degree problem for surfaces of revolution.

4.4.1 Definition and implicit equation of a surface of revolution

Let $\mathcal{C}$ be an algebraic irreducible plane affine curve (seen in the coordinate (y_2, y_3)–plane) defined by the irreducible polynomial $f(y_2, y_3) \in \mathbb{C}[y_2, y_3]$, and not equal to the line of equation $y_2 = 0$ (this is because we will rotate around this line; the construction is illustrated in Figure 4.3).

In order to introduce the revolution construction, we consider the following incidence diagram:

$$\begin{tikzcd} & \mathcal{B} \subset \mathbb{C}^5 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \pi_1(\mathcal{B}) \subset \mathbb{C}^3 & & \pi_2(\mathcal{B}) \subset \mathbb{C}^2 \times \mathbb{C} \end{tikzcd}$$

where the revolution incidence variety is

$$\mathcal{B} = \left\{ (r^o, \bar{y}^o, \lambda^o) \in \mathbb{C}^5 \;\middle|\; \begin{array}{l} f(r^o, y_3^o) = 0, \\ (r^o)^2 = (y_1^o)^2 + (y_2^o)^2, \\ (1 + (\lambda^o)^2) y_1^o = 2\lambda^o r^o, \\ (1 + (\lambda^o)^2) y_2^o = (1 - (\lambda^o)^2) r^o. \end{array} \right\}$$

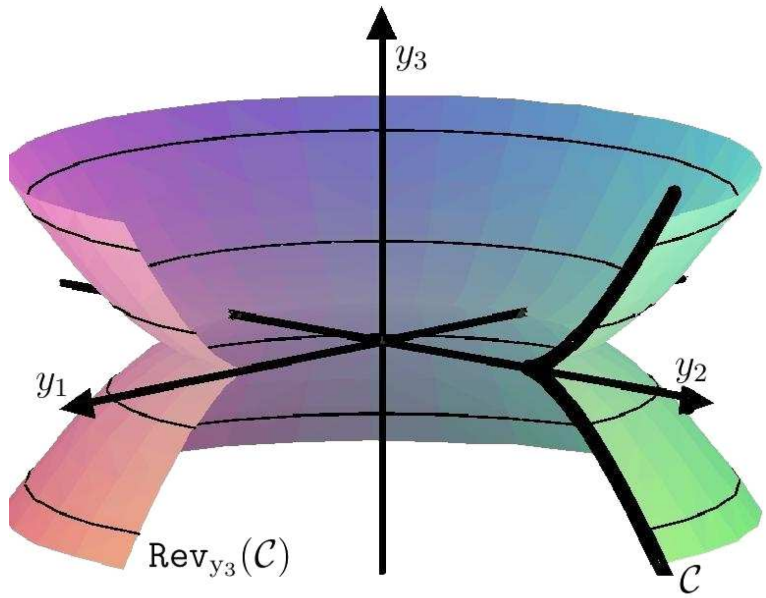

Figure 4.3: Construction of a Surface of Revolution

and the projection maps are:

$$\begin{array}{rccc} \pi_1 : & \mathbb{C}^5 & \longrightarrow & \mathbb{C}^3, \\ & (r, \bar{y}, \lambda) & \longmapsto & (\bar{y}) \end{array} \qquad \begin{array}{rccc} \pi_2 : & \mathbb{C}^5 & \longrightarrow & \mathbb{C}^2 \times \mathbb{C} \\ & (r, \bar{y}, \lambda) & \longmapsto & ((r, y_3), \lambda). \end{array}$$

Note that $\pi_2(\mathcal{B}) \subset \mathcal{C} \times \mathbb{C}$. With this notation we are ready for the formal definition:

Definition 4.49. *The* surface of revolution generated by rotating $\mathcal{C}$ around the y_3 *axis is the Zariski closure* $\pi_1(\mathcal{B})^*$ *of* $\pi_1(\mathcal{B})$. *We denote the surface of revolution of* $\mathcal{C}$ *by* $\mathtt{Rev}_{y_3}(\mathcal{C})$.

The following lemma lists the properties of the incidence diagram that we will need in the sequel:

Lemma 4.50. *Let* $\mathcal{C}$ *be irreducible, and not equal to the line of equation* $y_3 = 0$. *It holds that:*

(1) π_2 *is a birational map.*

(2) For all points $\bar{y}^o \in \pi_1(\mathcal{B})$, *with finitely many exceptions, the fiber* $\pi_1^{-1}(\bar{y}^o)$ *is zero-dimensional.*

Proof.

(1) First note that, for $((r, y_3), \lambda) \in \mathcal{C} \times (\mathbb{C} \setminus \{\pm\sqrt{-1}\})$, the inverse of π_2 is given by

$$\pi_2^{-1}((r, y_3), \lambda) = \left(r, \frac{2r\lambda}{\lambda^2+1}, \frac{(1-\lambda^2)r}{\lambda^2+1}, y_3, \lambda \right).$$

therefore, π_2 is birational.

(2) Let $\mathcal{D} = \{\bar{y}^o \in \pi_1(\mathcal{B}) / y_1^o = y_2^o = 0\}$. We observe that if $(0, 0, y_3^o) \in \mathcal{D}$, then $(0, y_3^o) \in \mathbb{C}$. Thus, since $\mathcal{C}$ is irreducible, and it is not the line of equation $y_2 = 0$, one deduces that $\mathcal{D}$ is finite. Now we prove that, for $\bar{y}^o \in \pi_1(\mathcal{B}) \setminus \mathcal{D}$ the fiber $\pi_1^{-1}(\bar{y}^o)$ is finite. Indeed, $\pi_1^{-1}(\bar{y}^o)$ is the set of points $(r^o, \bar{y}^o, \lambda^o)$ such that:

$$\begin{cases} f(r^o, y_3^o) = 0, \\ (r^o)^2 = (y_1^o)^2 + (y_2^o)^2, \\ (1 + (\lambda^o)^2) y_1^o = 2\lambda^o r^o, \\ (1 + (\lambda^o)^2) y_2^o = (1 - (\lambda^o)^2) r^o. \end{cases}$$

From the second equation, one has that there are, at most, two possible values for r^o. Now, if $\bar{y}_1^o \neq 0$, the third equation implies that there are, at most, two possible values for λ^o. On the other hand, if $y_1^o = 0$, then $y_2^o \neq 0$ and $r^o \neq 0$. Hence, the third equation implies that $\lambda^o = 0$. Therefore, $\pi_1^{-1}(\bar{y}^o)$ is finite.

□

The following proposition, shows that the above notion of revolution surface is well defined.

Proposition 4.51. *Let $\mathcal{C}$ be irreducible, and not equal to the line of equation $y_2 = 0$. Then $\mathtt{Rev}_{\mathrm{y}_3}(\mathcal{C})$ is an irreducible surface.*

Proof. Since $\pi_2(\mathcal{B})$ is irreducible and π_2 is birational (see Lemma 4.50(1)), then $\mathcal{B}$ is irreducible, and its dimension is $\dim(\pi_2(\mathcal{B})) = \dim(\pi_2(\mathcal{B}^*)) = \dim(\mathcal{C} \times \mathbb{C}) = 2$. Now, because of Lemma 4.50(2) and Lemma 1.5 (page 12), one concludes that $\dim(\mathtt{Rev}_{\mathrm{y}_3}(\mathcal{C})) = \dim(\pi_1(\mathcal{B})) = \dim(\mathcal{B}) = 2$. □

Remark 4.52. *If $\mathcal{C}$ is not irreducible, then its surface of revolution is defined as the union of the surfaces of revolution of its components.*

Our next goal is to derive an efficient method for computing the implicit equation of $\mathtt{Rev}_{\mathrm{y}_3}(\mathcal{C})$. For this purpose, first, collecting terms of odd and even degree in y_2, we write f (i.e. the implicit equation of $\mathcal{C}$) as follows:

$$f(y_2, y_3) = A(y_2^2, y_3) + y_2 B(y_2^2, y_3) \tag{4.46}$$

for some polynomials A and B.

There are two cases to consider:

- case (a): $B = 0$, and hence $f \in \mathbb{C}[y_2^2, y_3]$; that is f contains only even powers of y_2,
- case (b): $B \neq 0$, when f contains at least one odd power of y_3.

Then, the following theorem shows how the implicit equations of $\texttt{Rev}_{y_3}(\mathcal{C})$ and $\mathcal{C}$ are related by means of resultants. Furthermore, using this as theoretical foundation, the theorem shows how to obtain directly the implicit equation of $\texttt{Rev}_{y_3}(\mathcal{C})$, by means of a direct substitution in the polynomials A and B of Equation 4.46.

Theorem 4.53. *Let $\sigma(\bar{y})$ be the implicit equation of $\texttt{Rev}_{y_3}(\mathcal{C})$, and $f(y_2, y_3)$ the implicit equation of $\mathcal{C}$. Then there exists $\ell \in \mathbb{N}$ such that*

$$\sigma(\bar{y})^\ell = \mathrm{Res}_r(f(r, y_3), r^2 - (y_1^2 + y_2^2)).$$

Furthermore, if $R(\bar{y})$ is the above resultant, it holds that (with the notation of Equation 4.46):

1. *In* case (a), *$R(\bar{y}) = A(y_1^2 + y_2^2, y_3)^2$. Moreover, $\sigma(\bar{y}) = A(y_1^2 + y_2^2, y_3)$.*
2. *In* case (b), *$\sigma(\bar{y}) = R(\bar{y}) = A(y_1^2 + y_2^2, y_3)^2 - (y_1^2 + y_2^2)B(y_1^2 + y_2^2, y_3)^2$.*

Proof. We first consider the case where $f(y_2, y_3)$ is either $y_2 - 1$ or $y_2 + 1$. In this case

$$\mathrm{Res}_r(\pm 1 + r, r^2 - (y_1^2 + y_2^2)) = 1 - (y_1^2 + y_2^2).$$

On the other hand

$$\left\{ \left(\pm 1, \frac{2\lambda^o}{1 + (\lambda^o)^2}, y_3^o, \lambda^o \right) / \lambda^o \in \mathbb{C} \setminus \{\pm\sqrt{-1}, y_3^o \in \mathbb{C} \right\} \subset \mathcal{B}.$$

Hence,

$$\left\{ \left(\frac{2\lambda^o}{1 + (\lambda^o)^2}, y_3^o \right) / \lambda^o \in \mathbb{C} \setminus \{\pm\sqrt{-1}, y_3^o \in \mathbb{C} \right\} \subset \mathcal{B}.$$

From Proposition 4.51 (page 183), one gets that $\sigma = 1 - (y_1^2 + y_2^2)$. Moreover, observe that f is in case (b) of Equation 4.46, with $A = \pm 1$ and $B = 1$. So,

$$1 - (y_1^2 + y_2^2) = \sigma(\bar{y}) = R(\bar{y}) = A(y_1^2 + y_2^2, y_3)^2 - (y_1^2 + y_2^2)B(y_1^2 + y_2^2, y_3)^2,$$

and the theorem is proved in this case.

Now let us assume that $\mathcal{C}$ is not any of the lines defined by $y_2 \pm 1$. Since $R(\bar{y})$ equals the product of $f(r, y_3)$ evaluated at the roots of $r^2 - y_1^2 - y_2^2$ as a polynomial in r (see e.g. [55]), one has that

$$R(\bar{y}) = f(\sqrt{y_1^2 + y_2^2}, y_3) f(-\sqrt{y_1^2 + y_2^2}, y_3).$$

Therefore,

- in case (a), $R(\bar{y}) = A(y_1^2 + y_2^2, y_3)$.
- in case (b), $R(\bar{y}) = A(y_1^2 + y_2^2, y_3)^2 - (y_1^2 + y_2^2)B(y_1^2 + y_2^2, y_3)^2$.

We consider the ideal I generated in $\mathbb{C}[r, \bar{y}, \lambda]$ by the polynomials defining the incidence variety $\mathcal{B}$. That is,

$$I =< f(r, y_3), r^2 - y_1^2 - y_2^2, (1 + \lambda^2)y_1 - 2\lambda r, (1 + \lambda^2)y_3 - (1 - \lambda^2)r > .$$

Moreover, let J be the (r, λ)-elimination ideal of I. That is, $J = I \cap \mathbb{C}[\bar{y}]$. Note that $J =< \sigma(\bar{y}) >$, and that $R(\bar{y}) \in J$. Thus, $\sigma(\bar{y})$ divides $R(\bar{y})$. On the other hand, let Σ be the surface defined by $R(\bar{y})$. Let $\bar{y}^o \in \Sigma$, with $(y_1^o, y_2^o) \neq (0, -1)$. Note that we are excluding at most finitely many points of Σ, because

$$R(0, -1, y_3) = f(1, y_3)f(-1, y_3)$$

and we have assumed that $f(\bar{y})$ is not associated with $y_2 \pm 1$. Then, using that $r^2 - y_1^2 - y_2^2$ is monic in r, by the Extension Theorem for resultants (see [14], Theorem 5, page 161), one deduces that there exists $r^o \in \mathbb{C}$ such that

$$f(r^o, y_3^o) = 0 \text{ and } r_o^2 = (y_1^o)^2 + (y_2^o)^2.$$

We distinguish two cases, depending on wether r^o is equal to zero or not. If $r^o = 0$, then $(0, \rho^o, \pm\sqrt{-1}) \in \mathcal{B}$. If $r^o \neq 0$, then (y_1^o, y_2^o) belongs to the circle of equation $y_1^2 + y_2^2 = (r^o)^2$. Moreover, since $(y_1^o, y_2^o) \neq (0, -1)$, there exists $\lambda^o \in \mathbb{C}$ such that

$$\left(\frac{2\lambda^o r^o}{1 + (\lambda^o)^2}, \frac{((\lambda^o)^2 - 1)r^o}{1 + (\lambda^o)^2}\right) = (y_1^o, y_2^o),$$

and hence $(r^o, \bar{y}^o, \lambda^o) \in \mathcal{B}$. In either case, $\bar{y}^o \in \pi_1(\mathcal{B})$. From here we conclude that

$$\Sigma \setminus \{(0, -1, y_3^o)/y_3^o \in \mathbb{C}\} \subset \mathtt{Rev}_{\mathrm{y}_3}(\mathcal{C}).$$

Taking Zariski closures, $\Sigma \subset \mathtt{Rev}_{\mathrm{y}_3}(\mathcal{C})$. Therefore, since σ is irreducible, the square-free part of $R(\bar{y})$ equals $\sigma(\bar{y})$. Thus, there exists $\ell \in \mathbb{N}$ such that $\sigma(\bar{y})^\ell = R(\bar{y})$.

It only remains to prove the expressions of σ in terms of A and B in cases (a) and (b). In case (a), we have

$$\sigma(\bar{y})^\ell = R(\bar{y}) = A(y_1^2 + y_2^2, y_3).$$

Then

$$\sigma(0, y_2, y_3)^\ell = A(y_2^2, y_3) = f(y_2, y_3).$$

Since f is irreducible, it follows that $\ell = 1$, and the proof is finished in case (a). In case (b) one has

$$\sigma(\bar{y})^\ell = R(\bar{y}) = A(y_1^2 + y_2^2, y_3)^2 - (y_1^2 + y_2^2)B(y_1^2 + y_2^2, y_3)^2.$$

Then

$$\sigma(0, y_2, y_3)^\ell = A^2(y_2^2, y_3) - y_2^2 B^2(y_2^2, y_3) =$$
$$= (A(y_2^2, y_3) + y_2 B(y_2^2, y_3))(A(y_2^2, y_3) - y_2 B(y_2^2, y_3)) = f(y_2, y_3)f(-y_2, y_3).$$

Note that since $f(y_2, y_3)$ is irreducible, the same holds for $f(-y_2, y_3)$. Finally, since $f(-y_2, y_3) \neq f(y_2, y_3)$ (otherwise, we would be in case (a)), we have also in this case $\ell = 1$ and the proof is finished. □

Remark 4.54. *If $\mathcal{C}$ is not irreducible, the method described in this theorem still provides the implicit equation of its surface of revolution; in this case, the square-free part of the resultant factors into the implicit equations of the components of* $\mathtt{Rev}_{\mathrm{y}_3}(\mathcal{C})$.

The next corollary, which is a direct consequence of the previous theorem, gives a complete degree analysis of $\mathtt{Rev}_{\mathrm{y}_3}(\mathcal{C})$.

Theorem 4.55. *Let $\sigma(\bar{y})$ be the implicit equation of $\mathtt{Rev}_{\mathrm{y}_3}(\mathcal{C})$, and $f(y_2, y_3)$ the implicit equation of $\mathcal{C}$. Then, it holds that:*

1. *In* case (a), $\deg_{(\bar{y})}(\sigma) = \deg_{(y_2,y_3)}(f), \deg_{y_i}(\sigma) = \deg_{y_i}(f)$, *for* $i = 1, 2, 3$.
2. *In* case (b), $\deg_{(\bar{y})}(\sigma) = 2\deg_{(y_2,y_3)}(f), \deg_{y_i}(\sigma) = 2\deg_{y_i}(f)$, *for* $i = 1, 2, 3$.

We finish this section with an illustrating example.

Example 4.56. *Let $\mathcal{C}$ be the elliptic cubic defined by $f(y_2, y_3) = y_3^2 - y_2(y_2^2 - 1)$. Then $A(y_2, y_3) = y_3^2$ and $B(y_2, y_3) = -(y_2^2 - 1) \neq 0$, and so $f(y_2, y_3)$ is in case (b). Thus, the implicit equation of $\mathtt{Rev}_{\mathrm{y}_3}(\mathcal{C})$ is given by:*

$$\sigma(\bar{y}) = y_3^4 - (y_1^2 + y_2^2)((y_1^2 + y_2^2) - 1)^2$$

4.4.2 Offsets to revolution surfaces

In this subsection we apply the above results to analyze the offsetting process in the case of revolution surfaces. Let $\mathcal{C}$ be an irreducible curve, not equal to the line of rotation. The normal vectors to $\mathtt{Rev}_{\mathrm{y}_3}(\mathcal{C})$ have the following (geometrically intuitive) fundamental property.

Lemma 4.57. *Let $\tilde{p} \in \mathtt{Rev}_{y_3}(\mathcal{C})$ be obtained by rotating $p \in \mathcal{C}$ around the y_3 axis, and let us denote by θ the particular rotation carrying p to $\tilde{p}$. Then, $\tilde{N}(\tilde{p})$, the normal vector to $\mathtt{Rev}_{y_3}(\mathcal{C})$ at $\tilde{p}$, is parallel to the vector $\theta(N(p))$, obtained by applying the same rotation to the normal vector $N(p)$ to $\mathcal{C}$ at p.*

Proof. This can be seen by a straightforward computation, from the implicit equation of $\mathtt{Rev}_{y_3}(\mathcal{C})$ (see Theorem 4.53), and computing the gradient. □

In the following Theorem, we assume that both $\mathcal{C}$ and $\mathcal{O}_d(\mathcal{C})$ are in the (y_2, y_3)–plane.

Theorem 4.58. $\mathcal{O}_d(\mathtt{Rev}_{y_3}(\mathcal{C})) = \mathtt{Rev}_{y_3}(\mathcal{O}_d(\mathcal{C}))$.

Proof. This follows from Lemma 4.57 and the offset construction. □

This theorem allows the study of some properties $\mathcal{O}_d(\mathtt{Rev}_{y_3}(\mathcal{C}))$ by means of the related properties of $\mathcal{O}_d(\mathcal{C})$. In particular, and following the main line of work of this thesis, we will show that it allows us to give a complete and efficient solution of the degree problem for revolution surfaces. From Theorem 4.58, if we can decide in which case ((a) or (b)) of Theorem 4.53) the implicit equation of $\mathcal{O}_d(\mathcal{C})$ is, applying Theorem 4.55 as well as results in Chapters 2 and 3, we can provide formulae for the partial and total degree of $\mathcal{O}_d(\mathtt{Rev}_{y_3}(\mathcal{C}))$. Note that polynomials in $\mathbb{C}[y_2^2, y_3]$ (that is, the polynomials in case (b)) are characterized by the symmetry condition $f(-y_2, y_3) = f(y_2, y_3)$. Thus, we need to show that this symmetry condition is inherited by the offsetting construction. The answer is contained in the following two propositions. The first one analyzes the problem from the implicit point of view. The second one shows how to detect this property from the parametric point of view. This is useful e.g. if one is given a parametric representation of the generating curve, and wishes to obtain the offset surface degrees without implicitizing the curve.

In the following proposition, let $f(y_2, y_3)$ be the polynomial defining $\mathcal{C}$ and let $g(d, y_2, y_3)$ be the generic equation of the offset $\mathcal{O}_d(\mathcal{C})$ (see [43] for its definition and properties).

Proposition 4.59. *Let $f(y_2, y_3)$ and $g(d, y_2, y_3)$ be the polynomial defining $\mathcal{C}$ and $\mathcal{O}_d(\mathcal{C})$, respectively. Then, $g(d, y_2, y_3) = g(d, -y_2, y_3)$ if and only if $f(y_2, y_3) = f(-y_2, y_3)$.*

Proof. The right-left implication follows from the offset geometric construction, because the normal vector to $\mathcal{C}$ and the normal vector to its offset at the corresponding points are parallel. Conversely, suppose that $g(d, y_2, y_3) = g(d, -y_2, y_3)$. Now, let d^o be such that no coefficient w.r.t. $\{y_2, y_3\}$ of g vanishes when substituting d by d^o, and such that $g(d^o, y_2, y_3)$ is the implicit equation of $\mathcal{O}_{d^o}(\mathcal{C})$ (see Theorem 1.24, page 21). Then $\mathcal{O}_{d^o}(\mathcal{O}_{d^o}(\mathcal{C})) = \mathcal{C} \cup \mathcal{O}_{2d^o}(\mathcal{C})$ (see [50]). Let $\tilde{g}(d^o, y_2, y_3)$ be the defining polynomial of $\mathcal{O}_{d^o}(\mathcal{O}_{d^o}(\mathcal{C}))$. Then $\tilde{g}(d^o, y_2, y_3) = g(2d^o, y_2, y_3)f(y_2, y_3)$. Furthermore, because of the hypothesis and how d^o has been taken, $g(d^o, y_2, y_3) = g(d^o, -y_2, y_3)$. Moreover, because of the first implication $\tilde{g}(d^o, y_2, y_3)$ inherits this property. Now, from $\tilde{g} = gf$, it follows immediately that $f(y_2, y_3) = f(-y_2, y_3)$. □

Now we show how to detect this symmetry from a parametrization of $\mathcal{C}$. Assuming that $\mathcal{C}$ is rational, the symmetry condition can be translated as follows.

Proposition 4.60. *Let*

$$P_{\mathcal{C}}(t) = \left(\frac{P_1(t)}{P_0(t)}, \frac{P_2(t)}{P_0(t)}\right),$$

be a rational parametrization of $\mathcal{C}$*, such that* $\gcd(P_0, P_1, P_2) = 1$*. The condition* $f(y_2, y_3) = f(-y_2, y_3)$ *holds if and only if*

$$\tilde{P}_{\mathcal{C}}(t) = \left(-\frac{P_1(t)}{P_0(t)}, \frac{P_2(t)}{P_0(t)}\right)$$

parametrizes $\mathcal{C}$*.*

Proof. The left-right implication is trivial. Now, suppose that $\tilde{P}_{\mathcal{C}}(t)$ parametrizes $\mathcal{C}$, and suppose that $f(y_2, y_3) \neq f(-y_2, y_3)$. Since $f(y_2, y_3)$ is reducible, the same holds for $f(-y_2, y_3)$. Furthermore, $\deg_{(y_2,y_3)} f(y_2, y_3) = \deg_{(y_2,y_3)} f(-y_2, y_3)$. Let $g(y_2, y_3) = f(y_2, y_3) + f(-y_2, y_3)$. Because of our hypotheses, g is not zero, and $g(\tilde{P}_{\mathcal{C}}(t)) = 0$. Therefore, $f(y_2, y_3)$ divides $g(y_2, y_3)$, and hence $f(y_2, y_3)$ divides $f(-y_2, y_3)$. But the degree equality implies that $f(y_2, y_3) = f(-y_2, y_3)$, a contradiction. □

The next proposition shows how to check the symmetry condition directly from a parametrization of $\mathcal{C}$, without using the implicit equation of $\mathcal{C}$.

Proposition 4.61. *Let* $P(t)$ *be as in Proposition 4.60 (page 188). Then*

$$f(d, y_2, y_3) = f(d, -y_2, y_3)$$

iff

$$\gcd(P_0(t)P_1(s) + P_0(s)P_1(t), P_0(t)P_2(s) - P_0(s)P_2(t))$$

is non-trivial.

Proof. Let

$$M_1(s,t) = P_0(t)P_1(s) + P_0(s)P_1(t), \quad M_2(s,t) = P_0(t)P_2(s) - P_0(s)P_2(t),$$

and $D(s,t) = \gcd(M_1, M_2)$. We first observe that M_1 and M_2 can not be both simultaneously zero, since this would imply that P is not a parametrization. Moreover, note that if either $\frac{P_1}{P_0}$ or $\frac{P_2}{P_0}$ is constant, the result follows. Thus, in the rest of the proof we

assume that no component of P is constant.
Let $\mathcal{D} \subset \mathbb{C}^3$ be defined as follows:

$$\mathcal{D} := \{(t^0, s^o, u^o) \in \mathbb{C}^3 \,|\, u^o P_0(t^o) P_0(s^o) = 1,\, M_1(t^o, s^o) = M_2(t^o, s^o) = 0\}.$$

We consider the diagram:

$$\begin{tikzcd}
& \mathcal{D} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\pi_1(\mathcal{D}) \subset \mathbb{C} \arrow[dr, "\tilde{P}"] & & \pi_2(\mathcal{D}) \subset \mathbb{C} \arrow[dl, "P"'] \\
& \Delta := P(\mathbb{C}) \cap \tilde{P}(\mathbb{C}) &
\end{tikzcd}$$

where $\pi_1(t^0, s^o, u^o) = t^o, \pi_2(t^0, s^o, u^o) = s^o$ and $\tilde{P}_{\mathcal{C}}(t) = \left(-\frac{P_1(t)}{P_0(t)}, \frac{P_2(t)}{P_0(t)}\right)$ is as in Proposition 4.60. We observe that the diagram is commutative and that $P \circ \pi_1$ and $\tilde{P} \circ \pi_2$ are both surjective on Δ.

Let D be constant. Then $\mathcal{D}$ is either empty, or zero-dimensional. Thus, Δ is either empty or zero-dimensional. In particular, $\tilde{P}$ does not parametrize $\mathcal{C}$, and by Proposition 4.60 we conclude that $f(y_2, y_3) \neq f(-y_2, y_3)$.

If D is a non-constant polynomial, we first observe that $\gcd(D, P_0(t)) = \gcd(D, P_0(s)) = 1$. Indeed, if $\gcd(D, P_0(t)) \neq 1$ (similarly if $\gcd(D, P_0(s)) \neq 1$), then $M_1(t, s)$ and $M_2(t, s)$ have a non-trivial common factor depending only on t. Taking into account that no component of P is constant, that factor would then divide $P_0(t), P_1(t)$ and $P_2(t)$, which is impossible because $\gcd(P_0, P_1, P_2) = 1$. In this situation we have that $\dim(\mathcal{D}) = 1$, and that $\mathbb{C} \setminus \pi_1(\mathcal{D})$ is empty or finite. The same holds for $\mathbb{C} \setminus \pi_2(\mathcal{D})$. Thus, $\dim(\Delta) = 1$. This implies, by Proposition 4.60 that $f(y_2, y_3) = f(-y_2, y_3)$. □

Using these results, one derives the following algorithm for the solution of the offset degree problem in the case of surfaces of revolution.

Algorithm 4.62 (Offset Degree for the Surface of Revolution Generated by Curve $\mathcal{C}$).

- *Input: Either the defining polynomial f or a rational parametrization $P_{\mathcal{C}}(t)$, as above, of $\mathcal{C}$ ($\mathcal{C}$ is not the axes $y_2 = 0$)*

- *Output: The total and partial degrees of $\mathcal{O}_d(\mathtt{Rev}_{\mathtt{y_3}}(\mathcal{C}))$.*

1. *Apply either Prop. 4.59 or 4.61 to check whether the generic offset equation* $g(d, y_2, y_3)$ *of* $\mathcal{O}_d(\mathcal{C})$ *is in case(a) or case(b) (with the terminology introduced before Theorem 4.53).*

2. *Apply formulae in Chapters 2 and 3 to get*

$$\delta = \deg_{y_2,y_3}(g), \delta_2 = \deg_{y_2}(g), \delta_3 = \deg_{y_3}(g), \delta_d = \deg_d(g).$$

3. *Let* $G(d, \bar{x})$ *be the generic offset polynomial of* $\mathcal{O}_d(\mathtt{Rev}_{y_3}(\mathcal{C}))$*.*
In case (a) return

$$\{\deg_{(\bar{x})}(G) = \delta, \deg_{x_1}(G) = \deg_{x_2}(G) = \delta_2, \deg_{x_3}(G) = \delta_3, \deg_d(G) = \delta_d\},$$

and in case (b) return

$$\{\deg_{(\bar{x})}(G) = 2\delta, \deg_{x_1}(G) = \deg_{x_2}(G) = 2\delta_2, \deg_{x_3}(G) = 2\delta_3, \deg_d(G) = 2\delta_d\}.$$

Let us finish with some examples. In all of them, as usual, let $g(d, \bar{x})$ be the defining polynomial of $\mathcal{O}_d\left(\mathtt{Rev}_{y_3}(\mathcal{C})\right)$:

Example 4.63. *Consider the parabola defined by* $f(y_2, y_3) = y_3 - y_2^2$*. Then obviously* $f(-y_2, y_3) = f(y_2, y_3)$*, and so we are in case (a). Applying, e.g. the resultant-based formulae in Theorems 2.31 (page 68), 3.24 (page 96) and 3.36 (page 109), one has:* $\{\delta = 6, \delta_2 = 6, \delta_3 = 4, \delta_d = 6\}$*, and so the algorithm returns* $\{\deg_{(\bar{x})}(g(d, \bar{x})) = 6, \deg_{x_1}(g(d, \bar{x})) = \deg_{x_2}(g(d, \bar{x})) = 6, \deg_{x_3}(g(d, \bar{x})) = 4, \deg_d(g(d, \bar{x})) = 6\}$ *for the degrees of the offset of the circular paraboloid defined by* $y_3 - y_1^2 - y_2^2 = 0$*, which is the surface of revolution generated by* $\mathcal{C}$*.*

Example 4.64. *For the non-rational cubic* $\mathcal{C}$ *in Example 4.56, the formulae in Theorems 2.31 (page 68), 3.24 (page 96) and 3.36 (page 109), give:* $\{\delta = 14, \delta_2 = 14, \delta_3 = 12, \delta_d = 14\}$*, and so, since we are in case (b), the algorithm returns* $\{\deg_{(\bar{x})}(g(d, \bar{x})) = 28, \deg_{x_1}(g(d, \bar{x})) = \deg_{x_2}(g(d, \bar{x})) = 28, \deg_{x_3}(g(d, \bar{x})) = 24, \deg_d(g(d, \bar{x})) = 28\}$ *for the degrees of* $\mathcal{O}_d\left(\mathtt{Rev}_{y_3}(\mathcal{C})\right)$*.*

Example 4.65. *Let* $\mathcal{C}$ *be the lemniscate parametrized by*

$$P_{\mathcal{C}}(t) = \left(\frac{\sqrt{2}(t + t^3)}{1 + t^4}, \frac{\sqrt{2}(t - t^3)}{1 + t^4}\right)$$

then one has

$$\gcd(P_0(t)P_1(s) + P_1(t)P_0(s), P_0(t)P_2(s) - P_2(t)P_0(s)) = ts + 1$$

and it follows that we are in case (a). Using the formulae in Theorems 2.31 (page 68), 3.24 (page 96) and 3.36 (page 109), one has: $\{\delta = \delta_2 = \delta_3 = \delta_d = 12\}$*, and so the algorithm returns*

$$\{\deg_{(\bar{x})}(g(d,\bar{x})) = \deg_{x_1}(g(d,\bar{x})) = \deg_{x_2}(g(d,\bar{x})) = \deg_{x_3}(g(d,\bar{x})) = \deg_d(g(d,\bar{x})) = 12\}$$

for the degrees of $\mathcal{O}_d\left(\mathtt{Rev}_{\mathrm{y}_3}(\mathcal{C})\right)$*.*

Example 4.66. *Let* $\mathcal{C}$ *be the Folium parametrized by*

$$P_{\mathcal{C}}(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right)$$

then one has

$$\gcd(P_0(t)P_1(s) + P_1(t)P_0(s), P_0(t)P_2(s) - P_2(t)P_0(s)) = 1$$

and we are in case (b). Using the formulae in Theorems 2.31 (page 68), 3.24 (page 96) and 3.36 (page 109), one has: $\{\delta = \delta_2 = \delta_3 = \delta_d = 14\}$*, and so the algorithm returns*

$$\{\deg_{(\bar{x})}(g(d,\bar{x})) = \deg_{x_1}(g(d,\bar{x})) = \deg_{x_2}(g(d,\bar{x})) = \deg_{x_3}(g(d,\bar{x})) = \deg_d(g(d,\bar{x})) = 28\}$$

for the degrees of $\mathcal{O}_d\left(\mathtt{Rev}_{\mathrm{y}_3}(\mathcal{C})\right)$*.*

Appendices

Appendix A

Formulae Summary

For the convenience of the reader, we collect in this appendix the various offset degree formulae that have been obtained in this thesis

A.1 Formulae for curves

Let $\mathcal{C}$ be an algebraic irreducible affine plane curve over $\mathbb{C}$, and let $\mathcal{O}_d(\mathcal{C})$ denote the generic offset of $\mathcal{C}$. See Section 1.2 (page 15) for its definitions and properties, and specifically Subsection 1.2.2 (page 24) for the statement of the Degree Problem.

A.1.1 Implicit case

Let $\bar{x} = (x_1, x_2), \bar{y} = (y_1, y_2)$, and $\bar{x}_h = (x_0, x_1, x_2), \bar{y}_h = (y_0, y_1, y_2)$. Let $f(\bar{y}) \in \mathbb{C}[\bar{y}]$ be the defining irreducible polynomial of $\mathcal{C}$, and let $F(\bar{y}_h)$ be the homogenization of f w.r.t. y_0. The symbol F_i, for $i = 1, 2$ denotes the partial derivative of F w.r.t. y_i. In what follows k is considered as a parameter. Finally, let $H(\bar{y}_h) = F_1^2(\bar{y}_h) + F_1^2(\bar{y}_h)$.

Total Degree Formulae

All these formulae can be used to compute $\delta = \deg_{\bar{x}}(\mathcal{O}_d(\mathcal{C}))$. Let

$$S(d, k, \bar{y}_h) = H(\bar{y}_h)(y_1 - ky_2)^2 - d^2 y_0^2 (F_1(\bar{y}) - kF_2(\bar{y}))^2.$$

- First Formula. See Theorem 2.24, page 56.

$$\boxed{\deg_{\bar{x}}(\mathcal{O}_d(\mathcal{C})) = 2\left(\deg_{\bar{y}}(\mathcal{C})\right)^2 - \sum_{\bar{y}_h^o \in \mathcal{F}} \mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}_{(d^o,k^o)}}),}$$

where $\mathcal{F}$ is the set of fake points (see Definition 2.16, page 50).

- Hodograph-Based Formula. See Theorem 2.27, page 62. For $\bar{y}_h^o \in \mathcal{F}^\infty$, let

$$A_{\bar{y}_h^o} = \begin{cases} \min\left(\mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{H}}), \mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, y_0^2 F_1^2)\right) & \text{if } \bar{y}_2^o \neq 0 \\ \min\left(\mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{H}}), \mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, y_0^2 F_2^2)\right) & \text{if } \bar{y}_2^o = 0 \end{cases}$$

Then

$$\boxed{\delta = \deg_{\bar{x}}(\mathcal{O}_d(\mathcal{C})) = 2\deg_{\bar{y}}(\mathcal{C}) - \sum_{\bar{y}^o \in \mathcal{F}^a} \mathrm{mult}_{\bar{y}^o}(\overline{\mathcal{C}}, \overline{\mathcal{H}}) - \sum_{\bar{y}_h^o \in \mathcal{F}^\infty} A_{\bar{y}_h^o}.}$$

- Resultant-Based Formula. See Theorem 2.31, page 68.
Assuming $\mathcal{C}$ is not a line through the origin:

$$\boxed{\delta = \deg(\mathcal{O}_d(\mathcal{C})) = \deg_{\{y_1,y_2\}}\left(\mathrm{PP}_{\{d,k\}}\left(\mathrm{Res}_{y_0}(F(\bar{y}_h), S(d,k,\bar{y}_h))\right)\right)}$$

Partial Degree Formulae

These formulae can be used to compute $\delta_1 = \deg_{\bar{x}_1}(\mathcal{O}_d(\mathcal{C}))$. The necessary changes for computing $\delta_2 = \deg_{\bar{x}_2}(\mathcal{O}_d(\mathcal{C}))$ are evident. Let

$$S^1(d,k,\bar{y}_h) = H(\bar{y}_h)(y_2 - ky_0)^2 - y_0^2 d^2 F_2^2(\bar{y}_h).$$

- First partial degree formula. See Theorem 3.23, page 96. Let Ω_1 be as in Proposition 3.21. For every $(d^o,k^o) \in \Omega_1$, it holds that:

$$\boxed{\delta_1 = \deg_{x_1}(\mathcal{O}_{d^o}(\mathcal{C})) = 2\left(\deg(\mathcal{C})\right)^2 - \sum_{\bar{y}_h^o \in \mathcal{F}_1} \mathrm{mult}_{\bar{y}_h^o}(\overline{\mathcal{C}}, \overline{\mathcal{S}^1_{(d^o,k^o)}})}$$

- Resultant-Based Formula (see Theorem 3.24, page 96).
Assuming $\mathcal{C}$ is not a line through the origin:

$$\boxed{\delta_1 = \deg_{x_1}(\mathcal{O}_d(\mathcal{C})) = \deg_{\{\bar{y}\}}\left(\mathrm{PP}_{\{d,k\}}\left(\mathrm{Res}_{y_0}(F(\bar{y}_h), S^1(\bar{y}_h,d,k))\right)\right)}$$

Degree w.r.t the Distance

The formula below can be used to compute $\delta_d = \deg_d(\mathcal{O}_d(\mathcal{C}))$. Let

$$\mathrm{Nor}(\bar{x},\bar{y}_h) = F_2(\bar{y}_h)(x_1y_0 - y_1) - F_1(\bar{y}_h)(x_2y_0 - y_2).$$

- Degree formula for the distance. See Theorem 3.36, page 109.

$$\boxed{\delta_d = \deg_d(\mathcal{O}_d(\mathcal{C})) = 2\deg_{\{\bar{y}\}}\left(\mathrm{PP}_{\{\bar{x}\}}\left(\mathrm{Res}_{y_0}(F(\bar{y}_h), \mathrm{Nor}(\bar{x},\bar{y}_h))\right)\right)}$$

A.1.2 Parametric case

Let
$$P(t) = \left(\frac{X(t)}{W(t)}, \frac{Y(t)}{W(t)}\right)$$
be a proper rational parametrization of $\mathcal{C}$, with $\gcd(X, Y, W) = 1$. Construct the associated normal vector. First, compute the polynomials:
$$\begin{cases} A_1(t) = -(W(t)Y'(t) - W'(t)Y(t)) \\ A_2(t) = W(t)X'(t) - W'(t)X(t), \end{cases}$$
where $'$ denotes derivation w.r.t. t. Let $G = \gcd(A_1, A_2)$, and let
$$N(t) = (N_1(t), N_2(t)) := \left(\frac{A_1(t)}{G(t)}, \frac{A_2(t)}{G(t)}\right).$$
N is the associated normal vector of the parametrization $P(t)$. Also the parametric hodograph of P is:
$$H_P(t) = N_1^2(t) + N_2^2(t).$$
Then we have the following formulae:

- Total Degree Formula. See Theorem 2.40, page 75. Let
$$s_P(d, k, t) = H_P(t)\left(X(t) - kY(t)\right)^2 - d^2W^2(t)\left(N_1(t) - kN_2(t)\right)^2.$$
$$\boxed{\delta = \deg(\mathcal{O}_d(\mathcal{C})) = \deg_t\left(\mathrm{PP}_{(d,k)}(s_P(d, k, t))\right)}$$

- Partial Degree Formula. See Theorem 3.42, page 115. Let
$$s_P^{(1)}(d, k, t) = H_P(t)(Y(t) - kW(t))^2 - d^2N_2^2(t)W^2(t).$$
$$\boxed{\delta_1 = \deg_{x_1}(\mathcal{O}_d(\mathcal{C})) = \deg_t\left(\mathrm{PP}_{\{d,k\}}\left(s_P^{(1)}(d, k, t)\right)\right)}$$

- Degree in the Distance Formula. See Theorem 3.48, page 118. Let
$$\mathrm{nor}_P(\bar{x}, t) = N_2(t)\big(W(t)x_1 - X(t)\big) - N_1(t)\big(W(t)x_2 - Y(t)\big).$$
$$\boxed{\delta_d = \deg_d(\mathcal{O}_d(\mathcal{C})) = 2\deg_t\left(\mathrm{PP}_{\{\bar{x}\}}\left(\mathrm{nor}_P(\bar{x}, t)\right)\right)}$$

A.2 Formulae for Parametric Surfaces

General case

Let Σ be a parametric surface (not necessarily rational), and let (see Subsections 4.1.1, page 125, and 4.3.1, page 146)

$$P_h(\bar{t}_h) = \left(\frac{X(\bar{t}_h)}{W(\bar{t}_h)}, \frac{Y(\bar{t}_h)}{W(\bar{t}_h)}, \frac{Z(\bar{t}_h)}{W(\bar{t}_h)} \right)$$

be a parametrization of Σ, where $\bar{t}_h = (t_0 : t_1 : t_2)$, and $X, Y, Z, W \in \mathbb{C}[\bar{t}_h]$ are homogeneous polynomials P for which $\gcd(X, Y, Z, W) = 1$ holds. The associated normal vector is constructed as follows. Let X_i, Y_i, Z_i, W_i for $i = 1, 2$, denote the partial derivative w.r.t. t_i of X, Y, Z, W, respectively. Take $A^h = (A^h_1, A^h_2, A^h_3) \in (\mathbb{C}[\bar{t}_h])^3$ to be the polynomial vector defined by:

$$A^h = (X_1W - XW_1, Y_1W - YW_1, Z_1W - ZW_1) \wedge (X_2W - XW_2, Y_2W - YW_2, Z_2W - ZW_2)\,.$$

Let $G^h = \gcd(A^h_1, A^h_2, A^h_3)$, and define $N = (N_1, N_2, N_3) = \left(\frac{A^h_1}{G^h}, \frac{A^h_2}{G^h}, \frac{A^h_3}{G^h} \right)$. We also denote

$$H(\bar{t}_h) = (N_1(\bar{t}))^2 + (N_2(\bar{t}))^2 + (N_3(\bar{t}))^2.$$

Then, let $\bar{k} = (k_1, k_2, k_3), \bar{c} = (c_1, c_2, c_3)$ denote new variables and set:

$$\begin{cases} T_0(\bar{k}, \bar{t}_h) := \mathrm{PP}_{\bar{k}}\left(k_1(YN_3 - ZN_2) - k_2(XN_3 - ZN_1) + k_3(XN_2 - YN_1)\right), \\ T_1(d, \bar{k}, \bar{t}_h) := \mathrm{PP}_{(d,\bar{k})}\left(H(\bar{t}_h)(k_2Z - k_3Y)^2 - d^2W(\bar{t}_h)^2(k_2N_3 - k_3N_2)^2\right), \\ T_2(d, \bar{k}, \bar{t}_h) := \mathrm{PP}_{(d,\bar{k})}\left(H(\bar{t}_h)(k_1Z - k_3X)^2 - d^2W(\bar{t}_h)^2(k_1N_3 - k_3N_1)^2\right), \\ T_3(d, \bar{k}, \bar{t}_h) := \mathrm{PP}_{(d,\bar{k})}\left(H(\bar{t}_h)(k_1Y - k_2X)^2 - d^2W(\bar{t}_h)^2(k_1N_2 - k_2N_1)^2\right), \end{cases}$$

and

$$T(\bar{c}, d, \bar{k}, \bar{t}_h) = c_1T_1(d, \bar{k}, \bar{t}_h) + c_2T_2(d, \bar{k}, \bar{t}_h) + c_3T_3(d, \bar{k}, \bar{t}_h).$$

Then the following formula holds (see Theorem 4.45, page 172):

$$\boxed{m \cdot \delta = \deg_{\{\bar{t}\}}\left(\mathrm{PP}_{(d,\bar{k})}\left(\mathrm{Con}_{\bar{c}}\left(\mathrm{Res}_{t_0}\left(T_0(\bar{k}, \bar{t}_h), T(\bar{c}, d, \bar{k}, \bar{t}_h)\right)\right)\right)\right)}$$

where m is the tracing index of P (see Remark 4.3, page 126)

Surfaces of Revolution

Let $\mathcal{C}$ be an algebraic irreducible plane affine curve (seen in the coordinate (y_2, y_3)–plane), and not equal to the line of equation $y_2 = 0$. Let $\texttt{Rev}_{y_3}(\mathcal{C})$ be the surface of revolution generated by rotating $\mathcal{C}$ around the y_3-axis.

Implicit case Let $\mathcal{C}$ be defined by the irreducible polynomial $f(y_2, y_3) \in \mathbb{C}[y_2, y_3]$. We suppose that we have already solved the offset degree problem for $\mathcal{C}$, using e.g. the formulae in Subsection A.1.1 (page 195). Collecting terms of odd and even degree in y_2, we write f (i.e. the implicit equation of $\mathcal{C}$) as follows:

$$f(y_2, y_3) = A(y_2^2, y_3) + y_2 B(y_2^2, y_3)$$

for some polynomials A and B. There are two cases to consider:

- case (a): $B = 0$. Then it holds that:

$$\begin{cases} \deg_{\bar{y}}(\mathcal{O}(\texttt{Rev}_{y_3}(\mathcal{C}))) = \deg_{\bar{y}}(\mathcal{O}(\mathcal{C})), \\ \deg_{y_i}(\mathcal{O}(\texttt{Rev}_{y_3}(\mathcal{C}))) = \deg_{y_i}(\mathcal{O}(\mathcal{C})) \text{ for } (i = 1, 2, 3), \\ \deg_{d}(\mathcal{O}(\texttt{Rev}_{y_3}(\mathcal{C}))) = \deg_{d}(\mathcal{O}(\mathcal{C})). \end{cases}$$

- case (b): $B \neq 0$. Then it holds that:

$$\begin{cases} \deg_{\bar{y}}(\mathcal{O}(\texttt{Rev}_{y_3}(\mathcal{C}))) = 2\deg_{\bar{y}}(\mathcal{O}(\mathcal{C})), \\ \deg_{y_i}(\mathcal{O}(\texttt{Rev}_{y_3}(\mathcal{C}))) = 2\deg_{y_i}(\mathcal{O}(\mathcal{C})) \text{ for } (i = 1, 2, 3), \\ \deg_{d}(\mathcal{O}(\texttt{Rev}_{y_3}(\mathcal{C}))) = 2\deg_{d}(\mathcal{O}(\mathcal{C})). \end{cases}$$

Parametric case Let

$$P_{\mathcal{C}}(t) = \left(\frac{P_1(t)}{P_0(t)}, \frac{P_2(t)}{P_0(t)} \right),$$

be a rational parametrization of $\mathcal{C}$, such that $\gcd(P_0, P_1, P_2) = 1$. We suppose that we have already solved the offset degree problem for $\mathcal{C}$, using e.g. the formulae in Subsection A.1.2 (page 197). Then, if

$$\gcd(P_0(t)P_1(s) + P_0(s)P_1(t), P_0(t)P_2(s) - P_0(s)P_2(t))$$

is non-trivial, apply the formulae of case (a) in the previous paragraph. Otherwise (if the gcd is trivial), apply the formulae in case (b).

Appendix B

Computational Complements

Coefficients $c_j^{(i)}$ in Lemma 4.15 (page 138).

The polynomials s_i can be expressed as follows:

$$s_i = c_1^{(i)}\, b^P + c_2^{(i)}\, \mathrm{nor}^P_{(1,2)} + c_3^{(i)}\, \mathrm{nor}^P_{(1,3)} + c_4^{(i)}\, \mathrm{nor}^P_{(2,3)} + c_5^{(i)}\, w^P + c_6^{(i)}\, \ell_1 + c_7^{(i)}\, \ell_2 + c_8^{(i)}\, \ell_3$$

where $c_j^{(i)} \in \mathbb{C}[d, \bar{k}, l, r, \bar{t}, \bar{x}]$ for $i = 0, \ldots, 3$, $j = 1, \ldots, 8$ are the following polynomials:

$c_1^{(0)} = 0$

$c_2^{(0)} = k_3$

$c_3^{(0)} = -k_2$

$c_4^{(0)} = k_1$

$c_5^{(0)} = P_0\, n_2\, k_3 - P_0\, n_3\, k_2$

$c_6^{(0)} = -P_0\, n_1\, k_3 + P_0\, n_3\, k_1$

$c_7^{(0)} = P_0\, n_1\, k_2 - P_0\, n_2\, k_1$

$c_8^{(0)} = 0$

$c_1^{(1)} = -2\, P_0\, n_1^3\, n_2\, k_1\, k_2\, r + P_0\, n_1^2\, n_2^2\, k_1^2\, r - 2\, P_0\, n_1\, n_2^3\, k_1\, k_2\, r - 2\, P_0\, n_1\, n_2\, n_3^2\, k_1\, k_2\, r + P_0\, n_2^4\, k_1^2\, r + P_0\, n_2^2\, n_3^2\, k_1^2\, r + n_1^2\, k_2^2$

$c_2^{(1)} = 2\, P_1\, P_0\, n_1^3\, k_1\, k_2\, r - P_1\, P_0\, n_1^2\, n_2\, k_1^2\, r + 2\, P_1\, P_0\, n_1\, n_2^2\, k_1\, k_2\, r - P_1\, P_0\, n_2^3\, k_1^2\, r + P_1\, n_2\, k_2^2 - P_2\, P_0\, n_1^3\, k_1^2\, r - P_2\, P_0\, n_1\, n_2^2\, k_1^2\, r + P_2\, n_1\, k_2^2 - 2\, P_2\, n_2\, k_1\, k_2 + 2\, P_3\, P_0\, n_1^2\, n_3\, k_1\, k_2\, r + P_0^2\, n_1^3\, k_1^2\, r\, x_2 - 2\, P_0^2\, n_1^3\, k_1\, k_2\, r\, x_1 + P_0^2\, n_1^2\, n_2\, k_1^2\, r\, x_1 - 2\, P_0^2\, n_1^2\, n_3\, k_1\, k_2\, r\, x_3 + P_0^2\, n_1\, n_2^2\, k_1^2\, r\, x_2 - 2\, P_0^2\, n_1\, n_2^2\, k_1\, k_2\, r\, x_1 + P_0^2\, n_2^3\, k_1^2\, r\, x_1 - P_0\, n_1\, k_2^2\, x_2 + P_0\, n_2\, k_2^2\, x_1$

$c_3^{(1)} = 2\,P_1\,P_0\,n_1\,n_2\,n_3\,k_1\,k_2\,r - P_1\,P_0\,n_2^2\,n_3\,k_1^2\,r + P_1\,n_3\,k_2^2 - 2\,P_2\,P_0\,n_1^2\,n_3\,k_1\,k_2\,r - P_3\,P_0\,n_1\,n_2^2\,k_1^2\,r + P_3\,n_1\,k_2^2 - 2\,P_3\,n_2\,k_1\,k_2 + 2\,P_0^2\,n_1^2\,n_3\,k_1\,k_2\,r\,x_2 + P_0^2\,n_1\,n_2^2\,k_1^2\,r\,x_3 - 2\,P_0^2\,n_1\,n_2\,n_3\,k_1\,k_2\,r\,x_1 + P_0^2\,n_2^2\,n_3\,k_1^2\,r\,x_1 - P_0\,n_1\,k_2\,k_3\,x_2 + 2\,P_0\,n_2\,k_2\,k_3\,x_1 - P_0\,n_3\,k_2^2\,x_1$

$c_4^{(1)} = -2\,P_1\,n_3\,k_1\,k_2 + P_2\,P_0\,n_1^2\,n_3\,k_1^2\,r + P_2\,n_3\,k_1^2 + P_3\,P_0\,n_1^2\,n_2\,k_1^2\,r + P_3\,n_2\,k_1^2 - P_0^2\,n_1^2\,n_2\,k_1^2\,r\,x_3 - P_0^2\,n_1^2\,n_3\,k_1^2\,r\,x_2 - P_0\,n_2\,k_1\,k_3\,x_1 + P_0\,n_3\,k_1\,k_2\,x_1$

$c_5^{(1)} = 2\,P_1\,P_0\,n_1^2\,k_2^2 - 2\,P_1\,P_0\,n_2\,n_3\,k_2\,k_3 + 2\,P_1\,P_0\,n_3^2\,k_2^2 - 2\,P_2\,P_0\,n_1^2\,k_1\,k_2 + 2\,P_2\,P_0\,n_1\,n_2\,k_2^2 - 2\,P_2\,P_0\,n_2^2\,k_1\,k_2 + P_2\,P_0\,n_2\,n_3\,k_1\,k_3 - P_2\,P_0\,n_3^2\,k_1\,k_2 + 2\,P_3\,P_0\,n_1\,n_2\,k_2\,k_3 - P_3\,P_0\,n_2^2\,k_1\,k_3 - P_3\,P_0\,n_2\,n_3\,k_1\,k_2 + P_0^2\,n_1^2\,k_1\,k_2\,x_2 - P_0^2\,n_1^2\,k_2^2\,x_1 - 2\,P_0^2\,n_1\,n_2\,k_2^2\,x_2 - 2\,P_0^2\,n_1\,n_2\,k_3^2\,x_2 + P_0^2\,n_2^2\,k_1\,k_2\,x_2 + P_0^2\,n_2^2\,k_1\,k_3\,x_3 + P_0^2\,n_2^2\,k_2^2\,x_1 + 2\,P_0^2\,n_2\,n_3\,k_2\,k_3\,x_1 - P_0^2\,n_3^2\,k_2^2\,x_1$

$c_6^{(1)} = -2\,P_1\,P_0\,n_1^2\,k_1\,k_2 + P_1\,P_0\,n_1\,n_3\,k_2\,k_3 - 2\,P_1\,P_0\,n_3^2\,k_1\,k_2 + 2\,P_2\,P_0\,n_1^2\,k_1^2 - 2\,P_2\,P_0\,n_1\,n_2\,k_1\,k_2 + 2\,P_2\,P_0\,n_2^2\,k_1^2 + P_2\,P_0\,n_3^2\,k_1^2 - P_3\,P_0\,n_1^2\,k_2\,k_3 + P_3\,P_0\,n_2\,n_3\,k_1^2 - P_0^2\,n_1^2\,k_1^2\,x_2 + P_0^2\,n_1^2\,k_1\,k_2\,x_1 + P_0^2\,n_1^2\,k_2\,k_3\,x_3 + 2\,P_0^2\,n_1\,n_2\,k_1\,k_2\,x_2 - 2\,P_0^2\,n_1\,n_2\,k_1\,k_3\,x_3 + 2\,P_0^2\,n_1\,n_2\,k_3^2\,x_1 - P_0^2\,n_1\,n_3\,k_2\,k_3\,x_1 - P_0^2\,n_2^2\,k_1^2\,x_2 - P_0^2\,n_2^2\,k_1\,k_2\,x_1 - P_0^2\,n_2\,n_3\,k_1\,k_3\,x_1 + P_0^2\,n_3^2\,k_1\,k_2\,x_1$

$c_7^{(1)} = -P_1\,P_0\,n_1\,n_3\,k_2^2 + 2\,P_1\,P_0\,n_2\,n_3\,k_1\,k_2 - P_2\,P_0\,n_2\,n_3\,k_1^2 + P_3\,P_0\,n_1^2\,k_2^2 - 2\,P_3\,P_0\,n_1\,n_2\,k_1\,k_2 + P_3\,P_0\,n_2^2\,k_1^2 - P_0^2\,n_1^2\,k_2^2\,x_3 + 2\,P_0^2\,n_1\,n_2\,k_1\,k_2\,x_3 + 2\,P_0^2\,n_1\,n_2\,k_1\,k_3\,x_2 - 2\,P_0^2\,n_1\,n_2\,k_2\,k_3\,x_1 + P_0^2\,n_1\,n_3\,k_2^2\,x_1 - P_0^2\,n_2^2\,k_1^2\,x_3 - P_0^2\,n_2\,n_3\,k_1\,k_2\,x_1$

$c_8^{(1)} = 2\,P_1\,P_2\,n_1^2\,k_1\,k_2 - 2\,P_1\,P_0\,n_1^2\,k_1\,k_2\,x_2 - P_2^2\,n_1^2\,k_1^2 + 2\,P_2^2\,n_1\,n_2\,k_1\,k_2 - P_2^2\,n_2^2\,k_1^2 + 2\,P_2\,P_0\,n_1^2\,k_1^2\,x_2 - 2\,P_2\,P_0\,n_1^2\,k_1\,k_2\,x_1 - 4\,P_2\,P_0\,n_1\,n_2\,k_1\,k_2\,x_2 + 2\,P_2\,P_0\,n_2^2\,k_1^2\,x_2 + 2\,P_3^2\,n_1\,n_2\,k_1\,k_2 - P_3^2\,n_2^2\,k_1^2 - 4\,P_3\,P_0\,n_1\,n_2\,k_1\,k_2\,x_3 + 2\,P_3\,P_0\,n_2^2\,k_1^2\,x_3 - P_0^2\,n_1^2\,k_1^2\,x_2^2 + 2\,P_0^2\,n_1^2\,k_1\,k_2\,x_1\,x_2 - 2\,P_0^2\,n_1\,n_2\,k_1\,k_2\,d^2 + 2\,P_0^2\,n_1\,n_2\,k_1\,k_2\,x_2^2 + 2\,P_0^2\,n_1\,n_2\,k_1\,k_2\,x_3^2 + P_0^2\,n_2^2\,k_1^2\,d^2 - P_0^2\,n_2^2\,k_1^2\,x_2^2 - P_0^2\,n_2^2\,k_1^2\,x_3^2$

$c_1^{(2)} = P_0\,n_1^2\,n_2^2\,k_3^2\,r - 2\,P_0\,n_1^2\,n_2\,n_3\,k_2\,k_3\,r + P_0\,n_1^2\,n_3^2\,k_2^2\,r + P_0\,n_2^4\,k_3^2\,r - 2\,P_0\,n_2^3\,n_3\,k_2\,k_3\,r + P_0\,n_2^2\,n_3^2\,k_2^2\,r + P_0\,n_2^2\,n_3^2\,k_3^2\,r - 2\,P_0\,n_2\,n_3^3\,k_2\,k_3\,r + P_0\,n_3^4\,k_2^2\,r$

$c_2^{(2)} = -P_1\,P_0\,n_1^2\,n_2\,k_3^2\,r + 2\,P_1\,P_0\,n_1^2\,n_3\,k_2\,k_3\,r - P_1\,P_0\,n_2^3\,k_3^2\,r + 2\,P_1\,P_0\,n_2^2\,n_3\,k_2\,k_3\,r - P_2\,P_0\,n_1^3\,k_3^2\,r - P_2\,P_0\,n_1\,n_2^2\,k_3^2\,r + P_0^2\,n_1^3\,k_3^2\,r\,x_2 + P_0^2\,n_1^2\,n_2\,k_3^2\,r\,x_1 - 2\,P_0^2\,n_1^2\,n_3\,k_2\,k_3\,r\,x_1 + P_0^2\,n_1\,n_2^2\,k_3^2\,r\,x_2 + P_0^2\,n_2^3\,k_3^2\,r\,x_1 - 2\,P_0^2\,n_2^2\,n_3\,k_2\,k_3\,r\,x_1$

$c_3^{(2)} = -P_1\,P_0\,n_1^2\,n_3\,k_2^2\,r - P_1\,P_0\,n_2^2\,n_3\,k_2^2\,r - P_1\,P_0\,n_2^2\,n_3\,k_3^2\,r + 2\,P_1\,P_0\,n_2\,n_3^2\,k_2\,k_3\,r - P_1\,P_0\,n_3^3\,k_2^2\,r + 2\,P_2\,P_0\,n_1^3\,k_2\,k_3\,r + 2\,P_2\,P_0\,n_1\,n_2^2\,k_2\,k_3\,r - P_3\,P_0\,n_1^3\,k_2^2\,r - P_3\,P_0\,n_1\,n_2^2\,k_2^2\,r - P_3\,P_0\,n_1\,n_2^2\,k_3^2\,r + 2\,P_3\,P_0\,n_1\,n_2\,n_3\,k_2\,k_3\,r - P_3\,P_0\,n_1\,n_3^2\,k_2^2\,r + P_0^2\,n_1^3\,k_2^2\,r\,x_3 - 2\,P_0^2\,n_1^3\,k_2\,k_3\,r\,x_2 + P_0^2\,n_1^2\,n_3\,k_2^2\,r\,x_1 + P_0^2\,n_1\,n_2^2\,k_2^2\,r\,x_3 - 2\,P_0^2\,n_1\,n_2^2\,k_2\,k_3\,r\,x_2 + P_0^2\,n_1\,n_2^2\,k_3^2\,r\,x_3 - 2\,P_0^2\,n_1\,n_2\,n_3\,k_2\,k_3\,r\,x_3 + P_0^2\,n_1\,n_3^2\,k_2^2\,r\,x_3 + P_0^2\,n_2^2\,n_3\,k_2^2\,r\,x_1 + P_0^2\,n_2^2\,n_3\,k_3^2\,r\,x_1 - 2\,P_0^2\,n_2\,n_3^2\,k_2\,k_3\,r\,x_1 + P_0^2\,n_3^3\,k_2^2\,r\,x_1$

$c_4^{(2)} = 2\,P_2\,P_0\,n_1^2\,n_2\,k_2\,k_3\,r - P_2\,P_0\,n_1^2\,n_3\,k_2^2\,r + P_2\,P_0\,n_1^2\,n_3\,k_3^2\,r + 2\,P_2\,P_0\,n_2^3\,k_2\,k_3\,r - P_2\,P_0\,n_2^2\,n_3\,k_2^2\,r + 2\,P_2\,P_0\,n_2\,n_3^2\,k_2\,k_3\,r - P_2\,P_0\,n_3^3\,k_2^2\,r + P_2\,n_3\,k_3^2 - P_3\,P_0\,n_1^2\,n_2\,k_2^2\,r + P_3\,P_0\,n_1^2\,n_2\,k_3^2\,r - 2\,P_3\,P_0\,n_1^2\,n_3\,k_2\,k_3\,r - P_3\,P_0\,n_2^3\,k_2^2\,r - P_3\,P_0\,n_2\,n_3^2\,k_2^2\,r + P_3\,n_2\,k_3^2 - 2\,P_3\,n_3\,k_2\,k_3 + P_0^2\,n_1^2\,n_2\,k_2^2\,r\,x_3 - 2\,P_0^2\,n_1^2\,n_2\,k_2\,k_3\,r\,x_2 - P_0^2\,n_1^2\,n_2\,k_3^2\,r\,x_3 + P_0^2\,n_1^2\,n_3\,k_2^2\,r\,x_2 +$

$2\,P_0^2\,n_1^2\,n_3\,k_2\,k_3\,r\,x_3 - P_0^2\,n_1^2\,n_3\,k_3^2\,r\,x_2 + P_0^2\,n_2^3\,k_2^2\,r\,x_3 - 2\,P_0^2\,n_2^3\,k_2\,k_3\,r\,x_2 + P_0^2\,n_2^2\,n_3\,k_2^2\,r\,x_2 + P_0^2\,n_2\,n_3^2\,k_2^2\,r\,x_3 - 2\,P_0^2\,n_2\,n_3^2\,k_2\,k_3\,r\,x_2 + P_0^2\,n_3^3\,k_2^2\,r\,x_2 - P_0\,n_2\,k_3^2\,x_3 + P_0\,n_3\,k_3^2\,x_2$

$c_5^{(2)} = 0$

$c_6^{(2)} = 2\,P_2\,P_0\,n_1^2\,k_3^2 + 2\,P_2\,P_0\,n_2^2\,k_3^2 - 2\,P_3\,P_0\,n_1^2\,k_2\,k_3 - 2\,P_3\,P_0\,n_2^2\,k_2\,k_3 + 2\,P_3\,P_0\,n_2\,n_3\,k_3^2 - 2\,P_3\,P_0\,n_3^2\,k_2\,k_3 + P_0^2\,n_1^2\,k_2\,k_3\,x_3 - P_0^2\,n_1^2\,k_3^2\,x_2 + P_0^2\,n_2^2\,k_2\,k_3\,x_3 - P_0^2\,n_2^2\,k_3^2\,x_2 - 2\,P_0^2\,n_2\,n_3\,k_3^2\,x_3 + P_0^2\,n_3^2\,k_2\,k_3\,x_3 + P_0^2\,n_3^2\,k_3^2\,x_2$

$c_7^{(2)} = -2\,P_2\,P_0\,n_1^2\,k_2\,k_3 - 2\,P_2\,P_0\,n_2^2\,k_2\,k_3 + 2\,P_3\,P_0\,n_1^2\,k_2^2 + 2\,P_3\,P_0\,n_2^2\,k_2^2 - 2\,P_3\,P_0\,n_2\,n_3\,k_2\,k_3 + 2\,P_3\,P_0\,n_3^2\,k_2^2 - P_0^2\,n_1^2\,k_2^2\,x_3 + P_0^2\,n_1^2\,k_2\,k_3\,x_2 - P_0^2\,n_2^2\,k_2^2\,x_3 + P_0^2\,n_2^2\,k_2\,k_3\,x_2 + 2\,P_0^2\,n_2\,n_3\,k_2\,k_3\,x_3 - P_0^2\,n_3^2\,k_2^2\,x_3 - P_0^2\,n_3^2\,k_2\,k_3\,x_2$

$c_8^{(2)} = -P_2^2\,n_1^2\,k_3^2 - P_2^2\,n_2^2\,k_3^2 + 2\,P_2\,P_3\,n_1^2\,k_2\,k_3 + 2\,P_2\,P_3\,n_2^2\,k_2\,k_3 - 2\,P_2\,P_0\,n_1^2\,k_2\,k_3\,x_3 + 2\,P_2\,P_0\,n_1^2\,k_3^2\,x_2 - 2\,P_2\,P_0\,n_2^2\,k_2\,k_3\,x_3 + 2\,P_2\,P_0\,n_2^2\,k_3^2\,x_2 - P_3^2\,n_1^2\,k_2^2 - P_3^2\,n_2^2\,k_2^2 - P_3^2\,n_2^2\,k_3^2 + 2\,P_3^2\,n_2\,n_3\,k_2\,k_3 - P_3^2\,n_3^2\,k_2^2 + 2\,P_3\,P_0\,n_1^2\,k_2^2\,x_3 - 2\,P_3\,P_0\,n_1^2\,k_2\,k_3\,x_2 + 2\,P_3\,P_0\,n_2^2\,k_2^2\,x_3 - 2\,P_3\,P_0\,n_2^2\,k_2\,k_3\,x_2 + 2\,P_3\,P_0\,n_2^2\,k_3^2\,x_3 - 4\,P_3\,P_0\,n_2\,n_3\,k_2\,k_3\,x_3 + 2\,P_3\,P_0\,n_3^2\,k_2^2\,x_3 - P_0^2\,n_1^2\,k_2^2\,x_3^2 + 2\,P_0^2\,n_1^2\,k_2\,k_3\,x_2\,x_3 - P_0^2\,n_1^2\,k_3^2\,x_2^2 - P_0^2\,n_2^2\,k_2^2\,x_3^2 + 2\,P_0^2\,n_2^2\,k_2\,k_3\,x_2\,x_3 + P_0^2\,n_2^2\,k_3^2\,d^2 - P_0^2\,n_2^2\,k_3^2\,x_2^2 - P_0^2\,n_2^2\,k_3^2\,x_3^2 - 2\,P_0^2\,n_2\,n_3\,k_2\,k_3\,d^2 + 2\,P_0^2\,n_2\,n_3\,k_2\,k_3\,x_3^2 + P_0^2\,n_3^2\,k_2^2\,d^2 - P_0^2\,n_3^2\,k_2^2\,x_3^2$

$c_1^{(3)} = -2\,P_0\,n_1^3\,n_3\,k_1\,k_3\,r + P_0\,n_1^2\,n_3^2\,k_1^2\,r - 2\,P_0\,n_1\,n_2^2\,n_3\,k_1\,k_3\,r - 2\,P_0\,n_1\,n_3^3\,k_1\,k_3\,r + P_0\,n_2^2\,n_3^2\,k_1^2\,r + P_0\,n_3^4\,k_1^2\,r + n_1^2\,k_3^2$

$c_2^{(3)} = 2\,P_1\,P_0\,n_1\,n_2\,n_3\,k_1\,k_3\,r + P_1\,n_2\,k_3^2 - 2\,P_2\,P_0\,n_1^2\,n_3\,k_1\,k_3\,r + P_2\,n_1\,k_3^2 - 2\,P_3\,P_0\,n_1^2\,n_2\,k_1\,k_3\,r - 2\,P_3\,n_2\,k_1\,k_3 + 2\,P_0^2\,n_1^2\,n_2\,k_1\,k_3\,r\,x_3 + 2\,P_0^2\,n_1^2\,n_3\,k_1\,k_3\,r\,x_2 - 2\,P_0^2\,n_1\,n_2\,n_3\,k_1\,k_3\,r\,x_1 - P_0\,n_1\,k_3^2\,x_2 + P_0\,n_2\,k_3^2\,x_1$

$c_3^{(3)} = 2\,P_1\,P_0\,n_1^3\,k_1\,k_3\,r - P_1\,P_0\,n_1^2\,n_3\,k_1^2\,r + 2\,P_1\,P_0\,n_1\,n_3^2\,k_1\,k_3\,r - P_1\,P_0\,n_2^2\,n_3\,k_1^2\,r - P_1\,P_0\,n_3^3\,k_1^2\,r + P_1\,n_3\,k_3^2 + 4\,P_2\,P_0\,n_1^2\,n_2\,k_1\,k_3\,r - P_3\,P_0\,n_1^3\,k_1^2\,r - P_3\,P_0\,n_1\,n_2^2\,k_1^2\,r - P_3\,P_0\,n_1\,n_3^2\,k_1^2\,r + P_3\,n_1\,k_3^2 - 2\,P_3\,n_3\,k_1\,k_3 + P_0^2\,n_1^3\,k_1^2\,r\,x_3 - 2\,P_0^2\,n_1^3\,k_1\,k_3\,r\,x_1 - 4\,P_0^2\,n_1^2\,n_2\,k_1\,k_3\,r\,x_2 + P_0^2\,n_1^2\,n_3\,k_1^2\,r\,x_1 + P_0^2\,n_1\,n_2^2\,k_1^2\,r\,x_3 + P_0^2\,n_1\,n_3^2\,k_1^2\,r\,x_3 - 2\,P_0^2\,n_1\,n_3^2\,k_1\,k_3\,r\,x_1 + P_0^2\,n_2^2\,n_3\,k_1^2\,r\,x_1 + P_0^2\,n_3^3\,k_1^2\,r\,x_1 - P_0\,n_1\,k_3^2\,x_3 + P_0\,n_3\,k_3^2\,x_1$

$c_4^{(3)} = -P_2\,P_0\,n_1^2\,n_3\,k_1^2\,r + 2\,P_2\,P_0\,n_1\,n_2^2\,k_1\,k_3\,r + 2\,P_2\,P_0\,n_1\,n_3^2\,k_1\,k_3\,r - P_2\,P_0\,n_2^2\,n_3\,k_1^2\,r - P_2\,P_0\,n_3^3\,k_1^2\,r - P_3\,P_0\,n_1^2\,n_2\,k_1^2\,r - P_3\,P_0\,n_2^3\,k_1^2\,r - P_3\,P_0\,n_2\,n_3^2\,k_1^2\,r + P_0^2\,n_1^2\,n_2\,k_1^2\,r\,x_3 + P_0^2\,n_1^2\,n_3\,k_1^2\,r\,x_2 - 2\,P_0^2\,n_1\,n_2^2\,k_1\,k_3\,r\,x_2 - 2\,P_0^2\,n_1\,n_3^2\,k_1\,k_3\,r\,x_2 + P_0^2\,n_2^3\,k_1^2\,r\,x_3 + P_0^2\,n_2^2\,n_3\,k_1^2\,r\,x_2 + P_0^2\,n_2\,n_3^2\,k_1^2\,r\,x_3 + P_0^2\,n_3^3\,k_1^2\,r\,x_2$

$c_5^{(3)} = 2\,P_1\,P_0\,n_1^2\,k_3^2 + 2\,P_2\,P_0\,n_1\,n_2\,k_3^2 - 2\,P_3\,P_0\,n_1^2\,k_1\,k_3 + 2\,P_3\,P_0\,n_1\,n_3\,k_3^2 - 2\,P_3\,P_0\,n_2^2\,k_1\,k_3 - 2\,P_3\,P_0\,n_3^2\,k_1\,k_3 + P_0^2\,n_1^2\,k_1\,k_3\,x_3 - P_0^2\,n_1^2\,k_3^2\,x_1 - 2\,P_0^2\,n_1\,n_2\,k_3^2\,x_2 - 2\,P_0^2\,n_1\,n_3\,k_3^2\,x_3 + P_0^2\,n_2^2\,k_1\,k_3\,x_3 + P_0^2\,n_2^2\,k_3^2\,x_1 + P_0^2\,n_3^2\,k_1\,k_3\,x_3 + P_0^2\,n_3^2\,k_3^2\,x_1$

$c_6^{(3)} = 0$

$c_7^{(3)} = -2\,P_1\,P_0\,n_1^2\,k_1\,k_3 - 2\,P_2\,P_0\,n_1\,n_2\,k_1\,k_3 + 2\,P_3\,P_0\,n_1^2\,k_1^2 - 2\,P_3\,P_0\,n_1\,n_3\,k_1\,k_3 + 2\,P_3\,P_0\,n_2^2\,k_1^2 + 2\,P_3\,P_0\,n_3^2\,k_1^2 - P_0^2\,n_1^2\,k_1^2\,x_3 + P_0^2\,n_1^2\,k_1\,k_3\,x_1 + 2\,P_0^2\,n_1\,n_2\,k_1\,k_3\,x_2 +$

$2\,P_0^2\,n_1\,n_3\,k_1\,k_3\,x_3 - P_0^2\,n_2^2\,k_1^2\,x_3 - P_0^2\,n_2^2\,k_1\,k_3\,x_1 - P_0^2\,n_3^2\,k_1^2\,x_3 - P_0^2\,n_3^2\,k_1\,k_3\,x_1$

$c_8^{(3)} = 2\,P_1\,P_3\,n_1^2\,k_1\,k_3 - 2\,P_1\,P_0\,n_1^2\,k_1\,k_3\,x_3 + 2\,P_2\,P_3\,n_1\,n_2\,k_1\,k_3 - 2\,P_2\,P_0\,n_1\,n_2\,k_1\,k_3\,x_3 - P_3^2\,n_1^2\,k_1^2 + 2\,P_3^2\,n_1\,n_3\,k_1\,k_3 - P_3^2\,n_2^2\,k_1^2 - P_3^2\,n_3^2\,k_1^2 + 2\,P_3\,P_0\,n_1^2\,k_1^2\,x_3 - 2\,P_3\,P_0\,n_1^2\,k_1\,k_3\,x_1 - 2\,P_3\,P_0\,n_1\,n_2\,k_1\,k_3\,x_2 - 4\,P_3\,P_0\,n_1\,n_3\,k_1\,k_3\,x_3 + 2\,P_3\,P_0\,n_2^2\,k_1^2\,x_3 + 2\,P_3\,P_0\,n_3^2\,k_1^2\,x_3 - P_0^2\,n_1^2\,k_1^2\,x_3^2 + 2\,P_0^2\,n_1^2\,k_1\,k_3\,x_1\,x_3 + 2\,P_0^2\,n_1\,n_2\,k_1\,k_3\,x_2\,x_3 - 2\,P_0^2\,n_1\,n_3\,k_1\,k_3\,d^2 + 2\,P_0^2\,n_1\,n_3\,k_1\,k_3\,x_3^2 - P_0^2\,n_2^2\,k_1^2\,x_3^2 + P_0^2\,n_3^2\,k_1^2\,d^2 - P_0^2\,n_3^2\,k_1^2\,x_3^2$

Generic Offset Polynomial for Example 2.43 (page 76)

If $\mathcal{C}$ is the nodal cubic $\mathcal{C}$ (Descartes Folium) given by the parametrization:

$$P(t) = \left(\frac{t}{1+t^3}, \frac{t^2}{1+t^3}\right),$$

then is generic offset polynomial (computed with CoCoA, see [13]) is:
$g(d,\bar{x}) = 729x_1^{14} + 2916x_1^{12}x_2^2 - 5103x_1^{12}d^2 + 1458x_1^{11}x_2^3 + 4374x_1^{11}x_2d^2 + 4374x_1^{10}x_2^4 - 17496x_1^{10}x_2^2d^2 + 15309x_1^{10}d^4 + 5832x_1^9x_2^5 + 16038x_1^9x_2^3d^2 - 21870x_1^9x_2d^4 + 3645x_1^8x_2^6 - 24057x_1^8x_2^4d^2 + 45927x_1^8x_2^2d^4 - 26244x_1^8d^6 + 8748x_1^7x_2^7 + 26244x_1^7x_2^5d^2 - 78732x_1^7x_2^3d^4 + 43740x_1^7x_2d^6 + 3645x_1^6x_2^8 - 23328x_1^6x_2^6d^2 + 61236x_1^6x_2^4d^4 - 69984x_1^6x_2^2d^6 + 28431x_1^6d^8 + 5832x_1^5x_2^9 + 26244x_1^5x_2^7d^2 - 113724x_1^5x_2^5d^4 + 125388x_1^5x_2^3d^6 - 43740x_1^5x_2d^8 + 4374x_1^4x_2^{10} - 24057x_1^4x_2^8d^2 + 61236x_1^4x_2^6d^4 - 87480x_1^4x_2^4d^6 + 65610x_1^4x_2^2d^8 - 19683x_1^4d^{10} + 1458x_1^3x_2^{11} + 16038x_1^3x_2^9d^2 - 78732x_1^3x_2^7d^4 + 125388x_1^3x_2^5d^6 - 86022x_1^3x_2^3d^8 + 21870x_1^3x_2d^{10} + 2916x_1^2x_2^{12} - 17496x_1^2x_2^{10}d^2 + 45927x_1^2x_2^8d^4 - 69984x_1^2x_2^6d^6 + 65610x_1^2x_2^4d^8 - 34992x_1^2x_2^2d^{10} + 8019x_1^2d^{12} + 4374x_1x_2^{11}d^2 - 21870x_1x_2^9d^4 + 43740x_1x_2^7d^6 - 43740x_1x_2^5d^8 + 21870x_1x_2^3d^{10} - 4374x_1x_2d^{12} + 729x_2^{14} - 5103x_2^{12}d^2 + 15309x_2^{10}d^4 - 26244x_2^8d^6 + 28431x_2^6d^8 - 19683x_2^4d^{10} + 8019x_2^2d^{12} - 1458d^{14} - 1458x_1^{12}x_2 - 5832x_1^{10}x_2^3 + 5832x_1^{10}x_2d^2 - 1458x_1^9x_2^4 + 1458x_1^9d^4 - 8748x_1^8x_2^5 + 17496x_1^8x_2^3d^2 - 7290x_1^8x_2d^4 - 5832x_1^7x_2^6 + 5832x_1^7x_2^4d^2 + 5832x_1^7x_2^2d^4 - 5832x_1^7d^6 - 5832x_1^6x_2^7 + 17496x_1^6x_2^5d^2 - 11664x_1^6x_2^3d^4 - 8748x_1^5x_2^8 + 17496x_1^5x_2^6d^2 - 17496x_1^5x_2^2d^6 + 8748x_1^5d^8 - 1458x_1^4x_2^9 + 5832x_1^4x_2^7d^2 - 11664x_1^4x_2^3d^6 + 7290x_1^4x_2d^8 - 5832x_1^3x_2^{10} + 17496x_1^3x_2^8d^2 - 11664x_1^3x_2^6d^4 - 11664x_1^3x_2^4d^6 + 17496x_1^3x_2^2d^8 - 5832x_1^3d^{10} + 5832x_1^2x_2^7d^4 - 17496x_1^2x_2^5d^6 + 17496x_1^2x_2^3d^8 - 5832x_1^2x_2d^{10} - 1458x_1x_2^{12} + 5832x_1x_2^{10}d^2 - 7290x_1x_2^8d^4 + 7290x_1x_2^4d^8 - 5832x_1x_2^2d^{10} + 1458x_1d^{12} + 1458x_2^9d^4 - 5832x_2^7d^6 + 8748x_2^5d^8 - 5832x_2^3d^{10} + 1458x_2d^{12} - 1944x_1^{11}x_2 + 729x_1^{10}x_2^2 - 729x_1^{10}d^2 - 3888x_1^9x_2^3 + 9720x_1^9x_2d^2 - 972x_1^8x_2^4 - 18225x_1^8x_2^2d^2 + 2673x_1^8d^4 - 1944x_1^7x_2^5 + 15552x_1^7x_2^3d^2 - 19440x_1^7x_2d^4 - 3402x_1^6x_2^6 - 43254x_1^6x_2^4d^2 + 50058x_1^6x_2^2d^4 - 3402x_1^6d^6 - 1944x_1^5x_2^7 + 11664x_1^5x_2^5d^2 - 29160x_1^5x_2^3d^4 + 21384x_1^5x_2d^6 - 972x_1^4x_2^8 - 43254x_1^4x_2^6d^2 + 90882x_1^4x_2^4d^4 - 48114x_1^4x_2^2d^6 + 1458x_1^4d^8 - 3888x_1^3x_2^9 + 15552x_1^3x_2^7d^2 - 29160x_1^3x_2^5d^4 + 31104x_1^3x_2^3d^6 - 13608x_1^3x_2d^8 + 729x_1^2x_2^{10} - 18225x_1^2x_2^8d^2 + 50058x_1^2x_2^6d^4 - 48114x_1^2x_2^4d^6 + 15309x_1^2x_2^2d^8 + 243x_1^2d^{10} - 1944x_1x_2^{11} + 9720x_1x_2^9d^2 - 19440x_1x_2^7d^4 + 21384x_1x_2^5d^6 - 13608x_1x_2^3d^8 + 3888x_1x_2d^{10} - 729x_2^{10}d^2 + 2673x_2^8d^4 - 3402x_2^6d^6 + 1458x_2^4d^8 + 243x_2^2d^{10} - 243d^{12} - 216x_1^{11} + 648x_1^{10}x_2 + 4320x_1^9x_2^2 + 378x_1^9d^2 - 4536x_1^8x_2d^2 + 9720x_1^7x_2^4 - 5994x_1^7x_2^2d^2 + 648x_1^7d^4 + 4536x_1^6x_2^5 - 1134x_1^6x_2^3d^2 + 1134x_1^6x_2d^4 +$

$4536x_1^5x_2^6-4698x_1^5x_2^4d^2-8586x_1^5x_2^2d^4-1836x_1^5d^6+9720x_1^4x_2^7-4698x_1^4x_2^5d^2-15876x_1^4x_2^3d^4+7614x_1^4x_2d^6-1134x_1^3x_2^6d^2-15876x_1^3x_2^4d^4+14850x_1^3x_2^2d^6+2160x_1^3d^8+4320x_1^2x_2^9-5994x_1^2x_2^7d^2-8586x_1^2x_2^5d^4+14850x_1^2x_2^3d^6-3726x_1^2x_2d^8+648x_1x_2^{10}-4536x_1x_2^8d^2+1134x_1x_2^6d^4+7614x_1x_2^4d^6-3726x_1x_2^2d^8-1134x_1d^{10}-216x_2^{11}+378x_2^9d^2+648x_2^7d^4-1836x_2^5d^6+2160x_2^3d^8-1134x_2d^{10}+378x_1^{10}+432x_1^9x_2-972x_1^8x_2^2-1404x_1^8d^2-2052x_1^7x_2^3+4968x_1^7x_2d^2-54x_1^6x_2^4+1404x_1^6x_2^2d^2+1404x_1^6d^4-5832x_1^5x_2^5+15552x_1^5x_2^3d^2-7506x_1^5x_2d^4-54x_1^4x_2^6-432x_1^4x_2^4d^2+3510x_1^4x_2^2d^4-2052x_1^4d^6-2052x_1^3x_2^7+15552x_1^3x_2^5d^2-11880x_1^3x_2^3d^4-1620x_1^3x_2d^6-972x_1^2x_2^8+1404x_1^2x_2^6d^2+3510x_1^2x_2^4d^4-7128x_1^2x_2^2d^6+1458x_1^2d^8+432x_1x_2^9+4968x_1x_2^7d^2-7506x_1x_2^5d^4-1620x_1x_2^3d^6+3726x_1x_2d^8+378x_2^{10}-1404x_2^8d^2+1404x_2^6d^4-2052x_2^4d^6+1458x_2^2d^8+216d^{10}-144x_1^9-900x_1^8x_2-360x_1^7x_2^2+792x_1^7d^2-432x_1^6x_2^3-702x_1^6x_2d^2-612x_1^5x_2^4-1134x_1^5x_2^2d^2+306x_1^5d^4-612x_1^4x_2^5-3204x_1^4x_2^3d^2+3546x_1^4x_2d^4-432x_1^3x_2^6-3204x_1^3x_2^4d^2+4032x_1^3x_2^2d^4-252x_1^3d^6-360x_1^2x_2^7-1134x_1^2x_2^5d^2+4032x_1^2x_2^3d^4-1242x_1^2x_2d^6-900x_1x_2^8-702x_1x_2^6d^2+3546x_1x_2^4d^4-1242x_1x_2^2d^6-702x_1d^8-144x_2^9+792x_2^7d^2+306x_2^5d^4-252x_2^3d^6-702x_2d^8+32x_1^8+360x_1^7x_2+698x_1^6x_2^2-530x_1^6d^2+352x_1^5x_2^3-264x_1^5x_2d^2+1044x_1^4x_2^4-570x_1^4x_2^2d^2-33x_1^4d^4+352x_1^3x_2^5+272x_1^3x_2^3d^2-816x_1^3x_2d^4+698x_1^2x_2^6-570x_1^2x_2^4d^2-1008x_1^2x_2^2d^4+432x_1^2d^6+360x_1x_2^7-264x_1x_2^5d^2-816x_1x_2^3d^4+720x_1x_2d^6+32x_2^8-530x_2^6d^2-33x_2^4d^4+432x_2^2d^6+99d^8-8x_1^7-72x_1^6x_2-288x_1^5x_2^2+186x_1^5d^2-224x_1^4x_2^3+192x_1^4x_2d^2-224x_1^3x_2^4+354x_1^3x_2^2d^2-46x_1^3d^4-288x_1^2x_2^5+354x_1^2x_2^3d^2+12x_1^2x_2d^4-72x_1x_2^6+192x_1x_2^4d^2+12x_1x_2^2d^4-132x_1d^6-8x_2^7+186x_2^5d^2-46x_2^3d^4-132x_2d^6+x_1^6+16x_1^5x_2+48x_1^4x_2^2-37x_1^4d^2+74x_1^3x_2^3-86x_1^3x_2d^2+48x_1^2x_2^4-80x_1^2x_2^2d^2+20x_1^2d^4+16x_1x_2^5-86x_1x_2^3d^2+70x_1x_2d^4+x_2^6-37x_2^4d^2+20x_2^2d^4+16d^6-2x_1^4x_2-8x_1^3x_2^2+8x_1^3d^2-8x_1^2x_2^3+10x_1^2x_2d^2-2x_1x_2^4+10x_1x_2^2d^2-8x_1d^4+8x_2^3d^2-8x_2d^4+x_1^2x_2^2-x_1^2d^2-x_2^2d^2$

Generic Offset Polynomial for Example 3.49 (page 119)

$g(\bar{x},d) = 50096498540544\,{x_1}^{18} + 50096498540544\,{x_2}^{18} - 281474976710656\,d^{14} + 112717121716224\,{x_2}^{8} + 506342284460032\,{x_2}^{12} - 397988849516544\,{x_2}^{10} + 450868486864896\,{x_1}^{10} + 1414109392273408\,{x_1}^{16} + 3967381550399488\,{x_1}^{14} + 2857836879020032\,{x_1}^{15} - 219490008694784\,{x_2}^{14} - 51677046505472\,{x_2}^{16} + 2918121039986688\,{x_1}^{12} + 408193691811840\,{x_1}^{17} - 3300802626060288\,d^{10} + 1838383441641472\,d^{12} + 4040499073646592\,{x_1}^{13} + 1452798457675776\,{x_1}^{11} + 1803473947459584\,d^{8} + 17901148811821056\,{x_2}^{10}{x_1}^{3}d^{4} - 150683395662282752\,{x_2}^{4}{x_1}^{3}d^{6} + 49097248629325824\,{x_2}^{4}{x_1}^{3}d^{8} + 74471022060896256\,{x_2}^{2}{x_1}^{3}d^{8} - 11348059510276096\,{x_2}^{2}{x_1}^{3}d^{10} - 12364558010155008\,{x_2}^{12}{x_1}^{3}d^{2} - 400771988324352\,{x_2}^{4}{x_1}^{3}d^{10} - 88487424493092864\,{x_2}^{2}{x_1}^{3}d^{6} - 7013509795676160\,{x_2}^{4}{x_1}^{12}d^{2} + 3506754897838080\,{x_2}^{2}{x_1}^{12}d^{4} - 200385994162176\,{x_2}^{2}{x_1}^{6}d^{10} + 10520264693514240\,{x_2}^{4}{x_1}^{10}d^{4} + 47065969616486400\,{x_2}^{2}{x_1}^{10}d^{4} + 55700571868364800\,{x_2}^{8}{x_1}^{8} + 15665807312814080\,{x_1}^{8}d^{6} - 5009649854054400\,{x_2}^{12}d^{6} + 2554921425567744\,{x_1}^{8}d^{2} - 16135865713557504\,{x_1}^{8}d^{4} - 2898450089771008\,{x_1}^{10}d^{6} + 1105833819635712\,{x_1}^{6}d^{10} + 657542313148416\,{x_2}^{6}d^{8} - 4489907271630848\,{x_2}^{6}d^{6} + 19459843983147008\,{x_1}^{7}d^{6} - 6937849651789824\,{x_1}^{7}d^{8} - 133590662774784\,{x_1}^{7}d^{10} - 250482492702720\,{x_1}^{16}d^{2} + 500964985405440\,{x_1}^{14}d^{4} - 1766365430022144\,{x_1}^{15}d^{2} + 450868486864896\,{x_2}^{2}{x_1}^{16} +$

$$
\begin{aligned}
&24679844155490304\,{x_2}^2{x_1}^{12} + 31739189842149376\,{x_2}^4{x_1}^{12} + 4208105877405696\,{x_2}^6{x_1}^{12} - \\
&499195458879488\,{x_2}^8{x_1}^4 + 11218179699245056\,{x_2}^{12}{x_1}^4 + 19084566920691712\,{x_2}^2{x_1}^{13} + \\
&11429423370731520\,{x_2}^4{x_1}^{13} + 10247860687732736\,{x_2}^2{x_1}^{14} + 1803473947459584\,{x_2}^4{x_1}^{14} + \\
&3265549534494720\,{x_2}^2{x_1}^{15} + 1014454095446016\,{x_2}^4{x_1}^4 + 1803473947459584\,{x_2}^{14}{x_1}^4 - \\
&100192997081088\,{x_2}^6{x_1}^4 - 526459911274496\,{x_2}^{14}x + 1453142055059456\,{x_2}^{14}{x_1}^2 + \\
&644279454138368\,{x_2}^{10}{x_1}^2 + 535599601680384\,{x_2}^{10}x + 13175379116163072\,{x_2}^2{x_1}^9 + \\
&50599044073914368\,{x_2}^4{x_1}^9 + 73899894489743360\,{x_2}^6{x_1}^9 + 28573558426828800\,{x_2}^8{x_1}^9 - \\
&1052026469351424\,{x_2}^6{x_1}^3 + 1886349636403200\,{x_2}^8{x_1}^3 + 2163082969219072\,{x_2}^{12}{x_1}^3 + \\
&20874468771495936\,{x_2}^2{x_1}^{10} + 58073043083198464\,{x_2}^4{x_1}^{10} + 54420190577819648\,{x_2}^6{x_1}^{10} + \\
&6312158816108544\,{x_2}^8{x_1}^{10} - 4298209996308480\,{x_2}^6d^4 + 1198673832706048\,{x_2}^{10}{x_1}^4 + \\
&1352605460594688\,{x_2}^2{x_1}^7 + 25944419966386176\,{x_2}^4{x_1}^7 + 39093135925575680\,{x_2}^6{x_1}^7 + \\
&56978410538270720\,{x_2}^8{x_1}^7 + 22858846741463040\,{x_2}^{10}{x_1}^7 + 4508684868648960\,{x_2}^4{x_1}^5 + \\
&5954061622837248\,{x_2}^6{x_1}^5 + 5681107861241856\,{x_2}^8{x_1}^5 + 21570150394232832\,{x_2}^{10}{x_1}^5 + \\
&1352605460594688\,{x_1}^7d^2 - 12696679241220096\,{x_1}^7d^4 + 63820327800537088\,{x_2}^6{x_1}^8 + \\
&38069473420247040\,{x_2}^4{x_1}^8 + 5485566590189568\,{x_2}^2{x_1}^8 + 500964985405440\,{x_2}^{14}d^4 + \\
&1209737668460544\,{x_2}^6d^{10} + 6312158816108544\,{x_2}^{10}{x_1}^8 - 5209623531356160\,{x_1}^{14}d^2 + \\
&1014454095446016\,{x_1}^4d^4 - 6139952102375424\,{x_1}^4d^6 - 1239888338878464\,{x_1}^5d^4 + \\
&2504824927027200\,{x_2}^4{x_1}^6d^8 - 14027019591352320\,{x_2}^6{x_1}^{10}d^2 - 200385994162176\,{x_2}^6{x_1}^2d^{10} - \\
&67056534118465536\,{x_2}^6{x_1}^2d^6 - 19598382448246784\,{x_2}^8{x_1}^2d^6 + 1252412463513600\,{x_2}^8{x_1}^2d^8 + \\
&17735397433933824\,{x_2}^{10}{x_1}^2d^4 - 3005789912432640\,{x_2}^{10}{x_1}^2d^6 - 73712960333152256\,{x_2}^4{x_1}^2d^6 + \\
&57770986623205376\,{x_2}^4{x_1}^2d^8 + 61357971510132736\,{x_2}^2{x_1}^2d^8 - 21384126770708480\,{x_2}^2{x_1}^2d^{10} + \\
&1424967069597696\,{x_2}^2{x_1}^2d^{12} - 6449460330627072\,{x_2}^4{x_1}^2d^{10} - 142296733282467840\,{x_2}^6{x_1}^8d^2 - \\
&211321702452297728\,{x_2}^4{x_1}^8d^2 - 92739596734955520\,{x_2}^2{x_1}^8d^2 + \\
&127142267596046336\,{x_2}^2{x_1}^8d^4 + 12534088959262720\,{x_2}^8xd^4 - 2440022460465152\,{x_2}^{10}xd^2 + \\
&15592535170744320\,{x_2}^4xd^4 - 4270726500581376\,{x_2}^6xd^2 + 4360096180076544\,{x_2}^8xd^2 - \\
&32322515120422912\,{x_2}^6{x_1}^4d^2 - 17533774489190400\,{x_2}^8{x_1}^8d^2 - 3005789912432640\,{x_2}^2{x_1}^{10}d^6 + \\
&85537056838647808\,{x_2}^2{x_1}^9d^4 + 44752872029552640\,{x_2}^4{x_1}^9d^4 - 61822790050775040\,{x_2}^6{x_1}^9d^2 - \\
&12171593719480320\,{x_2}^2{x_1}^9d^6 + 223719280171024384\,{x_2}^4{x_1}^7d^4 + \\
&209223593748332544\,{x_2}^6{x_1}^4d^4 - 43421140110016512\,{x_2}^4{x_1}^4d^2 - \\
&33326695654096896\,{x_2}^2{x_1}^6d^2 + 166099373317423104\,{x_2}^2{x_1}^6d^4 - \\
&115590970233323520\,{x_2}^8{x_1}^6d^2 - 208529527033298944\,{x_2}^6{x_1}^6d^2 - \\
&136714478388510720\,{x_2}^4{x_1}^6d^2 + 159478664050769920\,{x_2}^2{x_1}^7d^4 - \\
&66299932679602176\,{x_2}^2{x_1}^7d^2 - 210380005102845952\,{x_2}^6{x_1}^7d^2 - \\
&31473520345088\,{x_2}^{12}xd^2 - 14527984576757760\,{x_2}^2xd^6 - 77977605160042496\,{x_2}^8{x_1}^4d^2 - \\
&49323404427264000\,{x_2}^{10}{x_1}^4d^2 + 161298458873954304\,{x_2}^4{x_1}^4d^4 + \\
&2983524801970176\,{x_2}^{12}xd^4 - 195077552023273472\,{x_2}^4{x_1}^7d^2 - 6754368648904704\,{x_2}^6xd^4 + \\
&2504824927027200\,{x_2}^6{x_1}^4d^8 - 7013509795676160\,{x_2}^{12}{x_1}^4d^2 + 56931539560169472\,{x_2}^2{x_1}^4d^4 - \\
&2028908190892032\,{x_2}^2{x_1}^4d^2 + 115079078851117056\,{x_2}^2{x_1}^5d^4 - 12286166267068416\,{x_2}^2{x_1}^5d^2 + \\
&264056376064475136\,{x_2}^4{x_1}^5d^4 - 82821572355686400\,{x_2}^4{x_1}^5d^2 - \\
&126454694871564288\,{x_2}^8{x_1}^5d^2 - 119205339831730176\,{x_2}^6{x_1}^5d^2 - \\
&62849459033210880\,{x_2}^2{x_1}^{11}d^2 - 1352605460594688\,{x_2}^4{x_1}^2d^2 + 18479766805938176\,{x_2}^6{x_1}^2d^4 - \\
&120377900263276544\,{x_2}^2{x_1}^6d^6 + 16373514843979776\,{x_2}^2{x_1}^6d^8 +
\end{aligned}
$$

$17533774489190400\,{x_2}^8{x_1}^6d^4 + 145051422226907136\,{x_2}^6{x_1}^6d^4 - 4374269572153344\,{x_1}^{11}d^2 +$
$11429423370731520\,{x_2}^{12}{x_1}^5 + 24927852747030528\,{x_2}^2{x_1}^{11} + 52028821506883584\,{x_2}^4{x_1}^{11} +$
$22858846741463040\,{x_2}^6{x_1}^{11} - 216844308840448\,{x_2}^{10}d^4 - 1699209321381888\,{x_2}^{10}d^2 +$
$2995224292884480\,{x_2}^8d^4 + 2286348530614272\,{x_2}^8d^2 - 676302730297344\,{x_2}^6{x_1}^2 +$
$935134639423488\,{x_2}^8{x_1}^2 - 2652022046195712\,{x_2}^{12}{x_1}^2 - 250482492702720\,{x_2}^{16}d^2 -$
$3306368903675904\,{x_1}^3d^6 - 8066601416916992\,{x_1}^{12}d^2 - 732313398804480\,{x_2}^4d^6 -$
$4508684868648960\,{x_2}^2d^6 + 348785704173568\,{x_2}^{12}d^2 - 200385994162176\,{x_2}^8x -$
$216947388055552\,{x_2}^{12}x + 408193691811840\,{x_2}^{16}x + 450868486864896\,{x_2}^{16}{x_1}^2 -$
$1127171217162240\,{x_2}^6d^2 + 3719665016635392\,{x_2}^4d^4 + 13576151104487424\,{x_2}^4{x_1}^6 +$
$19475993060179968\,{x_2}^6{x_1}^6 + 30090953192636416\,{x_2}^8{x_1}^6 + 33899180434915328\,{x_2}^{10}{x_1}^6 +$
$4208105877405696\,{x_2}^{12}{x_1}^6 + 3265549534494720\,{x_2}^{14}{x_1}^3 - 3917285051858944\,{x_2}^{10}{x_1}^3 +$
$131529078472704\,{x_2}^{14}d^2 - 581779090046976\,{x_2}^{12}d^4 - 925239034773504\,{x_1}^8d^8 -$
$50096498540544\,{x_1}^8d^{10} - 4517343522717696\,xd^{10} + 1442559255642112\,xd^{12} +$
$3406561900756992\,xd^8 + 284910950547456\,{x_1}^4d^{10} - 8342132158889984\,{x_1}^{13}d^2 +$
$6870504564588544\,{x_1}^4d^8 - 237494511599616\,{x_1}^4d^{12} - 4930485016854528\,{x_1}^5d^8 +$
$2852682918264832\,{x_1}^5d^{10} + 3255808548667392\,{x_1}^5d^6 - 2434318743896064\,{x_1}^{11}d^6 +$
$5928291819061248\,{x_1}^{11}d^4 + 250482492702720\,{x_2}^{10}d^8 + 1710427775959040\,{x_2}^{10}d^6 -$
$50096498540544\,{x_2}^8d^{10} - 2180743874740224\,{x_2}^8d^8 + 627237023907840\,{x_2}^8d^6 +$
$9328462808481792\,{x_1}^3d^8 - 3747685383274496\,{x_1}^3d^{10} - 316659348799488\,{x_1}^3d^{12} +$
$6750863955591168\,{x_1}^{12}d^4 - 500964985405440\,{x_1}^{12}d^6 - 237494511599616\,{x_2}^4d^{12} -$
$2587013421203456\,{x_2}^4d^{10} + 1671257674219520\,{x_2}^2d^{12} - 7277392586342400\,{x_2}^2d^{10} +$
$6442966340075520\,{x_2}^2d^8 + 8126559160369152\,{x_2}^4d^8 + 2983524801970176\,{x_1}^{13}d^4 -$
$10425741053263872\,{x_1}^6d^8 + 12473461200912384\,{x_1}^6d^6 - 8623744574554112\,{x_1}^2d^{10} +$
$11012880262496256\,{x_1}^2d^8 + 1424967069597696\,{x_1}^2d^{12} + 942556342910976\,{x_1}^9d^8 +$
$3957829543133184\,{x_1}^9d^6 + 250482492702720\,{x_1}^{10}d^8 - 4355264341868544\,{x_1}^6d^4 -$
$2705210921189376\,{x_1}^2d^6 - 11868197229690880\,{x_1}^9d^4 + 3343941277581312\,{x_1}^9d^2 -$
$2569249436467200\,{x_1}^{10}d^4 + 725935372369920\,{x_1}^{10}d^2 - 10019299708108800\,{x_2}^6{x_1}^6d^6 -$
$81173919821725696\,{x_2}^4{x_1}^6d^6 + 3770225371643904\,{x_2}^2{x_1}^7d^8 - 14027019591352320\,{x_2}^{10}{x_1}^6d^2 -$
$24343187438960640\,{x_2}^4{x_1}^7d^6 - 68408864701153280\,{x_2}^2{x_1}^7d^6 + 59670496039403520\,{x_2}^6{x_1}^7d^4 -$
$1766365430022144\,{x_2}^{14}xd^2 + 949978046398464\,{x_2}^2xd^{12} + 83510244591796224\,{x_2}^8{x_1}^4d^4 -$
$7514474781081600\,{x_2}^8{x_1}^4d^6 + 10520264693514240\,{x_2}^{10}{x_1}^4d^4 - 204529022695112704\,{x_2}^4{x_1}^4d^6 +$
$33342002917539840\,{x_2}^4{x_1}^4d^8 - 26343130370408448\,{x_2}^6xd^6 + 5124617538633728\,{x_2}^6xd^8 -$
$133590662774784\,{x_2}^6xd^{10} - 12364558010155008\,{x_2}^2{x_1}^{13}d^2 - 2003859941621760\,{x_2}^2{x_1}^{14}d^2 -$
$14502936327487488\,{x_2}^4{x_1}^3d^2 - 8428168943763456\,{x_2}^{12}{x_1}^2d^2 + 101918680580882432\,{x_2}^6{x_1}^3d^4 -$
$16939213576470528\,{x_2}^8{x_1}^3d^2 - 29606824479031296\,{x_2}^{10}{x_1}^3d^2 + 79588492773949440\,{x_2}^4{x_1}^3d^4 +$
$21566542621704192\,{x_2}^2{x_1}^3d^4 - 11558684706471936\,{x_2}^6{x_1}^3d^2 - 35133931992907776\,{x_2}^2{x_1}^{12}d^2 -$
$2003859941621760\,{x_2}^{14}{x_1}^2d^2 + 3506754897838080\,{x_2}^{12}{x_1}^2d^4 + 72066630649118720\,{x_2}^2{x_1}^4d^8 -$
$300578991243264\,{x_2}^4{x_1}^4d^{10} - 149850189186727936\,{x_2}^2{x_1}^4d^6 + 37034781438902272\,{x_2}^2{x_1}^5d^8 -$
$400771988324352\,{x_2}^2{x_1}^5d^{10} - 151412852907835392\,{x_2}^2{x_1}^5d^6 + 5655338057465856\,{x_2}^4{x_1}^5d^8 -$
$153416231813120000\,{x_2}^4{x_1}^5d^6 - 24343187438960640\,{x_2}^6{x_1}^5d^6 +$
$44752872029552640\,{x_2}^8{x_1}^5d^4 + 217218417671798784\,{x_2}^6{x_1}^5d^4 -$
$37093674030465024\,{x_2}^{10}{x_1}^5d^2 + 17901148811821056\,{x_2}^2{x_1}^{11}d^4 -$

$370936740304650 24\, {x_2}^4 {x_1}^{11} d^2 + 13862505164046336\, {x_2}^6 {x_1}^2 d^8 - 85774551350247424\, {x_2}^6 {x_1}^3 d^6 + 3770225371643904\, {x_2}^6 {x_1}^3 d^8 + 75785763089809408\, {x_2}^8 {x_1}^3 d^4 - 12171593719480320\, {x_2}^8 {x_1}^3 d^6 - 97393777915723776\, {x_2}^4 {x_1}^{10} d^2 - 83891362809774080\, {x_2}^2 {x_1}^{10} d^2 + 301774435260563456\, {x_2}^4 {x_1}^6 d^4 + 17533774489190400\, {x_2}^6 {x_1}^8 d^4 + 120173459819986944\, {x_2}^4 {x_1}^8 d^4 - 7514474781081600\, {x_2}^4 {x_1}^8 d^6 - 33425016045436928\, {x_2}^2 {x_1}^8 d^6 + 1252412463513600\, {x_2}^2 {x_1}^8 d^8 - 61822790050775040\, {x_2}^8 {x_1}^7 d^2 - 4725013781413888\, {x_2}^8 x d^6 + 942556342910976\, {x_2}^8 x d^8 + 2677860569448448\, {x_2}^{10} x d^4 - 2434318743896064\, {x_2}^{10} x d^6 - 16375834126319616\, {x_2}^4 x d^6 + 28273116874539008\, {x_2}^4 x d^8 + 22751850516185088\, {x_2}^2 x d^8 - 13249664870514688\, {x_2}^2 x d^{10} - 3469508941447168\, {x_2}^4 x d^{10} - 6553364179451904\, {x_2}^2 {x_1}^4 d^{10} - 71956164090265600\, {x_2}^6 {x_1}^4 d^6 - 11576465871077376\, {x_2}^6 {x_1}^2 d^2 + 3883595328389120\, {x_2}^8 {x_1}^2 d^2 + 36943831211442176\, {x_2}^8 {x_1}^2 d^4 - 4596233482010624\, {x_2}^{10} {x_1}^2 d^2 + 39109710204370944\, {x_2}^4 {x_1}^2 d^4 + 4734119112081408\, {x_2}^2 {x_1}^2 d^4 - 40242099806797824\, {x_2}^2 {x_1}^2 d^6 - 94745672879702016\, {x_2}^2 {x_1}^9 d^2 - 168007988064288768\, {x_2}^4 {x_1}^9 d^2$

Generic Offset Polynomial for Example 4.48 (page 178)

$g(d, x_1, x_2, x_3) = -16\, x_1^2\, x_2^{10}\, x_3^2 + 16\, x_1^2\, x_2^{10}\, d^2 - 48\, x_1^2\, x_2^8\, x_3^4 + 128\, x_1^2\, x_2^8\, x_3^2\, d^2 - 80\, x_1^2\, x_2^8\, d^4 - 48\, x_1^2\, x_2^6\, x_3^6 + 240\, x_1^2\, x_2^6\, x_3^4\, d^2 - 352\, x_1^2\, x_2^6\, x_3^2\, d^4 + 160\, x_1^2\, x_2^6\, d^6 - 16\, x_1^2\, x_2^4\, x_3^8 + 160\, x_1^2\, x_2^4\, x_3^6\, d^2 - 432\, x_1^2\, x_2^4\, x_3^4\, d^4 + 448\, x_1^2\, x_2^4\, x_3^2\, d^6 - 160\, x_1^2\, x_2^4\, d^8 + 32\, x_1^2\, x_2^2\, x_3^8\, d^2 - 176\, x_1^2\, x_2^2\, x_3^6\, d^4 + 336\, x_1^2\, x_2^2\, x_3^4\, d^6 - 272\, x_1^2\, x_2^2\, x_3^2\, d^8 + 80\, x_1^2\, x_2^2\, d^{10} - 16\, x_1^2\, x_3^8\, d^4 + 64\, x_1^2\, x_3^6\, d^6 - 96\, x_1^2\, x_3^4\, d^8 + 64\, x_1^2\, x_3^2\, d^{10} - 16\, x_1^2\, d^{12} - 16\, x_2^{12}\, x_3^2 + 16\, x_2^{12}\, d^2 - 64\, x_2^{10}\, x_3^4 + 160\, x_2^{10}\, x_3^2\, d^2 - 96\, x_2^{10}\, d^4 - 96\, x_2^8\, x_3^6 + 416\, x_2^8\, x_3^4\, d^2 - 560\, x_2^8\, x_3^2\, d^4 + 240\, x_2^8\, d^6 - 64\, x_2^6\, x_3^8 + 448\, x_2^6\, x_3^6\, d^2 - 1024\, x_2^6\, x_3^4\, d^4 + 960\, x_2^6\, x_3^2\, d^6 - 320\, x_2^6\, d^8 - 16\, x_2^4\, x_3^{10} + 208\, x_2^4\, x_3^8\, d^2 - 768\, x_2^4\, x_3^6\, d^4 + 1216\, x_2^4\, x_3^4\, d^6 - 880\, x_2^4\, x_3^2\, d^8 + 240\, x_2^4\, d^{10} + 32\, x_2^2\, x_3^{10}\, d^2 - 224\, x_2^2\, x_3^8\, d^4 + 576\, x_2^2\, x_3^6\, d^6 - 704\, x_2^2\, x_3^4\, d^8 + 416\, x_2^2\, x_3^2\, d^{10} - 96\, x_2^2\, d^{12} - 16\, x_3^{10}\, d^4 + 80\, x_3^8\, d^6 - 160\, x_3^6\, d^8 + 160\, x_3^4\, d^{10} - 80\, x_3^2\, d^{12} + 16\, d^{14} + 32\, x_1^4\, x_2^8\, x_3 + 744\, x_1^4\, x_2^6\, x_3^3 - 808\, x_1^4\, x_2^6\, x_3\, d^2 - 120\, x_1^4\, x_2^4\, x_3^5 - 1080\, x_1^4\, x_2^4\, x_3^3\, d^2 + 1232\, x_1^4\, x_2^4\, x_3\, d^4 + 32\, x_1^4\, x_2^2\, x_3^7 + 408\, x_1^4\, x_2^2\, x_3^5\, d^2 - 272\, x_1^4\, x_2^2\, x_3^3\, d^4 - 168\, x_1^4\, x_2^2\, x_3\, d^6 + 32\, x_1^4\, x_3^7\, d^2 - 352\, x_1^4\, x_3^5\, d^4 + 608\, x_1^4\, x_3^3\, d^6 - 288\, x_1^4\, x_3\, d^8 + 32\, x_1^2\, x_2^{10}\, x_3 + 968\, x_1^2\, x_2^8\, x_3^3 - 1032\, x_1^2\, x_2^8\, x_3\, d^2 + 720\, x_1^2\, x_2^6\, x_3^5 - 3176\, x_1^2\, x_2^6\, x_3^3\, d^2 + 2488\, x_1^2\, x_2^6\, x_3\, d^4 - 184\, x_1^2\, x_2^4\, x_3^7 - 392\, x_1^2\, x_2^4\, x_3^5\, d^2 + 2168\, x_1^2\, x_2^4\, x_3^3\, d^4 - 1592\, x_1^2\, x_2^4\, x_3\, d^6 + 32\, x_1^2\, x_2^2\, x_3^9 + 632\, x_1^2\, x_2^2\, x_3^7\, d^2 - 1672\, x_1^2\, x_2^2\, x_3^5\, d^4 + 1320\, x_1^2\, x_2^2\, x_3^3\, d^6 - 312\, x_1^2\, x_2^2\, x_3\, d^8 + 32\, x_1^2\, x_3^9\, d^2 - 512\, x_1^2\, x_3^7\, d^4 + 1344\, x_1^2\, x_3^5\, d^6 - 1280\, x_1^2\, x_3^3\, d^8 + 416\, x_1^2\, x_3\, d^{10} + 160\, x_2^{10}\, x_3^3 - 160\, x_2^{10}\, x_3\, d^2 + 224\, x_2^8\, x_3^5 - 736\, x_2^8\, x_3^3\, d^2 + 512\, x_2^8\, x_3\, d^4 - 32\, x_2^6\, x_3^7 - 256\, x_2^6\, x_3^5\, d^2 + 736\, x_2^6\, x_3^3\, d^4 - 448\, x_2^6\, x_3\, d^6 - 96\, x_2^4\, x_3^9 + 544\, x_2^4\, x_3^7\, d^2 - 928\, x_2^4\, x_3^5\, d^4 + 608\, x_2^4\, x_3^3\, d^6 - 128\, x_2^4\, x_3\, d^8 + 224\, x_2^2\, x_3^9\, d^2 - 1024\, x_2^2\, x_3^7\, d^4 + 1728\, x_2^2\, x_3^5\, d^6 - 1280\, x_2^2\, x_3^3\, d^8 + 352\, x_2^2\, x_3\, d^{10} - 128\, x_3^9\, d^4 + 512\, x_3^7\, d^6 - 768\, x_3^5\, d^8 + 512\, x_3^3\, d^{10} - 128\, x_3\, d^{12} - 16\, x_1^6\, x_2^6 - 2073\, x_1^6\, x_2^4\, x_3^2 + 873\, x_1^6\, x_2^4\, d^2 + 384\, x_1^6\, x_2^2\, x_3^4 - 324\, x_1^6\, x_2^2\, x_3^2\, d^2 + 2052\, x_1^6\, x_2^2\, d^4 - 16\, x_1^6\, x_3^6 + 504\, x_1^6\, x_3^4\, d^2 - 1728\, x_1^6\, x_3^2\, d^4 + 216\, x_1^6\, d^6 - 16\, x_1^4\, x_2^8 - 2797\, x_1^4\, x_2^6\, x_3^2 + 1277\, x_1^4\, x_2^6\, d^2 - 2529\, x_1^4\, x_2^4\, x_3^4 + 2602\, x_1^4\, x_2^4\, x_3^2\, d^2 + 2615\, x_1^4\, x_2^4\, d^4 + 560\, x_1^4\, x_2^2\, x_3^6 + 980\, x_1^4\, x_2^2\, x_3^4\, d^2 + 552\, x_1^4\, x_2^2\, x_3^2\, d^4 - 3372\, x_1^4\, x_2^2\, d^6 - 16\, x_1^4\, x_3^8 + 744\, x_1^4\, x_3^6\, d^2 - 3992\, x_1^4\, x_3^4\, d^4 + 3768\, x_1^4\, x_3^2\, d^6 - 504\, x_1^4\, d^8 - 620\, x_1^2\, x_2^8\, x_3^2 + 364\, x_1^2\, x_2^8\, d^2 - 1060\, x_1^2\, x_2^6\, x_3^4 + 252\, x_1^2\, x_2^6\, x_3^2\, d^2 + 1256\, x_1^2\, x_2^6\, d^4 - 824\, x_1^2\, x_2^4\, x_3^6 + 1436\, x_1^2\, x_2^4\, x_3^4\, d^2 + 2448\, x_1^2\, x_2^4\, x_3^2\, d^4 -$

$3252\,x_1^2\,x_2^4\,d^6 + 192\,x_1^2\,x_2^2\,x_3^8 + 1952\,x_1^2\,x_2^2\,x_3^6\,d^2 - 4032\,x_1^2\,x_2^2\,x_3^4\,d^4 + 608\,x_1^2\,x_2^2\,x_3^2\,d^6 +$
$1280\,x_1^2\,x_2^2\,d^8 + 224\,x_1^2\,x_3^8\,d^2 - 2432\,x_1^2\,x_3^6\,d^4 + 4544\,x_1^2\,x_3^4\,d^6 - 2688\,x_1^2\,x_3^2\,d^8 + 352\,x_1^2\,d^{10} +$
$8\,x_2^{10}\,x_3^2 - 8\,x_2^{10}\,d^2 - 480\,x_2^8\,x_3^4 + 296\,x_2^8\,x_3^2\,d^2 + 184\,x_2^8\,d^4 + 296\,x_2^6\,x_3^6 - 8\,x_2^6\,x_3^4\,d^2 + 152\,x_2^6\,x_3^2\,d^4 -$
$440\,x_2^6\,d^6 - 240\,x_2^4\,x_3^8 + 104\,x_2^4\,x_3^6\,d^2 + 424\,x_2^4\,x_3^4\,d^4 - 584\,x_2^4\,x_3^2\,d^6 + 296\,x_2^4\,d^8 + 672\,x_2^2\,x_3^8\,d^2 -$
$1792\,x_2^2\,x_3^6\,d^4 + 1600\,x_2^2\,x_3^4\,d^6 - 512\,x_2^2\,x_3^2\,d^8 + 32\,x_2^2\,d^{10} - 448\,x_3^8\,d^4 + 1408\,x_3^6\,d^6 - 1536\,x_3^4\,d^8 +$
$640\,x_3^2\,d^{10} - 64\,d^{12} + 2106\,x_1^8\,x_2^2\,x_3 - 216\,x_1^8\,x_3^3 + 1944\,x_1^8\,x_3\,d^2 + 2946\,x_1^6\,x_2^4\,x_3 + 3282\,x_1^6\,x_2^2\,x_3^3 +$
$54\,x_1^6\,x_2^2\,x_3\,d^2 - 312\,x_1^6\,x_3^5 + 4176\,x_1^6\,x_3^3\,d^2 - 5400\,x_1^6\,x_3\,d^4 + 760\,x_1^4\,x_2^6\,x_3 + 800\,x_1^4\,x_2^4\,x_3^3 +$
$56\,x_1^4\,x_2^4\,x_3\,d^2 + 1744\,x_1^4\,x_2^2\,x_3^5 + 2048\,x_1^4\,x_2^2\,x_3^3\,d^2 - 3696\,x_1^4\,x_2^2\,x_3\,d^4 - 96\,x_1^4\,x_3^7 + 2784\,x_1^4\,x_3^5\,d^2 -$
$8992\,x_1^4\,x_3^3\,d^4 + 5280\,x_1^4\,x_3\,d^6 - 16\,x_1^2\,x_2^8\,x_3 + 1362\,x_1^2\,x_2^6\,x_3^3 - 546\,x_1^2\,x_2^6\,x_3\,d^2 - 1560\,x_1^2\,x_2^4\,x_3^5 +$
$2880\,x_1^2\,x_2^4\,x_3^3\,d^2 - 2472\,x_1^2\,x_2^4\,x_3\,d^4 + 480\,x_1^2\,x_2^2\,x_3^7 + 2120\,x_1^2\,x_2^2\,x_3^5\,d^2 - 3408\,x_1^2\,x_2^2\,x_3^3\,d^4 +$
$1192\,x_1^2\,x_2^2\,x_3\,d^6 + 672\,x_1^2\,x_3^7\,d^2 - 5088\,x_1^2\,x_3^5\,d^4 + 6624\,x_1^2\,x_3^3\,d^6 - 2208\,x_1^2\,x_3\,d^8 - 72\,x_2^8\,x_3^3 +$
$72\,x_2^8\,x_3\,d^2 + 440\,x_2^6\,x_3^5 + 592\,x_2^6\,x_3^3\,d^2 - 1032\,x_2^6\,x_3\,d^4 - 320\,x_2^4\,x_3^7 - 864\,x_2^4\,x_3^5\,d^2 + 896\,x_2^4\,x_3^3\,d^4 +$
$288\,x_2^4\,x_3\,d^6 + 1120\,x_2^2\,x_3^7\,d^2 - 1440\,x_2^2\,x_3^5\,d^4 + 32\,x_2^2\,x_3^3\,d^6 + 288\,x_2^2\,x_3\,d^8 - 896\,x_3^7\,d^4 +$
$2176\,x_3^5\,d^6 - 1664\,x_3^3\,d^8 + 384\,x_3\,d^{10} - 729\,x_1^{10} - 1053\,x_1^8\,x_2^2 - 1377\,x_1^8\,x_3^2 + 2673\,x_1^8\,d^2 -$
$300\,x_1^6\,x_2^4 + 684\,x_1^6\,x_2^2\,x_3^2 + 3132\,x_1^6\,x_2^2\,d^2 - 888\,x_1^6\,x_3^4 + 5616\,x_1^6\,x_3^2\,d^2 - 3672\,x_1^6\,d^4 + 8\,x_1^4\,x_2^6 -$
$1500\,x_1^4\,x_2^4\,x_3^2 + 1072\,x_1^4\,x_2^4\,d^2 + 2232\,x_1^4\,x_2^2\,x_3^4 + 456\,x_1^4\,x_2^2\,x_3^2\,d^2 - 2576\,x_1^4\,x_2^2\,d^4 - 240\,x_1^4\,x_3^6 +$
$4384\,x_1^4\,x_3^4\,d^2 - 8272\,x_1^4\,x_3^2\,d^4 + 2336\,x_1^4\,d^6 + 276\,x_1^2\,x_2^6\,x_3^2 - 164\,x_1^2\,x_2^6\,d^2 - 1336\,x_1^2\,x_2^4\,x_3^4 +$
$1048\,x_1^2\,x_2^4\,x_3^2\,d^2 - 1024\,x_1^2\,x_2^4\,d^4 + 640\,x_1^2\,x_2^2\,x_3^6 + 480\,x_1^2\,x_2^2\,x_3^4\,d^2 - 112\,x_1^2\,x_2^2\,x_3^2\,d^4 +$
$272\,x_1^2\,x_2^2\,d^6 + 1120\,x_1^2\,x_3^6\,d^2 - 5760\,x_1^2\,x_3^4\,d^4 + 5088\,x_1^2\,x_3^2\,d^6 - 704\,x_1^2\,d^8 - 1\,x_2^8\,x_3^2 + 1\,x_2^8\,d^2 +$
$184\,x_2^6\,x_3^4 - 112\,x_2^6\,x_3^2\,d^2 - 72\,x_2^6\,d^4 - 240\,x_2^4\,x_3^6 - 896\,x_2^4\,x_3^4\,d^2 + 656\,x_2^4\,x_3^2\,d^4 + 480\,x_2^4\,d^6 +$
$1120\,x_2^2\,x_3^6\,d^2 - 480\,x_2^2\,x_3^4\,d^4 - 864\,x_2^2\,x_3^2\,d^6 + 224\,x_2^2\,d^8 - 1120\,x_3^6\,d^4 + 2080\,x_3^4\,d^6 - 1056\,x_3^2\,d^8 +$
$96\,d^{10} - 648\,x_1^8\,x_3 + 834\,x_1^6\,x_2^2\,x_3 - 968\,x_1^6\,x_3^3 + 2664\,x_1^6\,x_3\,d^2 - 336\,x_1^4\,x_2^4\,x_3 + 1352\,x_1^4\,x_2^2\,x_3^3 -$
$1408\,x_1^4\,x_2^2\,x_3\,d^2 - 320\,x_1^4\,x_3^5 + 3456\,x_1^4\,x_3^3\,d^2 - 3776\,x_1^4\,x_3\,d^4 + 2\,x_1^2\,x_2^6\,x_3 - 464\,x_1^2\,x_2^4\,x_3^3 +$
$176\,x_1^2\,x_2^4\,x_3\,d^2 + 480\,x_1^2\,x_2^2\,x_3^5 - 568\,x_1^2\,x_2^2\,x_3^3\,d^2 + 1176\,x_1^2\,x_2^2\,x_3\,d^4 + 1120\,x_1^2\,x_3^5\,d^2 -$
$3776\,x_1^2\,x_3^3\,d^4 + 2144\,x_1^2\,x_3\,d^6 + 8\,x_2^6\,x_3^3 - 8\,x_2^6\,x_3\,d^2 - 96\,x_2^4\,x_3^5 - 256\,x_2^4\,x_3^3\,d^2 + 352\,x_2^4\,x_3\,d^4 +$
$672\,x_2^2\,x_3^5\,d^2 - 64\,x_2^2\,x_3^3\,d^4 - 608\,x_2^2\,x_3\,d^6 - 896\,x_3^5\,d^4 + 1280\,x_3^3\,d^6 - 384\,x_3\,d^8 - 216\,x_1^8 +$
$132\,x_1^6\,x_2^2 - 456\,x_1^6\,x_3^2 + 720\,x_1^6\,d^2 - 1\,x_1^4\,x_2^4 + 376\,x_1^4\,x_2^2\,x_3^2 - 388\,x_1^4\,x_2^2\,d^2 - 240\,x_1^4\,x_3^4 +$
$1416\,x_1^4\,x_3^2\,d^2 - 856\,x_1^4\,d^4 - 32\,x_1^2\,x_2^4\,x_3^2 + 20\,x_1^2\,x_2^4\,d^2 + 192\,x_1^2\,x_2^2\,x_3^4 - 288\,x_1^2\,x_2^2\,x_3^2\,d^2 +$
$416\,x_1^2\,x_2^2\,d^4 + 672\,x_1^2\,x_3^4\,d^2 - 1472\,x_1^2\,x_3^2\,d^4 + 416\,x_1^2\,d^6 - 16\,x_2^4\,x_3^4 + 8\,x_2^4\,x_3^2\,d^2 + 8\,x_2^4\,d^4 +$
$224\,x_2^2\,x_3^4\,d^2 - 64\,x_2^2\,x_3^2\,d^4 - 160\,x_2^2\,d^6 - 448\,x_3^4\,d^4 + 512\,x_3^2\,d^6 - 64\,d^8 - 96\,x_1^6\,x_3 + 40\,x_1^4\,x_2^2\,x_3 -$
$96\,x_1^4\,x_3^3 + 320\,x_1^4\,x_3\,d^2 + 32\,x_1^2\,x_2^2\,x_3^3 - 8\,x_1^2\,x_2^2\,x_3\,d^2 + 224\,x_1^2\,x_3^3\,d^2 - 352\,x_1^2\,x_3\,d^4 + 32\,x_2^2\,x_3^3\,d^2 -$
$32\,x_2^2\,x_3\,d^4 - 128\,x_3^3\,d^4 + 128\,x_3\,d^6 - 16\,x_1^6 - 16\,x_1^4\,x_3^2 + 48\,x_1^4\,d^2 + 32\,x_1^2\,x_3^2\,d^2 - 48\,x_1^2\,d^4 -$
$16\,x_3^2\,d^4 + 16\,d^6.$

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