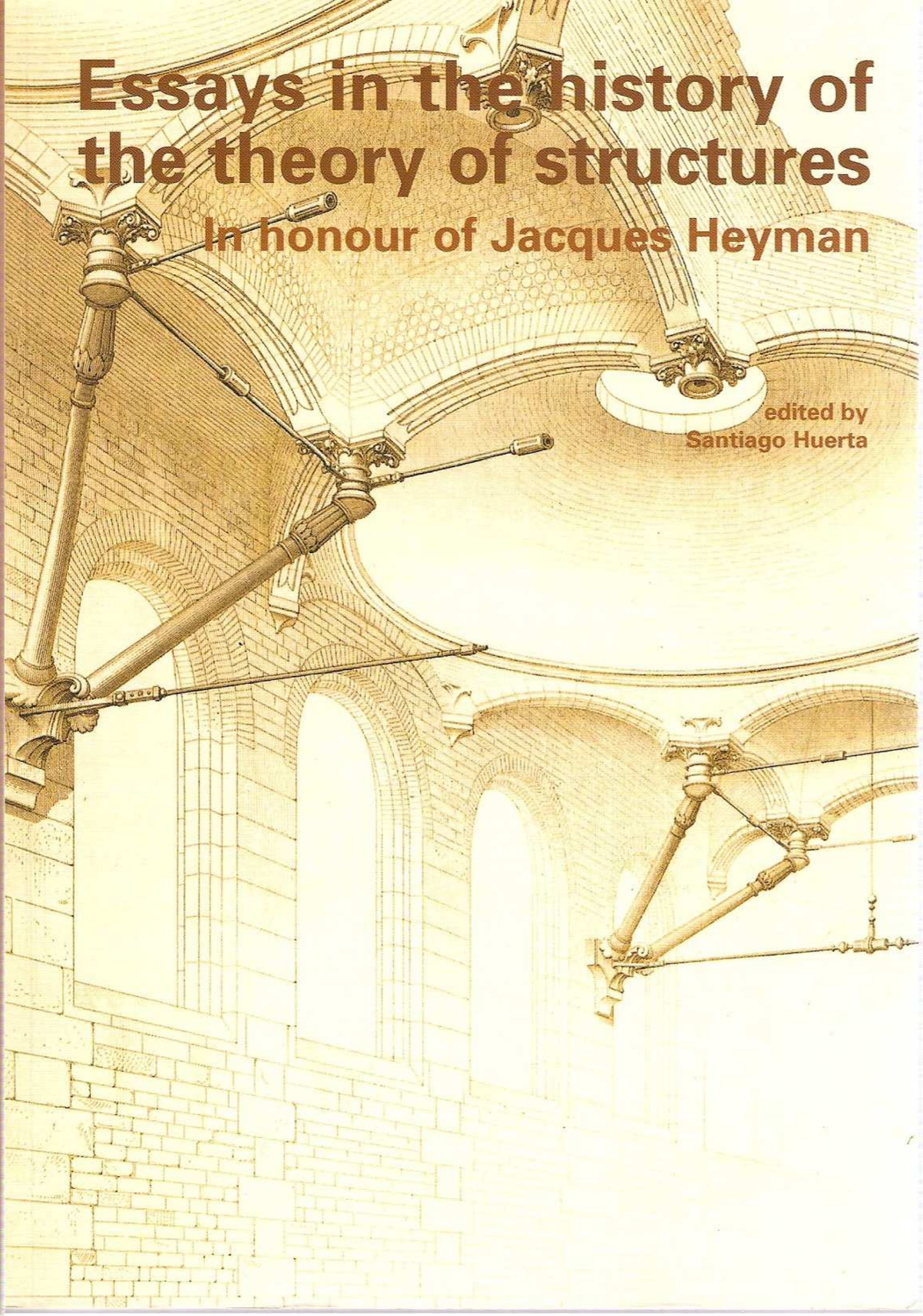
Essays in the history of the theory of structures

In honour of Jacques Heyman

edited by
Santiago Huerta



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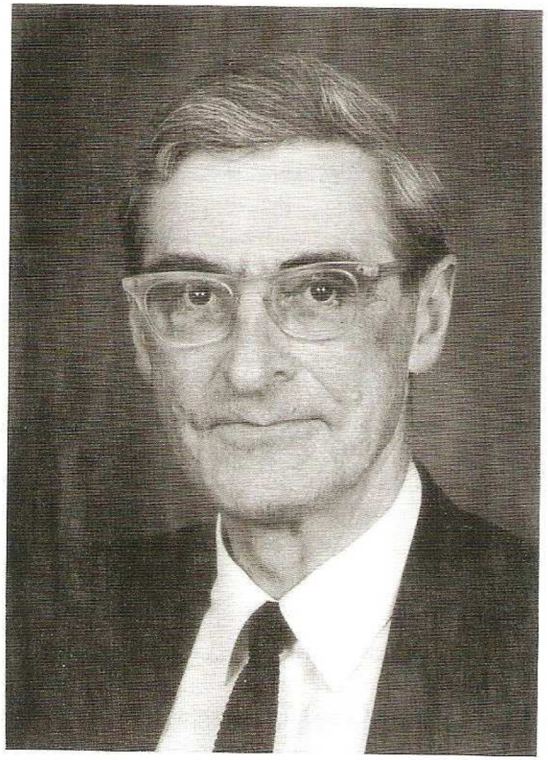

Professor Jacques Heyman

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Edited by
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Foreword by
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Introduction by
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Contents

Boscovich's contribution to the theory of domes: the expertise for the Tiburio of Milan cathedral

Gema López Manzanares

In 1763, Boscovich, one of the three authors of the famous *Parere di tre mattematici sopra i danni della cupola di San Pietro*, was in charge of studying the stability of the oval dome of the Caesarea Library in Vienna. This dome showed alarming cracks and although he did not include numerical calculations in his expertise, he made a detail study of the movements suffered by the counterparts and proposed to encircle the dome with iron rings. In this expertise he demonstrated himself as very knowledgeable about this type of constructions (López 2005).

Two years later, Boscovich must face up to a new problem: the construction of the spire over the dome of the Tiburio of Milan cathedral. The problem was different here than in the other expertises, so that the scholars had to predict the future behaviour of the dome charged with the weight of the spire. Once again, Boscovich made an analysis of great interest, where he included numerical calculations resulting from applying the Virtual Work principle to a collapse mechanism of the dome caused by sliding down of the keystone with the spire. Besides, he stated in brightness way his theory about vaults and domes, and he got ahead of Coulomb in the calculation of the worst location of the collapse hinges. Also, P. Regi was asked for his opinion, who applied La Hire's method in Bélidor's version to study the stability of the dome and the piers.

These contributions, in an intermediate period between the analysis of the stability of Saint Peter's dome and the construction of Sainte Genevieve's church or Pantheon of Paris, that also will be object of a great controversy about the stability of its piers, are of great importance to understand the development of the scientific theory of the domes that occurs along XVIII century.

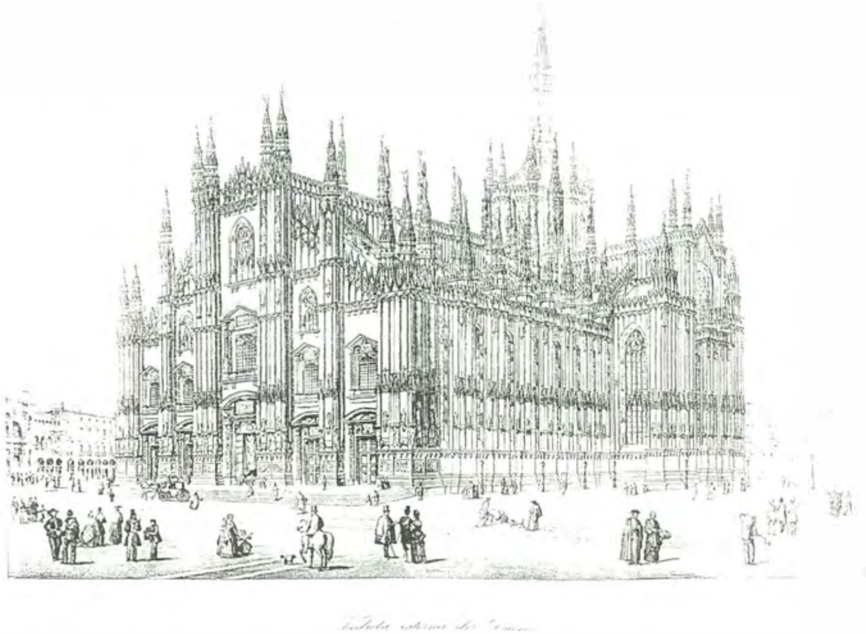

Figure 1
Elevation of the Duomo of Milan (Anonymous, s.a.)

The construction of the spire of the Tiburio in Milan

In 1762 the chapter of the Duomo of Milan decided to build a great spire that would crown the dome or Tiburio over the transept, projected and built by Amadeo at the beginning of the XVIth century. This spire was already envisaged in the initial projects but the construction of the dome stopped in 1640 and since then, there was no talk of it (Nava 1845, 11; Scaglia 1982).

The architect Francesco Croce was in charge of studying those projects and preparing the construction of the spire. Two years later, on 25th May 1764, Croce showed a wooden and wax model and a written plan about his project, but before its approval the project was submitted to the judgement of several experts. Between them there were two mathematicians, Boscovich and Regi, who had to analyse the effect of the weight of the spire over the stability of the building. The other experts were an anonymous one, Beccaria and Francesco Martinez, who signed off the last of these expertises on 13th May 1765. On 8th July the model proposed by Croce was approved and the spire was built in less than four years (Nava 1845; Benvenuto 1991, 371–374; López 1998, 522–556; Repishti 2003; Stolfi 2003).

Description of the dome

The dome or Tiburio rises over the transept of the Duomo of Milan (Fig. 2). Four main levels can be distinguished (Boscovich 1765, 56–8; Regi 1765, 69):

1° The first one corresponds to the piers and main arches until the springing of the dome itself. The main arches are pointed, with a radius equal to the span between the inner edges of the piers. Over them there are other incomplete arches that receive the weight of the eight ribs of the dome and concentrate it towards the piers; the octagonal plan is solved in the angles by a kind of rib pendentive.

The pier base is equivalent to a circle with a diameter of 5 arms (2.97 m) and their height until the springing of the main arches is greater than 51 arms (30.34 m). They form a square in plan of 32 arms side (19.04 m). Regi states that the piers are forty arms height in the minor naves and fifty three in the main nave.[1]

2° The dome rises over the described octagonal plan, at a height of one third of the span over the main arches keystone. Its eight ribs concentrate the weight towards these arches. The vertical drum rises outside, with 23 arms height (13.68 m) and 2 arms width (1.19 m), with a brick core and a marble revetment, and counterforts in the angles that increase its width until three arms (1.78 m). The

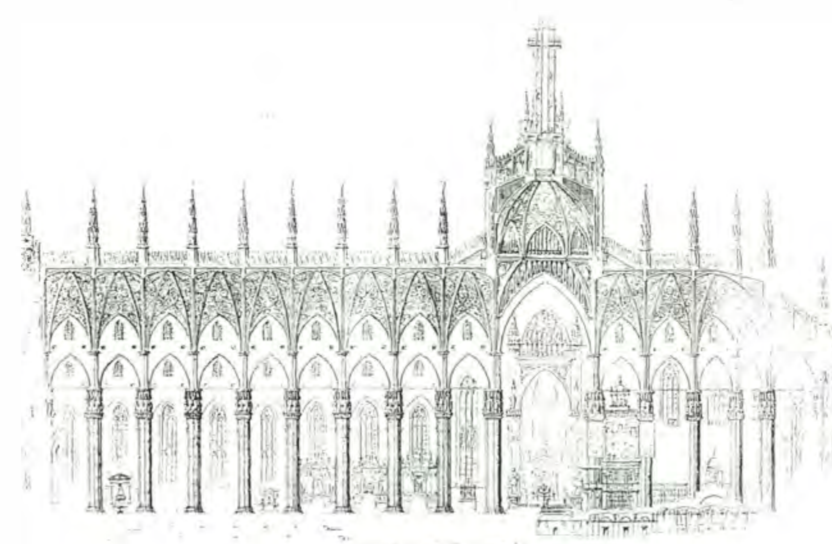

Figure 2
Longitudinal section of the Duomo of Milan (Anonymous, s.a.)

springing of the dome is located at the base of the drum, not at the upper edge; several transverse walls with passing openings join the dome and the drum. The dome is embedded in the drum until a height of 2 fi arms (1.5 m). These ribs are of marble and the joints between the great blocks of stone converge towards the corresponding centre of curvature. Between these ribs it was built an inner pointed shell made of brick and with 9 ounces width (23 cm); the extrados shell is nearly horizontal. The inner shell shows unloading pointed arches over the windows, in such way that it transmits its weight toward them and the ribs that concentrate it over the piers. In the upper part the ribs and the shells converge in a marble ring of great width that serves as a base for the lantern. In the angles, over the eight marble counterforts of the drum, there were to be built eight small spires and lattice masonry over the ribs that would join them to the lantern.

There were a lot of iron ties in the naves, but also two iron octagonal rings crossing through the windows of the dome: the one at the bottom part at a height

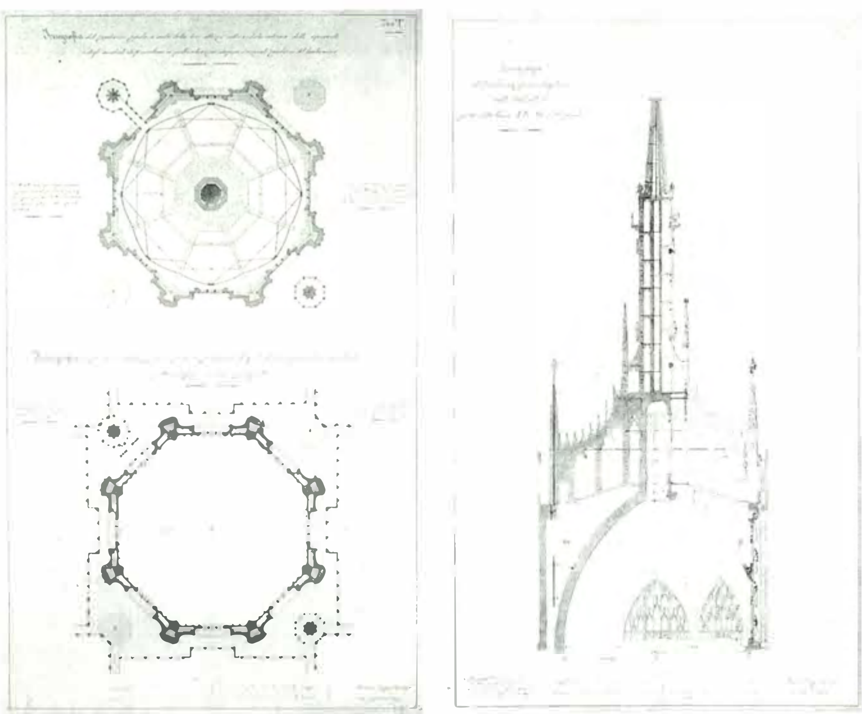

Figure 3
Plans and sections of the Tiburio by Pietro Pestagalli 1843 (Stolfi 2003)

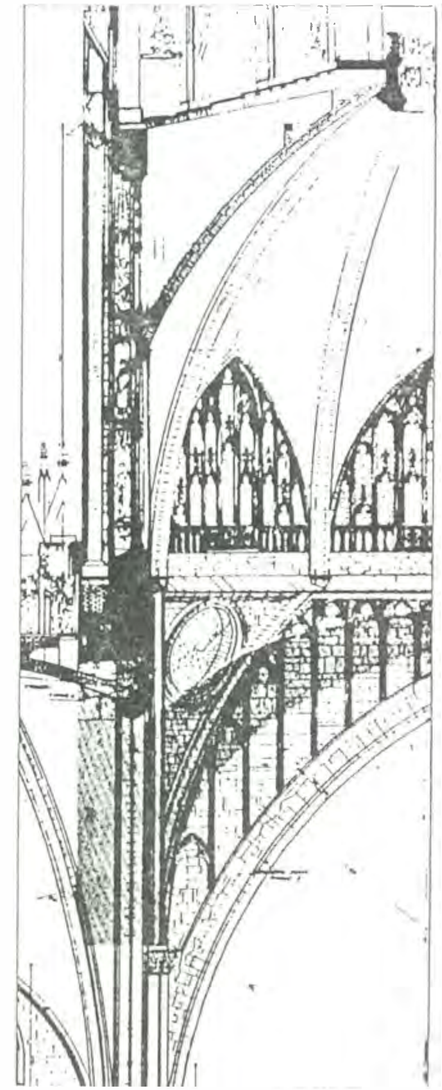

Figure 4
Detailed section of the Tiburio of Milan (Scaglia 1982)

of 8 1/2 arms (5.00 m) and the upper one, at a height of 14 (8.30 m). Their transversal section had 11×18 puntos^2 (4.5×7.5 cm^2). Besides there were a third hidden iron ring over the windows at a height of more than 30 arms (18 m).[2]

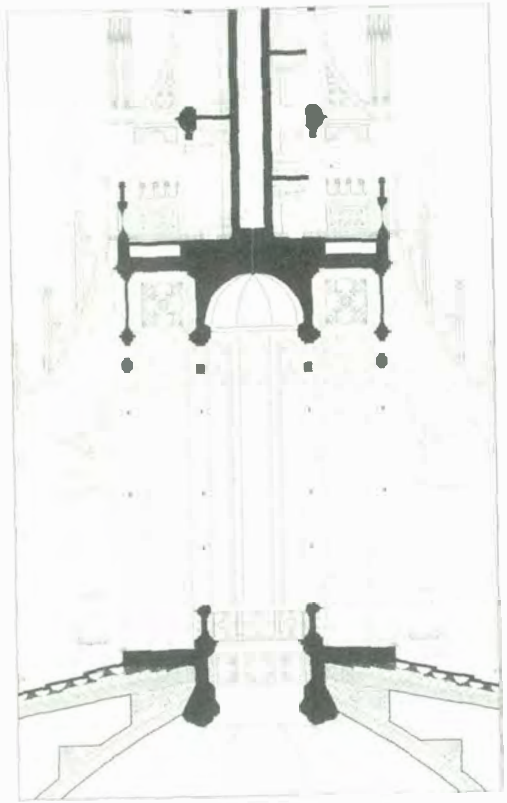

Figure 5
Section of the lantern (Ferrari da Passano 2003)

3° The third level corresponds to the lantern, which was not concluded, with 14 arms height (8.33 m). It rises over the marble ring where the ribs of the dome converge with a diameter of 5 fi (3.27 m). It has a drum with a perimeter gallery covered by a nearly flat annular vault that extends itself towards inside in the dome of the lantern.

4° Over the vault that covers the lantern, in the extrados, there were to be built other small spires joined by small arches to the central one projected by Croce, with 49 arms height (29 m). This one would be composed by eight slender pillars joined between them by a lattice masonry on the sides of the octagonal prism. There would be a stair inside the spire joined to the perimeter masonry by iron bars. The crown of the spire would be a massive pyramid and a great marble statue.

Boscovich

In 1764, Boscovich, Croatian jesuit and professor of Mathematics at the University of Pavia at that moment, received a letter from the people in charge of the Duomo of Milan. They asked him for an expertise about the stability of the spire projected by Croce that would crown the Tiburio, and also its repercussion on the global structure of the building. He delivered it on 24th February 1765 (Nava 1845, 13, 53–64), and it contained fifty five articles; Boscovich recognizes that he mainly applied the same method as in the expertise about the dome of Saint Peter, the famous *Parere*.[3]

Basic hypothesis

The survey of the dome

First, Boscovich studied in a detailed way the model of the projected spire and the model of the whole building. After this, he visited the dome with Croce's plans checking up the location of the different elements that the models showed. From these plans he deduced the main measurements and those that could be uncertain because of their small size were checked with Croce. Boscovich emphasizes the exactness of his procedure, that although it can not be like the accuracy of an astronomical observation, it is much more that the problem required (54).

Density of materials

Once measured the different elements, Boscovich calculated their weight. For doing this, he needed to know the density of materials, information that Croce also gave him. For instance, a marble cubic arm weights 800 pounds (2897 kg/m^3) and brick masonry with lime mortar, 600 (2173 kg/m^3).[4]

Iron rings

Boscovich also noted down the size and location of the iron rings that encircled the dome and calculated their resistance according to the values obtained by "the good experts".[5] He took into account the principle applied in the expertise about the dome of Saint Peter to calculate the resistance of an iron circle bar, six times greater than the resistance of a right bar and mentions Poleni, who confirmed three mathematicians' finding by means of the experiment of the octagonal thread.[6]

Virtual work principle and the properties of material

Boscovich applied the same tools used by him, Le Seur and Jacquier to evaluate the stability of the dome of Saint Peter. But we can notice that Boscovich considers a greater influence of the quality of materials when he applies the Virtual Work Principle.

my inquiries are based on the one hand on infallible and evident geometrical principles and on the other hand, on the physical properties of the used materials, that only can be known through experience and careful observation.[7]

Inasmuch as geometrical principles, Boscovich considers the displacements that the forces, that is, the weights and the thrusts exerted by the iron rings, would suffer "in those masonry buildings where the resistances are lower than the forces that thrust and press down".[8] That is, he applies the Principle of Virtual Work, "basis of all the Mechanics applied to machines"[9] that states, as it is well known, that if a system of forces is in equilibrium, the positive and negative work that these forces would make for a virtual displacement of the whole system would be equivalent. "A force exerts a thrust as great as the velocity of its initial movement in its own direction" and so, the works are calculated "multiplying the absolute forces by those distances that express these initial velocities".[10] In the case of masonry elements, the relation between the works made by the balanced forces and by the unbalanced forces must be equal or greater than 1 for a safe structure.

Now then, Boscovich points out that the hypothesis of the problem are slightly different from the assumptions made in the case of the dome of Saint Peter, since the Tiburio of Milan has no drum and instead of it, it has the counterpart of the lateral naves of the temple, that can not suffer displacements.

So, I consider that this is a very essential difference between this masonry building and that one, difference that has me compelled to investigate a particular theory, that could be adapted immediately to this case, but also to other with due reflection.[11]

Although he does not mention it in an explicit way, there is an important difference: in this case there were no damages in the Tiburio to be explained, as in the case of Saint Peter's dome. In this case the scholars had to find out if the Tiburio would be damaged when the spire would be built. So, Boscovich had to apply his practical knowledge about the pathology of vaults and so, he established a hypothetical mechanism of collapse; in the case of the dome of Saint Peter the mechanism was deduced from the real damages. This process of abstraction led the three mathematicians to make some mistakes about the location of the hinges like the one situated on the keystone extrados that they located in the intrados.

In the second place, Boscovich speaks about the physical properties of materials. In the case of the Tiburio, he does not take into account the compression of the marble, the prevalent material. In return, he considered this compression in the case of the dome of Saint Peter at Rome, where the inner core of the drum

was brick masonry; compression that "increases a lot the force that thrusts and diminishes the resistance".[12] Boscovich recognizes that taking into account the compression of material gives very disfavourable results. Nevertheless, it does not seem that Boscovich considers wrong, twenty years later, to have included compression in the calculation of the stability of Saint Peter's dome, that are slightly wrong by this hypothesis, López (1998, 214–234; 1998b).

He also explains that it is easy to check in an experimental way the marble toughness and its resistance to compression. The test consisted in placing one or two small columns of marble standing on an horizontal plate of the same material, imposing on them an increasing weight, and later, measuring from time to time the height variations in the test specimens to check if they had compressed or not. Boscovich affirms that the columns would not compress even if lateral fissures appeared, and this justified his hypothesis.

In relation to the materials, but rather with collapse mechanisms of vaulted structures and his observations on real structures, Boscovich deals with the formation of hinges and sliding:

When the masonry structures resent, this never happens without openings or detachments of one part with regard to the next one. That is, a surface slides along another one without leaving intermediate spaces or, that is more usual, the openings have a hinge form with an edge that does not slide along the contact plane. This can be deduced from experience, although it could be deduced also from theory, that is, from the nature of things, but I would be long-winded too much if I would start to explain all the principles that have guided me in my investigation and prove that are according to the known laws of the same nature.[13]

As Benvenuto (1991, 371–374) states, Boscovich got ahead of Coulomb (1773), who will expound in a methodical way the possible ways of collapse caused by sliding and formation of hinges (Heyman 1972). Monasterio, a Spanish engineer who analyzed Coulomb's work in a more detailed way in XIXth century, showed these mechanisms, figure 6.

Description of building and damages

Later on, Boscovich describes the building, specifically the crossing area over which the Tiburio raises. He distinguishes four basic levels: the first one until the springing of the dome, with the main arches and ribs that concentrate the efforts towards the main piers; over this part, the dome over an octagonal plan, that at the base springs from a false drum and it is composed by eight ribs or transverse walls that act as counterforts. Between these ribs were built the inner pointed

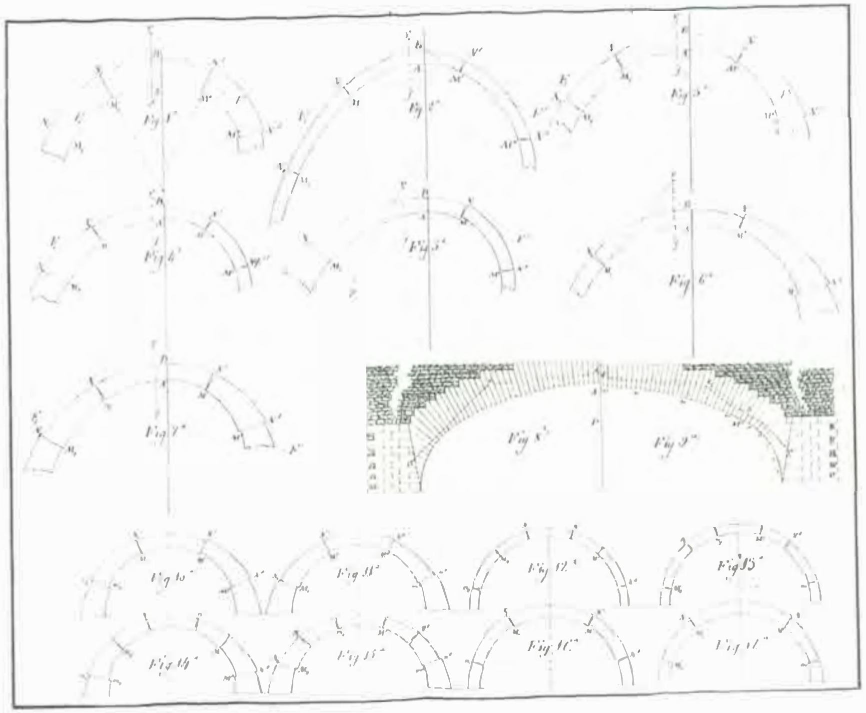

Figure 6
Different mechanisms of collapse by formation of hinges and sliding according to Monasterio (Huerta 2004)

shell and the nearly horizontal shell located outside; the third level corresponds to the lantern and the fourth one to the projected spire. He also describes the location and dimensions of the two octagonal rings that encircle the dome, and tells that the joints between the great marble pieces of the ribs converge on the centre of curvature of the dome. This detail is interesting to understand the collapse model that he will consider when calculating the stability of the dome (56–7).

Besides, although the aim of the expertise had no relation to the damages of the dome, Boscovich describes the light damages of the building just in case they would have influence on the stability. So, the masonry was undamaged except for one of the spires at the base of the dome because of a thunderbolt. At the base of the drum he could see some vertical fissures, with kite tail witnesses in the area next to the damaged spire, some of them broken and other sound. Boscovich also surveyed the dome from the inner side through the windows and he only discovered some detachments of the coating that let see the brick masonry. Finally, the people in charge of

the building told him that at the end of the building, near the cemetery, there were damages caused by a slight settlement of the foundation, but these did not have an effect on the vaults; this means that the works made until that moment had stopped the movement and, besides, the dome was far away from this area.

Calculation of weights and the thrust resisted by the rings

With the same criterion as in the building description, Boscovich calculates the weight of the four levels of the building, starting from the spire, and the thrust resisted by the iron rings.[14] It can be calculated that the repercussion of the weight of the spire over the total weight that support the foundation is 240.000 for 14.352.000 pounds, that is, 1/60, that can be neglected.

1° Spire projected by Croce	200.000 pounds	(65.360 kg)
The small spires at the base and the small arches that would join them	40.000 pounds	(13.072 kg)
Total	240.000 pounds	(78.432 kg)
2° Lantern	336.000 pounds	(109..805 kg)
Marble ring that acts as a base and join the ribs of the dome	40.000 pounds	(13.072 kg)
Total	376.000 pounds	(122.877 kg)
3° Dome itself, with drum, ribs and shells without the ring of the oculus	4.136.000 pounds	(1.351.645 kg)
(Without adding the filling and lattice of the windows and, by other side, without discounting the voids in the ribs)		
4° Piers	4 × 800.000 pounds	(1.045.760 kg)
Main arches and masonry from the impost till the springing of the dome	4 × 1.600.000 pounds	(2.091.520 kg)
Total aprox.	10.000.000 pounds	(3.268.000 kg)
5° Rings (each one of them)		
Resisted tension	164.000 pounds	(53.595 kg)
Total Horizontal thrust for an octagonal profile	1.131.000 pounds	(369.611 kg)

Table 1
Total weights and resisted thrust by the rings of the Tiburio of Milan

The stability of the spire itself

Once calculated the weights, Boscovich starts his analysis of the stability of the building studying the spire itself. Mainly he deals with the concept of resistance, although without giving concrete quantitative values.

First, if the base over which the eight small pillars of the spire rise, does not give up, these pillars are able to resist their own weight, just in the same way as the four great piers support the Tiburio without cracking. Inasmuch as their slenderness, it has to take into account the masonry and the iron bars that will join together and to the central stair, respectively, so that the proportion between the height and the width of the spire is actually between 5 and 6. According to

Boscovich, there are a lot of more slender towers like, for instance, the minarets that he could see in the mosques of the eastern countries.

After this, the base of the spire, that is the lantern, has to be considered. There is no lateral thrust because the spire pillars correspond with the inner pillars of the lantern and also the small spires that encircle the great one correspond with the outer pillars of the same lantern.[15] Only the inner part of the spire that rests upon the small lantern dome could suffer some damage. Now then, the false dome must open also towards outside to make possible this sliding and this is very difficult because of the weight that the lateral parts of the dome support.[16] That is, although it could be supposed that this dome act as a vault with lateral thrusts, because of the presence of a vertical key between the other horizontal courses, the counterparts are much greater proportionally and would prevent from overturning and subsequent sliding. In any case, Boscovich recommends putting granite slabs of great thickness that serve as a base for the spire to compress in a homogeneous way the lantern, without horizontal thrust.

Collapse mechanisms in vaulted structures

Before applying the Principle of Virtual Work to the dome it is necessary to choose in a correct way the possible collapse mechanism. Boscovich enumerates the different cases that can be found:

1° Arches and vaults. When they support an important weight on the key stone, they use to crack in the intrados at that zone, "remaining joined together the upper part like a hinge or a ball-and-socket joint. The same arch opens in the extrados, usually at a distance of one third, moving towards outside, in such way that the outer part cracks and the inner one remains joined as if there were a hinge there".[17] Finally, a third hinge appears at the base of the support, that turn around its extrados edge, but that, Boscovich explains, "does not use to move horizontally"[18] because of friction; for this to happen it would needed a horizontal force three times greater than the vertical one that acts upon the support.

> Therefore, the support, with a third part of the arch, turns towards outside around the immobile extrados angle at its base, and the two thirds of the upper arch turn with its bottom part towards outside and with the upper one that descends vertically, until finally the leaning supports overturn and falls the upper part of the arch divided in two parts with the whole weight that resting upon it was pushing it down.[19]

The placing of horizontal iron chains at a third height of the arch from the springing and also, the placing of counterforts prevent that the collapse happens.

2° Domes over drum without lantern.

> it happens that, for a homogeneous resistance, they open all around the perimeter like a shoot or pomegranate when the keystone of the vault descends and the rectilinear cylinder that supports it and it is usually called drum moves towards outside, which besides has to open vertically with cracks that widen when raising to the impost.[20]

3° Domes over drum with lantern.

As the ring over which the lantern rests is on the keystone of the dome, and resists compression, the cracks of the intrados appear slightly far away from here, in the transition between the dome and that ring and, so "the other opening has to produce lower down and near the impost".[21] Like the arches and barrel vaults, the dome yields out, and so a third hinge has to appear at the base of the drum, over which it will turn around and collapse.

For all the cases described, Boscovich explains that when the masonry is made of brick and lime the deformation usually consists of small compressions and dilatations of the material, instead of horizontal cracks, so that the structure bends instead of cracking. Indeed, the domes also show vertical cracks, although the horizontal cracks are easily unnoticed. This was the case of Saint Peter's dome.

4° Finally, Boscovich considers the particular case of the Tiburio, which dome with lantern has really no drum. He proposes two possible ways of collapse. The first one is based on the appearance of hinges in the keystone extrados, in the intrados of an intermediate area and in the extrados of the springing of the dome.

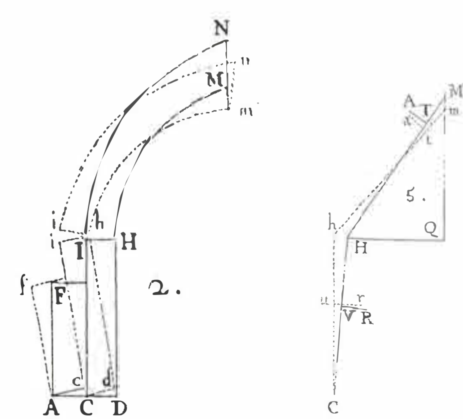

Figure 7
Hypothetical mechanism of collapse in Saint Peter's dome (Boscovich, Le Seur and Jacquier 1743)

The second one combines the presence of hinges with sliding: the ring of the keystone and part of the keystone detaches from the rest of the dome and this one, when turns outside around the extrados edge of the springing, lets the ring slide down with the lantern. It is the well known model of La Hire, but analysed in a different way and applied to a structure without buttresses or drum in this particular case (see figures 12 and 16 in figure 6).

> When the Dome has no drum over which its springing rests, but it raises at the same level as the lateral resistances that the naves exert, like happens in this case, can not occur the movement analogous to the mentioned above without an inner lower opening also in the arch, so that the bottom part of the same cracked arch turns around the extrados immobile edge, moving away its upper edge together with the lower edge of the upper part, and the upper edge of this one, opened towards inside and cracked makes descend with it the Lantern: or also one can imagine that instead of three hinges there are only two, one at the base of the impost, and the other one in the keystone next to the Lantern, remaining the extrados angle of the arch between these openings constant and turning outside its upper edge, while the Lantern, that has detached, descends through the space let by the opening of that edge as a receptacle of the wedge that, charged with the Lantern, descends with it.[22]

The application of Virtual Work Principle to the analysis of the dome

From the two mechanisms of collapse that we have just described for the domes with lantern but without drum, the first one he thinks it is not applicable to the case of the Tiburio.[23] The second one would be impossible if the joints between the marble pieces of the ribs that converge to the centre of curvature would have been more horizontal. But they are placed in a more vertical than horizontal position, and so it had sense to analyse this mechanism because it was possible that the lantern would break in the ring area and slide making the dome turn. Besides, he calculates the most disadvantageous position for this plane of sliding:

> I have found the way of calculating the forces that cause this type of movement and the resistances that stop it in the case that these same resistances are related to those forces with a minimum proportion and I have found the resistances much greater.[24]

The results obtained by Boscovich for the total weights are in table 2. As can be checked, the resistance forces produce a work four times greater than the unbalanced.[25] Boscovich draws our attention to certain factors that would increase even more that value: the toughness of the different parts, "which by itself has kept on foot the Dome of Saint Peter since so many years";[26] the work of the iron ring located over the windows at a distance of more than 30 arms height

	Unbalanced works	*Weights*	*Vertical displ*
Spire and lantern	10.000.000 pounds	616.000 pounds	(16.24)
	Balanced works	*Weights*	*Vertical displ*
Dome with the oculus ring	17.000.000 pounds	4.136.000 pounds	(4.11)
		Thrusts	*Horizontal displ.*
Lower iron ring	9.000.000 pounds	1.131.000 pounds	(7.96)
Upper iron ring	14.000.000 pounds	1.131.000 pounds	(12.38)
	40.000.000 pounds		

Table 2

Balanced and unbalanced works that result from the analysis of the Tiburio of Milan according to a collapse model of sliding

(17.85 m), that would have a value of 30 millions of pounds and, finally, the friction on the breaking plane between the lantern and the part of the dome near the keystone.

In any case, Boscovich, not in agreement with these calculations, checks that it would be a more disadvantageous mechanism if varying the location of the lower hinge and the sliding plane of the wedge. He also was anticipating Coulomb's memoir of 1773:

> Later I have considered what it would happen if these openings would appear in any other place instead of the keystone and the base of the ribs and I have found that, everywhere, the resistance would be greater in relation to the force than at the exposed places and considered in those calculations.[27]

The great stability of this dome allows Boscovich to understand why the gothic constructions have resisted in a better way along time than the old Greek and Roman buildings. In fact, the lantern was adding a double weight than the spire would add, without causing any problem in the dome. The slight damages were caused by thunderbolts or work imperfections, but the stability was too assured.[28]

Regi

P. Francesco de Regi was the other mathematician in charge of studying the stability of the spire. He was barnabite, regular cleric of St. Paul, and professor of Mathematics at the College of Saint Alexander in Milan, who signed out his ex-

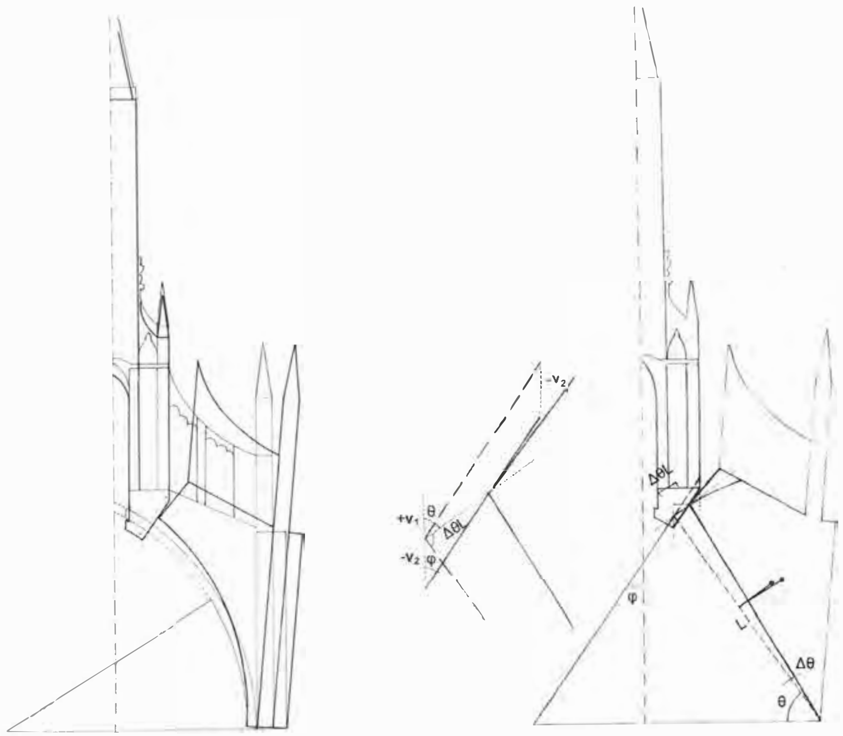

Figure 8
Hypothetical mechanism of collapse for the Tiburio of Milan analyzed by Boscovich. The dome profile is taken from Regi (1765)

pertise soon after Boscovich.[29] He analysed the stability of the spire projected by Croce and of the structure that had to support it: dome, main arches and piers.

The stability of the spire

First, Regi studied the stability of the spire, exactly the advantages of a pyramidal body compared to a prismatic one.

1º For the same slope and dimensions a prism overturns before than a pyramid, since that the gravity centre of the pyramid is lower, at a quarter part of the height from the base (see figure 1 in figure 9).

2º Regi studies the relation between the horizontal thrust that cause the overturning and the weight, both forces applied in the centre of gravity of the prism and the pyramid. The result obtained after establishing moment equilibrium is that the thrust that makes the pyramid overturn is greater, compared with the

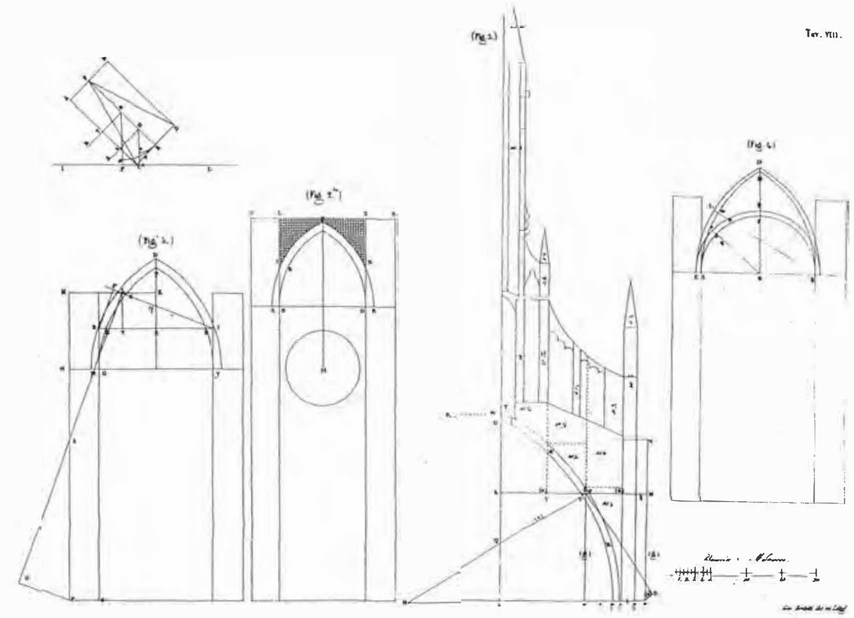

Figure 9
The stability of the Tiburio of Milan (Regi 1765)

weight, than in the case of the prism. That is, for a similar weight, the pyramid is more stable, since it requires a greater thrust to make it overturn.

3º Finally, Regi reflects on the real origin of the horizontal thrust that can suffer the prism and the pyramid: the wind. As the total force is proportional to the exposed surface and the incidence angle, the pyramid also is more stable because the wind thrust is smaller than in the case of the prism. (67) He states that the pyramidal volumes are especially suitable to counterpart the effect of hurricane winds that have an inverse figure than a pyramid. So, the projected pyramidal spire by Croce is very adequate, and it will be even more fit if the proportion between its base and height is greater.

The structure of the building: the main arches

After analysing the spire, Regi describes the Duomo. He insists on that the spire was foreseen in the original project and this was the reason why the building had the required structure to support its weight. Therefore a great number of iron ties were put through the naves and iron rings around the dome. In spite of it, he applied the theoretical principles to demonstrate that the different part of the transept, main arches, dome and piers, could resist safely the weight of the lantern.

The first problem studied by Regi is "if the collapse of an arch can happen and therefore, the fall of a vault when a great weight is placed on it, supposed that the supports are firm and immovable, and that the arches are well reinforced on the extrados", figure 2 in figure 9.[30] That is, he analyses the resistance of the main arches as a separated part from the buttresses. His conclusions are the following:

1° The ashlars of the vault are resistant enough to avoid the whole arch can collapse because of lack of resistance (71–2).

2° It is impossible that the intermediate wedges of the arch can slide if the lateral stones remain immovable, since that their joints converge towards the centre of curvature and so, they expand in the extrados.

3° Nor the vault can deform itself in such way that the point C of the keystone approaches the point B at the springing, because the material "can yield only in a unconscious way".[31]

4° Finally, an opening at I that would separate IABK from ICK can not happen, because there is no space to let the masonry move, counterparted by strong supports.

Moreover, Regi gives as a proof of his statement a test made by him. He built a wooden model of a pointed arch (*terzo acuto*). He ordered that small wooden wedges were prepared that he after put in place between two wooden supports in vertical position over a table. He used flour mixed with cold water as a mortar and a small plank that has the function of centering. He also put the wooden filling over the extrados and the model stood up at the moment of decentering, in such way that he could put over a brick like a point force that did not alter the equilibrium. The resisted weight of the brick was 49 ounces (1.3 kg) for supports eight times more light, of 6 ounces (0.16 kg).[32]

To guarantee the obtained results, Regi mentioned the Duomo belfry, that in spite of the strong weight of its three bells, it was in a perfect state because of the strong counterpart of the main nave.

Analysis of the stability of the dome

The second problem raised by Regi is if the dome with its false drum is stable when considered isolated, which is, separated from the main arches and the piers over which it rises.

Although not in an explicit way, Regi calculated the stability of one of the eight ribs of the dome, that is, of the transverse wall that joins the inner shell of the dome with the drum. He considers that all the parts are made of a homogeneous material, although they are actually made of marble and brick, and that the

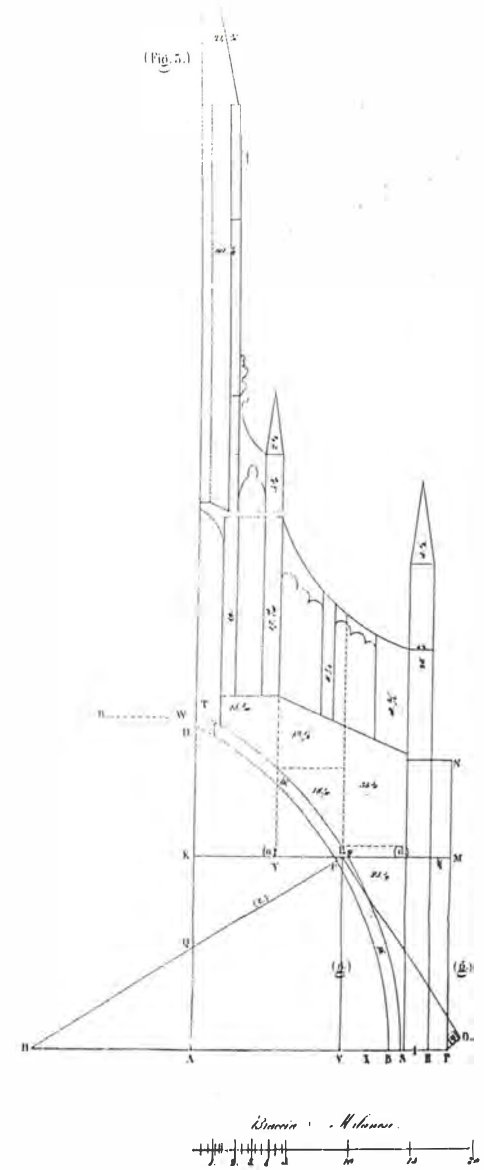

Figure 10
Analysis of the stability of the dome of the Tiburio of Milan (Regi 1765)

spire is massive. Instead of calculating the weights, he calculates volumes, taken as the surfaces of each element section, and he represented their value over the figure 10.[33]

The next step is to determine which parts of the dome thrust and which parts resist that thrust. He applied La Hire's method, according to Bélidor,[34] taking into account that the springing of the dome is at the same time the base of the ideal drum that overturns because of the thrust. So, from the centre of curvature of the intrados, H in the figure 10, it has to be drawed a line until the middle point of the arch BDST, that corresponds with the intrados nerve of the rib (also projection of the inner shell between the ribs).[35] The middle point of the joint of intersection, L, determines the vertical Lg that divides the section in two parts, on the left side, the part of the dome which weight causes the thrust and on the right side, the parts that counterpart it.

Once obtained the point L, it is drawed the perpendicular LO to the radius HL, that will be the direction of the thrust and after this, the perpendicular PO to LO from the extrados edge of the drum, that it is the lever arm of that thrust. The value of this thrust is obtained dividing the weight of the elements of the coin in two components, the horizontal one WR that acts on the keystone, in the middle of its width, and the one with direction LO, that is the one is being found out. Multiplying the thrust by the distance PO it is obtained the unbalanced moment. Inasmuch as the value of the balanced moments, it has to be calculated the sum of the products of the weight of each resistant element by the distance of P to the vertical that passes through its gravity centre.

Regi sets up certain geometrical relations to determine all those values and obtains the moments for the real dimensions of the Tiburio, which radius HB measures 28 arms (16.7 m), the angle LHV, 32º and the width BS, 1 arm (0.59 m.)

Balanced moments = 641 3/8

Unbalanced moments = 179 36/100 [36]

The proportion between them is 3,58 and, so, the dome with the spire will go on being very firm. As Boscovich, Regi also obtained balanced moments greater than unbalanced moments.[37]

Analysis of the stability of the piers

Finally, after analysing the behaviour of the dome, Regi considers necessary to analyse also the behaviour of the piers (74–77). He applies again Bélidor's version of La Hire's method, but introducing some changes. According to Regi, this method is very adequate to analysis arches or barrel vaults, but not pointed arches, like the main arches. He explains that he built another wooden model of a pointed arch (*terzo acuto*) and a semicircular one to check that the centre of gravity of the first one (semi arch) is nearer to the support than the centre of gravity of

the semicircular one, figure 4 in the figure 11. This is the reason why the breaking joint that separates the part of the arch that thrusts from the part that resists with the counterpart, forms in different places in both arches.[38]

So, for calculating the stability of the buttresses or piers that support a pointed arch, Regi divides the height of the pointed arch in three parts. The upper part will be the wedge that thrust and the two other thirds of the arch will help the support to counterpart it. As it can be seen in the figure 5, figure 11, to determine the direction of the thrust it has to be drawn the line ICF, where I is the point of intersection of the inner edge of the opposite pier with the horizontal one that passes at a distance of 1/3 from the springing of the arch, and C the middle point of the arch ET. By the middle point L of the intersection of the line ICF with the arch it has to be drawn a perpendicular line LO to ICF. The distance PO

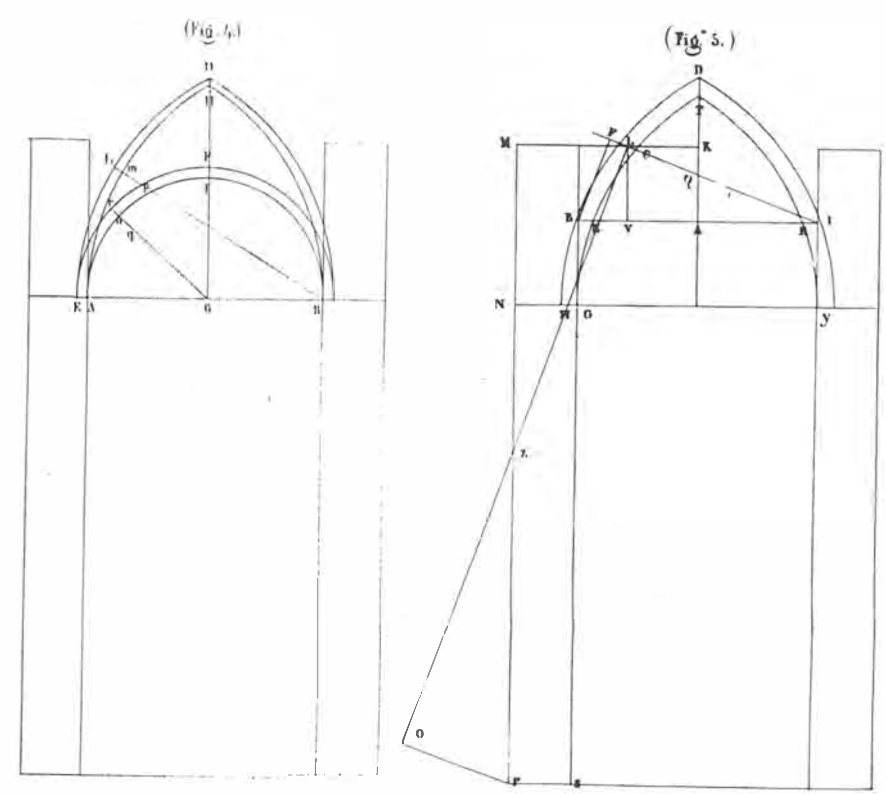

Figure 11
Analysis of the stability of the piers that support the Tiburio of Milan (Regi 1765)

from the extrados edge of the base to LO gives the lever arm of the thrust, which value is obtained like it was described above, decomposing the weight of the upper third of the semi arch in a horizontal component and other parallel one to LO. Once obtained the unbalanced moment, the balanced one is calculated adding up the moments produced by the weight of the pier and the 2/3 of the bottom part of the arch in relation to the edge P.

Regi does not give numerical results of the analysis of the piers because he admits that he did not have had time to take note of the needed information. Nevertheless, he gives approximate calculations about the stability of the piers loaded with the spire, the dome and the main arches, in the choir direction. Taking into account the weight of all the spires, the relation between the unbalanced and balanced moments is 6633/6884, that is, the structure is stable; but if the spires are not considered, the relation is worse, 6633/5988. In any case, the structure is not in a dangerous state, local nor global, since that friction and numerous iron ties of the naves are not considered.[39] He recommends that the spires and decorative walls are built as soon as possible and in this way they can help with their weight to increase the stability and also, that the arches of the naves are reinforced in the nearest area to the dome.

Conclusions

Boscovich, a famous scientist of his time, contributed to the development of the scientific theory of domes in a very remarkable way. He took part in the debates about Saint Peter's dome, the Bibliotheca Caesarea's oval dome and the Tiburio of Milan and anticipated some basic concepts to understand the behaviour of masonry vaults and domes, for instance, the possible mechanisms of collapse that Coulomb would explained in 1773 and the maximum and minimum method to find the real one. He was a theoretist but also a practitioner in the sense that he studied real cases and his theory was developed in this context. Although the famous *Parere* was written also by Le Seur and Jacquier, it seems clear that Boscovich played the most important role in the analysis of the dome. In contrast to Poleni, who applied for the first time the safe theorem or equilibrium theorem to the analysis of Saint Peter's dome, Boscovich studied domes from the point of view of a possible collapse, like La Hire, but making a careful study of the damages in real structures. So, he analyzed a real structure for the first time, Saint Peter's dome, applying the very powerful Virtual Work Principle to a simplified model of the dome. In the case of the Tiburio, he applied the theoretical principles to project a new structure, and proposed a new hypothetical mechanism of collapse. Nevertheless, he had little influence on his contemporaries and his method is not comprehensible to the scholars, even by Gauthey in his analysis of Saint Peter's dome in 1998. On the other side, Regi also studied the Tiburio in a scientific way applying La Hire-Bélidor's method for the first time to a dome, but not in a correct way, because he only analyzed the ribs as arches. This method will be used in the second middle of the XVIIIth century to analyze the dome of Sainte Geneviève, in Paris, by Bossut, Patte and Gauthey and it will be the reference to check the validity of the scientific theory of arches and vaults until XIXth century (Huerta 2004; Huerta y Hernando 1998). In this context, Boscovich's contribution shines as an inspired one and so, he can be considered as one of the fathers of the scientific theory of structures.

Notes

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1. An arm in Milan is 0.5949 m. It contains 24 ounces and 144 points, like in Florence. Parsons ([1939]1976, 625–640).
2. We have considered here that a point is equivalent to one twelfth of an ounce, that it is, as in Parma, one twelfth of a Milan arm. So, according to Parsons ([1939]1976, 630), a point is equivalent to 4 mm. We considered before that an arm in Milan is equivalent to two palms, as in Florence, and so, 24 ounces, according to Regi (1765, 69), but in the case of the iron rings, whose size gives Boscovich, would have a too small section, 2,25 x 3,75 cm^2.
3. Boscovich was born in Yugoslavia in 1711 and died in Milan in 1787 (see references about his biography and works). Along the expertise there are notes relating to a hypothetical attached document containing six articles where some questions would be explained in detail: a plan and two cross sections with elevation marks, weights calculation, the resistance of the iron rings, cracking patterns for arches and vaults and the analysis based on Virtual Work Principle. Nevertheless, in Nava's transcription this attached part is not included and the Archive of the Milan cathedral has confirmed us that it is not there. Perhaps Boscovich did not want to publish this part.
4. According to Parsons ([1939]1976, 630 y 636), a "*braccio*" in Milan was equivalent to 0.5949 m and a pound of 12 ounces weighed 0.3268 kg but like Boscovich mentions great pounds of 28 ounces, we have deduced that their weight was equivalent to 0.7625 kg.
5. "da buoni sperimentatori" (54).
6. Benvenuto (1991, 356) states that Boscovich was probably the mathematician who established the relation between the resistance of a straight iron bar and the resistance of a circular one, first exposed in the *Parere* at the end of 1742.
7. "le mie ricerche in parte sono appoggiate a principj geometrici infallibili ed evidenti ed in parte alle fisiche proprietà delle materie adoperate, le quali non ponno conoscersi, che colla esperienza e diligenti osservazioni". (55)
8. "nelle fabbriche nelle quali le resistenze sono inferiori alle forze spingenti e prementi". (54)
9. "fondamento di tutta la meccanica applicata alle macchine". (54)

10. "una forza esercita un conato tanto maggiore, quanto sarebbe maggiore la velocità del suo moto iniziale secondo la sua direzione, se vincesse, o contro di essa se fosse vinta; onde si ricavano i movimenti moltiplicando le forze assolute per quelle lineette, che esprimono queste iniziali velocità". (54)
11. "onde questa la stimo una differenza essenzialissima tra questa fabbrica e quella, la quale differenza mi ha costretto a cercare una teoria particolare, che si potesse adattare immediatamente a questo caso, benchè pur si potesse colle meditazioni dovute trasportare ad altri casi". (55)
12. "quale compressione accresce molto la forza che spinge e diminuisce la resistenza". (55)
13. "quando le fabbriche patiscono, ciò non succede mai senza una qualche apertura o distacco di una parte rispetto alla contigua. Non succede mai senza che una superficie si strisci lungo l'altra senza aprirsi per uno spazio di mezzo, e per l'ordinario ciò succede senza che neppur una punta strisci su di un piano, ma l'apertura si fa solo a modo di cerniera. Questra cosa si ricava dalle esperienze, ma si potrebbe ancora dedurre dalla teoria, ossia dalla natura delle cose, ma io mi dilungherei troppo se mi mettessi ad esporre tutti i principj, che mi hanno guidato nelle mie ricerche ed a provarne la conformità alla leggi conosciute dalla natura medesima". (55)
14. Boscovich adds the weight of the oculus ring to the lantern (58). Later, he gives the exact value of this weight, 40.000 pounds (60). Also it can be deduced that Boscovich multiplied by 6,9 the tension force exerted by the iron rings, which section is equal to 11 × 18 points^2 (4.5 × 7.5 cm^2). This is excessive because in an octagon the proportion between the total force that pull out the angles and the tension in each side is equal to 6.08, and in any case smaller than the proportion corresponding to a circular one, 2π. Nevertheless, he considers a resistance of approximately 1588 kg/cm^2. It is possible that he considered a tension resistance greater than 164.000 pounds, and this would explain the oversized horizontal thrust. In the expertise over Saint Peter's dome the iron rings had a section of 3 × 4 ounces^2 (5.6 × 7.4 cm^2), similar to the Tiburio's rings, but there it was considered a resistance of about 1713 kg/cm^2.
15. The small arches that would join the main spire to the perimeter smaller spires have a negligible horizontal effect. (60).
16. I speak of a false dome because this was built with horizontal courses, as it was already described. The collapse mechanism by sliding of the keystone is similar to La Hire's model, but Boscovich speaks about a breaking joint that appears before the sliding movement, that is, there is friction between the elements but this friction can be surpassed and breaking areas can appear. When the supports turn out, the keystone can slide inwards. La Hire does not explain the origin of the breaking joints in his model (Huerta 2004; Huerta and Hernando 1998).
17. "venendo giù unita la parte superiore a modo di una cerniera. Si apre lo stesso arco esteriormente per l'ordinario verso il suo terzo, dando ivi in fuora, in modo che la parte esteriore si distacca e l'interiore va unita come se la cerniera fosse ivi". (60)
18. "la quale non suole dare in fuori orizzontalmente". (60)
19. "Quindi il pilastro con una terza parte dell'arco gira in fuori sull'angolo esterno immobile del fondo del pilastro, ed il pezzo di due terzi dell'arco superiore gira colla sua parte inferiore in fuori e colla superiore scende verticalmente rovesciandosi ai fine i pilastri caduti in fuori, e cadendo a piombo la parte superiore dell'arco divisa in due pezzi con tutto il peso, che aggravandola la spingeva in giù". (60–1)

20. "conviene inoltre in parità di fortezza, che si aprano da tutte le parti intorno a modo di cannocchia, o di mela granata nell'abbassarsi la cima della volta e dare in fuori il cilindro rettilineo che la sostiene e suole chiamarsi tamburro, il quale inoltre deve aprirsi verticalmente con aperture che, andando in su verso l'imposta, si slarghino". (61)
21. "l'altra apertura dovrà farsi più giù e più vicina all'imposta". (61)
22. "Quando la Cupola non ha tamburro in cima al quale essa sia impostata, ma nasce immediatamente al pari delle resistenze laterali, che oppongono le navate, come accade qui, non può darsi il movimento analogo al detto di sopra se non col fare che la più bassa apertura interiore segua ancor essa nell'arco, sicchè una parte inferiore dell'arco medesimo apertosi ivi giri intorno il cantone esterno immobile, andando infuori la sua cima insieme col fondo dell'altra parte superiore e la cima di questa apertasi di dentro e distaccato al Cupolino scenda giù con esso: oppure si può concepire, che invece di tre distacchi ve ne siano due solamente, uno in fondo verso l'imposta, e l'altra in cima verso il Cupolino, rimanendo l'angolo esterno dell'arco che sta tra le medesime aperture al suo luogo e girando in fuori la sua cima, mentre il Cupolino che l'ha cacciato in su, viene giù pel luogo lasciatogli dall'apertura di quel come ricettacolo del cuneo, che, aggravato da esso Cupolino, discende con esso lui". (61–2)
23. In a dome without drum it is possible a collapse mechanism by formation of hinges (Heyman 1995, 43), but in this case it is not possible because of the width of the ribs that support the lantern, that is, we can draw a rectilinear line of thrust inside the ribs.
24. "Ho trovato modo di calcolare le forze che agiscono ad indurre un tale movimento e le resistenze che lo impediscono nel caso in cui le resistenze medesime hanno il minimo rapporto a quelle forze ed ho trovato le resistenze assai superiori". (62) It is the first time that a scientists applies the minimisation method to find out the most disadvantageous mechanism of collapse, that is, the location of the sliding joint in the keystone area for which the resistance is the smallest one compared to the unbalanced works, taking into account that the joints converge in the center of curvature and the base hinge is immovable (Fig. 8).
25. Boscovich explains later that he included in the balanced works the weight of the eight spires that would be built over the drum counterforts. But this work would be 100.000 pounds and he underrated 200.000 pounds in the whole resistance of the dome to round off the work value to 17.000.000. Besides the iron rings work had to be added (63).
26. "la quale sola ha tenuta per tanti anni in piedi la Cupola di S. Pietro". (63)
27. "Ho poi considerato che cosa accaderebbe se queste due aperture invece di farsi in cima e in fondo a'costoloni si facesse in qualunque altro luogo, ed ho trovato che dappertutto la resistenza sarebbe maggiore rispetto alla forza che ne'due siti esposti e considerati in que' calcoli". (63) See also Benvenuto (1991, 374) about the significance of this trial and error method.
28. As in the *Parere* about Saint Peter's dome, Boscovich gives balanced and unbalanced works in pounds, but they should be multiplied by a linear measure. Boscovich does not give the displacements value in a explicit way (as he did in the *Parere*). I have deduced these displacements dividing the virtual works by the corresponding weights, and I have compared them with the displacements that would suffer an hypothetical mechanism like Boscovich's one and they are in the same proportion (Fig. 8). These are the relations between the displacements: $v_1 = \Delta\theta L \cos\theta$; $v_2 = \Delta\theta L \sin(\theta-\varphi)/\sin\varphi$ and $\Delta\theta L = K$. For $\varphi = 0.608$ rad and $\theta = 0.958$ rad, $v_1 = K\ 0.575$ and $v_2 = K\ 0.600$ (the

lantern with the spire movement); the gravity center of the ribs moves K 0.158 upward (I have simplified this part and considered the gravity center of the ribs, but not the gravity center of the rib with the two shells. It is not clear how Boscovich obtained this point because he calculated total weights and works as in the *Parere.* He does not explain if he considered 1/8 of the dome to calculate gravity centers). Rings movement is proportional to the height they are placed, Äèh, that is, Kh/L. (Fig. 8 is based on Regi's dome section, Fig. 10).

29. Nava (1845, 65–77) with 1 plate. According to Nava (1845, 13), this writing was delivered soon after Boscovich's one, on 10th March 1765.
30. "se possa succedere la rottura di un arco, e quindi la caduta de una volta per un grave peso sovrappostogli, supposto che gli sostegni sieno immobili ed invincibili, e che gli archi siano dalla parte convessa ben rinfiancati". (71)
31. "non può che insensibilmente cedere". (72)
32. It is possible that the paste used by Regi would help to resist the weight of the brick. He does not indicate the dimensions of the supports. (72)
33. Regi does not explain how the double shell built between the ribs is supported. From the numerical information I have deduced that he does not take it into account. That is, he analyzed the rib arches, that support their own weight and the lantern with one of the eight pillars of the spire. So, he considers the surfaces of the different elements instead of volumes.
34. Belidor (1729). Regi considers Belidor as "theorist and, also practitioner". (72)
35. Although all the rib resists the thrust, Regi places the thrust in relation to the granite nerve or projection of the inner shell.
36. They are not really moments because Regi considered surfaces and not weights, but they are proportional.
37. Boscovich analyzed the relation between balanced and unbalanced works, not moments.
38. Regi applied Bélidor's method to a pointed dome.
39. The lack of accuracy is about the calculation of gravity centers and the division between resistant and no resistant elements. Regi also explains that he did not take into account the heterogeneous resistant, that is, the voids that would diminish the balanced moments. (76 7)

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