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Influence of G. Kazinczy in the Structural Analysis of Long Cylindrical Roof Shells

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Abstract: This work is devoted to the study of the beam method in cylindrical shells from a mathematical point of view. We mainly focus on the methodological contribution by the engineer G. Kazinczy, well known by his influence on the Theory of Plasticity, in order to illustrate the structural calculation of long cylindrical reinforced concrete roof shells. Kazinczy's methodological contribution to their structural calculation was developed at a time when attempts were being made to make computations in a rigorous manner, applying the mathematical mechanism of the theory of elasticity. The method is more feasible than the analytical method from the computational point of view, and it is based on the study of various states of equilibrium used by Kazinczy in his analysis of these structures. DOI: 10.1061/(ASCE)SC.1943-5576.0000627. © 2021 American Society of Civil Engineers.

Author keywords: G. Kazinczy; Shells; Beam method; Structural calculation of long cylindrical roof shells; Reinforced concrete shells; Study of states of equilibrium.

3 Introduction

The study of shell structures and their structural mechanics is nowadays of great interest both in Mathematics and architecture (Malek and Williams 2017; Levy 2017). The nineteenth century saw extraordinary changes in construction technology. The method used in structural analysis, based on trial and error, was replaced by the scientific method. This caused a transformation on the knowledge regarding the behaviour of structures. From then on, the structural engineers of the time were interested in determining and calculating the real state of structures, prior to their execution, and the theory of elasticity was based on this idea. As a matter of fact, the development of mathematical calculations of structures has been related directly to the development of Mathematics (Falter 2015).

In 1826, the French engineer and physicist Claude-Louis Navier published his *Leçon*, which laid the foundations for the Theory of Elasticity, and provided the contemporary engineers with a precise mathematical and logical instrument for carrying out structural calculations, while providing some certainty in the results. As a result,

the elastic-linear philosophy was unchallengeable for a century and a half, despite the fact that although it was finally possible to solve the complex systems of equations resulting from the application of the analytical theory, the solutions could not represent the correct behaviour of the structure, and in short, were not meaningful. This was all because the new material which had emerged in the construction sphere in the late nineteenth century, reinforced concrete, did not comply with the assumptions of Hooke's law, and because it was impossible to know the "real" state of the structure beforehand.

Although the hypotheses of elastic theory seemed reasonable, by the early twentieth century some engineers were beginning to doubt whether the trivial and unpredictable faults and imperfections of the structures could really affect their strength. Based on this reasoning, the accuracy of the calculation of elastic stresses began to be called into question when the "real" strength of a structure was being calculated. More precisely, the point requiring consideration was being addressed, since what needed to be prevented were the conditions in which the structure collapsed.

In this context, more than two hundred years after the tests performed on fixed-end beams by the French physicist Edme Mariotte, the Hungarian engineer Gábor Kazinczy appeared on the scene in 1914. Kazinczy based his calculations not only on theories, but also on empirical data obtained from tests. Studying the results of the deformations, he defined the concept of the "hinge" as a residual break that occurs in the embedded beam. Three of them were required for the structure to collapse. As a result, and without diminishing the importance of the application of analytical theory, the calculation of elastic stresses became less important when predicting the real strength of a structure. In the case that the implementation was made using a ductile material, then no dependence on the elastic stress threshold at a point could be observed. Instead, the dependence appeared on an unacceptable increase in the deformations due to the action of the loads.

Based on these arguments, scholars in the field such as Kazinczy, Kist, Maier-Leibnitz and Melan, began to investigate in the 1930s other calculation methods, not as the product of theories based on elastic calculations, but as a consequence of significant inconsistencies between elastic calculations and the results of tests carried out on real structures.

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At the same time, in the construction sphere after the First World War, long cylindrical roof shells became increasingly widespread as types of structures capable of covering large spans with minimal costs of the material used - reinforced concrete. A new construction system thereby emerged, which had a geometry which was ideal for covering utilitarian spaces, such as stations, warehouses, factories and hangars, etc.; in short, spaces with large spans that had previously been executed in steel. The first long cylindrical reinforced concrete shell appeared in Germany in 1924, in the roof of a building that was to be used as a factory by the company Zeiss.

Research Significance

Although much has been written about the contribution by the Hungarian engineer G.V. Kazinczy as regards of his studies of plasticity, this work aims to deal with his simultaneous contribution to the structural calculation of long cylindrical roof shells. This study will not attempt to undermine the analytical method applied to these types of structures, but instead to fill a gap in the existing knowledge of the application of other simpler calculation methods, and specifically the so-called Beam Method. This method, which was based on the study of different states of equilibrium, enabled G. Kazinczy to carry out a clear, simple and safe structural analysis of these types in 1949, as we will outline below.

Throughout the history of structures several studies have emerged that address this problem (Assan 2002; Suanno et al. 2003; Chen 2007; Chandrasekaran, Gupta and Carannante 2009; Jawad 2015; Pophare and Jadhav 2017).

The present work tries to deepen in the methodological contribution of G. Kazinczy to the structural calculation of these typologies in a moment in which they were tried to be calculated, in a rigorous and “real” way, applying the mathematical mechanism of the theory of elasticity. It will be necessary to explain and demonstrate how the long cylindrical roof shells were analyzed using other simpler methods, as reliable as the analytical method. One of them is the so-called beam method, based on the study of different equilibrium states, and used by Kazinczy in the analysis of these structures.

Studies of States of Equilibrium in Reinforced Concrete Structures

Until the 1920s, and in Germany in particular, the structural behaviour of reinforced concrete shells were studied as if they were “membranes”. These studies found that if the stresses in a thin but sufficiently rigid shell were only compressive, tensile and tangential, were all contained within its thickness and there were no bending stresses at any point, then the sheet could be very thin - with a thickness of only a few centimetres - for its shape and support conditions to meet certain basic conditions. Reinforced concrete was obviously the material that complied with this mathematical and construction model, as it contained reinforcements to counteract tensile and shear stresses. Subsequently, given the construction requirements of long cylindrical shells, it became necessary to develop the mathematical theory taking into account the effects of bending. Elastic Theory was applied due to its widespread use in the structural calculations of the time. The mathematical formulation provided by the analytical theory was also applied to the structural calculation of the long cylindrical shells, without any analysis of the characteristics of the new construction material used.

However, it was impossible to apply elastic theory to shells in practice, as this involved solving complex eighth-order differential

equations, based on unreal hypotheses about the surrounding conditions and the structural material used. All these hypotheses involved either assuming real conditions that were impossible to ascertain beforehand, or referred to an ideal, homogeneous and isotropic material, when reinforced concrete has none of those properties. In any case, it was impossible to guarantee that the state of stress obtained in the shell represented the “real state” of the structure. As a consequence, some insurmountable inconsistencies appeared when comparing the results obtained from the elastic calculation and the results under real conditions and in tests.

G. Kazinczy began to perform tests on reinforced concrete structures in the 1930s. In 1933, he published an article on the plasticity of reinforced concrete (Kazinczy 1933), in which he proposed the concept of redistribution at moments of uniaxial bending, based on the plastic behaviour of both steel and concrete.

The first Congress of the IABSE (International Association for Bridge and Structural Engineering) was held in Paris in 1932, and was followed by the Second Congress LABSE, in 1936, held in Berlin. The latter event turned out to be decisive, as eight papers on the theory of plasticity were presented, including those by the engineers Melan (1936) and Maier-Leibnitz (1936), which cited the studies previously carried out by Kazinczy (1913, 1933). In the first lecture, Melan explained the non-linear stress-deformation relationship for an ideal plastic material, and in the second, Maier-Leibnitz presented the results obtained from tests on continuous beams with three supports subjected to the action of an external load, while the central support was moved upwards or downwards. As Kazinczy had done three years earlier, Maier-Leibnitz showed that the collapse of the structure did not depend on the different levelling of the supports, as elastic theory maintained, but instead on the appearance of a sufficient number of hinges to constitute a mechanism. The ultimate strength therefore did not depend on possible imperfections that could arise during the construction, such as poor levelling in the supports.

Kazinczy attended the Berlin Congress and published a short article for the final proceedings (Kazinczy 1936). Kazinczy insisted on the importance of considering plastic deformation in order to determine the true load-bearing capacity of statically indeterminate structures, as it was greater than that obtained according to the theory of elasticity (Kazinczy 1936). He referred to this new calculation method, which he called the “theory of plastic equilibrium” as a method in which the permanent stresses are taken into account, in contrast to the theory of elasticity, in which only elastic stresses are taken into consideration.

Despite the importance of these studies based on the equilibrium of stresses, structures - and long cylindrical shells in particular - continued to be calculated according to the elastic theory. This is apparent in the papers which were also contributed to this congress by the engineers Dischinger (1936) and Finsterwalder (1936).

States of Equilibrium in the Structural Analysis of Long Cylindrical Roof Shells

The fundamental theorems of plasticity presented by Gvozdev (1936), and published in 1952 by Prager, preceded the work by Gilman who used a distribution of normal forces in reinforced concrete that was different to the distribution established by elastic theory in a long cylindrical shell (Gilman 1938). Gilman proposed applying a simple calculation method consisting of likening the shell to a reinforced concrete beam, and using three different elasticity moduli instead of just one: one for the concrete when it is

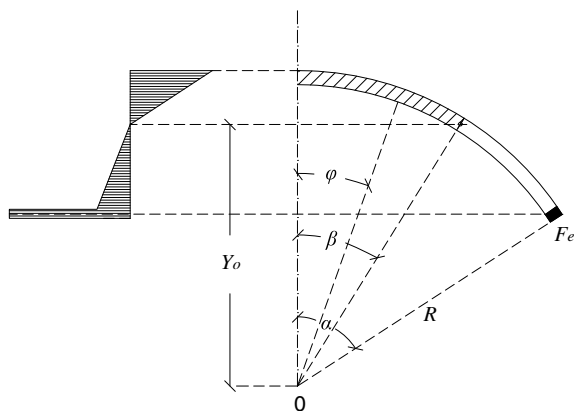


Fig. 1. Diagram of stresses in a long cylindrical shell. Diagram of similarity between a reinforced concrete beam and a long cylindrical shell. (Adapted from [Gilman 1938](#).)

subjected to compression stresses, another when it is under tensile stresses, and finally, a third for the reinforcement (Fig. 1).

Other investigations of the structural analysis of long cylindrical roof shells using a method based on the study of states of equilibrium were carried out by the Danish engineers K.W. Johansen and H. Lundgren in 1944 and 1949.

The literature related to these types of structural calculation before the publication of the book *Cylindrical Shell* by H. Lundgren in 1949, contained very few studies of the problem, apart from membrane theory and above all, the theory of elasticity. However, in view of the practically unsolvable nature of the eighth-order differential equations that arose from the application of elastic theory, different patterns emerged which were linked to achieving some degree of simplification of the analytical tools. These included those proposed between 1936 and 1940, by W. Flügge, R. Vallette,

H. Schorer, U. Finsterwalder and A.A. Jakobsen. In all of them, the authors proposed the beam method as a means of simplification, but within the theoretical framework of the theory of elasticity. It was assumed that ideal shells were being used, i.e., they had been constructed using a material that conformed to Hooke's law, and that the entire shell therefore had a single modulus of linear elasticity.

The major change took place in 1944, when the engineer Knud Winstrup Johansen published an article in which he analysed the structure of long cylindrical shells (Johansen 1944). The shell, built in 1938, was part of the roof of the restaurant of the General Broadcasting Corporation used by the film industry in Copenhagen (Statsradiofonien 1946), and its calculation was based solely on the equilibrium equations approach, enabling a simple and accurate calculation.

Johansen performed the calculation for the shell using the beam method, based on the hypothesis that the shell acts as if it were a reinforced concrete beam with a straight, hollow and circular cross-section. The novelty of this method lies in its proposed linear distribution of stresses for the compressions in the concrete, for a neutral axis determined in the section, and another for the tensile stresses; it only considers the stresses arising from the reinforcement of the concrete section which are concentrated in the two areas located on the edges of the shell, which have an identical value (Martínez 2019). With these assumptions in mind, Johansen therefore moved away from the theory of elasticity used for an ideal, homogeneous and isotropic material.

Longitudinally, the state of equilibrium in the shell is achieved by transferring the stress from the areas under most strain to the areas under least strain (Martínez 2019); this all depends on the transverse geometry of the shell, the location of the neutral axis and the various provisions chosen for the reinforcement. The lever arm is obtained by selecting a neutral line in the cross-section of the shell, and the internal stresses are counteracted with the moments due to the loads (Fig. 2). The reinforcement area required for this

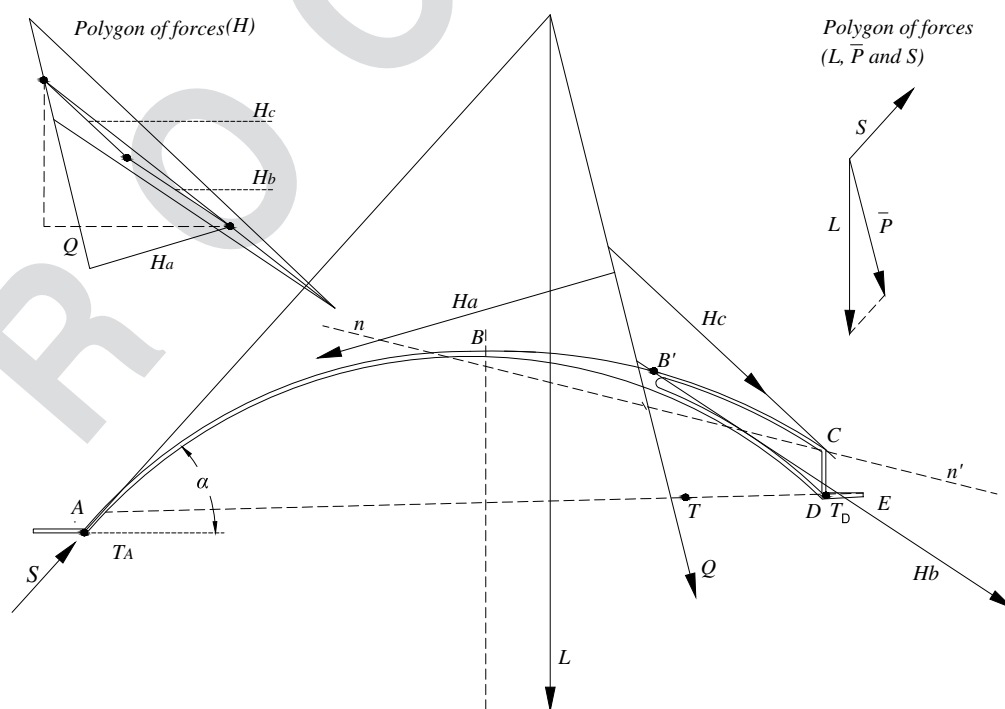


Fig. 2. Representation of external loads and internal forces in the cross-section of the cylindrical shell for the positive moment and polygons of forces. (Adapted from [Johansen 1944](#).)

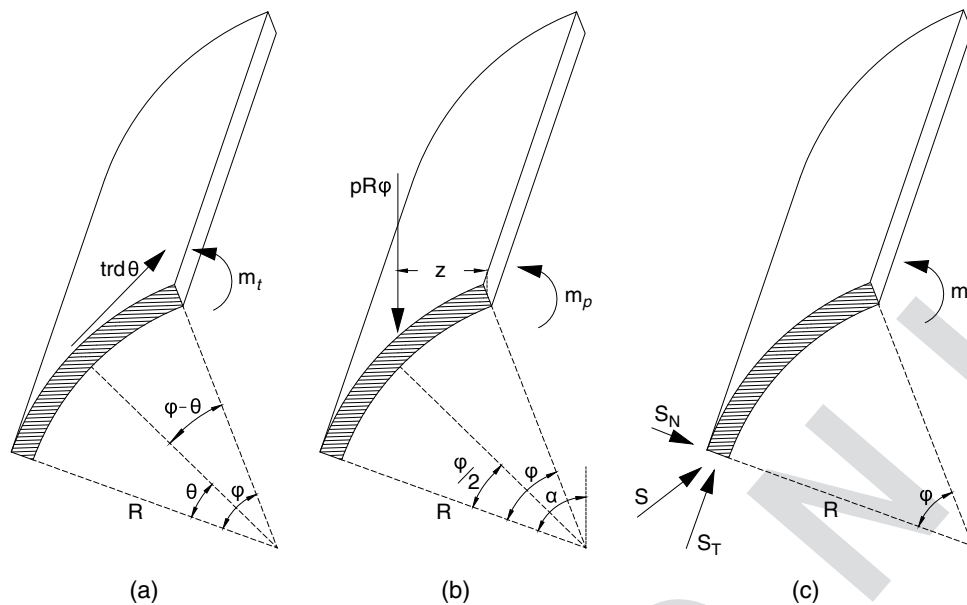


Fig. 3. Obtaining tangential moments: (a) tangential moments due to shear forces; (b) tangential moments due to the external load P ; and (c) tangential moments due to reactions in the pillars. (Adapted from Johansen 1944.)

particular state of equilibrium is computed following direct arguments. This is one solution to the problem, albeit not the only one. Given a cross-sectional geometry, it is therefore possible to study any state of the structure in which equilibrium between the acting forces occurs.

Similarly, both the shear stresses and the reinforcement necessary are calculated in the same way as in a concrete beam in the cross-sectional analysis of the shell. The transverse moments due to tangential stresses are obtained by statics, considering the balance in a transverse strip of uniform length subject to the vertical loads acting on it (Fig. 3); and the difference between the shear forces in both cross-sections that border the edge. This difference is considered to be unitary forces tangential to the cross-section (Martínez 2019).

Johansen succeeded in calculating a situation of equilibrium in the shell which was a safe solution satisfying the material yield condition. It was possible because it was constructed with a ductile material, and in the absence of problems of instability.

The studies begun by Johansen (1944, 1948) were followed by the publication of the book *Cylindrical Shell* by H. Lundgren in 1949, in which the author formulated a theory based on the equilibrium approach that was practical, clear and simple to apply (Lundgren 1949). Lundgren, an engineer who designed laminar structures at the Danish company Christiant & Nielsen, had noted the difficulty in the contemporary literature with understanding the calculation methods applied to these types of structure, due to the mathematical complexity involved in the analytical method. For this reason, he considered the beam method to be the best approach for calculating a long reinforced concrete cylindrical shell, in which the degree of accuracy depends on the magnitude of the transverse bending moments compared to normal forces. In other words, if the load acting on the shell is distributed in such a way as to provide moments with small values, the approach is quite accurate in comparison with the results obtained when using analytical theory. Like Johansen's approach (Martínez 2019), the mathematical process used by Lundgren is based on the study of systems of equilibrium of forces to resolve internal stresses, considering the characteristic yield of reinforced concrete.

Lundgren's theory was disseminated in Spanish in 1973, in the book published by Juan Antonio Tonda Magallón (Tonda 1973; Martínez 2018), an architect who worked with Felix Candela, who also used this method (Martínez 2018).

Calculation Methodology: Kazinczy and the Application of the Beam Method to Cylindrical Shells

In 1949, G. Kazinczy was designing sawtooth cylindrical shells for the Construction Department at the Swedish studio *Kooperativa Förbundets Arkitektkontor* (Office of the Cooperative Association of Architecture and Engineering). It was at this point when he applied the method formulated by K.W. Johansen and subsequently developed by H. Lundgren to his designs.

In the same year, Kazinczy published an article on the calculation of long cylindrical reinforced concrete shells, in which he applied the beam method based on the study of possible states of equilibrium (Kazinczy 1949). In this paper, Kazinczy argues that analytical methods are responsible for the beginning of the end of the construction of long cylindrical roof shells. These types were replaced by others that were less economical from a constructive point of view, but which nevertheless required less mathematical knowledge to solve the equations arising from the mathematical development based on the theoretical framework of the theory of elasticity.

Kazinczy proposes the following assumptions as the basis for calculating these typologies:

- Concrete does not bear tensile stresses.
- In elements subjected to bending, the compressive stresses are proportional to the distance to the neutral axis at each point in the section.
- In areas with shear stresses, some tensile stresses are permitted. Moreover, these are actual oblique tensile stresses.
- If the tensile forces are greater than the admissible forces, they are entirely absorbed by the reinforcement.

When justifying the calculation, Kazinczy states that if the length of the shell is between 2 and 2.5 times its width, the error

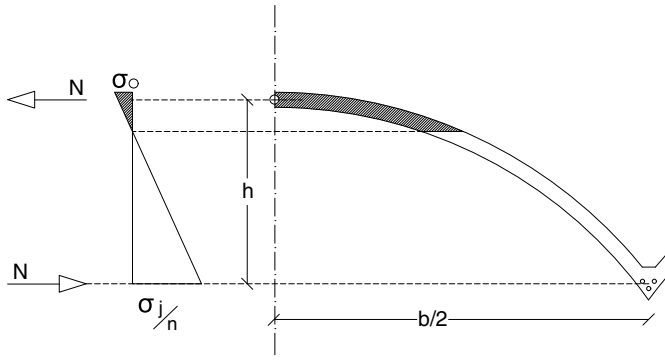


Fig. 4. Distribution of normal stresses in the shell. (Adapted from Kazinczy 1949.)

caused by likening the shell to a concrete beam and comparing the results with those obtained analytically is negligible. On the other hand, the proposed method is not sensitive to deviations from the assumed distribution in terms of compressive stresses.

Like Johansen and Lundgren, Kazinczy based the method on the search for an ideal state of equilibrium between the stresses and loads acting on the shell; this means that it is necessary to determine the location of the neutral axis. When the deformations and the stresses in the reinforcements are equal, the neutral axis is horizontal, whereas in sawtooth shells the neutral axis will be inclined. The compressive stresses are above the neutral axis, while the tensile stresses are below it, absorbed by the reinforcements (Fig. 4); it is more economical to locate them as far as possible from the neutral axis; and of course, to consider different elasticity moduli in the shell.

The distribution of the reinforcement required in both areas of the shell will be such that the sum of the compressive forces is equal to the sum of the tensile forces, and its resulting torque will in turn be equal to the moment of the external forces; in the same way as it occurs in a concrete beam:

$$\int \sigma_b dA_b = \sigma_{arm} A_{arm} \quad (1)$$

$$M = \sigma_{arm} A_{arm} h_t \quad (2)$$

where the total area of the reinforcement is:

$$A_{arm, sup} + A_{arm, inf} = \frac{M}{\sigma_{arm} h_t} = \frac{N_x}{\sigma_{arm}} \quad (3)$$

In order to determine h it is necessary to know the value of the height, as well as the position of the neutral axis (Fig. 4). Although the precise calculation to obtain the position of the neutral axis was described by Johansen (1944), the value of the compressive stresses is always well below the admissible level. Its impact on the position of the centre of mass of the compressions can be considered negligible, and a position for the neutral axis that is considered valid can be assumed. If an optimum result is finally not obtained, the axis can always be displaced again in order to achieve a more optimal and safer structure. During the first attempt, in order to avoid major errors, the same compressive stresses should be assumed to occur as those that would appear in the shell if it were a rectangular beam; with the same height, but with a width of (Fig. 4): $2100 \frac{b^2}{b} \text{ cm}$.

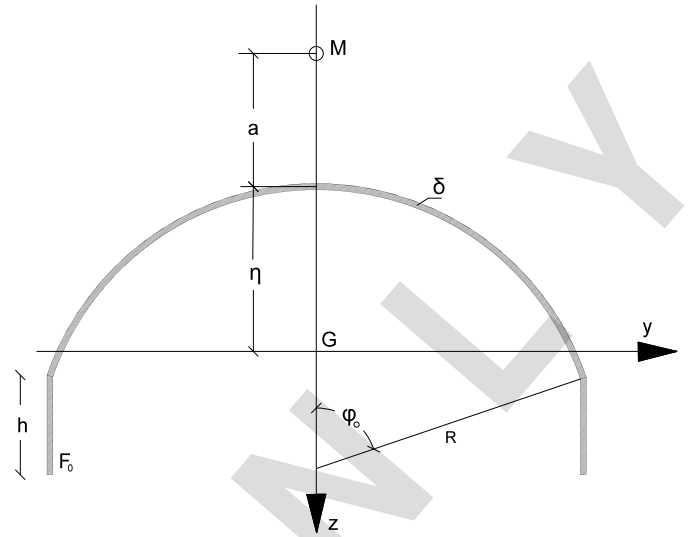


Fig. 5. Modelling of the cross-section of shell.

Longitudinal Calculation of the Shell

By modelling the cross section of the shell (Fig. 5), following the shell-beam analogy and having obtained the value of the total external load, q_z , the moment value and maximum shear would be as follows:

$$M_z = \frac{1}{8} q_z l^2, \quad Q_z = \frac{q_z l}{2} \quad (4)$$

The value of the maximum stress in the concrete section is given by:

$$\sigma_x = \frac{M_z}{I_{zz}} Z \quad (5)$$

The value of this bending moment is resisted by the internal stresses; as such, it is necessary to calculate the magnitude of the stresses in order to determine the amount of reinforcement required in the shell. The values of these stresses depend in turn on the lever arm, the distance between the centre of compressions and tensions (Fig. 5).

The result of the normal stresses in the longitudinal direction of the shell, N_x , is given by the expression:

$$N_x = \frac{M_z}{I_{zz}} S_z(S_0) \quad (6)$$

$S_z(S_0)$ represents the static moment of the area determined with respect to the axial axis. Accordingly, the value of the necessary area for reinforcement would be immediate after the operation:

$$A = \frac{N_x}{\sigma_{adm \text{ steel}}} \quad (7)$$

The following abbreviations are considered in Fig. 5 (see *Notation List*):

$2\phi_0$: opening angle of the shell.

η : distance between the crown, or upper point of the shell, and the centre of gravity of the shell, which is given by the expression (Lundgren 1949, 67):

$$\eta = R \left[\frac{1}{6} \varphi_0^2 \left(1 - \frac{1}{20} \varphi_0^2 \right) + \frac{C_1 C_2}{1 + C_2} \right] \quad (8)$$

where: $C_1 = 1/3 \varphi_0^2 (1 - 1/10 \varphi_0^2) + h/(2R)$, $C_2 = F_0/(R \delta \varphi_0)$, and the moment of inertia of the transverse section of the beam, I_{zz} , with respect to the axial axis y , is determined by the following expression:

$$I_{zz} = 2R^3 \delta \left[\frac{1}{45} \varphi_0^5 \left(1 - \frac{1}{7} \varphi_0^2 \right) + \frac{C_1^2 C_2 \varphi_0}{1 + C_2} + \frac{1}{12} \left(\frac{h}{R} \right)^2 C_2 \varphi_0 \right] \quad (9)$$

The resultant amount for the tractions is located in the centre of masses in the steel's reinforcement area; while the result of the compressive forces would be located at a distance of $1/5\eta$ (Fig. 5). The values of η and I_{zz} follow classical integration techniques for computing the centre of gravity of an object by means of parametrizations. It is worth noticing the appearance of two terms in the sum for η which correspond to the two geometrically different natures in the shell: the first term considers the arc of circle whilst the second is such that it tends to 0 when the radius R becomes larger. As a consequence, η has no dependence on the second term of the sum (and hence on h), at the limit.

The calculation of the theoretical distance of the lever arm in the shell and the location of the centre of stresses is thus determined by:

$$Z = \frac{I_{zz}}{2S_z(s_0)} \quad (10)$$

The state of equilibrium in the shell is therefore achieved by transferring the stress from the areas under most strain to the areas under least strain; depending on the transverse geometry of the shell, the location of the neutral axis and the various provisions made for the reinforcement. The lever arm is obtained by selecting a neutral axis in the cross-section of the shell, to which end the internal stresses are counteracted with the moments due to loads. The area for reinforcement required for this particular state of equilibrium is obtained by basic arithmetic operations.

The state of equilibrium thereby obtained is one solution to the problem, but not the only one; in other words, given a cross-sectional geometry, it is possible to study any state of the structure in which equilibrium between the acting forces occurs.

Transverse Calculation of the Shell

After determining the neutral axis and the normal forces, N_x , Kazinczy determines the tangential forces, $T_{\varphi x}$, and the transverse bending moments, M_{φ} , taking purely geometric and equilibrium considerations into account.

The maximum value of the tangential stress is determined by:

$$T_{\varphi x} = \frac{Q_z}{I_{zz}} S_z(S) \quad (11)$$

where Q_z represents the force of the shear stress on the beam; and its value can be calculated in a similar way to the procedure for a concrete beam. As a result, the reinforcement area necessary would also be obtained immediately [7].

When analysing an element of the shell (Fig. 6), the equilibrium condition must be such that the difference between the tangential forces acting in the same direction, T_{xn} and $T_{x(n+1)}$ are of equal value to the difference between the forces N_x ; or in other words:

$$N_{xn} - N'_{xn} = T_{xn} - T_{x(n+1)} \quad (12)$$

As the result that N_x and T_x is a pair of forces, equilibrium will only occur in the event that two tangential forces appear on surfaces 1 and 2 (Fig. 6), $T_{\varphi n}$ and $T'_{\varphi n}$. As the values of these two forces are also not equal, two other normal forces must therefore appear, $N_{\varphi n}$ and $N_{\varphi(n+1)}$, acting on sections "Studies of States Of Equilibrium in Reinforced Concrete Structures" and "States of Equilibrium in the Structural Analysis of Long Cylindrical Roof Shells", which are not parallel since the bands are curved. An external force with a specific value, position and direction must therefore appear to fulfil the equilibrium. As such a load does not exist, since the pre-existing load generally does not meet these conditions, normal shear forces Q_{φ} , and the tangential moments, M_{φ} , they occur in sections "Studies of States Of Equilibrium in Reinforced Concrete Structures" and "States of Equilibrium in the Structural Analysis of Long Cylindrical Roof Shells" of the shell.

At each point, given the shear stresses, τ_x , $\tau_x = \tau_{\varphi}$, the resulting T can therefore be calculated from the result of the forces T_x . To do this, when representing a shell element (Fig. 7) in which section "Introduction" is the same as that of the maximum moment in the shell span, and section "Research Significance" is the same as the span located at a distance of 1 m from the previous one, the forces

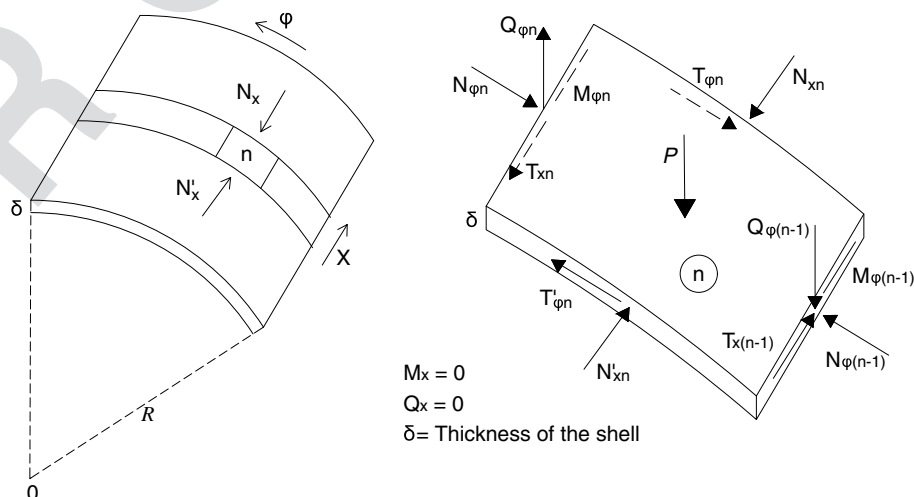


Fig. 6. State of stress in the shell. (Adapted from Lundgren 1949.)

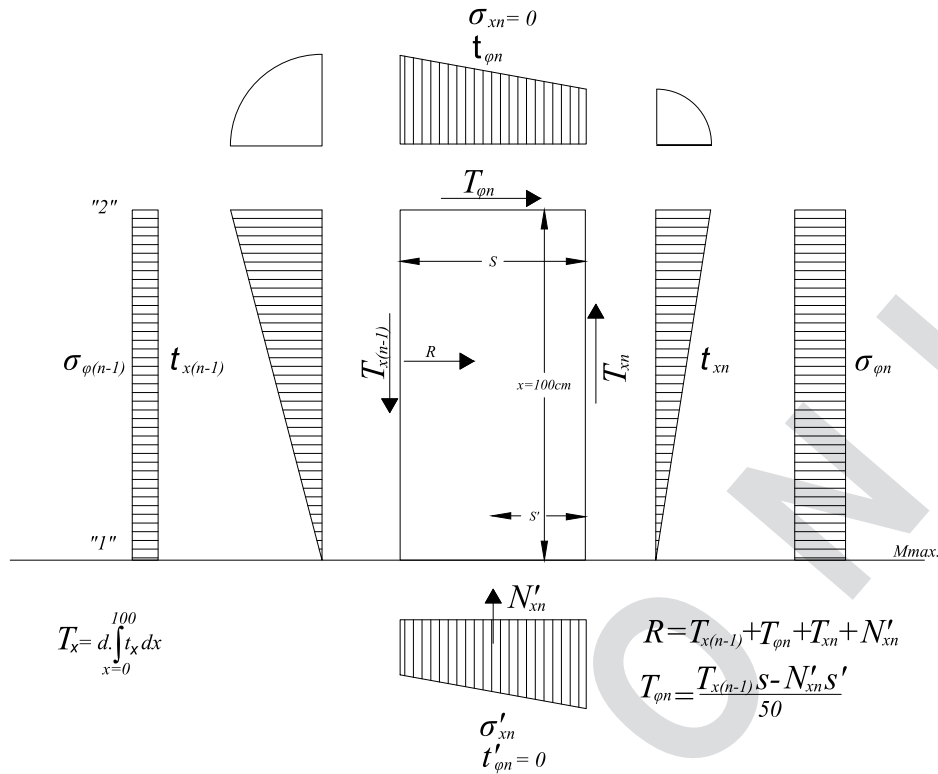


Fig. 7. State of stress for a moment in the maximum span.

T_{φ} on the surface in section “Introduction” are equal to 0, while the shear stresses, τ_x , have a triangular distribution.

Accordingly, in section “Introduction”: $\tau_{xn} = \tau_{\varphi n} = 0$, while in section “Research Significance”: $\tau_{xn} = \tau_{\varphi n} = 2(T_{xn}/100d)$, where d is the thickness of the shell.

The shear stress, τ_{xn} , is obtained in a similar way to that of a beam; where in the upper axis of the transversal edge $\tau_x = 0$. In the case of shells, the beginning is at the point where the traction reinforcements are located, i.e.

$$T_x = A_{arm} \sigma_{arm} - A'_{arm} \sigma'_{arm} \quad (13)$$

From this point, and as in the studies carried out by H. Lundgren (Lundgren 1949), the cross-section of the shell can be divided into segments of identical length. The area of the interval is assumed to be concentrated at the centre of the interval, as are the tangential stresses. Instead of finding a single stress, Kazinczy thereby achieves a more accurate approximation than the approach used by Johansen in 1944.

The tangential forces are determined in the same way as in the elementary theory of reinforced concrete for beams, in which the shear stresses τ_x increase on a linear basis from the point where the moment reaches its maximum, and they reach their maximum value in the supports.

The value of the moment of inertia, I_{zz} , for the neutral line and the centre of gravity G of the cross-section is obtained by:

$$I_{zz} = 2z^2 \cdot \delta\Delta \quad (14)$$

where z is the height of each of the intervals in which the cross-section of the shell has been divided; and $\delta\Delta$ is the area of each interval (Fig. 8).

As the area of each interval in the arc, $\delta\Delta$, has been considered as being concentrated at the midpoint, the value of the static

moment will also be constant and concentrated at the midpoint. Likewise, the load q is considered to be uniformly distributed in each interval.

The value of the tangential forces in each interval of the arc, $T_{\varphi x}$, is obtained by means of expression [11] and the vertical and horizontal components:

$$(T_{\varphi x})_y = T_{\varphi x} \Delta y, \quad (T_{\varphi x})_z = T_{\varphi x} \Delta z \quad (15)$$

where $(\Delta y, \Delta z)$ are the differences in elevations between the coordinates of the odd points in each interval in which the arc of the shell is divided (Fig. 8).

Having obtained the components of the tangential forces which arise at each point, adding all of them together from interval to interval gives the values for the horizontal and vertical components of the internal forces T_y and T_z , applied at the intermediate points of each range. The bending moments caused by these forces and their increases in each interval are easily found by determining the different lever arms for each interval, Δz and Δy .

As a result, the increase in the value of the transverse moment in each interval will be defined by the expression:

$$\Delta M_{\varphi} = T_y \Delta z - T_z \Delta y \quad (16)$$

The sum of the various increments, ΔM_{φ} , is used to obtain the value of the transverse moments in each interval, starting from a null value at the start of the shell.

In short, both the shear forces and the corresponding necessary reinforcement are calculated in the same way as in a concrete beam. The transverse moments due to tangential stresses are obtained by statics, considering the balance in a transverse strip of uniform length subject to the vertical loads acting on it; and the difference between the shear stresses in both cross-sections that border the

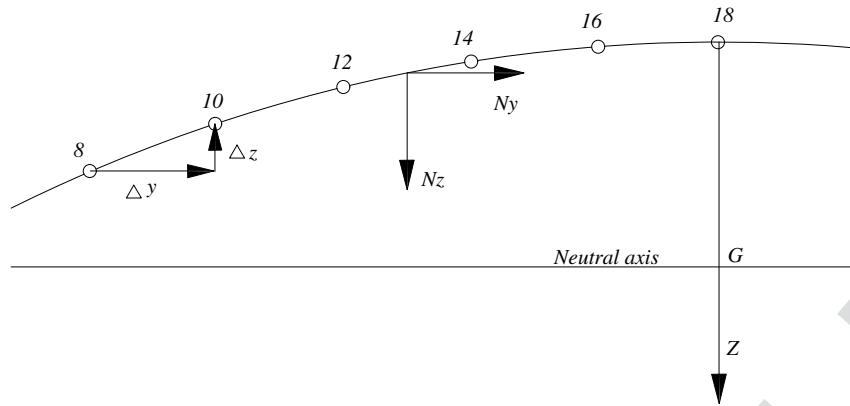


Fig. 8. Geometry of the intervals in which the section of the shell is divided.

strip. This difference is considered to be unitary forces tangential to the cross-section.

The results obtained by Kazinczy for the horizontal moments and forces in different types of long cylindrical shells (I, II, III and IV) are shown (Fig. 9).

Long cylindrical sawtooth shells have a lever arm with very small internal forces, in comparison with the total overhang, to counteract longitudinal bending. This requires increased reinforcement, and diagonal bars near the supports to absorb shear forces greater than those in a normal long cylindrical shell. Meanwhile, the free horizontal upper edge cannot be left unsupported, because it would buckle; it must be supported by columns with a small cross-section supported on the lower edge beam of the adjacent shell, with the increase in reinforcement that this entails.

In summary, the theory of elasticity was first used in the structural calculation of long cylindrical shells in Germany in the 1930s, as a result of the work done by the engineers U. Finsterwalder (Finsterwalder 1928, 1932, 1936) and Dischinger (1928, 1930, 1935, 1936), and later the Norwegian A. Aas Jakobsen (1937, 1939, 1940, 1941).

These structures, at first, were calculated conservatively using the theory of elasticity. The mathematical formulation provided by the analytical theory was applied to the structural calculation of the long cylindrical shells, without any analysis of the

characteristics of the new construction material used: reinforced concrete.

However, it was practically impossible to apply elastic theory to shells, as this involved solving complex eighth-order differential equations, based on unreal hypotheses about the surrounding conditions and the structural material used. All these hypotheses involved either assuming real conditions that were impossible to ascertain beforehand, or referred to an ideal, homogeneous and isotopic material, when reinforced concrete has none of those properties. In any case, it was always impossible to guarantee that the state of stress obtained in the shell represented the “real state” of the structure. The extreme difficulty involved in mathematically solving all these equations also contradicted the way the shell and its reinforcement were calculated; as such, all the detailed and complex mathematical work that had been rendered irrelevant.

As a consequence, some insurmountable inconsistencies appeared between the results obtained from the elastic calculation and the results under real conditions and in tests, a question that has been tried to demonstrate through the methodological contribution of G. Kazinczy.

Therefore, the methodology used by G. Kazinczy turned out to be an appropriate, safe and simple analysis method to the calculation of this type of structures in comparison with the elastic analysis.

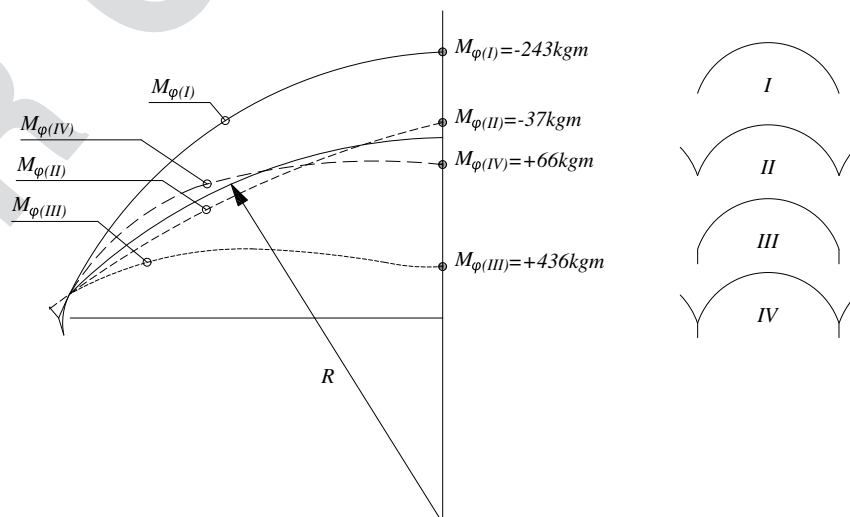


Fig. 9. Horizontal moments and forces in different types of long cylindrical shells. (Adapted from Kazinczy 1949.)

From 1950, until the appearance of computers, thanks to the contribution of engineers such as G. Kazinczy, the application of limit analysis with the relevant study of equilibrium systems was what really facilitated the calculation and construction of long cylindrical roof shells. This is how some engineers and architects worked when calculating this type of structure. This was the case of the architect Félix Candela Outeriño (Martínez 2019) and the engineer Juan Antonio Tonda (Martínez 2018).

Conclusions

1. The beam method, which Kazinczy applied to the structural analysis of long cylindrical shells, is therefore based on the study of different states of equilibrium. These states of equilibrium, caused by the adjustment between stresses, are the result of the transverse geometry of the shell that is selected, the location of the neutral axis and the various layouts chosen for the reinforcement. By arbitrarily correcting the slope of the neutral axis and correctly choosing the reinforcement amount, the ideal state of equilibrium is obtained, complying with safety criteria and overcoming compatibility and deformation considerations.
2. In the absence of a single solution to the problem, the structure reacts against the action of all the possible loads that may appear, or the possible shortcomings in their foundations, if there are any. The structure will adopt the most appropriate solution involving a distribution of stresses and strains, enabling it to continue to comfortably withstand the external forces which it is subject to in each case. Any state of the structure in which the equilibrium of forces occurs can be studied, meaning that the calculating engineer could focus on studying the safety of the shell in each one. On the other hand, first calculating the state of stresses using the homogeneous and elastic material hypothesis, and then replacing the tensile areas with stress-absorbing reinforcement, up to the admissible value of the stress, can result in inefficient shells which are more expensive, since they require excessive amounts of reinforcement.
3. The beam method provides an equilibrium solution, which is a safe solution in cases where the shell is made of a ductile material, and in the absence of problems of instability; without any consideration given to the conditions surrounding the shell, which are variable and impossible to determine in many cases.

Data Availability Statement

All data, models, and code generated or used during the study appear in the published article.

Notation

The following symbols are used in this paper:

- A = area of traction reinforcement;
- $A_{arm,inf}$ = reinforcement necessary at the bottom of the shell;
- $A_{arm,sup}$ = reinforcement necessary at the top of the shell;
- F_e = reinforcement on the edge of the shell (Fig. 1);
- G = centre of gravity;
- I_{zz} = moment of inertia of the transverse section of the beam, for the neutral line and the centre of gravity G ;
- H = resultant of the shear stresses;
- L = value of the resultant of external forces;

- l = span of the shell;
- M = positive moment;
- M_{emp} = moment of abutment;
- M_z = bending moment due to the load q_z ;
- M_φ = transverse bending moments;
- m_P = transverse moments due to the external load (P);
- m_S = transverse moments due to reactions in the pillars;
- m_t = transverse moments due to shear forces (t);
- m_φ = resulting transverse moment;
- N = resultant of the compressive stresses;
- N_X = result of the normal stresses in the longitudinal direction of the shell;
- $N_{x\varphi}$ = tangential forces in the cross-section of the shell;
- P = value of the external forces acting on the shell;
- \bar{P} = oblique resultant of the external forces;
- Q_z = resulting the force of the shear stress on the beam;
- Q_φ = normal shear force;
- qz = value of the vertical resulting from the load per unit of length on the axial axis x ;
- R = radius of the circumference that defines the shell;
- S = value of reactions in the direction tangent to the curve;
- S_t = horizontal component of the reaction;
- S_v = vertical component of the reaction;
- S_N = normal component of the reactions in the pillars;
- S_T = tangential component of the reactions in the pillars;
- $S_z(S_o)$ = static moment of the area determined with respect to the axial axis;
- T_A = centre of mass of tensile part (Fig. 2);
- T_D = centre of mass of tensile part (Fig. 2);
- T_X = tangential forces in the longitudinal direction of the shell;
- $T_{\varphi x}$ = tangential forces;
- t = value of the shear stresses;
- τ_x = shear stresses;
- Y_0 = distance from centre to neutral axis (Fig. 1);
- Z = specific distance from the point to the neutral axis of the cross section of the shell;
- α = angle the arc;
- β = angle the arc that delimits the compressed area of the concrete. Neutral axis position (Fig. 1);
- δ = thickness of the shell;
- η = distance between the crown, or upper point of the shell, and the centre of gravity of the shell;
- $\delta\Delta$ = area of each interval in the arc;
- σ = maximum compressive stress in the concrete; and
- $2\varphi_0$ = opening angle of the shell.

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