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Computation of the singularities of parametric plane curves

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Abstract

Given an algebraic plane curve $\mathcal{C}$ defined by a rational parametrization $\mathcal{P}(t)$, we present formulae for the computation of the degree of $\mathcal{C}$, and the multiplicity of a point. Using the results presented in [Sendra, J.R., Winkler, F., 2001. Tracing index of rational curve parametrizations. Computer Aided Geometric Design 18 (8), 771–795], the formulae simply involve the computation of the degree of a rational function directly determined from $\mathcal{P}(t)$. Furthermore, we provide a method for computing the singularities of $\mathcal{C}$ and analyzing the non-ordinary ones without knowing its defining polynomial. This approach generalizes the results in [Abhyankar, S., 1990. Algebraic geometry for scientists and engineers. In: Mathematical Surveys and Monographs, vol. 35. American Mathematical Society; van den Essen, A., Yu, J.-T., 1997. The *D*-resultants, singularities and the degree of unfaithfulness. Proceedings of the American Mathematical Society 25, 689–695; Gutierrez, J., Rubio, R., Yu, J.-T., 2002. *D*-Resultant for rational functions. Proceedings of the American Mathematical Society 130 (8), 2237–2246] and [Park, H., 2002. Effective computation of singularities of parametric affine curves. Journal of Pure and Applied Algebra 173, 49–58].

Keywords: Rational curve parametrization; Algebraic curve; Degree of an algebraic curve; Singularities of an algebraic curve; Multiplicity of a point

1. Introduction

Parametrizations, in particular parametrizations of rational curves, play an important role in many practical applications in computer aided geometric design where objects are often given

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and manipulated parametrically, such as in visualization (see Hoffmann et al. (1997), Hoschek and Lasser (1993)) or rational parametrization of offsets (see Arrondo et al. (1997)). Also, there exists an implicitization approach based on resultants (see Cox et al. (1998) and Sendra and Winkler (2001)).

Many authors have studied different problems related to rational curves (i.e. algebraic curves that can be parametrized rationally) assuming that the original curve was given implicitly. Now we consider some of these problems but from another point of view, namely, we assume that the curve is given in parametric form.

In this paper, we first deal with the determination of the degree of a rational plane curve (i.e. the total degree of the implicit equation) defined by a rational parametrization. We will use this approach to compute the multiplicity of a given point.

This question, i.e. the determination of the degree, is especially interesting since the degree of the curve can be used to approach some other problems, for instance the construction of implicitization algorithms based on interpolation techniques (see Kotsireas (2004), Marco and Martínez (2001)). Motivated by this fact, some authors have addressed this problem. For instance, in Sendra and Winkler (2001) the authors present some formulae for the computation of the partial degrees of the implicit equation defining a plane curve. For the surface case, interesting contributions have also been presented. In Pérez-Díaz and Sendra (2005), the results for curves of Sendra and Winkler (2001) are generalized for the case of parametric surfaces. In Schicho (1999), one may find bounds for the degree of the parametrization of a rational surface, in Chionh and Goldman (1992) a method based on the degree of rational maps and on the base points of the parametrization is presented, in Buse et al. (2003) and Cox (2001) one computes the degree by analyzing the base points and by means of syzygies, etc. In addition, the degree problem has also been studied for the case of special computer aided geometric constructions such as offsetting (see San Segundo and Sendra (2005)).

The formula presented in this paper is obtained from the results in Sendra and Winkler (2001), and it is based on the computation of the degree of a rational function directly obtained from the given parametrization. Using this result, a formula for computing the multiplicity of a given point is provided.

Moreover, we focus on the problem of computing and analyzing all the singularities of a rational plane curve without knowing its defining polynomial. Several approaches solving this problem when the algebraic curve is given implicitly are presented (see for instance Sakkalis and Farouki (1990)). Many authors have addressed the problem when the curve is defined by a rational parametrization, but up to the present only partial answers are provided. In Abhyankar (1990), the author develops the notion of the Taylor resultant to deal with this problem for rational curves parametrized polynomially. This concept was generalized as the D-resultant in van den Essen and Yu (1997) which works over an arbitrary field. In Park (2002), the author extends this result to a curve in affine n-space parametrized polynomially. Finally, in Gutierrez et al. (2002) the authors introduce the D-resultant of two rational functions, and show how it can be used to find the singularities of a parametric algebraic curve.

In this paper, we generalize these results. More precisely, we show how to find the singularities and their corresponding multiplicities of a parametric algebraic curve simply by analyzing the degree of a rational function directly obtained from the given parametrization. Moreover, we describe the singularities without directly introducing algebraic numbers in the computations. More precisely, we simulate the decomposition of singularities for plane curves defined implicitly (see Sendra and Winkler (1997)) for the case of curves defined parametrically. We also analyze the character of the singularity and we compute the neighboring singularities. The results presented here provide effective methods that can be used to solve the problem of curves in affine n-space. We will deal with this problem in a future work.

The structure of the paper is as follows. In Section 2, we introduce the terminology that will be used throughout this paper as well as some previous results. In Section 3, we present a formula that computes the degree of a plane curve defined parametrically. Section 4 generalizes the results of Section 3, and presents some formulae for computing the multiplicity of a given point, affine or projective. Section 5 is devoted to the computation of singularities. In addition, we outline the algorithm, and we illustrate it with an example. Finally, non-ordinary singularities are treated specially in Section 6.

2. Notation and preliminaries

In this section we introduce the notation and terminology that will be used throughout this paper. In addition, we recall some basic results on parametric curves. These results will be used throughout the subsequent sections.

2.1. Notation and basic notions

Throughout this paper, we assume that $\mathbb{K}$ is a computable field of characteristic zero and K is its algebraic closure. Let $F(x_1, x_2) \in \mathbb{K}[x_1, x_2]$ be the defining polynomial of a rational affine irreducible curve $\mathcal{C}$. In the following, we assume that $\mathcal{C}$ is not a line. Observe that in this case, the problems that we deal with in this paper are trivial.
Let $\mathbb{L}$ be a subfield of K such that $\mathbb{K} \subset \mathbb{L} \subset K$, and let

$$\mathcal{P}(t) = (p_1(t),\ p_2(t)) \in \mathbb{L}(t)^2,$$

be a rational parametrization of $\mathcal{C}$, where $p_i(t) = \frac{p_{i,1}(t)}{p(t)}, i = 1, 2$, and $\gcd(p_{1,1}, p_{2,1}, p) = 1$. Sometimes, we will need the parametrization $\mathcal{P}$ in reduced form. We write it as

$$\mathcal{P}(t) = \left(\frac{q_{1,1}(t)}{q_{1,2}(t)}, \frac{q_{2,1}(t)}{q_{2,2}(t)}\right), \quad \gcd(q_{i,1},\ q_{i,2}) = 1,\ i = 1, 2.$$

Under these conditions, we immediately get that the corresponding projective curve $\mathcal{C}^\star$ is defined by the homogenization $F^\star(x_1, x_2, x_3)$ of $F(x_1, x_2)$. Therefore, if we write

$$F(x_1, x_2) = F_d(x_1, x_2) + F_{d-1}(x_1, x_2) + \cdots + F_0(x_1, x_2),$$

where $F_k(x_1, x_2)$ is a homogeneous polynomial of degree k, and $F_d \neq 0$, then

$$F^\star(x_1, x_2, x_3) = F_d(x_1, x_2) + F_{d-1}(x_1, x_2)x_3 + \cdots + F_0(x_1, x_2)x_3^d.$$

Thus, a projective parametrization of $\mathcal{C}^\star$ is given by $\mathcal{P}^\star(t) = \big(p_{1,1}(t),\ p_{2,1}(t),\ p(t)\big)$.

Every affine point (a, b) on $\mathcal{C}$ corresponds to a point $(a : b : 1)$ of the projective plane $\mathbb{P}^2$ on the curve $\mathcal{C}^\star$, and every additional point on $\mathcal{C}^\star$ is a point at infinity. In other words, the first two coordinates of the additional points are the nontrivial solutions of $F_d(x_1, x_2)$. Thus, the curve $\mathcal{C}^\star$ has only finitely many points at infinity.

On the other hand, associated with every projective curve there are infinitely many affine curves. If $\mathcal{C}^\star$ is the projective curve defined by the form $F^\star(x_1, x_2, x_3)$, we denote by $\mathcal{C}_{x_3}$ the affine plane curve defined by $F(x_1, x_2) = F^\star(x_1, x_2, 1)$. Similarly, we consider $\mathcal{C}_{x_2}$, and $\mathcal{C}_{x_1}$. Observe that

$$\mathcal{P}_{x_1}(t) = \left(\frac{p_{2,1}(t)}{p_{1,1}(t)}, \frac{p(t)}{p_{1,1}(t)}\right), \qquad \mathcal{P}_{x_2}(t) = \left(\frac{p_{1,1}(t)}{p_{2,1}(t)}, \frac{p(t)}{p_{2,1}(t)}\right),$$
$$\mathcal{P}_{x_3}(t) = \left(\frac{p_{1,1}(t)}{p(t)}, \frac{p_{2,1}(t)}{p(t)}\right)$$

parametrize $\mathcal{C}_{x_1}$, $\mathcal{C}_{x_2}$, and $\mathcal{C}_{x_3}$, respectively (note that $\mathcal{C}$ is not a line and then $p_{i,1}(t) \neq 0$ for $i = 1, 2$).

Associated with the rational functions of $\mathcal{P}(t)$, we consider the induced rational map $\phi_{\mathcal{P}} : K \longrightarrow \mathcal{C} \subset K^2; t \longmapsto \mathcal{P}(t)$, and the polynomials

$$G_i^{\mathcal{P}}(s, t) = q_{i,1}(t)\, q_{i,2}(s) - q_{i,1}(s)\, q_{i,2}(t) \in \mathbb{L}[s, t], \quad i \in \{1, 2\},$$
$$G^{\mathcal{P}}(s, t) = \gcd(G_1^{\mathcal{P}}, G_2^{\mathcal{P}}).$$

The polynomials $G_i^{\mathcal{P}}$ play an important role in deciding whether a parametrization $\mathcal{P}(t)$ is proper by means of the degree of $\phi_{\mathcal{P}}$; i.e. in studying whether the parametrization is injective for almost all parameter values (see Sendra and Winkler (2001)).

Related to the problem we are dealing with, we will use different concepts of degree:

(1) For a polynomial $G \in K[x_1, x_2, x_3]$ we denote by $\text{tdeg}(G)$ the total degree of G and by $\deg_{x_i}(G)$ the degree of G w.r.t. x_i.
(2) We denote by $\deg(\mathcal{C})$ the degree of $\mathcal{C}$, that is $\deg(\mathcal{C}) = \text{tdeg}(F)$.
(3) We define the partial degree of $\mathcal{C}$ w.r.t. x_i as the partial degree of its implicit equation, and we denote it by $\deg_{x_i}(\mathcal{C})$.
(4) We define the degree of the parametrization $\mathcal{P}(t)$ as the maximum of the degrees of its rational components, in reduced form, and we denote it by $\deg(\mathcal{P}(t))$.
(5) We denote by $\deg(\phi_{\mathcal{P}})$ the degree of the rational map $\phi_{\mathcal{P}}$ (for further details see e.g. Shafarevich (1994, pp. 143), or Harris (1995, pp. 80)). As an important result, we recall that the birationality of $\phi_{\mathcal{P}}$, i.e. the properness of $\mathcal{P}(t)$, is characterized by $\deg(\phi_{\mathcal{P}}) = 1$ (see Harris (1995) and Shafarevich (1994)). Also, we recall that the degree of a rational map can be seen as the cardinality of the fibre of a generic element (see Theorem 7, pp. 76 in Shafarevich (1994)). We will use this characterization in our reasoning. For this purpose, we denote by $\mathcal{F}_{\mathcal{P}}(P)$ the fibre of a point $P \in \mathcal{C}$; that is $\mathcal{F}_{\mathcal{P}}(P) = \mathcal{P}^{-1}(P) = \{t \in K \mid \mathcal{P}(t) = P\}$.

2.2. *Singular points*

Singular points play an important role in the theory of algebraic curves. In the following, some basic notions and results are reviewed (see Brieskorn and Knoerrer (1986), Fulton (1989), Shafarevich (1994) or Walker (1950)).

Definition 1. Let $P = (a, b) \in \mathcal{C}$. We say that P is of multiplicity ℓ on $\mathcal{C}$ if and only if all the derivatives of F up to and including the $(\ell - 1)$-th vanish at P but at least one ℓ-th derivative does not vanish at P. We denote it by $\text{mult}_P(\mathcal{C})$.

P is called a simple point on $\mathcal{C}$ iff $\text{mult}_P(\mathcal{C}) = 1$. If $\text{mult}_P(\mathcal{C}) = \ell > 1$, then we say that P is a multiple or singular point (or singularity) of multiplicity ℓ on $\mathcal{C}$ or an ℓ-fold point. Furthermore if $P \notin \mathcal{C}$ then $\text{mult}_P(\mathcal{C}) = 0$.

Observe that the multiplicity of $\mathcal{C}$ at P is given as the order of the Taylor expansion of F at P. The tangents to $\mathcal{C}$ at P are the irreducible factors of the first nonvanishing form in the Taylor expansion of F at P, and the multiplicity of a tangent is the multiplicity of the corresponding factor.

For analyzing a singular point P on a curve $\mathcal{C}$ we need to know its multiplicity but also the multiplicities of the tangents at P. If all the ℓ tangents at the ℓ-fold point P are different, then this singularity is called ordinary, and it is called non-ordinary otherwise. Thus, we say that the character of P is either ordinary or non-ordinary.

Now, as far as projective curves are concerned all these definitions also apply, since every point at infinity can be transformed to a point at finite distance by a change of coordinates.

In order to compute the affine singularities one just has to find the finite solutions of the system of algebraic equations $\{F = 0, \frac{\partial F}{\partial x_1} = 0, \frac{\partial F}{\partial x_2} = 0\}$, and to determine the singularities at infinity one can dehomogenize $F^\star(x_1, x_2, x_3)$ with respect to another variable. Also, one can look for the non-zero solutions of $\{\frac{\partial F^\star}{\partial x_1} = 0, \frac{\partial F^\star}{\partial x_2} = 0, \frac{\partial F^\star}{\partial x_3} = 0\}$. We remark that every curve without multiple components has only finitely many singularities.

2.3. Computation of the degree of a rational map

In this subsection, we compute the degree of the rational map induced by $\phi_{\mathcal{P}}$. Intuitively speaking, the degree measures the number of times the parametrization traces the curve when the parameter takes values in K. All results in this subsection are included in Sendra and Winkler (2001). Therefore, we omit proofs.

Theorem 2. *Let* $T(s) = \text{Resultant}_t(G_1^{\mathcal{P}}/G^{\mathcal{P}}, G_2^{\mathcal{P}}/G^{\mathcal{P}})$ *if* $\mathcal{P}$ *does not have constant components, and* $T(s) = 1$ *otherwise. Let* $M(s) = \gcd(\text{lc}(G_1^{\mathcal{P}}, t), \text{lc}(G_2^{\mathcal{P}}, t))$*, where* $\text{lc}(G_i^{\mathcal{P}}, t)$ *denotes the leading coefficient of* $G_i^{\mathcal{P}}$ *w.r.t. the variable* t*. Then, for* $\alpha \in \mathbb{K}$ *such that* $q_{1,2}(\alpha)q_{2,2}(\alpha)T(\alpha)M(\alpha) \neq 0$*, we have*

(1) $\deg(\phi_{\mathcal{P}}) = \text{Card}(\mathcal{F}_{\mathcal{P}}(\mathcal{P}(s))) = \deg(G^{\mathcal{P}}(\alpha, t)) = \deg_t(G^{\mathcal{P}}(s, t))$,
(2) $\mathcal{F}_{\mathcal{P}}(\mathcal{P}(\alpha)) = \{\beta \in \mathbb{K} \mid G^{\mathcal{P}}(\alpha, \beta) = 0\}$. □

Since a parametrization is proper if and only if it defines a birational mapping between the affine line and the curve, it is clear that a parametrization is proper if and only if $\deg_t(G^{\mathcal{P}}) = 1$.

The next theorem characterizes the properness of a parametrization by means of the degree of the implicit equation of the curve.

Theorem 3. $\mathcal{P}(t)$ *is proper if and only if* $\deg(\mathcal{P}(t)) = \max\{\deg_{x_1}(F), \deg_{x_2}(F)\}$*. Furthermore, if* $p_1(t)$ *is non-zero then* $\deg_{x_2}(F) = \frac{\deg(p_1(t))}{\deg(\phi_{\mathcal{P}})}$*; similarly if* $p_2(t)$ *is non-zero then* $\deg_{x_1}(F) = \frac{\deg(p_2(t))}{\deg(\phi_{\mathcal{P}})}$. □

2.4. Normality of a rational parametrization

Any rational parametrization $\mathcal{P}(t)$ induces a natural dominant rational mapping $\phi_{\mathcal{P}}$ from the affine line onto the curve. In fact, when we study the properness of a parametrization $\mathcal{P}(t)$ we analyze the injectivity of $\phi_{\mathcal{P}}$ over almost all values in K. Now, we focus on the surjectivity. The mapping $\phi_{\mathcal{P}}$ is dominant; thus, in general, it might not be surjective, and hence some points of the algebraic set are missed. In this situation, we introduce the following notion. All the results in this subsection are included in Sendra (2002).

Definition 4. A rational affine parametrization $\mathcal{P}(t)$ is normal iff the rational mapping $\phi_{\mathcal{P}}$ is surjective, or equivalently iff for all $P \in \mathcal{C}$ there exists $t_0 \in K$ such that $\mathcal{P}(t_0) = P$.

In Sendra (2002), it is proved that any affine rational parametrization generates, when the parameter takes values in an algebraically closed field, all affine points on the curve with the exception of at most one point. In fact, it is shown that any affine parametrization can always be reparametrized into a normal one. More precisely, one has the following theorem.

Theorem 5. *Let $\mathcal{P}$ be the given parametrization in reduced form. Let $n = \deg(q_{1,1})$, $m = \deg(q_{1,2})$, $r = \deg(q_{2,1})$, $s = \deg(q_{2,2})$, and let $\ell_1 = \mathrm{coeff}(q_{1,1}, m)$, $\ell_2 = \mathrm{coeff}(q_{1,2}, m)$, $\ell_3 = \mathrm{coeff}(q_{2,1}, s)$, $\ell_4 = \mathrm{coeff}(q_{2,2}, s)$. Then, it holds that:*

(1) *If $n > m$ or $r > s$ then $\mathcal{P}(t)$ is normal.*
(2) *If $n \le m$ and $r \le s$ then $\mathcal{P}(t)$ is normal if and only if*

$$\deg(\gcd(\ell_1 q_{1,2}(t) - \ell_2 q_{1,1}(t), \ell_3 q_{2,2}(t) - \ell_4 q_{2,1}(t))) \ge 1.$$

Furthermore, if $\mathcal{P}(t)$ is not normal, all points in $\mathcal{C}$ are generated by $\mathcal{P}(t)$ with the exception of $(\ell_1/\ell_2,\ \ell_3/\ell_4)$ which is a point on $\mathcal{C}$. □

In the following, we will refer to the only possible missing point of the parametrization, $(\ell_1/\ell_2,\ \ell_3/\ell_4)$, as the critical point of $\mathcal{P}(t)$.

Remark 1. Let $P = (a_1 : a_2 : a_3) \in \mathbb{P}^2$ be a point of $\mathcal{C}^\star$ not generated by $\mathcal{P}^\star$. Let $a_i \neq 0$, for some $i = 1, 2, 3$. Then $(a_j/a_i, a_k/a_i)$, where $j, k \in \{1, 2, 3\}$, and $j \neq i \neq k$, is the critical point of the parametrization $\mathcal{P}_{x_i}$ in reduced form.

3. The degree of a rational plane curve

The computation of the degree of an algebraic curve is an important problem. To approach this problem one may for instance compute the maximum number of intersection points of the curve with a line in general position.

In this section, we introduce an additional method that solves this problem. The approach presented will be a necessary tool for the next section. For this purpose, we apply the result on partial degrees presented in Theorem 3, to compute the degree of a given algebraic plane curve $\mathcal{C}$ defined by a parametrization

$$\mathcal{P}(t) = (p_1(t),\ p_2(t)) \in \mathbb{L}(t)^2,$$

where $p_i(t) = p_{i,1}(t)/p(t),\ i = 1, 2$, and $\gcd(p_{1,1}, p_{2,1}, p) = 1$.

We remark that $p_i \notin K$. Observe that in this case, the problem is trivial since $\mathcal{C}$ is a line and then, $\deg(\mathcal{C}) = 1$.

Theorem 6. *Let $(a, b) \notin \mathcal{C}$. Then,*

$$\deg(\mathcal{C}) = \frac{\deg\left(\frac{p_{2,1}(t)-bp(t)}{p_{1,1}(t)-ap(t)}\right)}{\deg(\phi_{\mathcal{P}})}.$$

Proof. We consider the curve $\mathcal{D}$ defined by the polynomial $G(x_1, x_2) = F(x_1 + a, x_2 + b)$. Observe that

$$\mathcal{Q}(t) = \left(\frac{p_{1,1}(t) - ap(t)}{p(t)},\ \frac{p_{2,1}(t) - bp(t)}{p(t)}\right)$$

parametrizes $\mathcal{D}$, and $(0, 0) \notin \mathcal{D}$. Thus, we write

$$G^\star(x_1, x_2, x_3) = G_d(x_1, x_2) + G_{d-1}(x_1, x_2)x_3 + \cdots + G_0(x_1, x_2)x_3^d,$$

and since $(0, 0) \notin \mathcal{D}$, we deduce that

$$d = \deg(\mathcal{D}) = \deg_{x_3}(G^\star(x_1, x_2, x_3)) = \deg_{x_3}(G^\star(1, x_2, x_3)).$$

Now, we consider the parametrization $\mathcal{Q}_{x_1}(t)$ of the curve $\mathcal{D}_{x_1}$ (see Section 2), and we apply Theorem 3 (one reasons similarly if one considers $\mathcal{Q}_{x_2}$). We get that

$$\deg(\mathcal{D}) = \frac{\deg\left(\frac{p_{2,1}(t)-bp(t)}{p_{1,1}(t)-ap(t)}\right)}{\deg(\phi_{\mathcal{Q}_{x_1}})}.$$

Observe that since the degree is multiplicative with respect to composition (see (Shafarevich, 1994)), and taking into account that $\mathcal{Q}_{x_1} = R \circ \mathcal{P}$, where $R = ((y - b)/(x - a), 1/(x - a))$, we deduce that $\deg(\phi_{\mathcal{Q}_{x_1}}) = \deg(\phi_R) \cdot \deg(\phi_{\mathcal{P}})$. Clearly $\deg(\phi_R) = 1$, and then $\deg(\phi_{\mathcal{Q}_{x_1}}) = \deg(\phi_{\mathcal{P}})$. Finally, we also note that the total degree of a curve is invariant under linear changes of coordinates, and hence $\deg(\mathcal{D}) = \deg(\mathcal{C})$. □

Remark 2. In order to check whether $(a, b) \in \mathcal{C}$, one computes

$$\gcd(q_{1,1}(t) - aq_{1,2}(t),\ q_{2,1}(t) - bq_{2,2}(t)).$$

If the degree of the above polynomial is greater than or equal to 1, then $(a, b) \in \mathcal{C}$. Otherwise, one analyzes whether (a, b) is the critical point of $\mathcal{P}$. In the affirmative case $(a, b) \in \mathcal{C}$, otherwise $(a, b) \notin \mathcal{C}$.

In the following result, we generalize Theorem 6. More precisely, we avoid the condition of taking a point $(a, b) \notin \mathcal{C}$.

Theorem 7. *Let* $d_k = \deg_{x_k}(F),\ k = 1, 2,$ *and let* $(a, c_i) \in K^2,\ i = 1, \ldots, d_1 + d_2 + 1,$ *be such that* $c_i \neq c_j$ *for* $i \neq j$. *Then,*

$$\deg(\mathcal{C}) = \frac{\max_{1\le i\le d_1+d_2+1}\left\{\deg\left(\frac{p_{2,1}(t)-c_i p(t)}{p_{1,1}(t)-ap(t)}\right)\right\}}{\deg(\phi_{\mathcal{P}})}.$$

Proof. First we note that since $\mathcal{C}$ is not the line $x = a$ ($p_i \notin K$) and $\mathcal{C}$ is irreducible, by the Bézout Theorem (see Walker (1950)), we get that the line $x - a$ and the curve $\mathcal{C}$ intersect in $\deg(\mathcal{C})$ points. Taking into account that $\deg(\mathcal{C}) \le d_1 + d_2$, we deduce that there exists at least a point $(a, c_{i_0}) \in K^2$ not in $\mathcal{C}$. Therefore, by Theorem 6, we get that

$$\deg(\mathcal{C}) = \frac{\deg\left(\frac{p_{2,1}(t)-c_{i_0} p(t)}{p_{1,1}(t)-ap(t)}\right)}{\deg(\phi_{\mathcal{P}})} \qquad \text{(I)}.$$

Now, we note that if $(a, c_j) \in \mathcal{C}$, then

$$\deg(\mathcal{C}) > \frac{\deg\left(\frac{p_{2,1}(t)-c_j p(t)}{p_{1,1}(t)-ap(t)}\right)}{\deg(\phi_{\mathcal{P}})} \qquad \text{(II)}.$$

Indeed, let $\mathcal{D}$ be the curve defined by the polynomial $G(x_1, x_2) = F(x_1 + a, x_2 + c_j)$. Observe that $G(0, 0) = F(a, c_j) = 0$. Thus, we write

$$G^\star(x_1, x_2, x_3) = G_d(x_1, x_2) + G_{d-1}(x_1, x_2)x_3 + \cdots + G_0(x_1, x_2)x_3^d,$$

and since $(0, 0) \in \mathcal{D}$, we deduce that

$$d = \deg(\mathcal{D}) = \deg(\mathcal{C}) > \deg_{x_3}(G^\star(x_1, x_2, x_3)) = \deg_{x_3}(G^\star(1, x_2, x_3)).$$

Reasoning similarly to in the proof of Theorem 6, we get that

$$\deg_{x_3}(G^\star(x_1, x_2, x_3)) = \frac{\deg\left(\frac{p_{2,1}(t)-c_j p(t)}{p_{1,1}(t)-ap(t)}\right)}{\deg(\phi_{\mathcal{P}})}.$$

Therefore, from (I) and (II), we conclude that

$$\deg(\mathcal{C}) = \frac{\max_{1\leq i\leq d_1+d_2+1}\left\{\deg\left(\frac{p_{2,1}(t)-c_i p(t)}{p_{1,1}(t)-ap(t)}\right)\right\}}{\deg(\phi_{\mathcal{P}})}. \quad \square$$

Example 1. Let $\mathcal{C}$ be the rational curve over $\mathbb{C}$ defined by the parametrization

$$\mathcal{P}(t) = (p_1(t), p_2(t)) = \left(\frac{p_{1,1}(t)}{p(t)}, \frac{p_{2,1}(t)}{p(t)}\right) = \left(\frac{t^4}{(t^2+1)t^3}, \frac{(1+3t^3)(t^2+1)}{(t^2+1)t^3}\right).$$

First, we apply Theorem 2 to compute $\deg(\phi_{\mathcal{P}})$. We get that $\mathcal{P}$ is proper, that is $\deg(\phi_{\mathcal{P}}) = 1$. In addition, by Theorem 3, we get that

$$\deg_{x_2}(F) = \frac{\deg(p_1(t))}{\deg(\phi_{\mathcal{P}})} = 2, \qquad \deg_{x_1}(F) = \frac{\deg(p_2(t))}{\deg(\phi_{\mathcal{P}})} = 3.$$

Now, we consider $P = (0, 0)$, and we observe that $P \notin \mathcal{C}$ (see Remark 2, and note that $\gcd(t, 1 + 3t^3) = 1$, and P is not the critical point). Thus, we apply Theorem 6, and we get that

$$\deg(\mathcal{C}) = \frac{\deg\left(\frac{(1+3t^3)(t^2+1)}{t^4}\right)}{\deg(\phi_{\mathcal{P}})} = 5.$$

Now, we apply Theorem 7. For this purpose, we consider $a = 1/2$, and $c_i = i$, for $i = 1, \ldots, 6$. Then,

$$\deg(\mathcal{C}) = \frac{\max_{1\leq i\leq 6}\left\{\deg\left(\frac{(1+3t^3)(t^2+1)-i(t^2+1)t^3}{t^4-1/2(t^2+1)t^3}\right)\right\}}{\deg(\phi_{\mathcal{P}})} = 5.$$

4. Computation of the multiplicity of a point from a rational parametrization

In this section, we compute the multiplicity of a given point (see Definition 1), directly from a given rational parametrization of a plane curve. A direct approach to this problem could consist in implicitizing the parametrization (see González-Vega (1997), Marco and Martínez (2001), Sederberg et al. (1997) or Sendra and Winkler (2001)) to apply afterwards algorithms developed for instance in Fulton (1989), Harris (1995), Sakkalis and Farouki (1990), Sendra and Winkler (1991) or Walker (1950), to the implicit equation. This solution might be too time-consuming and then, we would like to approach the problem without implicitizing.

For this purpose, we first observe that (0, 0) is a point of multiplicity ℓ on $\mathcal{C}$, if and only if

$$F^{\star}(x_1, x_2, x_3) = F_d(x_1, x_2) + F_{d-1}(x_1, x_2)x_3 + \cdots + F_{\ell}(x_1, x_2)x_3^{d-\ell}.$$

That is, $d - \ell = \deg_{x_3}(F^{\star})$, where $d = \deg(\mathcal{C})$. In this section, we generalize this result to any point, affine or projective. We start with the affine case.

We recall that $\mathcal{C}$ is not a line. Observe that in this case, the problem is trivial since all the points in a line are simple points.

Theorem 8. *Let $(a, b) \in K^2$. Then,*

$$\mathrm{mult}_{(a:b:1)}(\mathcal{C}^{\star}) = \mathrm{mult}_{(a,b)}(\mathcal{C}) = \deg(\mathcal{C}) - \frac{\deg\left(\frac{p_{2,1}(t)-bp(t)}{p_{1,1}(t)-ap(t)}\right)}{\deg(\phi_{\mathcal{P}})}.$$

Proof. Let $\mathcal{D}$ be the curve defined by the polynomial $G(x_1, x_2) = F(x_1 + a, x_2 + b)$. Observe that

$$\mathcal{Q}(t) = \left(\frac{p_{1,1}(t) - ap(t)}{p(t)}, \frac{p_{2,1}(t) - bp(t)}{p(t)}\right)$$

parametrizes $\mathcal{D}$. We write

$$G^{\star}(x_1, x_2, x_3) = G_d(x_1, x_2) + G_{d-1}(x_1, x_2)x_3 + \cdots + G_0(x_1, x_2)x_3^d,$$

and since $\mathrm{mult}_{(a,b)}(\mathcal{C}) = \mathrm{mult}_{(0,0)}(\mathcal{D})$, we deduce that

$$\deg(\mathcal{D}) - \mathrm{mult}_{(a,b)}(\mathcal{C}) = \deg_{x_3}(G^{\star}(x_1, x_2, x_3)) = \deg_{x_3}(G^{\star}(1, x_2, x_3)).$$

Now, we consider the parametrization $\mathcal{Q}_{x_1}(t)$ of the curve $\mathcal{D}_{x_1}$ (see Section 2), and we apply Theorem 3 (one reasons similarly if one considers $\mathcal{Q}_{x_2}$). We get that

$$\deg_{x_3}(G^{\star}(1, x_2, x_3)) = \frac{\deg\left(\frac{p_{2,1}(t)-bp(t)}{p_{1,1}(t)-ap(t)}\right)}{\deg(\phi_{\mathcal{Q}_{x_1}})}.$$

Reasoning as in the proof of Theorem 6, we get that $\deg(\phi_{\mathcal{Q}_{x_1}}) = \deg(\phi_{\mathcal{P}})$, and $\deg(\mathcal{D}) = \deg(\mathcal{C})$. □

In the following theorem, we deal with the problem for the projective case.

Theorem 9. *It holds that*

$$\mathrm{mult}_{(1:k:0)}(\mathcal{C}^{\star}) = \deg(\mathcal{C}) - \frac{\deg\left(\frac{p(t)}{p_{2,1}(t)-kp_{1,1}(t)}\right)}{\deg(\phi_{\mathcal{P}})},$$

$$\mathrm{mult}_{(0:1:0)}(\mathcal{C}^{\star}) = \deg(\mathcal{C}) - \frac{\deg\left(\frac{p(t)}{p_{1,1}(t)}\right)}{\deg(\phi_{\mathcal{P}})}.$$

Proof. First, we prove the formula for the point $(1 : k : 0)$. For this purpose, we consider the curve $\mathcal{C}_{x_1}$ defined by the polynomial $F^{\star}(1, x_2, x_3)$, and by the parametrization $\mathcal{P}_{x_1}(t)$ (see Section 2). By applying Theorem 8, we get that

$$\mathrm{mult}_{(1:k:0)}(\mathcal{C}^{\star}) = \mathrm{mult}_{(k,0)}(\mathcal{C}_{x_1}) = \deg(\mathcal{C}_{x_1}) - \frac{\deg\left(\frac{p(t)}{p_{2,1}(t)-kp_{1,1}(t)}\right)}{\deg(\phi_{\mathcal{P}_{x_1}})}.$$

Observe that since the degree is multiplicative with respect to composition, and taking into account that $\mathcal{P}_{x_1} = R \circ \mathcal{P}$, where $R = (y/x, 1/x)$, we deduce that $\deg(\phi_{\mathcal{P}_{x_1}}) = \deg(\phi_R) \cdot \deg(\phi_{\mathcal{P}})$. Clearly $\deg(\phi_R) = 1$, and then $\deg(\phi_{\mathcal{P}_{x_1}}) = \deg(\phi_{\mathcal{P}})$. Finally, we prove that $\deg(\mathcal{C}_{x_1}) = \deg(\mathcal{C})$. Indeed, since

$$F^\star(x_1, x_2, x_3) = F_d(x_1, x_2) + F_{d-1}(x_1, x_2)x_3 + \cdots + F_0(x_1, x_2)x_3^d, \quad F_d \neq 0,$$

if $\deg(\mathcal{C}_{x_1}) \neq \deg(\mathcal{C})$, then $\deg_{x_2}(F_k) < k$, for $k = 1, \ldots, d$, and $F_0 = 0$. Taking into account that F_k is an homogeneous polynomial, we get that x_1 divides F_k, for $k = 0, \ldots, d$ which is impossible since F is irreducible.

For the point $(0 : 1 : 0)$, we reason similarly to above with the curve $\mathcal{C}_{x_2}$ defined by $F^\star(x_1, 1, x_3)$, and the parametrization $\mathcal{P}_{x_2}(t)$ (see Section 2). □

In the following, we illustrate Theorems 8 and 9 with an example.

Example 2. Let $\mathcal{C}$ be the rational curve over $\mathbb{C}$ defined by the parametrization $\mathcal{P}$ introduced in Example 1. We showed that $\deg(\phi_{\mathcal{P}}) = 1$, and $\deg(\mathcal{C}) = 5$. Now, we consider the point $P = (a, b) = (1, 2)$, and we apply Theorem 8. We get that

$$\begin{aligned}\operatorname{mult}_P(\mathcal{C}) &= \deg(\mathcal{C}) - \frac{\deg\left(\frac{p_{2,1}(t)-bp(t)}{p_{1,1}(t)-ap(t)}\right)}{\deg(\phi_{\mathcal{P}})} \\ &= 5 - \deg\left(\frac{(1+3t^3)(t^2+1) - 2(t^2+1)t^3}{t^4 - (t^2+1)t^3}\right) = 2.\end{aligned}$$

Now, let $P = (1 : k : 0) = (1 : 0 : 0)$, and we apply Theorem 9, statement (1). Then,

$$\operatorname{mult}_P(\mathcal{C}^\star) = \deg(\mathcal{C}) - \frac{\deg\left(\frac{p(t)}{p_{2,1}(t)-kp_{1,1}(t)}\right)}{\deg(\phi_{\mathcal{P}})} = 5 - \deg\left(\frac{(t^2+1)t^3}{(1+3t^3)(t^2+1)}\right) = 2.$$

Finally, let $P = (0 : 1 : 0)$, and we apply Theorem 9, statement (2). We get that

$$\operatorname{mult}_P(\mathcal{C}^\star) = \deg(\mathcal{C}) - \frac{\deg\left(\frac{p(t)}{p_{1,1}(t)}\right)}{\deg(\phi_{\mathcal{P}})} = 5 - \deg\left(\frac{(t^2+1)t^3}{t^4}\right) = 3.$$

5. Computation of singularities from a rational parametrization

In this section, we present a method for computing the singularities of a rational plane curve (not being a line) directly from a given rational parametrization; that is, without implicitizing. This new approach generalizes the results in Abhyankar (1990), van den Essen and Yu (1997), Gutierrez et al. (2002) and Park (2002) in the sense that we describe the singularities and their multiplicities without introducing algebraic numbers in the computations.

We start with a theorem that analyzes when a point not being at infinity, and generated by the given parametrization, is a simple point. For this purpose, in the following we consider the polynomials

$$T(s) = \operatorname{Resultant}_t(G_1^{\mathcal{P}}/G^{\mathcal{P}},\ G_2^{\mathcal{P}}/G^{\mathcal{P}}), \quad \text{and} \quad M(s) = \gcd(\operatorname{lc}(G_1^{\mathcal{P}}, t),\ \operatorname{lc}(G_2^{\mathcal{P}}, t))$$

introduced in Theorem 2.

Theorem 10. *Let $s_0 \in K$ be such that $T(s_0)M(s_0)p(s_0) \neq 0$. Let $P = (p_{1,1}(s_0) : p_{2,1}(s_0) : p(s_0)) \in \mathbb{P}^2$. Then, $\mathrm{mult}_P(\mathcal{C}^\star) = 1$.*

Proof. First, note that from Theorem 2, we get that $\deg(G^{\mathcal{P}}(s_0,t)) = \deg(\phi_{\mathcal{P}})$. In addition, taking into account the behavior of the gcd's under a homomorphism (see e.g. Lemma 3 in Sendra and Winkler (2001)), we deduce that

$$\begin{aligned} G^{\mathcal{P}}(s_0,t) &= \gcd(G_1^{\mathcal{P}}(s_0,t),\ G_2^{\mathcal{P}}(s_0,t)) \\ &= \gcd(p_{1,1}(t)p(s_0) - p_{1,1}(s_0)p(t),\ p_{2,1}(t)p(s_0) - p_{2,1}(s_0)p(t)), \end{aligned}$$

and then

$$\deg(\gcd(p_{1,1}(t)p(s_0) - p_{1,1}(s_0)p(t),\ p_{2,1}(t)p(s_0) - p_{2,1}(s_0)p(t))) = \deg(\phi_{\mathcal{P}}) \qquad \text{(I)}.$$

In these conditions, we consider $(a,b) \notin \mathcal{C}$, and we observe that $\gcd(p_{1,1}(t) - ap(t),\ p_{2,1}(t) - bp(t)) = 1$. Thus, using (I), we deduce that

$$\deg\left(\frac{p_{2,1}(t)p(s_0) - p_{2,1}(s_0)p(t)}{p_{1,1}(t)p(s_0) - p_{1,1}(s_0)p(t)}\right) = \deg\left(\frac{p_{2,1}(t) - bp(t)}{p_{1,1}(t) - ap(t)}\right) - \deg(\phi_{\mathcal{P}}).$$

Theorem 6 implies that

$$\deg\left(\frac{p_{2,1}(t) - bp(t)}{p_{1,1}(t) - ap(t)}\right) - \deg(\phi_{\mathcal{P}}) = \deg(\mathcal{C})\deg(\phi_{\mathcal{P}}) - \deg(\phi_{\mathcal{P}}),$$

and then, from Theorem 8, we get that $\mathrm{mult}_P(\mathcal{C}^\star) = 1$. □

The previous theorem provides a sufficient condition for a point not being at infinity, and generated by the parametrization $\mathcal{P}^\star$, is a simple point. Additionally, one also has to analyze points at infinity generated by $\mathcal{P}^\star$, and the critical point (if it exists). Therefore, from Theorem 10, and taking into account Theorem 2, we deduce the following result.

Theorem 11. *If $P = (a_1 : a_2 : a_3) \in \mathbb{P}^2$ is a singularity, then one of the following statements holds:*

1. *Let $i \in \{1,2,3\}$ be such that $a_i \neq 0$. Then $(a_j/a_i, a_k/a_i)$, where $j,k \in \{1,2,3\}$, and $j \neq i \neq k$, is the critical point of the parametrization $\mathcal{P}_{x_i}$, in reduced form.*
2. *$P = \mathcal{P}^\star(\alpha)$, where $p(\alpha) = 0$.*
3. *$P = \mathcal{P}^\star(\alpha)$, where $T(\alpha) = 0$.*
4. *$P = \mathcal{P}^\star(\alpha)$, where $M(\alpha) = 0$.* □

Once the singularity is determined, one computes its multiplicity by applying Theorem 8 or 9. We illustrate this process with the following example.

Example 3. Let $\mathcal{C}$ be the rational curve over $\mathbb{C}$ defined by the parametrization $\mathcal{P}$ introduced in Example 1. We proved that $\deg(\phi_{\mathcal{P}}) = 1$, and $\deg(\mathcal{C}) = 5$ (see Example 1), and we computed the multiplicity of some given points (see Example 2). Now, we deal with the problem of computing the singularities. For this purpose, we apply Theorem 11. First, we analyze whether the parametrizations

$$\mathcal{P}_{x_3}(t) = \mathcal{P}(t) = \left(\frac{t}{t^2+1}, \frac{1+3t^3}{t^3}\right),$$

$$\mathcal{P}_{x_1}(t) = \left(\frac{(1+3t^3)(t^2+1)}{t^4}, \frac{t^2+1}{t}\right), \qquad \mathcal{P}_{x_2}(t) = \left(\frac{t^4}{(1+3t^3)(t^2+1)}, \frac{t^3}{1+3t^3}\right)$$

have a critical point (see Theorem 5). We get that $(0, 3)$ is the critical point of $\mathcal{P}_{x_3}$. Thus, let $P_1 = (0 : 3 : 1)$. Now, we determine the polynomials

$$G_1^{\mathcal{P}} = -ts^2 - t + st^2 + s, \quad G_2^{\mathcal{P}} = -s^3 + t^3, \quad G^{\mathcal{P}} = t - s,$$

$$T(s) = \text{Resultant}_t\left(\frac{G_1^{\mathcal{P}}}{G^{\mathcal{P}}}, \frac{G_2^{\mathcal{P}}}{G^{\mathcal{P}}}\right) = s^4 + s^2 + 1,$$

$$M(s) = \gcd(\text{lc}(G_1^{\mathcal{P}}, t), \text{lc}(G_2^{\mathcal{P}}, t)) = 1.$$

For each root α of the polynomials T and $p(t) = (t^2 + 1)t^3$, we compute $\mathcal{P}^\star(\alpha)$. We get the points

$$P_2 = (0 : 1 : 0), \qquad P_3 = (-1 : 4 : 1), \qquad P_4 = (1 : 0 : 0), \qquad P_5 = (1 : 2 : 1).$$

In Example 2, we showed that $\text{mult}_{P_4}(\mathcal{C}^\star) = \text{mult}_{P_5}(\mathcal{C}^\star) = 2$, and $\text{mult}_{P_2}(\mathcal{C}^\star) = 3$. By applying Theorem 8, we get that $\text{mult}_{P_3}(\mathcal{C}^\star) = 2$, and $\text{mult}_{P_1}(\mathcal{C}^\star) = 1$. Therefore, the points P_2, P_3, P_4, and P_5 are the singular points of the curve $\mathcal{C}$.

In a direct method, in order to compute the singularities, one would introduce algebraic numbers during the computations. However, in the following we present a method based on a notion that generalizes the concept of a family of conjugate points (see Sendra and Winkler (1991) or Sendra and Winkler (1997)). This new notion allows us to determine the singularities of a curve without directly introducing algebraic numbers in the computations. For this purpose, we recall that $\mathbb{K}$ is a computable field of characteristic zero containing the coefficients of the defining polynomial of $\mathcal{C}$, and K is its algebraic closure. Furthermore, $\mathbb{L}$ is a subfield of K such that $\mathbb{K} \subset \mathbb{L} \subset K$.

We adapt the notion of a *family of conjugate points* to the parametric case. The idea is to collect points whose coordinates depend algebraically on all conjugate roots of the same polynomial $m(t)$. This will imply that the computations on such families can be carried out by using the defining polynomial $m(t)$ of these algebraic numbers.

Definition 12. Let

$$\mathcal{F} = \{(q_1(\alpha) : q_2(\alpha) : q_3(\alpha)) \mid m(\alpha) = 0\} \subset \mathbb{P}^2.$$

The set $\mathcal{F}$ is called a *family of conjugate parametric points* over $\mathbb{L}$ if the following conditions are satisfied:

(1) $q_1, q_2, q_3, m \in \mathbb{L}[t]$, and $\gcd(q_1, q_2, q_3) = 1$.
(2) m is square-free.
(3) $\deg(q_i) < \deg(m)$, for $i = 1, 2, 3$.

We denote such a family by $\{(q_1(s) : q_2(s) : q_3(s))\}_{m(s)}$.

In the following lemma, we analyze the different points of $\mathbb{P}^2$ containing a family of conjugate parametric points.

Lemma 13. *Let $\mathcal{F} = \{\,(q_1(s) : q_2(s) : q_3(s))\,\}_{m(s)}$ be a family of conjugate parametric points over $\mathbb{L}$. Let*

$$H_{\mathcal{F}}(x_1, x_2, x_3) = \text{sqrfree}(\text{Resultant}_s(q_1(s)x_1 + q_2(s)x_2 + q_3(s)x_3,\ m(s))),$$

where sqrfree *denotes the square-free part of* $\text{Resultant}_s(q_1(s)x_1 + q_2(s)x_2 + q_3(s)x_3,\ m(s))$. *Then $\mathcal{F}$ contains* $\text{tdeg}(H_{\mathcal{F}})$ *different points of $\mathbb{P}^2$.*

Proof. Taking into account the properties of the resultants (see Brieskorn and Knoerrer (1986)), we deduce that up to constants in K,

$$\begin{aligned}&\text{Resultant}_s(q_1(s)x_1 + q_2(s)x_2 + q_3(s)x_3,\ m(s))\\ &\quad= \prod_{\{\alpha\in K \mid m(\alpha)=0\}} (q_1(\alpha)x_1 + q_2(\alpha)x_2 + q_3(\alpha)x_3).\end{aligned}$$

Observe that by the properties of the resultants (see Brieskorn and Knoerrer (1986)), we have that $H_{\mathcal{F}}$ is an homogeneous polynomial in $\mathbb{L}[x_1, x_2, x_3]$. Furthermore, the above equality immediately yields that $\mathcal{F}$ contains $\text{tdeg}(H_{\mathcal{F}})$ different points of $\mathbb{P}^2$. □

Remark 3. Definition 12 generalizes the usual notion of a family of conjugate points (see e.g. Sendra and Winkler (1997)). More precisely, conditions (1), (2) and (3) are the same. However, a family of conjugate points contains exactly $\deg(m)$ different points of $\mathbb{P}^2$.

In the following, we state a definition and some properties concerning a family of conjugate parametric points. These properties are essentially the same as the properties satisfied by a family of conjugate points. We recall that these two notions only differ in the cardinality of the set.

Definition 14. We say that a family $\mathcal{F}$ of conjugate parametric points over $\mathbb{L}$ is a family of conjugate ℓ-fold parametric points on $\mathcal{C}^\star$ over $\mathbb{L}$ iff $\text{mult}_P(\mathcal{C}) = \ell$ for all $P \in \mathcal{F}$.

Taking into account Theorems 8 and 9, and Definition 14, one gets the following lemma.

Lemma 15. *Let $\mathcal{F} = \{\,(q_1(s) : q_2(s) : q_3(s))\,\}_{m(s)}$ be a family of conjugate parametric points over $\mathbb{L}$. The following statements are equivalent:*

1. *$\mathcal{F}$ is a family of conjugate ℓ-fold parametric points on $\mathcal{C}^\star$ over $\mathbb{L}$.*
2. *ℓ is the greatest non-negative integer such that all partial derivatives of $F^\star$ of order less than ℓ vanish at $(q_1(s) : q_2(s) : q_3(s))$ modulo $m(s)$.*
3. *It holds that:*
   3.1. *If $q_3 \neq 0$, then*
   $$(\deg(\mathcal{C}) - \ell)\deg(\phi_{\mathcal{P}}) = \deg_t\left(\frac{p_{2,1}(t)q_3(s) - q_2(s)p(t)}{p_{1,1}(t)q_3(s) - q_1(s)p(t)}\right) \mod m(s).$$
   3.2. *If $q_1 \neq 0$ and $q_3 = 0$, then*
   $$(\deg(\mathcal{C}) - \ell)\deg(\phi_{\mathcal{P}}) = \deg_t\left(\frac{p(t)}{p_{2,1}(t)q_1(s) - q_2(s)p_{1,1}(t)}\right) \mod m(s).$$
   3.3. *If $q_1 = q_3 = 0$, then*
   $$(\deg(\mathcal{C}) - \ell)\deg(\phi_{\mathcal{P}}) = \deg\left(\frac{p(t)}{p_{1,1}(t)}\right). \quad \square$$

Now, we analyze whether singularities can be structured in families of conjugate parametric points.

Theorem 16. *The singularities of the projective curve $\mathcal{C}^\star$ can be decomposed as a finite union of families of conjugate parametric points over $\mathbb{L}$ such that all points in the same family have the same multiplicity and character.*

Proof. Taking into account Remark 3, and the properties of families of conjugate points (see Sendra and Winkler (1991) and Sendra and Winkler (1997)), we have that if $\mathcal{F} = \{(q_1(s) : q_2(s) : q_3(s))\}_{m(s)}$ is a family of conjugate ℓ-fold parametric points of $\mathcal{C}^\star$ over $\mathbb{L}$, and $m(s)$ is irreducible over $\mathbb{L}$, then all points in $\mathcal{F}$ have the same character (see Section 2). Thus, it is enough to prove that singularities can be distributed in conjugate families over $\mathbb{L}$ of the same multiplicity. For each irreducible factor $m(s)$ of the square-free part of $T(s)M(s)p(s) \in \mathbb{L}[s]$ over $\mathbb{L}$, we consider $\mathcal{F} := \{(q_1(s) : q_2(s) : q_3(s))\}_{m(s)}$, where $q_i(s)$ is the remainder of $p_{i,1}(s)$ modulo $m(s)$, and $q_3(s)$ is the remainder of $p(s)$ modulo $m(s)$. Thus, by Theorem 11 and Lemma 15, we conclude that $\mathcal{F}$ is a family of conjugate ℓ-fold parametric points on $\mathcal{C}^\star$ over $\mathbb{L}$. □

Algorithm and example

Results obtained in this section and in Section 4 allow us to compute the singularities of a given curve defined by a parametrization, and their multiplicities without computing the implicit equation and without introducing algebraic numbers. In particular, the ideas described in the proof of Theorem 16 immediately yield the following algorithm.

Algorithm Parametric-Decomposition-Singularities.
Given the parametrization $\mathcal{P}(t) = \left(p_{1,1}(t)/p(t),\ p_{2,1}(t)/p(t)\right) \in \mathbb{L}(t)^2$ of a non-linear irreducible curve $\mathcal{C}$, the algorithm computes a set $\mathcal{S}$ containing the decomposition of singularities of $\mathcal{C}^\star$.

1. Compute the polynomials
$$G_i^{\mathcal{P}}(s,t) = p_{i,1}(t)\,p(s) - p_{i,1}(s)\,p(t),\ i \in \{1,2\}, \quad G^{\mathcal{P}}(s,t) = \gcd(G_1^{\mathcal{P}}, G_2^{\mathcal{P}}),$$
$$M(s) = \gcd(\mathrm{lc}(G_1^{\mathcal{P}},t), \mathrm{lc}(G_2^{\mathcal{P}},t)), \quad \text{and} \quad T(s) = \mathrm{Resultant}_t(G_1^{\mathcal{P}}/G^{\mathcal{P}}, G_2^{\mathcal{P}}/G^{\mathcal{P}}).$$
Set $\mathcal{S} = \emptyset$.
2. Let $\deg(\phi_{\mathcal{P}}) = \deg_t(G^{\mathcal{P}})$. Apply Theorem 6 or 7 to compute $\deg(\mathcal{C})$.
3. For every irreducible factor $m(s)$ of the square-free part of $T(s)M(s)p(s)$ over $\mathbb{L}$ consider the family $\mathcal{F} := \{(q_1(s) : q_2(s) : q_3(s))\}_{m(s)}$, where $q_i(s)$ is the remainder of $p_{i,1}(s)$ modulo $m(s)$, and $q_3(s)$ is the remainder of $p(s)$ modulo $m(s)$. Compute ℓ such that:
   3.1. If $q_3 \neq 0$, then
   $$(\deg(\mathcal{C}) - \ell)\deg(\phi_{\mathcal{P}}) = \deg_t\left(\frac{p_{2,1}(t)q_3(s) - q_2(s)p(t)}{p_{1,1}(t)q_3(s) - q_1(s)p(t)}\right) \mod m(s).$$
   3.2. If $q_1 \neq 0$ and $q_3 = 0$, then
   $$(\deg(\mathcal{C}) - \ell)\deg(\phi_{\mathcal{P}}) = \deg_t\left(\frac{p(t)}{p_{2,1}(t)q_1(s) - q_2(s)p_{1,1}(t)}\right) \mod m(s).$$
   3.3. If $q_1 = q_3 = 0$, then
   $$(\deg(\mathcal{C}) - \ell)\deg(\phi_{\mathcal{P}}) = \deg\left(\frac{p(t)}{p_{1,1}(t)}\right).$$
   (see Theorems 8 and 9). If $\ell > 1$ do $\mathcal{S} = \mathcal{S} \cup \{(q_1(s) : q_2(s) : q_3(s))\}_{m(s)}$.
4. Check whether some of the parametrizations $\mathcal{P}_{x_i}$, $i = 1, 2, 3$, in reduced form, have a critical point that is a singularity (see Theorem 11, and apply Theorem 8 or 9). In the affirmative case construct $\mathcal{S} = \mathcal{S} \cup P$, where P is the critical point in projective coordinates.
5. Return $\mathcal{S}$.

Remark 4. In order to determine the different points containing every family $\mathcal{F}$, one applies Lemma 13.

In the following, we illustrate Algorithm Parametric-Decomposition-Singularities with an example.

Example 4. Let $\mathcal{C}$ be the rational curve over $\mathbb{C}$ defined by the parametrization

$$\mathcal{P}(t) = \left(\frac{-50t^3 - 12t^2 - 18t + 31}{56t^5 + 49t^4 + 63t^3 + 57t^2 - 59t + 46}, \frac{-55t + 25}{56t^5 + 49t^4 + 63t^3 + 57t^2 - 59t + 46}\right).$$

Let us apply the previous algorithm to compute the singularities and their corresponding multiplicities for the curve $\mathcal{C}$. For this purpose, first we determine the polynomials

$$\begin{aligned}
G_1^{\mathcal{P}} &= 1001t - 1001s - 1519t^4 - 2319t^2 - 4253t^3 + 2319s^2 - 1734ts^2 + 1734st^2 \\
&\quad + 4253s^3 + 1519s^4 - 1736t^5 - 2094t^3s^2 + 4084t^3s + 1736s^5 - 672t^2s^5 \\
&\quad + 2094t^2s^3 - 1008ts^5 - 882ts^4 - 4084ts^3 + 2800t^5s^3 + 672t^5s^2 + 1008t^5s \\
&\quad + 2450t^4s^3 + 588t^4s^2 + 882t^4s - 588t^2s^4 - 2800t^3s^5 - 2450t^3s^4, \\
G_2^{\mathcal{P}} &= -3080ts^5 - 2695ts^4 - 3465ts^3 - 3135ts^2 - 1055t + 1400s^5 + 1225s^4 \\
&\quad + 1575s^3 + 1425s^2 + 1055s + 3080t^5s - 1400t^5 + 2695t^4s \\
&\quad - 1225t^4 + 3465t^3s - 1575t^3 + 3135st^2 - 1425t^2, \\
G^{\mathcal{P}} &= t - s, \qquad M(s) = 1, \\
T &= 2672\,999\,980\,000s^8 + 2980\,394\,977\,700s^7 - 60\,064\,612\,566\,476s^6 \\
&\quad - 25\,004\,640\,526\,824s^5 - 286\,359\,608\,760\,660s^4 + 31\,132\,294\,531\,008s^3 \\
&\quad + 109\,487\,085\,267\,360s^2 + 41\,560\,865\,358\,829s - 32\,588\,733\,598\,663.
\end{aligned}$$

From Theorem 2, we obtain that $\deg(\phi_{\mathcal{P}}) = 1$, and by Theorem 6 we get that $\deg(\mathcal{C}) = 5$. Now, we apply Step 3 of the algorithm. For this purpose, let

$$m_1(s) = p(s) = 56s^5 + 49s^4 + 63s^3 + 57s^2 - 59s + 46, \quad \text{and} \quad m_2(s) = T(s).$$

Now, we consider the family $\mathcal{F}_1 := \{(q_1(s) : q_2(s) : q_3(s))\}_{m_1(s)}$, where $q_i(s)$ is the remainder of $p_{i,1}(s)$ modulo $m_1(s)$, and $q_3(s)$ is the remainder of $p(s)$ modulo $m_1(s)$. We get

$$\mathcal{F}_1 := \{(-50s^3 - 12s^2 - 18s + 31 : -55s + 25 : 0)\}_{m_1(s)}.$$

Since $q_3 = 0$, and $q_1 \neq 0$, we apply Step 3.2, and we compute ℓ such that

$$(\deg(\mathcal{C}) - \ell)\deg(\phi_{\mathcal{P}}) = \deg_t\left(\frac{p(t)}{p_{2,1}(t)q_1(s) - q_2(s)p_{1,1}(t)}\right) \mod m_1(s).$$

We obtain that $\ell = 1$, and then $\mathcal{F}_1$ is a family of simple points.

Now, we reason similarly for $m_2(s)$, and we get the family

$$\mathcal{F}_2 := \{(p_{1,1}(s) : p_{2,1}(s) : p(s))\}_{m_2(s)}.$$

Since $q_3 \neq 0$, we apply Step 3.1, and we compute ℓ such that

$$(\deg(\mathcal{C}) - \ell)\deg(\phi_{\mathcal{P}}) = \deg_t\left(\frac{p_{2,1}(t)q_3(s) - q_2(s)p(t)}{p_{1,1}(t)q_3(s) - q_1(s)p(t)}\right) \mod m_2(s).$$

We get that $\ell = 2$, and then $\mathcal{F}_2$ is a family of double points. In fact, by applying Lemma 13, we get that $\mathcal{F}_2$ contains exactly $\mathrm{tdeg}(H_{\mathcal{F}_2}) = 4$ different double points of $\mathbb{P}^2$. Hence,

$$\mathcal{S} = \{(p_{1,1}(s) : p_{2,1}(s) : p(s))\}_{m_2(s)}.$$

Finally, in Step 4 of the algorithm, we check whether some of the parametrizations $\mathcal{P}_{x_i}$, $i = 1, 2, 3$, in reduced form, have a critical point that is a singularity. By applying Theorem 5, we get that (0, 0) is the critical point of the parametrization $\mathcal{P}_{x_3} = \mathcal{P}$. From Theorem 8, we have that $\mathrm{mult}_{(0:0:1)}(\mathcal{C}^\star) = 2$. Therefore,

$$\mathcal{S} = \{(p_{1,1}(s) : p_{2,1}(s) : p(s))\}_{m_2(s)} \cup (0 : 0 : 1).$$

6. Analysis of non-ordinary singularities

Non-ordinary singularities have to be treated specially since a non-ordinary singularity might have other singularities in its "neighborhood". The analysis of such neighborhoods is the topic of the field of resolution of singularities (see e.g. Zariski (1939)). Here we treat the determination of the so-called neighboring singularities and their multiplicities.

In order to check whether a singularity is ordinary or not, one has to analyze the tangents (see Section 2). For this purpose, in the following, we consider the polynomial

$$H^{\mathcal{P}}(t) = \gcd(G_1^{\mathcal{P}}(s_0, t),\ G_2^{\mathcal{P}}(s_0, t)), \quad \text{where } s_0 \in K, \text{ and}$$
$$G_i^{\mathcal{P}}(s_0, t) = q_{i,1}(t)\, q_{i,2}(s_0) - q_{i,1}(s_0)\, q_{i,2}(t) \in K[t], \quad i \in \{1, 2\}$$

(see Section 2). We remark that if $T(s_0)M(s_0) \neq 0$, then $H^{\mathcal{P}}(t) = G^{\mathcal{P}}(s_0, t)$ (see Lemma 3 in Sendra and Winkler (2001)).

In the next theorem, we show that the tangents of an affine point generated from the given parametrization can be computed by means of a gcd.

Theorem 17. *Let $s_0 \in K$ be such that $p(s_0) \neq 0$, and $P = (p_{1,1}(s_0) : p_{2,1}(s_0) : p(s_0)) \in \mathbb{P}^2$. Let*

$$H^{\mathcal{P}}(t) = \prod_{j=1}^{n} (b_j t - a_j)^{m_j}, \quad a_j,\ b_j \in K,\ b_j \neq 0.$$

Then, the tangents of $\mathcal{C}$ at P are n lines over K each of multiplicity m_j, $j = 1, \ldots, n$.

Proof. First, we note that since $p(s_0) \neq 0$ then, $q_{1,2}(s_0)q_{2,2}(s_0) \neq 0$. Thus, for $i = 1, 2$, we may write

$$G_i^{\mathcal{P}}(s_0, t) = q_{i,2}(s_0)q_{i,2}(t)\left(\frac{q_{i,1}(t)}{q_{i,2}(t)} - \frac{q_{i,1}(s_0)}{q_{i,2}(s_0)}\right) = H^{\mathcal{P}}(t)N_i^{\mathcal{P}}(t), \quad N_i^{\mathcal{P}}(t) \in K[t],$$

and then,

$$\frac{q_{i,1}(t)}{q_{i,2}(t)} - \frac{q_{i,1}(s_0)}{q_{i,2}(s_0)} = H^{\mathcal{P}}(t)\bar{N}_i^{\mathcal{P}}(t), \quad \bar{N}_i^{\mathcal{P}}(t) \in K(t),\ i = 1, 2 \qquad \text{(I)}.$$

Let

$$\ell_{i,k}(t) = \frac{\partial^k (q_{i,1}/q_{i,2})}{\partial^k t}, \quad \text{for } i = 1, 2, \text{ and } k = 1, \ldots, m_j.$$

From (I), we get that

$$\ell_{1,k}(a_j/b_j) = \ell_{2,k}(a_j/b_j) = 0, \quad \text{for } j = 1, \dots, n, \text{ and, } k = 1, \dots, m_j - 1,$$

and

$$\ell_{1,m_j}(a_j/b_j) \cdot \ell_{2,m_j}(a_j/b_j) \neq 0, \quad \text{for } j = 1, \dots, n.$$

Therefore, there exist n places with center at $q := (\frac{q_{1,1}(s_0)}{q_{1,2}(s_0)}, \frac{q_{2,1}(s_0)}{q_{2,2}(s_0)})$, given by

$$\begin{cases} x_1(t) = \frac{q_{1,1}(s_0)}{q_{1,2}(s_0)} + t^{m_j}\ell_{1,m_j}(a_j/b_j) + \cdots \\ x_2(t) = \frac{q_{2,1}(s_0)}{q_{2,2}(s_0)} + t^{m_j}\ell_{2,m_j}(a_j/b_j) + \cdots, \quad j = 1, \dots, n \end{cases}$$

(see Brieskorn and Knoerrer (1986) or Walker (1950)). Since the curves tangent to $\mathcal{C}$ at q consist of the tangents to the places of the curve that are centered at q (see Hoffmann (1989)), we conclude that the tangents to $\mathcal{C}$ at q are n lines defined by

$$\begin{cases} x_1(t) = \frac{q_{1,1}(s_0)}{q_{1,2}(s_0)} + t^{m_j}\ell_{1,m_j}(a_j/b_j) \\ x_2(t) = \frac{q_{2,1}(s_0)}{q_{2,2}(s_0)} + t^{m_j}\ell_{2,m_j}(a_j/b_j), \quad j = 1, \dots, n, \end{cases}$$

or equivalently by

$$\left(\ell_{2,m_j}(a_j/b_j)\left(x_1 - \frac{q_{1,1}(s_0)}{q_{1,2}(s_0)}\right) - \ell_{1,m_j}(a_j/b_j)\left(x_2 - \frac{q_{2,1}(s_0)}{q_{2,2}(s_0)}\right)\right)^{m_j}.$$

Note that the multiplicity of each line, i.e. the multiplicity of each tangent, is m_j. □

From the proof of the previous theorem, and taking into account that $\text{mult}_P(\mathcal{C}^\star) = \sum_{j=1}^{n} m_j$ (see Walker (1950, pp. 113)), we get the next corollary that provides an alternative method for computing the multiplicity of an affine point generated by the parametrization.

Corollary 1. *Let $s_0 \in K$ be such that $p(s_0) \neq 0$, and $P = (p_{1,1}(s_0) : p_{2,1}(s_0) : p(s_0)) \in \mathbb{P}^2$. Then,* $\text{mult}_P(\mathcal{C}^\star) = \deg(H^{\mathcal{P}}(t))$.

Theorem 17 provides a method for computing the tangents to $\mathcal{C}$ at an affine point generated by the parametrization. In order to determine the tangents at critical points, at points at infinity, and at a family of conjugate parametric points, we reason as follows.

(1) Let P be the critical point of $\mathcal{P}(t)$. We consider a change of variable in $\mathcal{P}(t)$ such that P is generated by the new parametrization (note that the tangents of $\mathcal{C}$ at P are invariant under changes of variable in the parametrization). For instance, one may take $\mathcal{Q}(t) = \mathcal{P}(1/(t-a))$, where $a \in \mathbb{L}$ is such that $p(a) \neq 0$. Note that $\mathcal{Q}(a) = P$. Then, we apply Theorem 17 to $\mathcal{Q}(t)$, and $s_0 = a$.
(2) For points at infinity, $P = (a : b : 0)$, we consider the parametrizations $\mathcal{P}_{x_1}$ or $\mathcal{P}_{x_2}$, depending on whether $a \neq 0$ or $b \neq 0$, respectively. Then, we apply Theorem 17 to the new parametrization.
(3) Let $\mathcal{F} = \{(q_1(s) : q_2(s) : q_3(s))\}_{m(s)}$ be a family of conjugate parametric points. We consider a generic point of $\mathcal{F}$, say $P = (q_1(s) : q_2(s) : q_3(s))$, and we compute the polynomial $H^{\mathcal{P}}(t) \in (\mathbb{L}[s])[t]$ modulo $m(s)$. Then, the tangents are given as

$$\left(\ell_{2,m_j}(\lambda)(x_1q_3(s) - q_1(s)) - \ell_{1,m_j}(\lambda)(x_2q_3(s) - q_{2,1}(s))\right)^{m_j}, \quad \text{where } H^{\mathcal{P}}(\lambda) = 0.$$

We recall that all points in $\mathcal{F}$ have the same multiplicity and character (see Theorem 16).

In the following, we illustrate Theorem 17 with an example.

Example 5. Let $\mathcal{C}$ be a rational curve over $\mathbb{C}$ defined by the parametrization

$$\mathcal{P}(t) = \left(\frac{p_{1,1}(t)}{p(t)}, \frac{p_{2,1}(t)}{p(t)}\right) = \left(1 - t^2, t - t^3\right).$$

We apply Algorithm Parametric-Decomposition-Singularities,to compute the singularities and their corresponding multiplicities for the curve $\mathcal{C}$. For this purpose, first we determine the polynomials

$$G_1^{\mathcal{P}}(s,t) = -(t-s)(t+s), \qquad G_2^{\mathcal{P}}(s,t) = -(t-s)(t^2 + st - 1 + s^2),$$
$$G^{\mathcal{P}}(s,t) = t - s, \qquad M(s) = 1, \qquad T(s) = -(s-1)(s+1).$$

From Theorem 2, we obtain that $\deg(\phi_{\mathcal{P}}) = 1$, and by Theorem 6 we get that $\deg(\mathcal{C}) = 3$. Now, we apply Step 3 of the algorithm. For this purpose, let

$$m_1(s) = s - 1, \quad \text{and} \quad m_2(s) = s + 1.$$

In this case, we may compute the roots of the polynomials, and we have that $P := \mathcal{P}(1) = \mathcal{P}(-1) = (0,0)$ is a possible singular point of the curve. Now, we apply Step 3.1 of the algorithm, and we compute ℓ such that

$$(\deg(\mathcal{C}) - \ell)\deg(\phi_{\mathcal{P}}) = \deg_t\left(\frac{p_{2,1}(t)}{p_{1,1}(t)}\right).$$

We obtain that $\ell = 2$, and then P is a double point. In addition, P is the only singular point of $\mathcal{C}$ since in Step 4 of the algorithm, we do not get more points.

Now, we apply Theorem 17 to compute the tangents of the curve $\mathcal{C}$ at P. For this purpose, first we determine the polynomials

$$G_1^{\mathcal{P}}(1,t) = p_{1,1}(t) = 1 - t^2, \qquad G_2^{\mathcal{P}}(1,t) = p_{2,1}(t) = t - t^3,$$
$$H^{\mathcal{P}}(t) = \gcd(G_1^{\mathcal{P}}, G_2^{\mathcal{P}}) = 1 - t^2.$$

Thus, from Theorem 17, we get that the tangents of $\mathcal{C}$ at P are given by two different lines each of multiplicity 1. Thus, P is an ordinary singularity. Moreover, these tangents are defined parametrically by

$$\begin{cases} x_1(t) = 0 + t\,\frac{\partial(p_{1,1}/p)}{\partial t}(1) = -2t \\ x_2(t) = 0 + t\,\frac{\partial(p_{2,1}/p)}{\partial t}(1) = -2t, \end{cases} \qquad \begin{cases} x_1(t) = 0 + t\,\frac{\partial(p_{1,1}/p)}{\partial t}(-1) = 2t \\ x_2(t) = 0 + t\,\frac{\partial(p_{2,1}/p)}{\partial t}(-1) = -2t, \end{cases}$$

and implicitly by the lines: $-x_1 + y_1, \quad x_1 + y_1$.

Finally, we remark that P is a singular point of the curve (one may check that in fact $\mathcal{C}$ is the cubic defined implicitly by $x^3 - x^2 + y^2$), but in parametric form, P does not perform like a singular point. Note that $\frac{\partial(p_{i,1}/p)}{\partial t}(1) = -2$, $i = 1, 2$, and we can define a tangent at $t = 1$ without any problem.

In the following, we treat any non-ordinary singularities. The problem with these singularities is that they have multiple tangents. We will resolve these multiple tangents by "blowing up" the singularity, reasoning similarly to in the case where $\mathcal{C}^\star$ is given implicitly (see Brieskorn and Knoerrer (1986), Fulton (1989) or Walker (1950)). More precisely, we achieve the blow-ups by quadratic transformations of the plane that are special birational maps of the projective plane onto itself (see Fulton (1989)).

In the sequel, we briefly summarize the sequence of quadratic transformations resolving the singularities of a given irreducible curve $\mathcal{C}^\star$ defined implicitly. The method consists in recursively "blowing up" $\mathcal{C}^\star$ at the non-ordinary singularities (see Walker (1950)):

(1) Let P be a non-ordinary singularity of $\mathcal{C}^\star$. Apply a linear change of coordinates, $\mathcal{L}$, such that P is moved to $(0 : 0 : 1)$, none of its tangents is an irregular line (i.e. a line $x_1 = 0, x_2 = 0$ or $x_3 = 0$), and no other point on an irregular line is a singular point.
(2) Apply the quadratic transformation $\mathcal{T} = (x_2x_3 : x_1x_3 : x_1x_2)$ to $\mathcal{C}^\star$, getting the transform curve $\mathcal{D}^\star$. Outside of the irregular lines, this transform preserves the multiplicity of points and their tangents. New ordinary singularities might be created at the points $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$ (called the fundamental points). The new curve $\mathcal{D}^\star$ might have singularities, also non-ordinary ones.
(3) Check whether there exists a point $(a : b : 0) \in \mathcal{D}^\star$, with $ab \neq 0$, that is a non-ordinary singularity. In the affirmative case, apply steps (1) and (2) recursively to $\mathcal{D}^\star$, and this non-ordinary singularity. Otherwise, choose any other non-ordinary singularity of $\mathcal{C}$, and repeat the process until no non-ordinary singularity is left.

This method always achieves an irreducible curve having only ordinary singularities in a finite number of steps (see Fulton (1989)).

In order to describe this blowing-up process in more detail, we introduce the concept of *neighboring points* (see Fulton (1989) or Walker (1950)). Let P be an r-fold point of the irreducible curve $\mathcal{C}^\star$. Assuming that P was moved to $(0 : 0 : 1)$ by the corresponding change of coordinates, the point P is replaced on the transformed curve $\mathcal{D}^\star$ by points $\{P_1', \ldots, P_s'\}$ of multiplicities $\{r_1', \ldots, r_s'\}$, where $P_i' = (a_i : b_i : 0)$, and $a_ib_i \neq 0$. We say that the *first neighborhood* of P is defined by the points $\{P_1', \ldots, P_s'\}$. With this convention, each point of $\mathcal{D}^\star$ is the transform of one or more points of $\mathcal{C}^\star$, including those in the first neighborhood of P. Also, every point of $\mathcal{C}^\star$ is transformed into a point of $\mathcal{D}^\star$ except P, which disappears. This terminology is easily extended to $\mathcal{D}^\star$ and neighborhoods of arbitrarily high order. More precisely, if $\{P_1', \ldots, P_s'\}$ is the first neighborhood of P, we get the second neighborhood of P as the union of the first neighborhoods of P_k', $k = 1, \ldots, s$. The points in the second neighborhood of P are called the *neighboring points of P at its second neighborhood*. The multiplicity and character of points at the second neighborhood are defined in a way analogous to the one for points in the first neighborhood. In general, we will call any point in one of the neighborhoods of P a *neighboring point* of P.

It is proved that there are at most a finite number of singular points in the neighborhoods of any point of an irreducible curve (see Walker (1950, pp. 82)). Hence the analysis of a singularity in terms of neighboring singularities is a finite process, and leads to a complete classification of all singular points.

With this terminology, the next theorem proves that neighboring points can also be structured in families of conjugate parametric points.

Theorem 18. *Let $\mathcal{F}$ be a conjugate family of non-ordinary singularities on $\mathcal{C}^\star$. The singularities at each neighborhood of $\mathcal{F}$ can be decomposed as a finite union of families of conjugate parametric points over $\mathbb{L}$ such that all points in the same family have the same multiplicity and character as neighboring points.*

Proof. Let

$$\mathcal{F} = \{(q_1(s) : q_2(s) : q_3(s))\}_{m(s)},$$

and let $q_s = (q_1(s) : q_2(s) : q_3(s))$ be a generic element of $\mathcal{F}$. We apply a change of projective coordinates $\mathcal{L}_s$, defined over $\mathbb{L}$, such that q_s is moved to $(0 : 0 : 1)$, none of its tangents is an irregular line, and no other point on an irregular line is singular. Let $\mathcal{G}_s$ be the composition of $\mathcal{T} = (x_2x_3 : x_1x_3 : x_1x_2)$ with $\mathcal{L}_s$, and let $\mathcal{Q}^\star(s, t)$ be the defining parametrization of the transformed curve $\mathcal{D}_s^\star$ of $\mathcal{C}^\star$ under $\mathcal{G}_s$; that is $\mathcal{Q}^\star = \mathcal{G}_s(\mathcal{P}^\star)$. Then, the first neighborhood of q_s is $\{(h : 1 : 0)\}_{U(s,h,1)}$. Furthermore, applying Theorem 16, the above family can be decomposed into families of conjugate parametric points over $\mathbb{L}$ such that all elements in the same family have the same multiplicity and character as points in $\mathcal{D}_s^\star$. Thus, the first neighborhood of q_s can be expressed as

$$\bigcup_{i\in I}\{(h : 1 : 0)\}_{m_i(s,h)},$$

where within a family multiplicity and character are the same.

Repeating this process through all levels of neighborhoods and for all families of non-ordinary singularities, one reaches a representation, in families of conjugate parametric points, of the singularities at each neighborhood of $\mathcal{C}^\star$. □

In the following, we illustrate Theorems 17 and 18 with an example.

Example 6. Let $\mathcal{C}$ be a rational curve over $\mathbb{C}$ defined by the parametrization

$$\mathcal{P}(t) = \left(\frac{p_{1,1}(t)}{p(t)}, \frac{p_{2,1}(t)}{p(t)}\right) = \left(\frac{3t^2 + 3t + 1}{-3t - 1 + t^6 - 2t^4}, \frac{t^2(t^4 - 2t^2 + 2)}{-3t - 1 + t^6 - 2t^4}\right).$$

We apply Algorithm Parametric-Decomposition-Singularities, and we get that $P_1 := (-1 : 0 : 1)$ is a singularity of multiplicity 2, $P_2 := (0 : 1 : 1)$ is a singularity of multiplicity 4 (in fact, P_2 is the critical point), and

$$\mathcal{F} := \{(-81(3s^2 + 3s + 1) : -4(35s^2 + 3s + 1) : 11(21s + 7 + 2s^2))\}_{m(s)},$$

where $m(s) = 9s^4 + 6s^3 - 16s^2 - 12s - 2$, is a family of double points. Moreover, by Lemma 13, we get that $\mathcal{F}$ contains exactly 2 different double points of $\mathbb{P}^2$.

Now, we apply Theorem 17 to compute the tangents of the curve $\mathcal{C}$ at the singular points. We start with P_1. For this purpose, first we determine the polynomials

$$G_1^{\mathcal{P}} = p_{1,1}(t) + p(t) = t^2(3 + t^4 - 2t^2), \qquad G_2^{\mathcal{P}} = p_{2,1}(t) = t^2(t^4 - 2t^2 + 2),$$
$$H^{\mathcal{P}}(t) = \gcd(G_1^{\mathcal{P}}, G_2^{\mathcal{P}}) = t^2.$$

Thus, from Theorem 17, we get that the tangents of $\mathcal{C}$ at P_1 are given by one line of multiplicity 2. Thus, P_1 is a non-ordinary singularity. Moreover, these tangents are defined parametrically by

$$\begin{cases} x_1(t) = -1 + t^2\, \frac{\partial^2(p_{1,1}/p)}{\partial^2 t}(0) = -1 - 6t^2 \\ x_2(t) = 0 + t^2\, \frac{\partial^2(p_{2,1}/p)}{\partial^2 t}(0) = -4t^2, \end{cases}$$

and implicitly by $(-2x_1 - 2 + 3x_2)^2$.

In order to compute the tangents at P_2, since it is the critical point, we first consider a change of variable in $\mathcal{P}$ such that P_2 is generated by the new parametrization. For instance, we take $\mathcal{Q}(t) = \mathcal{P}(1/t)$. Note that $\mathcal{Q}(0) = P_2$. Now, reasoning similarly to above, we get that the tangents of $\mathcal{C}$ at P_2 are given by $(-2x_1 - 3 + 3x_2)^4$. Thus, P_2 is a non-ordinary singularity.

Finally, for the family $\mathcal{F}$, we consider a generic point of $\mathcal{F}$, say

$$P_3 := \left(-81(3s^2+3s+1) : -4(35s^2+3s+1) : 11(21s+7+2s^2)\right),$$

and we compute the polynomial $H^{\mathcal{P}}(t)$ modulo $m(s)$. We get that

$$H^{\mathcal{P}}(t) = -177ts^2 + 18t^2s^2 - 59s^2 - 36ts + 189t^2s - 12s + 63t^2 - 4 - 12t.$$

Then, the tangents of $\mathcal{C}$ at P_3 are given by two different lines each of multiplicity 1. Thus, the singular point P_3 is ordinary. Moreover, the tangents are defined by

$$\ell_{2,1}(\lambda)(11x_1(21s+7+2s^2)+81(3s^2+3s+1)) \\ -\ell_{1,1}(\lambda)(11x_2(21s+7+2s^2)+4(35s^2+3s+1)),$$

modulo $H^{\mathcal{P}}(\lambda)$, where $\ell_{i,1} = \frac{\partial(p_{i,1}/p)}{\partial t}$.

Now, we structure the neighboring points in families of conjugate points. For this purpose, we apply the ideas developed in the proof of Theorem 18. We start with the point P_1. We take the change of variables $\mathcal{L}_1 = (x_1 + x_3 : x_2 : x_3)$ such that P_1 is moved to $(0 : 0 : 1)$, none of its tangents is an irregular line, and no other point on an irregular line is singular. Then, we consider the parametrization

$$\mathcal{Q}_1^{\star} = \mathcal{T}(\mathcal{L}_1(\mathcal{P}^{\star})) = (t^2(t^4-2t^2+2)(-3t-1+t^6-2t^4), \\ t^2(3+t^4-2t^2)(-3t-1+t^6-2t^4), t^4(3+t^4-2t^2)(t^4-2t^2+2)),$$

where $\mathcal{T}$ is the quadratic transformation $\mathcal{T} = (x_2x_3 : x_1x_3 : x_1x_2)$. Let

$$\mathcal{Q}_1(t) = \left(\frac{-3t-1+t^6-2t^4}{t^2(3+t^4-2t^2)}, \frac{-3t-1+t^6-2t^4}{t^2(t^4-2t^2+2)}\right).$$

We apply Algorithm Parametric-Decomposition-Singularities, and we get that $Q_1 := (2 : 3 : 0)$ is a singularity of multiplicity 2, the points $Q_2 := (1 : 0 : 0)$, $Q_3 = (0 : 1 : 0)$, $Q_4 = (1 : 1 : 1)$ are singularities of multiplicity 4, $Q_5 := (0 : 0 : 1)$ is a singularity of multiplicity 6, and

$$\mathcal{F} := \{(-22(508+1524s+203s^2) : -11(4749s+1583+568s^2) : \\ 4(7753s^2+1095s+365))\}_{m(s)},$$

where $m(s) = 9s^4 + 6s^3 - 16s^2 - 12s - 2$, is a family of double points. Moreover, by 13, we get that $\mathcal{F}$ contains exactly 2 different double points of $\mathbb{P}^2$.

The first neighborhood of P_1 is given by Q_1 which is an ordinary double point of $\mathcal{Q}_1$ (we apply Theorem 17). Thus, the process finishes with the point P_1.

Now, we reason similarly for P_2. We take the change of variables $\mathcal{L}_2 = (x_1 : x_2 - x_3 : x_3)$, and we consider the parametrization

$$\mathcal{Q}_2^{\star} = \mathcal{T}(\mathcal{L}_2(\mathcal{P}^{\star})) = (t^2(t^4-2t^2+2)(-3t-1+t^6-2t^4), \\ t^2(3+t^4-2t^2)(-3t-1+t^6-2t^4), t^4(3+t^4-2t^2)(t^4-2t^2+2)).$$

Let

$$\mathcal{Q}_2(t) = \left(\frac{-3t-1+t^6-2t^4}{t^2(3+t^4-2t^2)}, \frac{-3t-1+t^6-2t^4}{t^2(t^4-2t^2+2)}\right).$$

We apply Algorithm Parametric-Decomposition-Singularities, and we get that there does not exist a point $(a : b : 0)$ in the new curve, with $ab \neq 0$, that is a singularity. Thus, the first neighborhood of P_2 is given by a simple point, and then the process finishes.

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