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Properness and Inversion of Rational Parametrizations of Surfaces*

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Abstract

In this paper we characterize the properness of rational parametrizations of hypersurfaces by means of the existence of intersection points of some additional algebraic hypersurfaces directly generated from the parametrization over a field of rational functions. More precisely, if V is a hypersurface over an algebraically closed field $\mathbb{K}$ of characteristic zero and $\mathcal{P}(\overline{t}) = \left(\frac{p_1(\overline{t})}{q_1(\overline{t})}, \ldots, \frac{p_n(\overline{t})}{q_n(\overline{t})}\right)$ is a rational parametrization of V, then the characterization is given in terms of the intersection points of the hypersurfaces defined by $x_i q_i(\overline{t}) - p_i(\overline{t})$, $i = 1, \ldots, n$ over the algebraic closure of $\mathbb{K}(V)$. In addition, for the case of surfaces we show how these results can be stated algorithmically. As a consequence we present an algorithmic criteria to decide whether a given rational parametrization is proper. Furthermore, if the parametrization is proper, the algorithm also computes the inverse of the parametrization. Moreover, for surfaces the auxiliary hypersurfaces turn to be plane curves over $\mathbb{K}(V)$, and hence the algorithm is essentially based on resultants. We have implemented these ideas, and we have empirically compared our method with the method based on Gröbner basis.

Keywords: Proper rational parametrization, parametrization inverse, unirrationality.

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Introduction

Unirational algebraic varieties, specially rational curves and surfaces, play an important role in the frame of practical applications (see [10],[11],[12]). Many authors have addressed problems related to the construction of conversion algorithms for these type of varieties; i.e. algorithmic methods that change from the implicit representation to the parametric one, and vice versa (see [5], [8], [9], [11], [14], [15], [18], [21], [22], etc).

In addition, if one considers rational parametrizations as rational mappings from an affine space onto the variety, two natural questions appear. First, deciding whether the mapping is birational (i.e. whether the parametrization is proper); secondly, in case of birationality, the question of computing the inverse of the parametrization is considered. More precisely, let $\mathcal{P}(t_1, \ldots, t_r)$ be a rational parametrization of a variety V over an algebraically closed field $\mathbb{K}$ of characteristic zero. Then $\mathcal{P}$ defines the rational mapping

$$\begin{array}{rccc} \mathcal{P}: & \mathbb{K}^r & \longrightarrow & V. \\ & (t_1, \ldots, t_r) & \longmapsto & \mathcal{P}(t_1, \ldots, t_r) \end{array}$$

In this situation, the following two problems are considered:

(1) Decide whether $\mathcal{P}(t_1, \ldots, t_r)$ is proper, i.e. $\mathcal{P}$ is invertible (properness problem).

(2) If $\mathcal{P}(t_1, \ldots, t_r)$ is proper, compute its rational inverse; i.e a rational mapping assigning to every point, in a non-empty Zariski subset of V, the corresponding parameter value (inversion problem).

Before reporting on the state of the art of these two problems, and before describing the contributions of the paper, we motivate the problem with some applications. Both of these problems are important by their own right. However, for the case of curves and surfaces, they appear in many applications. For instance:

(i) One of the first applications of parametric representations is in computer graphics. However, if the parametrization is not proper the injectivity is lost. This implies that the variety is traced more than once when giving values to the parameters, and this affects the computing real time.

(ii) Computing the intersection of two varieties is also an application of parametrizations, since they allow to reduce the number of variables. But, once more, if the properness of the parametrization is not guaranteed, the degrees of the polynomials in the system of algebraic equations increase. Similarly the injectivity of the parametrization affects the efficiency of methods for line (or surface) integrals, since it provides the optimal degrees before integrating.

(iii) Parametrizations of curves and surfaces arise in the algebraic manipulation of offsets (see [2], [10], [19]), and therefore they are applicable to tolerance analysis, geometric control, robot path-planning, numerical–control machining problems, etc. One of the most relevant problems in this frame is to guarantee the computability of data structures and algorithms, and therefore rational parametrizations of offset varieties are required. In [2] it is shown how to solve this problem by means of the properness analysis of certain hypersurfaces.

On the other hand, inverting rational parametrizations of offsets provides the necessary information on the original variety to reach this point. Note that this is connected to kinematic problem in robotics.

(iv) Most of the parametrizing algorithms, specially for the case of surfaces, output parametrizations over some algebraic extension of $\mathbb{Q}$, not necessarily real. Thus, the feasibility of applications requires parametrization techniques over real algebraic extensions of $\mathbb{Q}$. In [1],[20] this problem is reduced to the Weil's descente variety of the parametrization. Furthermore, parametrizations of the descente variety are generated from the inverse of the parametrization. Thus, producing inverse of parametrizations provides a necessary tool to approach this problem.

(v) Another interesting application of inverses is that they can be used to compute the implicit equation of a variety. More precisely, if $M(\bar{x})$ is the inverse of $\mathcal{P}(\bar{t})$, one can use the fact that $\mathcal{P}(M(\bar{x})) = \bar{x}$ modulo the implicit equation of V.

An algorithmic approach to a more general statement of these two problems (namely, rational maps between algebraic varieties), based on Gröbner Basis, can be found in [16]. We, in this paper, deal with the case of hypersurfaces. For plane curves, the problem is directly related to Lüroth's theorem, that is valid over any field, and different algorithmic procedures to solve the problem can be found in [3], [4], [17], [23], [25]. For surfaces, although it is known from Castelnuovo's theorem that unirationality and rationality are equivalent over algebraically closed fields, algorithmic questions and approaches are still required.

In this paper we present a criteria for deciding the properness of rational parametrizations of hypersurfaces, based on the existence of intersection points of some algebraic hypersurfaces over the field of rational functions $\mathbb{K}(V)$ of the given hypersurface. For the case of surfaces, these auxiliary hypersurfaces turn to be plane curves, and therefore the problem is solved by means of resultants. Furthermore, the general criteria can be stated algorithmically. As a consequence of these results we present an algorithm that decides whether a given rational parametrization is proper, and in the affirmative case computes the inverse. Moreover, we have implementated these ideas in Maple. We have designed two prototypes of the algorithm, one deterministic and the second heuristic, and actual computing times in comparison to the method based on Gröbner bases are analyzed.

The structure of the paper is as follows: In Section 1 we prove the characterization of the properness of rational parametrizations of hypersurfaces. In Section 2 we show how these results can be improved in order to derive an algorithmic approach for the case of surfaces. Section 3 is devoted to outline the algorithm and examples. Section 4 focuses on the actual computing times of our implementations. In the appendix, we show the data parametrizations used in Section 4.

1 Characterization of the Properness

In this section we show how to characterize the birationality of a rational parametrization of a hypersurface over an algebraically closed field of characteristic zero by means of some auxiliary hypersurfaces constructed directly from the parametrization. This result provides a constructive method for checking the properness of rational parametrizations of hypersurfaces, and for computing the inverse, if a method for determining intersection points is given. Applications of this statement can be seen in the next section.

We start with a lemma that characterizes the birationality of parametrizations in terms of the injectivity.

Lemma 1. Let $\mathbb{F}$ be an algebraically closed field of characteristic zero, let

$$\begin{array}{rccc} \mathcal{Q}: & \mathbb{F}^{n-1} & \longrightarrow & V \subset \mathbb{F}^n \\ & \overline{t} & \longmapsto & (\mathcal{Q}_1(\overline{t}), \ldots, \mathcal{Q}_n(\overline{t})) \end{array}$$

be a rational parametrization of a hypersurface V, and let

$$\begin{array}{rccc} \mathcal{U}: & V & \longrightarrow & \mathbb{F}^{n-1} \\ & \overline{x} & \longmapsto & (\mathcal{U}_1(\overline{x}), \ldots, \mathcal{U}_{n-1}(\overline{x})) \end{array}$$

be a rational map, where the denominators of $\mathcal{U}$ do not belong to the ideal of V. The following statements are equivalent:

(1.) $\mathcal{U}$ is the inverse of $\mathcal{Q}$.

(2.) For almost all points $\overline{x} \in V$, it holds that $\overline{x} = \mathcal{Q}(\mathcal{U}(\overline{x}))$.

(3.) For almost all points $\overline{t} \in \mathbb{F}^{n-1}$, it holds that $\overline{t} = \mathcal{U}(\mathcal{Q}(\overline{t}))$.

Proof.
By definition, (2) is an explicit way of saying that the composition $\mathcal{Q} \circ \mathcal{U} : V \to V$ is defined and is the identity as a rational map, and (3) is an explicit way of saying that $\mathcal{U} \circ \mathcal{Q} : \mathbb{F}^{n-1} \to \mathbb{F}^{n-1}$ is defined and is the identity. This shows that (1) implies both (2) and (3), and that the conjunction of (2) and (3) implies (1).

By definition of "parametrization", $\mathcal{Q}$ is a *dominant* rational map from $\mathbb{F}^{n-1}$ to V, i.e. its image is Zariski dense. We show indirectly that either (2) or (3) implies that also $\mathcal{U}$ is dominant. Indeed: if $\mathcal{U}$ would not be dominant, then we would have a rational map (either $\mathcal{Q} \circ \mathcal{U}$ or $\mathcal{U} \circ \mathcal{Q}$) between two varieties of dimension $n-1$ factoring through a set of strictly lower dimension. But on the other hand, this map is supposed to be the identity, which is clearly dominant. Thus this is a contradiction, and then $\mathcal{U}$ is dominant.
It is well-known (see [7], p. 77) that dominant maps induce field embeddings. Hence we have field embeddings $\mathcal{Q}^* : \mathbb{F}(V) \to \mathbb{F}(\bar{t})$ and $\mathcal{U}^* : \mathbb{F}(\bar{t}) \to \mathbb{F}(V)$.
Next we show that (2) implies (3). Let us assume (2). Then the composite field embedding $\mathcal{U}^* \circ \mathcal{Q}^* : \mathbb{F}(V) \to \mathbb{F}(V)$ is the identity. It follows that

$$\mathcal{U}^* \circ \mathcal{Q}^* \circ \mathcal{U}^* = \mathcal{U}^* \circ id_{\mathbb{F}(\bar{t})}.$$

Because of $\mathcal{U}^*$ is injective and therefore left cancelable, it follows that $\mathcal{Q}^* \circ \mathcal{U}^* = id_{\mathbb{F}(\bar{t})}$, which is equivalent to (3).
The proof of (3) implies (2) is analogous. □

We observe that from Lemma 1 one might derive two methods to compute the inverse. The first one uses statement (3), and deals with linear conditions, provided some assumptions on the degree. A second approach may be based on statement (2), and leads to algebraic equations over the field of rational functions of the hypersurface. Even though, the second method seems to be more complicated, we will see that it leads to better algorithms where no assumptions on the degree have to be taken. Furthermore, if one works with surfaces, the algebraic equations that appear in the method can be resolved by means of resultants.

Once we have this general statement (Lemma 1) we introduce the notation that will be use in the sequel. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $F \in \mathbb{K}[x_1, x_2, \ldots, x_n]$ an irreducible polynomial defining a unirational hypersurface V, and let

$$\mathcal{P}(t_1, \ldots, t_{n-1}) = \left(\frac{p_1(t_1, \ldots, t_{n-1})}{q_1(t_1, \ldots, t_{n-1})}, \ldots, \frac{p_n(t_1, \ldots, t_{n-1})}{q_n(t_1, \ldots, t_{n-1})} \right) \in \mathbb{K}(t_1, \ldots, t_{n-1})^n$$

be a rational parametrization of V, such that $\gcd(p_i, q_i) = 1 \ \forall i \in \{1, 2, \ldots, n\}$.
Also let K the algebraic closure of the field $\mathbb{K}(V)$; i.e $K = \overline{\mathbb{K}(V)}$.

In this situation, we consider the affine hypersurfaces $V_1, V_2, \ldots, V_n$ defined, respectively, over K by the polynomials

$$G_i(t_1, \ldots, t_{n-1}) = x_i q_i(t_1, \ldots, t_{n-1}) - p_i(t_1, \ldots, t_{n-1}) \in \mathbb{K}(V)[t_1, \ldots, t_{n-1}], \ \ i = 1, \ldots, n.$$

Finally let us also consider the affine hypersurface V_{n+1} defined, over K by the polynomial

$$G_{n+1}(t_1, \ldots, t_{n-1}) = lcm(q_1(t_1, \ldots, t_{n-1}), \ldots, q_n(t_1, \ldots, t_{n-1})) \in \mathbb{K}(V)[t_1, \ldots, t_{n-1}].$$

The next theorem characterizes the properness of the parametrization $\mathcal{P}(t_1, \ldots, t_{n-1})$ by means of the intersection points of the hypersurfaces $V_1, \ldots, V_n$ defined above.

Theorem 1. The following statements are equivalent:

(1.) The parametrization $\mathcal{P}(t_1, \ldots, t_{n-1})$ is proper.

(2.) There exists exactly one point $A = (A_1, A_2, \ldots, A_{n-1}) \in [(V_1 \cap V_2 \cap \cdots \cap V_n) \setminus (V_1 \cap V_2 \cap \cdots \cap V_n \cap V_{n+1})] \cap K^{n-1}$.

Furthermore, if (2) holds, then A is the inverse of $\mathcal{P}$.

Proof.

$\boxed{(1)\Rightarrow(2)}$ Let $\overline{x} = (x_1, \ldots, x_n)$, and $M = (M_1(\overline{x}), M_2(\overline{x}), \ldots, M_{n-1}(\overline{x}))$ be the inverse of the rational proper parametrization $\mathcal{P}(t_1, \ldots, t_{n-1})$. Then,

$$\mathcal{P}(M) = \overline{x} \ mod \ I(V)$$

Therefore, crossing out denominators one gets that

$$M \in V_1 \cap V_2 \cap \cdots \cap V_n \cap K^{n-1}$$

Furthermore, since M is the inverse of the parametrization one has that $q_i(M)$ is not zero $\forall i \in \{1, \ldots, n\}$, and then $M \notin V_{n+1}$. Hence, $M \in [(V_1 \cap V_2 \cap \cdots \cap V_n) \setminus (V_1 \cap V_2 \cap \cdots \cap V_n \cap V_{n+1})] \cap K^{n-1}$.

Finally, let us see that M is unique. Let $M^* \in K^{n-1} \cap [(V_1 \cap V_2 \cap \cdots \cap V_n) \setminus (V_1 \cap V_2 \cap \cdots \cap V_n \cap V_{n+1})]$. Taking into account that $M^* \notin V_{n+1}$, one obtains that $q_i(M^*)$ is not zero $\forall i \in \{1, \ldots, n\}$. Therefore the equalities $G_i(M^*) = 0 \ \ \forall i \in \{1, \ldots, n\}$ imply that

$$\mathcal{P}(M^*) = \overline{x} \ mod \ I(V)$$

Thus, $\mathcal{P} \circ M^* = id_V$. Hence, left composing by $\mathcal{P}^{-1}$, one concludes that $M^* = \mathcal{P}^{-1} = M$.

$\boxed{(2)\Rightarrow(1)}$ Since there exists a unique point in $[(V_1 \cap V_2 \cap \cdots \cap V_n) \setminus (V_1 \cap V_2 \cap \cdots \cap V_n \cap V_{n+1})] \cap K^{n-1}$, A is fixed under the action of the Galois group. Therefore, $A \in \mathbb{K}(V)^{n-1}$. Reasoning as we did for the uniqueness in the previous implication, one gets that $\mathcal{P} \circ A = id_V$. Now applying Lemma 1 one gets (1). □

Example 1. Let V be the hypersurface in $\mathbb{C}^4$ defined by the irreducible polynomial

$F(x, y, z, w) = -8\,x\,y + 2\,y^2 + 4\,y^2 z - 3\,y^2 w + 16\,x\,y^2 - 24\,x^2\,y + 16\,x^2\,y^2 + 16\,x^3 + 16\,x^2\,z - 14\,x^2\,w + 12\,x\,y\,w - 16\,x\,y\,z + 8\,x^2 + 42\,x\,y^2\,w + 16\,x\,y^2\,z - 4\,y^3 - 3\,y^2 z\,w - 8\,x\,z^2\,y + 12\,x\,y\,z\,w + 8\,x^2\,z^2 + 3\,w^2 x^2 - 14\,w\,x^2\,z - 16\,x^3\,y + 16\,x^3\,z + 36\,x^3\,w - 76\,x^2\,y\,w - 24\,x^2\,y\,z + 8\,x^4 + 2\,y^2 z^2 + 2\,y^4 - 4\,y^3\,z - 8\,y^3\,x - 6\,y^3\,w,$

and the rational parametrization

$$\mathcal{P}(t_1,t_2,t_3) = \left(\frac{t_1-1}{t_3+t_1^2+1}, \frac{2\,t_1-t_2}{t_3+t_1^2+1}, \frac{2\,t_2-t_3}{t_3+t_1^2+1}, \frac{t_2^2}{t_3+t_1^2+1}\right)$$

Let us check whether $\mathcal{P}(t_1,t_2,t_3)$ is a proper parametrization.
Taking into account Theorem 1, we have to analyze the intersection points of the four curves V_1, V_2, V_3, V_4, defined respectively over $\overline{\mathbb{C}(V)}$ by the polynomials

$$\left\{\begin{array}{rl} G_1(t_1,t_2,t_3) = & x\,t_3 + x\,t_1^2 + x - t_1 + 1, \\ G_2(t_1,t_2,t_3) = & y\,t_3 + y\,t_1^2 + y - 2\,t_1 + t_2, \\ G_3(t_1,t_2,t_3) = & z\,t_3 + z\,t_1^2 + z - 2\,t_2 + t_3, \\ G_4(t_1,t_2,t_3) = & w\,t_3 + w\,t_1^2 + w - t_2^2. \end{array}\right.$$

Let us consider the point $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) \in \mathbb{C}(V)$ where

$$\mathcal{A}_1 = \frac{-2\,x + 3\,y^2 - 2\,x^2 - 4\,x\,y + 3\,x\,w - 2\,x\,z}{10\,x^2 + 3\,y^2 - 12\,x\,y},$$

$$\mathcal{A}_2 = \frac{-2\,y^2 + 2\,y + 4\,x\,y - 3\,y\,w + 2\,z\,y + 6\,x\,w - 4\,x - 4\,x^2 - 4\,x\,z}{10\,x^2 + 3\,y^2 - 12\,x\,y},$$

$$\mathcal{A}_3 = \frac{-4\,y^2 - 6\,y\,w - 4\,z\,y + 8\,x\,y + 4\,y - 8\,x - 8\,x^2 - 3\,z\,w + 12\,x\,w + 4\,x\,z + 2\,z + 2\,z^2}{10\,x^2 + 3\,y^2 - 12\,x\,y}.$$

It holds that (this can be check, for instance using Gröbner Basis)

$$\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) \in V_1 \cap V_2 \cap V_3 \cap V_4,$$

and that

$$\mathcal{A}(\mathcal{P}(t_1,t_2,t_3)) = (t_1,t_2,t_3).$$

Thus, applying Theorem 1, we deduce that $\mathcal{P}$ is proper and the inverse mapping is $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$. $\square$

2 Application to Surfaces

In this section, we show how to apply Theorem 1 to the special case of surfaces. The particular case of planes curves can be treated similarly, but we do not deal with it in this paper. For surfaces, Theorem 1 can be improved such that algorithmic characterizations based on resultants can be stated. As a consequence of these results an algorithm for deciding the properness and for computing the inverse, if it exists, is derived.

In this situation, the notation introduced in Section 1 is adapted as follows: $F \in \mathbb{K}[x, y, z]$ is an irreducible polynomial defining a rational surface V, and

$$\mathcal{P}(t_1, t_2) = \left(\frac{p_1(t_1, t_2)}{q_1(t_1, t_2)}, \frac{p_2(t_1, t_2)}{q_2(t_1, t_2)}, \frac{p_3(t_1, t_2)}{q_3(t_1, t_2)} \right) \in \mathbb{K}(t_1, t_2)^3$$

is a rational parametrization of V, where $\gcd(p_i, q_i) = 1, \ \forall i \in \{1, 2, 3\}$. Furthermore, V_1, V_2, V_3 are the plane curves defining, respectively, over K by the polynomials

$$\begin{aligned} G_1(t_1, t_2) &= xq_1(t_1, t_2) - p_1(t_1, t_2), \\ G_2(t_1, t_2) &= yq_2(t_1, t_2) - p_2(t_1, t_2), \\ G_3(t_1, t_2) &= zq_3(t_1, t_2) - p_3(t_1, t_2). \end{aligned}$$

Moreover, we introduce a new curve V_4 that is defined over K by the polynomial

$$G_4(t_1, t_2) = lcm(q_1(t_1, t_2), q_2(t_1, t_2), q_3(t_1, t_2)).$$

Note that V_4 is empty if $\mathcal{P}(t_1, t_2)$ is a polynomial parametrization, and otherwise it is a plane curve. For simplicity, we assume w.l.o.g. that $\mathcal{P}(t_1, t_2)$ does not parametrize a plane. Note that the problem for the case of planes is trivial. Moreover, one can check whether $\mathcal{P}(t_1, t_2)$ corresponds to a plane by introducing undermined coefficients a, b, c, d and solving the linear system of equations generated by the equality

$$a \frac{p_1(t_1, t_2)}{q_1(t_1, t_2)} + b \frac{p_2(t_1, t_2)}{q_2(t_1, t_2)} + c \frac{p_3(t_1, t_2)}{q_3(t_1, t_2)} = d.$$

In the next lemma we analyze the intersection points of the auxiliary curves V_1, V_2, V_3, V_4.

Lemma 2. The intersection points of the curves V_1, V_2, V_3, V_4, are in $\mathbb{K}^2$.

Proof. Let $(\alpha, \beta) \in V_1 \cap V_2 \cap V_3 \cap V_4$. Then, $G_4(\alpha, \beta) = 0$. Thus, there exists a denominator q_i that vanishes on (α, β); let us say w.l.o.g that it is q_1. Therefore, since $(\alpha, \beta) \in V_1$ it holds that

$$p_1(\alpha, \beta) = q_1(\alpha, \beta) = 0 \ \ mod \ \ I(V).$$

Now, since $\gcd(p_1, q_1) = 1$, it holds that the resultant of p_1, and q_1 w.r.t. t_2 is not identically zero. Furthermore, this resultant is in $\mathbb{K}[t_1]$, and its roots are the t_1-coordinates of the intersection points of the curves given by p_1 and q_1 over $\mathbb{K}$. Hence, since $\mathbb{K}$ is algebraically closed one has that α is in $\mathbb{K}$; similarly for the t_2-coordinate. $\square$

Now, we apply Lemma 2, to show how Theorem 1 can be improved for the case of surfaces.

Theorem 2. The following statements are equivalent:

(1.) The parametrization $\mathcal{P}(t_1, t_2)$ is proper.

(2.) There exists exactly one point $A = (A_1, A_2) \in (V_1 \cap V_2 \cap V_3) \cap (K^2 \setminus \mathbb{K}^2)$.

(3.) There exists exactly one point $A = (A_1, A_2) \in (V_1 \cap V_2 \cap V_3) \cap (K \setminus \mathbb{K})^2$.

(4.) There exists exactly one point $A = (A_1, A_2) \in (V_1 \cap V_2 \cap V_3) \setminus (V_1 \cap V_2 \cap V_3 \cap V_4)$.

Furthermore, if either (2) or (3) or (4) holds then A is the inverse of $\mathcal{P}$.

Proof. (1) $\Leftrightarrow$ (4) follows from Theorem 1. In order to prove that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) we see that $S_1 = S_2 = S_3$ where

$$S_1 = (V_1 \cap V_2 \cap V_3) \cap (K^2 \setminus \mathbb{K}^2)$$

$$S_2 = (V_1 \cap V_2 \cap V_3) \setminus (V_1 \cap V_2 \cap V_3 \cap V_4)$$
$$S_3 = (V_1 \cap V_2 \cap V_3) \cap (K \setminus \mathbb{K})^2.$$

From Lemma 2 one has that $S_1 \subset S_2$. Now, take $A \in S_2$. Then, reasoning as the in proof of the uniqueness of in the implication (1) $\Rightarrow$ (2) of Theorem 1, one gets that A is a right inverse of $\mathcal{P}$. Thus, no component of A can be constant, and therefore $A \in S_3$. Finally, it is clear that $S_3 \subset S_1$. □

In the following we show statement (2), (3) and (4) of Theorem 2, can be decided computationally. We start with a technical result where we state a one to one relation between the intersection points of any three planes curves, without common components, and the roots of a univariate polynomial computed by a resultant.

For this purpose, in the next proposition we assume w.l.o.g. that the gcd of each two of the leading coefficients of the defining polynomials of the curves, w.r.t. one the variables, is trivial. Note that this condition can always be achieved by a linear change of coordinates.

Proposition 1. Let $\mathbb{L}$ be a subfield of an algebraically closed field $\mathbb{F}$ of characteristic zero, and let C_1, C_2, C_3 be plane algebraic curves over $\mathbb{F}$, with no common components, defined by the polynomials F_1, F_2, $F_3 \in \mathbb{L}[t_1, t_2]$, respectively. Let F_1, F_2, F_3 be such that each two of their leading coefficients, w.r.t. one of the variables, have trivial gcd. Let F_1 do not have a factor in $\mathbb{L}[t_1]$. Then, the t_1-coordinates of the intersection points of C_1, C_2, C_3 are the roots of the content w.r.t Z of the resultant w.r.t. t_2 of the polynomials $F_1, F_2 + ZF_3$.

Proof. Let $R(t_1, Z)$ be the resultant w.r.t. t_2 of F_1, and $F_2 + ZF_3$, and let $S(t_1)$ be the content of $R(t_1, Z)$ w.r.t Z. First of all note that, since C_1, C_2, C_3 do not have common components, then $R(t_1, Z)$ is not identically zero. Now we prove that the t_1-coordinates of the points in $C_1 \cap C_2 \cap C_3$ are the roots of $S(t_1)$. Indeed: let α be a root of $S(t_1)$ (note that α does not depend on Z). Then $R(\alpha, Z) = 0$. Thus, since we have assume that the gcd of each two of the leading coefficients of F_1, F_2, F_3, w.r.t. t_2 is trivial, one has that there exists $\beta \in \mathbb{F}$ such that $F_1(\alpha, \beta) = F_Z(\alpha, \beta) = 0$. Let us see that β does not depend on Z. If $\beta = \beta(Z)$ depends on Z, there exists infinitely many values of Z in $\mathbb{F}$ such that $F_1(\alpha, \beta(Z)) = 0$. This implies that $m(t_1)$ divides F_1, where $m(t_1)$ is the minimal polynomial of α over $\mathbb{L}$, which is impossible by hypothesis. Therefore, β does not depend on Z, and hence. $F_1(\alpha, \beta) = F_2(\alpha, \beta) = F_3(\alpha, \beta) = 0$.

Reciprocally, if one assume that (α, β) is a common point of C_1, C_2, C_3, one obtains that $F_1(\alpha, \beta) = F_Z(\alpha, \beta) = 0$. Thus $R(\alpha, Z) = 0$. Furthermore, since (α, β) does not depend on Z, one has that $S(\alpha) = 0$. □

In the next propositions, we prove that our auxiliary curves V_1, V_2, V_3 are in the conditions of Proposition 1.

Proposition 2. The curves V_1, V_2, V_3, do not have common components.

Proof. Let us assume that G_1, G_2, G_3 have a common component. Then, there exist $M, N_i \in \mathbb{K}(V)[t_1, t_2]$ such that

$$G_i(t_1, t_2) = M(t_1, t_2) N_i(t_1, t_2), \text{ for } i = 1, 2, 3$$

The coefficients of M and N_i are rational functions over V. Thus, taking common denominators, the equality above can be written in $K[t_1, t_2]$ as

$$T_i(x, y, z) G_i = M^* N_i^* + A_i(x, y, z) F(x, y, z)$$

where $A_i, T_i \in \mathbb{K}[x, y, z]$, $M^*, N_i^* \in \mathbb{K}[x, y, z][t_1, t_2]$ and F is the implicit equation of V. Now we consider the set Ω consisting in all $P \in V$ such that $T_i(P) \neq 0$, and $M^*(P, t_1, t_2)$ is not constant, and $N_i^*(P, t_1, t_2)$ is not zero. Note that Ω is a non-empty open subset of V since T_i represents a denominator of an element in $\mathbb{K}(V)$, M^* corresponds to a numerator of a non constant polynomial over $\mathbb{K}(V)$, and N_i^* corresponds to a numerator of a non zero polynomial over $\mathbb{K}(V)$.
In this situation, for $i = 1, 2, 3$ and for all $P = (a_1, a_2, a_3) \in \Omega$ it holds that

$$T_i(P)(q_i(t_1, t_2) a_i - p_i(t_1, t_2)) = M^*(P, t_1, t_2) N_i^*(P, t_1, t_2)$$

Observe that $\gcd(q_i(t_1, t_2), M^*(P, t_1, t_2)) = 1$ for $i = 1, 2, 3$, since otherwise it would imply that $\gcd(q_i, p_i) \neq 1$ for some i. For each $P \in \Omega$ we consider the set

$$\Sigma_P = \{Q \in \mathbb{K}^2 \,/\, M^*(P, Q) = 0,\ q_i(Q) \neq\ 0 \text{ for } i = 1, 2, 3\}.$$

Note that since $\gcd(q_i(t_1, t_2), M^*(P, t_1, t_2)) = 1$ one has that Σ_P is a non-empty open of the curve $M^*(P, t_1, t_2)$. Hence $Card(\Sigma_P) = \infty$. Finally, using the Theorem of the dimension of fibres (see [24], pp.76) there exists an open set Ω' of V such that for every $P \in \Omega'$ the fibres $\mathcal{P}^{-1}(P)$ are zero dimensional. Now take $P \in \Omega \cap \Omega'$ (note that $\Omega \cap \Omega' \neq \emptyset$ because V is irreducible). Then one has that $\Sigma_P \subset \mathcal{P}^{-1}(P)$ which is impossible since $Card(\Sigma_P) = \infty$, and $\mathcal{P}^{-1}(P)$ is zero dimensional. □

Proposition 3. At least one of the polynomials G_1, G_2, G_3 does not have factors in $\mathbb{K}(V)[t_1]$ neither in $\mathbb{K}(V)[t_2]$.

Proof. We distinguish two cases depending on whether there exists or not one component of $\mathcal{P}$ depending simultaneously on both parameters. If none component of $\mathcal{P}$ depends simultaneously on t_1, t_2, we may assume w.l.o.g. that $\mathcal{P}$ is of the form $(\frac{p_1(t_1)}{q_1(t_1)}, \frac{p_2(t_2)}{q_2(t_2)}, \frac{p_3(t_2)}{q_3(t_2)})$. Therefore, the implicit equation of V depends only on y, z, and hence it is a cylinder. Thus, the subfield of $\mathbb{K}(V)$ generated by the class of x is algebraically closed in $\mathbb{K}(V)$. Let us see that irreducibility over $\mathbb{K}(x)$ is equivalent to irreducibility over $\mathbb{K}(V)$. Indeed, if G_1 does not factor over $\mathbb{K}(x)$, but it factors over the algebraic closure, the coefficients of the factors are algebraic over $\mathbb{K}(x)$, and hence they do not belong to $\mathbb{K}(V)$. Thus, it suffices to prove that G_1 is irreducible over $\mathbb{K}(x)$, but this is obvious since G_1 is irreducible over $\mathbb{K}$.

Let us assume at least one component of $\mathcal{P}$ depends on t_1, t_2. Let us say that it is the first one. Observe that G_1 has a factor in $\mathbb{K}(V)[t_1]$ (similarly in $\mathbb{K}(V)[t_2]$) if and only if the content w.r.t. t_2 is not trivial; i.e. the content depends on t_1. Since the euclidian algorithm does not extend the ground field, one deduces that the above condition is equivalent to ask G_1 not to have a trivial content w.r.t. t_2 in $\mathbb{K}(x)[t_1]$, where x can be seen as a transcendental element over $\mathbb{K}$, since we have assumed that V is not a plane. Let us now assume that there exists a non–trivial content. It implies that there exists polynomials $A(x, t_1), B(x), D(x, t_1, t_2) \in \mathbb{K}[x, t_1, t_2]$ such that $deg_{t_1}(A) \geq 1$ and

$$B(x)G_1(x, t_1, t_2) = A(x, t_1)D(x, t_1, t_2).$$

Obviously, we may assume that A is primitive w.r.t. x. Thus, A divides G_1 as polynomials in $\mathbb{K}[x, t_1, t_2]$. Therefore, $deg_x(A) \leq 1$. If $deg_x(A) = 0$ one gets that $\gcd(q_1, p_1) \neq 1$ which is impossible by hypothesis. If $deg_x(A) = 1$, then the cofactor of A in G_1 does not depend on x. Therefore, it must be constant because $\gcd(q_1, p_1) = 1$. Thus, G_1 does not depend on t_2, which is impossible. □

From Propositions $1, 2, 3$, one can derive an algorithmic method to compute the t_1 and the t_2 coordinates of the points in $V_1 \cap V_2 \cap V_3$. We state the result for the t_1–coordinates, and we assume w.l.o.g. that G_1 is the polynomial without factors in $\mathbb{K}(V)[t_1]$ (see Proposition 3). A similar result holds for the t_2–coordinates. For this purpose, if $\mathbb{A}$ is a U.F.D we denote by

$$\mathrm{cont}_Z(H),$$

the content of the polynomial $H \in \mathbb{A}[\bar{x}, Z]$ w.r.t. Z, and we denote by

$$\mathrm{res}(H_1, H_2, t) \text{ or } \mathrm{res}_{\mathbb{A}[t]}(H_1, H_2, t)$$

the resultant of $H_1, H_2 \in \mathbb{A}[t]$ w.r.t. t.

Corollary 1. If each two of the leading coefficients of G_1, G_2, G_3, w.r.t. t_2 has trivial gcd, the t_1-coordinates of the intersection points of V_1, V_2, V_3 are the roots of the polynomial

$$S(t_1) = \text{cont}_Z(\text{res}(G_1, G_2 + ZG_3, t_2)) \in \mathbb{K}(V)[t_1]$$

Remark. In the following, we will refer to the polynomial $S(t_1)$, introduced in Corollary 1, as the polynomial defining the t_1-coordinates of the intersection points of the curves V_1, V_2, V_3. □

The next proposition states that the defining polynomial of the t_1-coordinates of $V_1 \cap V_2 \cap V_3$ does not have non-constant multiple roots; similarly for the polynomial defining of the t_2–coordinates.

Proposition 4. Let $\mathcal{P}$ proper, and let $S(t_1)$ be the polynomial

$$S(t_1) = \text{cont}_Z(\text{res}(G_1, G_2 + ZG_3, t_2)).$$

If $S(t_1)$ has a multiple root, then it is constant.

Proof. Let $M(x, y, z) = (A(x, y, z), B(x, y, z))$ be the inverse of $\mathcal{P}(t_1, t_2)$. Now, we observe that, from Theorem 2, one has that $M(x, y)$ is the unique non-constant point in $V_1 \cap V_2 \cap V_3$. We claim that for generic Z, M is a point of transversal intersection of the two plane curves with equation $G_1 = 0$ (i.e. V_1) and $G_2 + ZG_3 = 0$. To show this, we have to show that M is a simple point of both curves, and that the two tangents at M are distinct.

Note that V_1 is a generic curve of the pencil of plane curves spanned by $p_1(t_1, t_2) = 0$ and $q_1(t_1, t_2) = 0$. By Bertini's theorem (see [6], p 137), V_1 is smooth outside the common zero set of p_1 and q_1. Since M is not in this zero set, M is a smooth point of V_1. Similarly, M is a smooth point of V_2 and V_3. Therefore, the gradients of G_2 and G_3 are not zero at M, and therefore the gradient of $G_2 + ZG_3$ at M also is not zero for generic Z.

Now, assume indirectly that the gradients at M of G_1 and of $G_2 + ZG_3$ are parallel for all Z. This is only possible if all three curves V_1, V_2, V_3 have common tangent. Because M is not in the zero set of q_i, the gradient of G_i is parallel to the gradient of the rational function $p_i(t_1, t_2)/q_i(t_1, t_2)$. Therefore the common tangent assumption implies that the Jacobi matrix of $\mathcal{P}$ is rank deficient at M. Now, let Ω be the non–empty open subset of $\mathbb{K}^2$ where the parametrization $\mathcal{P}$ is defined and moreover the inverse M is defined on the image of $\mathcal{P}$. Then, for every $(a, b) \in \Omega$ one has that the rank of the jacobian matrix of $\mathcal{P}$ at $M(\mathcal{P}(a, b)) = (a, b)$ is rank deficient. But then the image of the parametrization is only a curve, contradicting our assumption. Therefore, we have proved that M is a transversal intersection of G_1 and $G_2 + ZG_3$.

By well-known properties of resultants, it follows that $\mathrm{res}(G_1, G_2 + ZG_3, t_2)$ has only a simple root at $M(x, y, z)$. Since S is a divisor of $\mathrm{res}(G_1, G_2 + ZG_3, t_2)$, M is a simple root of S. $\square$

From Theorem 2 and Proposition 4 one deduces the following characterization.

Theorem 3. Let $\overline{S}(t_1)$ and $\overline{S^*}(t_2)$ be the defining polynomials of the t_1-coordinates and the t_2-coordinates of the points in $(V_1 \cap V_2 \cap V_3) \cap (K \setminus \mathbb{K})^2$, respectively. The following statements are equivalent:

(1.) $\mathcal{P}$ is proper.

(2.) $\overline{S}(t_1)$ is linear. Furthermore, if $A_1 \in K$ is the root of $\overline{S}(t_1)$, then the polynomial

$$M(t_2) = \frac{\gcd_{\mathbb{K}(V)[t_2]}(G_1(A_1, t_2), G_2(A_1, t_2), G_3(A_1, t_2))}{\gcd_{\mathbb{K}(V)[t_2]}(G_1(A_1, t_2), G_2(A_1, t_2), G_3(A_1, t_2), G_4(A_1, t_2))},$$

only has one distinct root.

(3.) $\overline{S}(t_1)$ and $\overline{S}^*(t_2)$ are linear.

Proof.
$\boxed{1 \Rightarrow 2}$ If $\mathcal{P}$ is proper, taking into account Theorem 2, one has that $(V_1 \cap V_2 \cap V_3) \setminus (V_1 \cap V_2 \cap V_3 \cap V_4)$ only has one point (α, β) in K^2. Since the roots of $\overline{S}(t_1)$ correspond to the t_1-coordinates of $(V_1 \cap V_2 \cap V_3) \setminus (V_1 \cap V_2 \cap V_3 \cap V_4)$, one deduces that α is the unique root of the polynomial $\overline{S}(t_1)$. Furthermore, taking into account that $M(t_2)$ defines the t_2-coordinates of $(V_1 \cap V_2 \cap V_3) \setminus (V_1 \cap V_2 \cap V_3 \cap V_4)$, one also has that β is the only different root of $M(t_2)$.

$\boxed{2 \Rightarrow 3}$ If follows from the fact that $M(t_2)$ corresponds to the t_2-coordinate in $(V_1 \cap V_2 \cap V_3) \setminus (V_1 \cap V_2 \cap V_3 \cap V_4)$ over K.

$\boxed{3 \Rightarrow 1}$ Using that $\overline{S}(t_1)$ and $\overline{S}^*(t_2)$ define the t_1-coordinates and t_2-coordinates of the points in $(V_1 \cap V_2 \cap V_3) \setminus (V_1 \cap V_2 \cap V_3 \cap V_4)$ respectively, one deduces that $(V_1 \cap V_2 \cap V_3) \setminus (V_1 \cap V_2 \cap V_3 \cap V_4)$ only has one different point $(A_1, A_2) \in K$. Therefore, from Theorem 2 one concludes that $\mathcal{P}$ is proper. $\square$

3 Algorithm and Examples

The results in Section 2 can be applied to derive an algorithm that decides the properness of a given rational parametrization of a surface, and that (in the affirmative case) determines the inverse. In this section, we outline the algorithm and we illustrate it by examples.

Algorithm INVERSION($\mathcal{P}, F$)

- **Input:** $F \in \mathbb{K}[x, y, z]$ irreducible and defining a surface (not being a plane) V over $\mathbb{K}$, and $\mathcal{P}(t_1, t_2) = (\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$ a rational parametrization of V, where $\gcd(p_i, q_i) = 1 \ \forall i \in \{1, 2, 3\}$.
- **Output:** the message "$\mathcal{P}$ `is not proper`" or the inverse of $\mathcal{P}$.

(1) Compute the polynomials

$$\begin{aligned} G_1(t_1, t_2) &= xq_1(t_1, t_2) - p_1(t_1, t_2), \\ G_2(t_1, t_2) &= yq_2(t_1, t_2) - p_2(t_1, t_2), \\ G_3(t_1, t_2) &= zq_3(t_1, t_2) - p_3(t_1, t_2), \end{aligned}$$

that define the curves V_1, V_2, V_3, respectively.

(2) Check whether each two of the leading coefficients of G_1, G_2, G_3, w.r.t. t_2 is trivial, (resp. w.r.t. t_1). If not, do a suitable change of coordinates.

(3) Compute the polynomial $S(t_1)$ defining the t_1-coordinates of the intersection points of V_1, V_2, V_3 (see Corollary 1).

(4) Compute the polynomial $\overline{S}(t_1)$ obtained by crossing out the constant roots of the polynomial $S(t_1)$.

(5) If $\bar{S}(t_1)$ is not linear then return "$\mathcal{P}$ `is not proper`". Else let α the root of $\bar{S}(t_1)$

(6) Compute the polynomial $S^*(t_2)$ defining the t_2-coordinates of the intersection points of V_1, V_2, V_3 (see Corollary 1).

(7) Compute the polynomial $\overline{S}^*(t_2)$ obtained by crossing out the constant roots of the polynomial $S^*(t_2)$.

(8) If $\bar{S}^*(t_2)$ is not linear then return "$\mathcal{P}$ `is not proper`". Else let β the root of $\bar{S}^*(t_2)$.

(9) Return "(α, β) `is the inverse of` $\mathcal{P}$".

Correctness. The algorithm follows essentially from Theorem 3. In addition note that, taking into account Proposition 1, when computing S and S^* in steps (2) and (6) respectively, the correspondent resultant can not be identically zero. □

Remark. The computation of the polynomial $\overline{S}(t_1)$ (and similarly $\overline{S}^*(t_2)$), can be performed as follows: substitute $\mathcal{P}(s_1, s_2)$ in $S(x, y, z)$. The new variables s_1, s_2 are necessary to avoid coincidences with t_1, t_2. Then clear denominators by multiplying with a polynomial in $\mathbb{K}[s_1, s_2]$ and compute the content with respect to (s_1, s_2). The resulting polynomial $C(t_1)$ corresponds to the constant roots. Now, $\overline{S}(t_1)$ is obtained as the quotient of S by C. $\square$

Finally we illustrate the algorithm by three examples. In examples 2 and 4 the parametrizations turn to be proper and their inverse is determine. However in example 3 the algorithm shows that the given parametrization is not proper. The surfaces in examples 3 and 4, correspond the offset surfaces and pipe surfaces, respectively.

Example 2. Let V be the surface defined by the irreducible polynomial $F = 1 - 2x + y + 2x^2 + x^2z^2 - 2xy + yx^2 - 2xz - yzx + 2zx^2 + zyx^2 \in \mathbb{C}[x, y, z]$. We consider the rational parametrization of V:

$$\mathcal{P}(t_1, t_2) = \left(\frac{t_1}{t_2 + t_1}, \frac{t_1^2 + 1}{t_2}, \frac{1 + t_2}{t_1} \right).$$

Let us check whether $\mathcal{P}(t_1, t_2)$ is a proper parametrization. For this purpose, we apply the algorithm. First we consider the polynomials

$$G_1(t_1, t_2) = x(t_1 + t_2) - t_1, \quad G_2(t_1, t_2) = yt_2 - (t_1^2 + 1),$$

$$G_3(t_1, t_2) = zt_1 - (1 + t_2), \quad G_4(t_1, t_2) = t_1t_2(t_1 + t_2).$$

Doing the computation one has that

$$R = \mathrm{res}_{\mathbb{C}(V)[t_1,Z][t_2]}(G_1, G_2 + ZG_3, t_2) = (-t_1yx - x + yt_1 - {t_1}^2x) + (-x + xzt_1 - t_1 + xt_1)Z$$

In addition, $S = \mathrm{cont}_Z(R) = t_1zx - t_1 - x + t_1x$. Note that S does not have constant roots, hence $\overline{S}(t_1) = S(t_1)$. Moreover, since $\bar{S}(t_1)$ is linear we obtain the first coordinate of the inverse

$$\alpha(x, y, z) = \frac{-x}{-xz + 1 - x}.$$

Reasoning similarly with the variable t_2, we obtain

$$R^* = \mathrm{res}_{\mathbb{C}(V)[t_2,Z][t_1]}(G_1, G_2 + ZG_3, t_1) = (-x^2t_2^2 - 1 + 2x - x^2 + yt_2 - 2yt_2x + yt_2x^2) +$$

$$+ (xt_2z - x^2t_2z - t_2 + 2xt_2 - t_2x^2 - 1 + 2x - x^2)Z$$

In addition, $S^* = \mathrm{cont}_Z(R^*) = xt_2z - x^2t_2z - t_2 + 2xt_2 - t_2x^2 - 1 + 2x - x^2$. Note that S^* does not have constant roots, hence $\overline{S}^*(t_2) = S^*(t_2)$. Moreover, since $\bar{S}^*(t_2)$ is linear we obtain the second coordinate of the inverse

$$\beta(x, y, z) = \frac{-x + 1}{xz - 1 + x}.$$

Thus, applying the algorithm, we deduce that $\mathcal{P}$ is proper and its inverse mapping is:

$$\mathcal{P}^{-1}(t_1,\, t_2) = \,(\alpha, \beta) = \,(\frac{-x}{-xz + 1 - x}, \frac{-x + 1}{xz - 1 + x})$$

Now, let us compute the inverse applying statement (2) of Theorem 3. Reasoning as above, we compute the first coordinate α of the inverse. Then, we substitute α in $\{G_1, G_2, G_3\}$, and we compute the gcds

$$\frac{\gcd_{\mathbb{C}(V)[t_2]}(G_1(\alpha, t_2), G_2(\alpha, t_2), G_3(\alpha, t_2))}{\gcd_{\mathbb{C}(V)[t_2]}(G_1(\alpha, t_2), G_2(\alpha, t_2), G_3(\alpha, t_2), G_4(\alpha, t_2))} = \frac{x(-1+x-t_2+xt_2+t_2zx)}{-1+x+zx}.$$

Hence we get

$$\beta(x, y, z) = \frac{1-x}{xz-1+x}.$$

Thus, applying Corollary 1, we deduce that $\mathcal{P}$ is proper and its inverse mapping is:

$$\mathcal{P}^{-1}(t_1, t_2) = \left(\frac{-x}{-xz+1-x}, \frac{1-x}{xz-1+x}\right). \qquad \square$$

Example 3. Let V be the offset to the elliptic paraboloid $-x+y^2+z^2=0$, defined by the irreducible polynomial

$F = -25+40x+9x^2-40x^3+16x^4+28y^2+6xy^2-32y^2x^3-47y^4-40xy^4+16x^2y^4+16y^6+28z^2+6xz^2-32x^3z^2-94y^2z^2-80y^2xz^2+32z^2x^2y^2+48z^2y^4-47z^4-40xz^4+16x^2z^4+48z^4y^2+16z^6.$

We consider the rational parametrization of V (see [19], Theorem 2):

$$\mathcal{P}(t_1, t_2) = \Big(\frac{1-2t_2^2-6t_1^2-4t_2^4t_1^4+8t_2^4t_1^2+4t_2^2t_1^2+2t_2^2t_1^4+4t_2^4+15t_1^4+4t_2^4t_1^{12}-4t_2^4t_1^8+8t_2^4t_1^{10}-16t_2^4t_1^6}{(t_2^2+2t_2^2t_1^2+2t_1^2-t_1^4+t_2^2t_1^4-1)^2(1-2t_1^2+t_1^4+t_2^2+2t_2^2t_1^2+t_2^2t_1^4)} +$$

$$+\frac{-8t_2^2t_1^6+2t_2^2t_1^8-20t_1^6+15t_1^8+t_1^{12}-6t_1^{10}-t_2^6-6t_2^6t_1^2-15t_2^6t_1^4-20t_2^6t_1^6-15t_2^6t_1^8-6t_2^6t_1^{10}-t_2^6t_1^{12}+4t_1^{10}t_2^2-2t_1^12t_2^2}{(t_2^2+2t_2^2t_1^2+2t_1^2-t_1^4+t_2^2t_1^4-1)^2(1-2t_1^2+t_1^4+t_2^2+2t_2^2t_1^2+t_2^2t_1^4)},$$

$$\frac{-(-1+t_1^2)^2t_2(-1+2t_1^2-t_1^4+3t_2^2+6t_2^2t_1^2+3t_2^2t_1^4)}{(t_2^2+2t_2^2t_1^2+2t_1^2-t_1^4+t_2^2t_1^4-1)(1-2t_1^2+t_1^4+t_2^2+2t_2^2t_1^2+t_2^2t_1^4)}, \frac{-2(-1+t1^2)t1t2(-1+2t1^2-t1^4+3t2^2+6t2^2t1^2+3t2^2t1^4)}{(t_2^2+2t_2^2t_1^2+2t_1^2-t_1^4+t_2^2t_1^4-1)(1-2t_1^2+t_1^4+t_2^2+2t_2^2t_1^2+t_2^2t_1^4)}\Big).$$

Let us analyze whether $\mathcal{P}(t_1, t_2)$ is a proper parametrization. For this purpose, we apply the algorithm. First we consider the polynomials

$G_1(t_1, t_2) = -1+x-xt_1^{12}t_2^2+2xt_1^{10}t_2^2+xt_2^6t_1^{12}+6xt_2^6t_1^{10}+15xt_2^6t_1^8+20xt_2^6t_1^6+15xt_2^6t_1^4+xt_2^6t_1^2+t_2^6+xt_2^2t_1^8-4xt_2^2t_1^6+4xt_2^4t_1^6-2xt_2^4t_1^{10}-xt_2^4t_1^{12}+2t_2^2+6t_1^2+4t_2^4t_1^4-8t_2^4t_1^2-4t_2^2t_1^2-2t_2^2t_1^4-4t_2^4-15t_1^4-4t_2^4t_1^{12}+4t_2^4t_1^8-8t_2^4t_1^{10}+16t_2^4t_1^6+8t_2^2t_1^6-2t_2^2t_1^8+20t_1^6-15t_1^8-t_1^{12}+6t_1^{10}+6t_2^6t_1^2+15t_2^6t_1^4+20t_2^6t_1^6+15t_2^6t_1^8+6t_2^6t_1^{10}+t_2^6t_1^{12}-2xt_2^4t_1^2-4t_1^{10}t_2^2+2t_1^{12}t_2^2-xt_2^4+2xt_2^2t_1^2+xt_2^2t_1^4-t_2^2x+xt_2^4t_1^8+xt_2^4t_1^4+xt_2^6-20xt_1^6+15xt_1^4-6xt_1^2+15xt_1^8+xt_1^{12}-6xt_1^{10}$,

$G_2(t_1, t_2) = -y+4yt_1^2+6yt_2^4t_1^4+4yt_2^4t_1^2+yt_2^4-6yt_1^4+yt_2^4t_1^8+4yt_2^4t_1^6+4yt_1^6-yt_1^8-t_2+4t_2t_1^2-6t_2t_1^4+3t_2^3-6t_2^3t_1^4+4t_1^6t_2-t_1^8t_2+3t_2^3t_1^8$,

$G_3(t_1, t_2) = -z+4zt_1^2+6zt_2^4t_1^4+4zt_2^4t_1^2+zt_2^4-6zt_1^4+zt_2^4t_1^8+4zt_2^4t_1^6+4zt_1^6-zt_1^8+2t_1t_2-6t_1^3t_2+6t_1^5t_2-6t_1t_2^3-6t_1^3t_2^3+6t_2^3t_1^5-2t_1^7t_2+6t_1^7t_2^3$.

Doing the computation one has

$$R = \mathrm{res}_{\mathbb{C}(V)[t_1,Z][t_2]}(G_1, G_2 + ZG_3, t_2) =$$

$= (64(t_1-1)^{16}(t_1+1)^{16}(t_1^2+1)^{18})((-z^2(t_1^2+1)^2(2t_1y - zt_1^2+z)^4) + Z(4t_1(t_1-1)^2(t_1+1)^2(2t_1y-zt_1^2+z)^3(2t_1^4xz^2+z^2t_1^4-18t_1^2-18xt_1^2+4t_1^2xz^2+2z^2t_1^2+2xz^2+z^2))+Z^2(-(t_1-1)^4(t_1+1)^4(2t_1y-zt_1^2+z)^2(3z^4t_1^8-59z^2t_1^6+16t_1^6z^2x^2-16t_1^6xz^2+12z^4t_1^6+32t_1^4z^2x^2+384t_1^4-118z^2t_1^4-336x^2t_1^4-32t_1^4xz^2+48xt_1^4+18z^4t_1^4-59z^2t_1^2+16t_1^2z^2x^2-16t_1^2xz^2+12z^4t_1^2+3z^4))+Z^3(-2t_1(t_1-1)^6(t_1+1)^6(4t_1^8xz^4+7z^4t_1^8-16t_1^6z^2x^2+16t_1^6xz^4-8t_1^6xz^2-73z^2t_1^6+28z^4t_1^6-16t_1^4xz^2+24t_1^4xz^4-32t_1^4z^2x^2-146z^2t_1^4-252xt_1^4+42z^4t_1^4+340t_1^4+256t_1^4x^3-16t_1^2z^2x^2+16t_1^2xz^4-8t_1^2xz^2-73z^2t_1^2+28z^4t_1^2+4z^4x+7z^4)(2t_1y-zt_1^2+z))+Z^4((t_1-1)^8(t_1+1)^8(24t_1^4xz^2+112z^2t_1^4-47z^4t_1^{10}-47z^4t_1^2-188z^4t_1^8-282z^4t_1^6-400t_1^6+24z^6t_1^2+48t_1^6xz^2+60z^6t_1^8+4z^6+16t_1^2z^4x^2+64t_1^4z^4x^2+80z^6t_1^6+64t_1^8z^4x^2+16t_1^{10}z^4x^2+96t_1^6z^4x^2-40t_1^{10}z^4x-128t_1^8z^2x^3+24t_1^8z^2x-128t_1^4z^2x^3+24z^6t_1^{10}+4z^6t_1^{12}-188z^4t_1^4-640t_1^6x^3+640xt_1^6+144t_1^6x^2+60z^6t_1^4+256t_1^6x^4+112z^2t_1^8+224z^2t_1^6-256t_1^6z^2x^3-40t_1^2xz^4-160t_1^8xz^4-240t_1^6xz^4-160t_1^4xz^4)))$

In addition

$$S(t_1) = \mathrm{cont}_Z(R) = 64(t_1-1)^{16}(t_1+1)^{16}(t_1^2+1)^{18}(2t_1y - zt_1^2 + z).$$

Note that S has the constant roots corresponding to the polynomial $64(t_1-1)^{16}(t_1+1)^{16}(t_1^2+1)^{18}$. We cross out these roots, and we obtain the polynomial $\bar{S}$:

$$\bar{S}(t_1) = 2t_1y - zt_1^2 + z.$$

Moreover, since $\bar{S}(t_1)$ is not linear, one deduces that $\mathcal{P}$ is not proper. □

Example 4. Let V be the pipe surface (with rational spine $(t, t^2-1, 0)$) defined by the irreducible polynomial

$F = 2y+56x^2yz^2+32x^2z^2y^2+16x^6+48x^2z^4-32x^2y^3+2x^2-8x^4y+16x^4y^2+48x^4z^2+z^2-90x^2y-52x^2z^2-24z^2y-71x^4-15y^2-96x^2y^2+72y^2z^2--8z^4+24y^3+16y^2z^4+32y^3z^2+64z^4y+16z^6+16y^4.$

We consider the rational parametrization of V (see [13]):

$$\mathcal{P}(t_1,t_2) = \left(\frac{t_1(4t_2+4t_1^2t_2^2+1+t_2^2)}{4t_1^2t_2^2+1+t_2^2}, \frac{-2t_2+4t_1^4t_2^2+t_1^2-3t_1^2t_2^2-1-t_2^2}{4t_1^2t_2^2+1+t_2^2}, \frac{4t_1^2t_2^2-1+t_2^2}{4t_1^2t_2^2+1+t_2^2}\right).$$

Let us check whether $\mathcal{P}(t_1,t_2)$ is a proper parametrization. For this purpose, we apply the algorithm. First we consider the polynomials

$$G_1(t_1,t_2) = 4xt_1^2t_2^2 + x + xt_2^2 - 4t_1t_2 - 4t_1^3t_2^2 - t_1 - t_1t_2^2,$$
$$G_2(t_1,t_2) = 4yt_1^2t_2^2 + y + yt_2^2 + 2t_2 - 4t_1^4t_2^2 - t_1^2 + 3t_1^2t_2^2 + 1 + t_2^2,$$
$$G_3(t_1,t_2) = 4zt_1^2t_2^2 + z + zt_2^2 - 4t_1^2t_2^2 + 1 - t_2^2.$$

Doing the computation one has

$$R = \mathrm{res}_{\mathbb{C}(V)[t_1,Z][t_2]}(G_3, G_2 + ZG_1, t_2) =$$

$= (2y + 16yt_1^4 + 16y^2t_1^4 - 32yt_1^6 + y^2 + 14yt_1^2 + t_1^4 + z^2 + 4z^2t_1^2 + 2t_1^2 + 16t_1^8 + 8y^2t_1^2 - 24t_1^6) + (-16t_1^3z^2 - 2yt_1 - 16yt_1^3 + 2t_1^3 - 16t_1^5 - 32t_1^6x + 2yx + 32t_1^7 - 32yt_1^5 + 14xt_1^2 - 4t_1z^2 + 2t_1 + 32yt_1^4x + 16yxt_1^2 + 16t_1^4x + 2x)Z + (8x^2t_1^2 - 2xt_1 + 4z^2t_1^2 + 16z^2t_1^4 - 32t_1^5x + x^2 - 16t_1^3x - 8t_1^4 + 16x^2t_1^4 - 3t_1^2 + 16t_1^6)Z^2.$

In addition, one obtains that

$$S = \mathrm{cont}_Z(R) = (4t_1^2+1)((-8x^4-16x^2z^2+12yx^2+27x^2-8z^4+6z^2-12z^2y+5y-1-4y^2)t_1+$$
$$+4yx^3 + 5x^3 - 4xz^2 + 4xz^2y - 4xy^2 - 13yx - x).$$

$S(t_1)$ has the constant roots corresponding to the polynomial $4t_1^2 + 1$ in $\mathbb{C}$. We cross out this roots of the polynomial S, and we obtain the polynomial $\overline{S}(t_1)$:

$\overline{S}(t_1) = (-8x^4 - 16x^2z^2 + 12yx^2 + 27x^2 - 8z^4 + 6z^2 - 12z^2y + 5y - 1 - 4y^2)t_1 + 4yx^3 + 5x^3 - 4xz^2 + 4xz^2y - 4xy^2 - 13yx - x.$

Moreover, since $\bar{S}(t_1)$ is linear we obtain the first coordinate of the inverse

$$\alpha(x, y, z) = \frac{(4yx^2 + 5x^2 - 4z^2 + 4z^2y - 4y^2 - 13y - 1)x}{8x^4 + 16x^2z^2 - 12yx^2 - 27x^2 + 8z^4 - 6z^2 + 12z^2y - 5y + 1 + 4y^2}.$$

Reasoning similarly with the variable t_2, we obtain

$$R^* = \mathrm{res}_{\mathbb{C}(V)[t_2,Z][t_1]}(G_1, G_2 + ZG_3, t_1) =$$

$= 64t_2^4(-48zt_2^5 + 96t_2^6z^2 + 16t_2^5 + 16t_2^6 - 16t_2^3 - 64t_2^6z - 64t_2^6z^3 + 4z^2t_2^2 + 16zt_2^3 - 16z^3t_2^3 + 48z^2t_2^5 + 16t_2^6z^4 - 4t_2^2 + 16z^2t_2^3 - 16z^3t_2^5 - 32x^2t_2^4z + 24t_2^4z - 12t_2^4 - 32t_2^4z^3 + 16t_2^4z^4 + 16x^2t_2^4z^2 + 16x^2t_2^4 + 4z^2t_2^4)Z^2 + 64t_2^4(-96xt_2^5z - 64yt_2^4zx - 8xt_2^2 + 32yt_2^4x - 80xt_2^4z + 8z^2t_2^2x + 32xt_2^5 + 32yt_2^4z^2x + 40xt_2^4 + 40xt_2^4z^2 + 96xt_2^5z^2 - 32xt_2^5z^3)Z + 64t_2^4(1 + 40yt_2^4z^2 + 96yt_2^5z^2 - 32yt_2^5z^3 - 80yt_2^4z + 8z^2t_2^2y - 32y^2t_2^4z - 96yt_2^5z - 16y^2t_2^4z^2 + 16y^2t_2^4 + 2z + z^2 + 25t_2^4 + 16t_2^6 + 40t_2^5 - 8t_2^3 - 10t_2^2 + 25z^2t_2^4 + 8zt_2^3 - 64t_2^6z^3 - 64t_2^6z + 96t_2^6z^2 + 16t_2^6z^4 + 32yt_2^5 - 8yt_2^2 + 40yt_2^4 - 50t_2^4z + 10z^2t_2^2 + 8z^2t_2^3 - 120zt_2^5 + 120z^2t_2^5 - 8z^3t_2^3 - 40z^3t_2^5).$

In addition, one obtains that

$$S^* = \mathrm{cont}_Z(R^*) =$$

$= t_2^4(-16 - 848y + 128yzx^2 + 256y^2zx^2 - 16z - 384z^2 - 896y^2 - 624x^2 + 128yx^2 + 320z^2y + 512x^2z^2 + 256x^4 + 256z^4 - 384z^3 - 624zx^2 - 256y^3 + 256z^5 - 256y^3z - 896y^2z + 256y^2x^2 +$

$256y^2z^3+256y^2z^2-848yz+320yz^3+256x^4z+512x^2z^3+(2816y^2z^2+1024y^3z^2-16-2816y^2-688z^2+1024yz^4+448z^2y-1024yx^2+1280x^2z^2-1024y^3+1024yx^2z^2-1280x^2+704z^4-1472y)t_2)$.

$S^*(t_2)$ has the constant roots corresponding to the polynomial t_2^4 in $\mathbb{C}$. We cross out this roots of the polynomial S^*, and we obtain the polynomial $\overline{S}^*$:

$\overline{S}^*(t_2) = -16-848y+128yzx^2+256y^2zx^2-16z-384z^2-896y^2-624x^2+128yx^2+320z^2y+512x^2z^2+256x^4+256z^4-384z^3-624zx^2-256y^3+256z^5-256y^3z-896y^2z+256y^2x^2+256y^2z^3+256y^2z^2-848yz+320yz^3+256x^4z+512x^2z^3+(2816y^2z^2+1024y^3z^2-16-2816y^2-688z^2+1024yz^4+448z^2y-1024yx^2+1280x^2z^2-1024y^3+1024yx^2z^2-1280x^2+704z^4-1472y)t_2$.

Moreover, since $\bar{S}^*$ is linear we obtain the second coordinate of the inverse $\beta(x,y,z)$:

$$-\frac{16z^4+16y^2z^2+20z^2y-24z^2+32x^2z^2-1-53y+8x^2y+16x^2y^2-39x^2-56y^2-16y^3+16x^4}{64z^3y+44z^3-44z^2-64z^2y+176y^2z+64y^3z+80x^2z+64yx^2z+92zy+z-1-92y-176y^2-80x^2-64x^2y-64y^3}\ .$$

Thus, applying the algorithm, we deduce that $\mathcal{P}$ is proper and its inverse mapping is:

$$\mathcal{P}^{-1}(t_1,\, t_2) = \,(\alpha, \beta).$$

$\square$

4 Practical Implementation

This section is devoted to the experimental computing times of the previous algorithm. It focuses on the implementation in Maple of two prototypes of the algorithm. Actual computing times, running on a PC INTEL PENTIUM 350 MHz and 64 MB of RAM, are given in seconds of CPU.

The first implementation assumes that the implicit equation of the surface is known. We use the implicit equation of the surface to carry out the arithmetic over $\mathbb{K}(V)$. Note that, since $I(V) = (F)$, basic arithmetic on $\mathbb{K}[V]$ can be carried out by computing polynomial remainders. Therefore the quotient field $\mathbb{K}(V)$ is computable. In addition, we remark that we compute resultants of polynomials in $\mathbb{K}(V)[t_1, t_2]$ that is a UFD, and we also calculate gcds of univariate polynomials over $\mathbb{K}(V)$, and hence in an euclidean domain. It can be seen clearly that the algorithm outperforms implizitization by Gröbner bases in most cases. However, we note that this comparison is not really fair because the Gröbner bases method allows to compute the implicitization rather than assuming it.

The second implementation avoids the requirement on the implicit equation. For this purpose, elements are represented (not uniquely) as function of polynomials in the variables x, y, z. In order to check zero equality one may use the parametrization.

However, this can be too time consuming. Instead, we test zero-equality by substituting a random point on the surface. The result of this zero test is correct with probability almost one. In addition, if the output is a rational inverse, we also test its correctness by checking it on a randomly chosen point on the surface. It can be seen that the run-time behavior of this algorithm is similar to the deterministic one. Hence we can avoid computing the implicit equation if we accept a probabilistic answer.

In the following table we illustrate the performance of our two implementations, showing times for some parametrizations. In the table we also show the degree of each parametrization. In the appendix, we give the parametrizations considered in this analysis.

Parametrization	Degree	Time of the Deterministic implentation	Time of the Probabilistic implentation	Time using Gröbner basis
P_1	2	0.92	0.74	> 3000
P_2	2	0.11	0.95	> 3000
P_3	2	1.62	1.98	> 2000
P_4	4	0.23	0.17	4.24
P_5	2	6.89	22.68	0.14
P_6	2	10.69	5.85	0.58
P_7	2	0.56	0.58	0.51
P_8	3	0.51	0.36	2.26
P_9	3	0.09	0.14	0.72
P_{10}	3	0.20	0.14	> 3000
P_{11}	3	0.04	0.09	0.59
P_{12}	2	1.50	0.64	> 4000
P_{13}	3	3.50	3.95	> 3000
P_{14}	3	11.23	43.86	> 3000
P_{15}	3	1.21	2.07	6.22
P_{16}	2	1.91	2.50	> 3000
P_{17}	2	21.49	56.92	> 4000
P_{18}	7	0.41	0.40	4.53
P_{19}	8	0.52	0.55	7.44
P_{20}	8	0.43	0.38	6.65
P_{21}	7	0.4	0.52	> 4000

5 Appendix: Parametrizations in Section 4

$P_1 = \left(\frac{t_1}{t_1+t_2}, \frac{{t_1}^2-t_1+1}{t_2+1}, {t_1}^2 + t_2\right)$

$P_2 = \left(\frac{t_1^2+t_2}{t_1+t_2}, \frac{t_1^2-t_1+1}{t_2^2+1}, \frac{t_1^2+t_2}{t_1}\right)$

$P_3 = \left(\frac{2t_2^2-3t_1^2}{-2t_1^2-t_1t_2}, \frac{-2t_1t_2}{2t_1-t_2^2}, \frac{t_2^2+2t_1t_2}{3t_2+3t_1}\right)$

$P_4 = \left(\frac{-3-3t_1^3t_2}{t_2}, \frac{3t_1t_2-2t_1^2}{2t_2}, \frac{-2+3t_1}{3t_2}\right)$

$P_5 = (3 - t_2 - t_1 - 3t_1t_2 + 2t_2^2 + 3t_1^2, -1 - 2t_2 + t_1 - 3t_1t_2 + 2t_2^2 + 3t_1^2, -2 - 2t_2 - 3t_1^2)$

$P_6 = (t_2 + 2t_1t_2 - 3t_1^2 - t_2^2, 3 + t_2 + 2t_1 + 2t_1t_2 + 3t_1^2, 1 + 2t_2 + 2t_1 - 2t_1t_2 - 2t_1^2)$

$P_7 = \left(t_1 + t_2t_1^2 - 1, \frac{t_2^2}{t_2+t_1t_2+1}, t_1 + t_2t_1^2 - t_2\right)$

$P_8 = \left(\frac{t_1t_2+1+t_1}{t_2^2+t_1^2}, \frac{1}{t_2^2+t_1^2}, \frac{t_2^2+t_1^2+1}{(t_2^2+t_1^2)t_2}\right)$

$P_9 = \left(\frac{t_1(t_1+t_1t_2-1)}{t_1-t_1^2+t_2}, \frac{-t_1+t_2}{t_1-t_1^2+t_2}, \frac{t_1+t_2}{t_1-t_1^2+t_2}\right)$

$P_{10} = \left(\frac{t_1t_2+1}{t_1+1}, \frac{t_2}{t_1-1}, \frac{t_1+t_2}{(t_1^2-1)t_2}\right)$

$P_{11} = \left(\frac{t_1}{t_1^2+t_1t_2-3}, \frac{t_1+t_2^3}{t_1+t_2}, \frac{1}{t_1+t_2}\right)$

$P_{12} = \left(\frac{t_1+t_2^2}{t_1+t_2-1}, \frac{t_1^2+t_2+3t_1}{t_1}, \frac{t_1}{t_2+t_1t_2+1}\right)$

$P_{13} = \left(\frac{t_1^2+t_2^2+t_1^3}{t_1^2}, t_2 + t_1^3, \frac{t_1}{t_2^2+1}\right)$

$P_{14} = \left(\frac{t_1^2+t_2^2+t_1^3}{t_1^2}, \frac{t_2+t_1^3}{t_1+t_2-4}, \frac{t_1}{t_2^2+1}\right)$

$P_{15} = (3t_2 + 3t_2t_1^2 - t_2^3, 3t_1 + 3t_2^2t_1 - t_1^3, 3t_2^2 - 3t_1^2)$

$P_{16} = \left(\frac{-3-t_1t_2+t_2-2t_1}{2}, \frac{1-2t_2-3t_2^2+t_1^2}{-1+2t_1}, \frac{-3-3t_1t_2-2t_1+3t_2^2-t_1^2}{3}\right)$

$P_{17} = \left(\frac{2t_1t_2+2t_2^2-3t_1^2}{t_1(t_2+1)}, -\frac{t_1(2t_2+1)}{t_2+3t_1}, \frac{2+t_1t_2+3t_2^2}{t_2(-2t_1+t_2)}\right)$

$$P_{18} = \Big(\frac{-t_1(2+t_1(24t_2^5-113t_2^4+15t_2^3-133t_2^2-110t_2+42))}{t_1(-19t_2^5-47t_2^4+68t_2^3-72t_2^2-87t_2+79)+1+t_1^2(80t_2^5+72t_2^4+66t_2^3-29t_2^2-91t_2-53)},$$

$$\frac{-1-t_1(43t_2^5-66t_2^4-53t_2^3-61t_2^2-23t_2-37)+t_1^2(80t_2^5+72t_2^4+66t_2^3-29t_2^2-91t_2-53)}{t_1(-19t_2^5-47t_2^4+68t_2^3-72t_2^2-87t_2+79)+1+t_1^2(80t_2^5+72t_2^4+66t_2^3-29t_2^2-91t_2-53)},\ t_2\Big)$$

$$P_{19} = \Big(\frac{(-18t_2^6-58t_2^4-54t_2^3+98t_2^2-191t_2+18-45t_2^5)t_1^2-2t_1}{t_1(-59t_2^6+45t_2^5-8t_2^4-93t_2^2+92t_2+43)+1+t_1^2(97t_2^6+50t_2^5+79t_2^4+56t_2^3+49t_2^2+63t_2)},$$

$$\frac{t_1^2(97t_2^6+50t_2^5+79t_2^4+56t_2^3+49t_2^2+63t_2)+(5t_2^2-77t_2^6-66t_2^4-54t_2^3-99t_2+61)t_1-1}{t_1(-59t_2^6+45t_2^5-8t_2^4-93t_2^2+92t_2+43)+1+t_1^2(97t_2^6+50t_2^5+79t_2^4+56t_2^3+49t_2^2+63t_2)},\ t_2\Big)$$

$$P_{20} = \Big(\frac{-t_1(2+t_1(-31t_2^6+70t_2^5-85t_2^3-104t_2^2-79t_2+126))}{t_1(-4t_2^6+38t_2^5-61t_2^3-58t_2^2-91t_2+45)+1+t_1^2(58t_2^6-92t_2^5-8t_2^3+36t_2^2-90t_2+96)},$$

$$\frac{t_1^2(58t_2^6-92t_2^5-8t_2^3+36t_2^2-90t_2+96)-t_1(-27t_2^6+32t_2^5-24t_2^3-46t_2^2+12t_2+81)-1}{t_1(-4t_2^6+38t_2^5-61t_2^3-58t_2^2-91t_2+45)+1+t_1^2(58t_2^6-92t_2^5-8t_2^3+36t_2^2-90t_2+96)}, t_2\Big)$$

$$P_{21} = \Big(\frac{-t_1(t_1(-37+155t_2^5+100t_2^4-121t_2^3+122t_2^2+58t_2)+2)}{t_1(63t_2^5+57t_2^4-59t_2^3+45t_2^2-8t_2-91)+1+t_1^2(28t_2^5+154t_2^4-9t_2^3+124t_2^2+48t_2-43)},$$

$$\frac{t_1^2(-6-127t_2^5+54t_2^4+112t_2^3+2t_2^2-10t_2)-1+t_1(-56-92t_2^5-43t_2^4+62t_2^3-77t_2^2-66t_2)}{t_1(63t_2^5+57t_2^4-59t_2^3+45t_2^2-8t_2-91)+1+t_1^2(28t_2^5+154t_2^4-9t_2^3+124t_2^2+48t_2-43)},\ t_2\Big)\,.$$

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