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Computing all parametric solutions for blending parametric surfaces

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Abstract

In this paper we prove that, for a given set of parametric primary surfaces and parametric clipping curves, all parametric blending solutions can be expressed as the addition of a particular parametric solution and a generic linear combination of the basis of a free module of rank 3. As a consequence, we present an algorithm that outputs a generic expression for all the parametric solutions for the blending problem. In addition, we also prove that the set of all polynomial parametric solutions (i.e. solutions that have polynomial parametrizations) for a parametric blending problem can also be expressed in terms of the basis of a free module of rank 3, and we prove an algorithmic criterion to decide whether there exist parametric polynomial solutions. As a consequence we also present an algorithm that decides the existence of polynomial solutions, and that outputs (if this type of solution exists) a generic expression for all polynomial parametric solutions for the problem.

1. Introduction

Computing blending and modelling surfaces is one of the central problems in computer aided geometric design (see e.g. Hoffmann, 1993; Hoschek and Lasser, 1993). In many applications, objects are modelled as a collection of several surfaces whose pieces join smoothly. This situation leads directly to the blending problem in the sense that a blending surface is a surface that provides a smooth transition between distinct geometric features of an object (see e.g. Hartmann, 1995; Hoffmann and Hopcroft, 1986, 1987; Warren, 1986).

More precisely, if one is given a collection of primary surfaces $V_1, \ldots, V_n$ (surfaces to be blended), and a collection of auxiliary surfaces $U_1, \ldots, U_n$ (clipping surfaces), then the blending problem deals with the computation of a surface V containing the space curves

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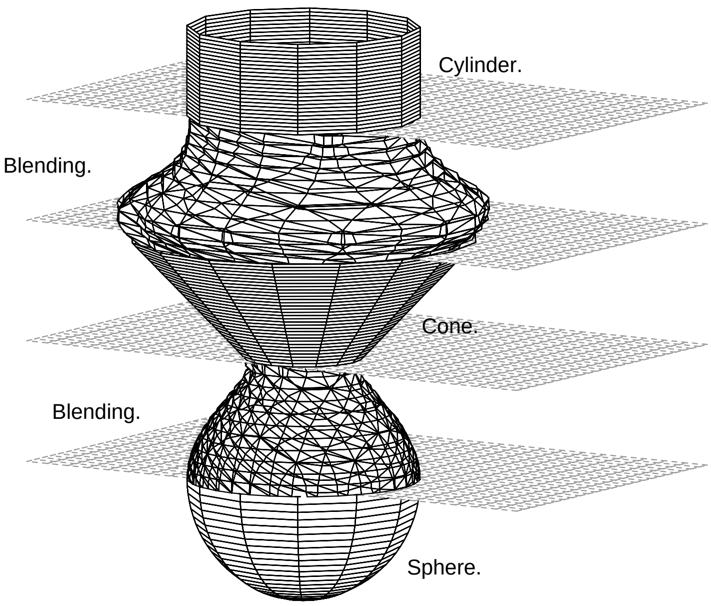

Fig. 1. Primary surfaces (cylinder, cone, sphere), clipping surfaces (planes parallel to the floor), and blending surface.

$C_i = U_i \cap V_i$, and such that V meets each V_i at C_i with "certain" smooth conditions (G^k-continuity, see DeRose, 1985) . Intuitively speaking the G^k-continuity consists in requiring that the Taylor expansions at C_i of the different pieces of the object agree till certain order with the corresponding Taylor expansion of the blending surface. In Fig. 1, we illustrate an example of a blending where the primary surfaces are a cylinder, a cone and a sphere, and the clipping surfaces are planes parallel to the floor.

The blending problem can be approached from two different points of view, namely, implicitly (see Hoffmann and Hopcroft, 1987; Warren, 1989), where an implicit expression of the solution is computed, or parametrically (see Filip, 1989; Hartmann, 2001a,b; Pérez-Díaz and Sendra, 2001; Pottman and Wallner, 1997; Vida et al., 1994) where parametric outputs are reached.

In addition, one may also consider two different types of statements for the parametric version of the problem. On one hand, one may work with global parametrizations of the geometric objects, i.e. with rational curves and surfaces, and, on the other, one may deal with local parametrizations, which implies that the set of possible data is bigger (see e.g. Hoffmann, 1993; Vida et al., 1994). Furthermore, a second consideration, depending on whether either symbolic or numerical techniques are used, can be made (see Bajaj et al., 1993; Hartmann, 1998; Hoschek and Lasser, 1993 for numerical techniques, and Hoschek and Lasser, 1993; Vida et al., 1994 for symbolic techniques).

In this paper, we are interested in the symbolic global parametric version of the problem. That is, we consider that surfaces and curves are rational and that they are given by global parametrizations, and we develop symbolic methods to derive global parametrizations of the solutions. In the following we will refer to this problem as the parametric blend problem. As an interesting open problem, one may consider the extension of these ideas to the case where geometric objects are given by local parametrizations, and therefore an additional effort has to be done to control the domains of definitions.

For the implicit blending, Hoffmann and Hopcroft proved that using the potential method (see Hoffmann and Hopcroft, 1987) one may compute all possible implicit solutions of degree 4 for the case of two quadrics and with G^1 geometric continuity. Afterwards, Warren (see Warren, 1986) extended Hoffmann and Hopcroft's results to the general case, stating that all solutions are in the intersection of some polynomial ideals generated by the implicit equations of V_i, and powers of the equations of U_i. This result (that we will refer as Hoffmann–Warren's theorem) gives a description of the space of surface solutions (non-necessarily rational) for the blending problem.

For the parametric blending, although there exist algorithmic achievements (see Filip, 1989; Hartmann, 2001a; Pérez-Díaz and Sendra, 2001; Pottman and Wallner, 1997; Vida et al., 1994), they only provide partial answers in the sense that only partial families of rational blending surfaces are computed. In many cases, these approaches can be used satisfactorily for applied purposes as modelling surfaces. Nevertheless, from a theoretical point of view there is no "parametric version" of Hoffmann–Warren's theorem that algebraically structures the space of all parametric solutions of the blending problem. Of course, one may try a straightforward approach that first computes the space of all implicit equations of the blending problem to afterwards apply parametrization algorithms to derive the parametric solution (see Abhyankar and Bajaj, 1989; Schicho, 1998). However, parametrization algorithms are time consuming (see Mňuk et al., 1997), and on the other hand deciding which implicit solutions are rational is a very hard problem that would require the development of parametrization algorithms for families of surfaces depending on parameters.

Another interesting open problem in this context is the computation and characterization of existence of polynomial parametric solutions (note that the generation of polynomial blendings, i.e. polynomial parametrizations that are blendings, is important in applications; for instance one avoids the unstable numerical behaviour of the denominators when tracing the surface), as well as the theoretical study of the corresponding set of solutions.

In this paper we deal with these problems, and we give theoretical and algorithmic answers. We prove that for a given set of parametric primary surfaces and parametric clipping curves the set of all parametric solutions can be directly related to a free module of rank 3 (see Section 4). More precisely, we prove that any parametric solution of a parametric blending problem can be expressed as the addition of a particular parametric solution and a generic linear combination of the basis of the module. Furthermore, since the basis of the module of solutions is explicitly computed, this result provides an algorithm that outputs a generic expression for all the parametric solutions for the problem (see Section 7). Moreover, in order to have a complete algorithm one needs to determine a single particular parametric solution. Therefore, an *auxiliary* algorithm for computing any particular parametric solution is required. For this purpose, we extend Hartmann's method in Hartmann (2001a) to the case of n surfaces and our method in Pérez-Díaz and Sendra (2001) to the case of G^k geometric continuity (see Section 5). Also a comparison analysis of these two methods is presented. This comparison analysis focuses on different aspects such as: algebraic manipulation required in the algorithms, upper bounds of the degrees of the output parametrizations, capability of the methods to provide polynomial parametrizations as outputs and actual computing times in the implementation; the particular parametric inputs taken in the real time analysis appear in the Appendix.

In addition, we also prove that the set of all polynomial parametric solutions for a parametric blending problem can also be expressed in terms of a free module of rank 3, in this case over a bivariate polynomial ring. More precisely, we prove that any polynomial parametric solution of a parametric blending problem can be expressed as the addition of a particular polynomial parametric solution and a generic linear combination of the basis of the module; which is explicitly obtained. Moreover, we state an algorithmic criterion to decide whether there exist parametric polynomial solutions and we prove that the extension of the method in Pérez-Díaz and Sendra (2001) always reaches a polynomial parametrization if there exists any (see Section 6). As a consequence we present an algorithm that decides the existence of polynomial solutions, and that outputs (if this type of solution exists) a generic expression for all polynomial parametric solutions for the problem (see Section 7).

Throughout this paper, $\mathbb{K}$ is a field of characteristic zero (in practical applications, $\mathbb{K}$ can be taken as a computable subfield of the field of the real numbers). Surfaces and curves are seen as affine varieties over the algebraic closure of $\mathbb{K}$, but implicit equations and parametrizations are taken over $\mathbb{K}$. Also, all rational functions are supposed to be expressed in reduced form; i.e. where numerators and denominators are coprime.

2. Preliminaries on blending surfaces

This section is preliminary and we report on the basic definitions and results that will be used throughout the paper. We start with the concept of blending surface for a family $\mathcal{S}$ of finitely many irreducible surfaces. Intuitively speaking, a blending surface is a surface meeting the elements in $\mathcal{S}$ with certain "smoothness" at some prescribed curves.

The precise meaning of "smoothness" is formalized in the concept of G^k-continuity (geometric continuity). The geometric continuity provides information on how smoothly two irreducible surfaces V_1, V_2 meet at a given space curve C. Thus, zero geometric continuity requires that $C \subset V_1 \cap V_2$, G^1-continuity imposes that tangent planes at V_1, V_2 agree along C, and for $k \geq 1$ the concept is equivalent to asking that the multiplicity of intersection of V_1, V_2 at C is at least $k+1$ (see Garrity and Warren, 1991). More precisely, the notion of G^k-continuity can be defined as follows (see e.g. Garrity and Warren, 1991; Warren, 1986).

Definition 1. Let V_1, V_2 be irreducible surfaces, and let $C \subset V_1 \cap V_2$ be an irreducible curve such that V_1, V_2 are smooth at all but finitely many points on C. Then, we say that *V_1 meets V_2 at C with G^k-continuity* if there exist two polynomials $A, B \in \mathbb{K}[x_1, x_2, x_3]$, not identically zero along C, such that all derivatives of $AF_1 - BF_2$ up to order k vanish along C, where F_1, and F_2 are the implicit equations of V_1, and V_2 respectively.

For the case of rational surfaces (that is the one we are interested in) the notion of G^k-continuity can be characterized as follows (see DeRose, 1985; Garrity and Warren, 1991; Liang et al., 1995).

Proposition 1. *Let V_1, V_2 be rational surfaces, and let $C \subset V_1 \cap V_2$ be an irreducible curve such that V_1, V_2 are smooth at all but finitely many points on C. Then, the following statements are equivalent:*

(1) *V_1 meets V_2 at C with G^k-continuity.*
(2) *There exist rational parametrizations $\mathcal{P}_1(t,h)$, $\mathcal{P}_2(t,h)$ of V_1, V_2 respectively such that all partial derivatives of $\mathcal{P}_1(t,h)$, and $\mathcal{P}_2(t,h)$ up to order*[1] *k agree along C.* □

In this situation the notion of blending surfaces is defined as follows.

Definition 2. Let $\overline{V} = (V_1, \ldots, V_n)$, $n \geq 2$, be an n-tuple of irreducible surfaces, and let $\overline{C} = (C_1, \ldots, C_n)$ be an n-tuple of irreducible curves such that $C_i \subset V_i$ and V_i is smooth at all but finitely many points on C_i. Then, we say that a surface W is a *G^k-blending surface for* $(\overline{V}, \overline{C})$ if for $i = 1, \ldots, n$ it holds that

(1) W is smooth at all but finitely many points on C_i,
(2) W and V_i meet at C_i with G^k-continuity.

A pair $(\overline{V}, \overline{C})$ as above is called a *blending data*. Furthermore, $\overline{V}$ is called the *vector of primary surfaces*, and $\overline{C}$ the *vector of clipping curves*. We will refer to the coordinate surfaces of $\overline{V}$ as the *primary surfaces* and to the coordinate curves of $\overline{C}$ as the *clipping curves*.

The following theorem is proved in Hoffmann and Hopcroft (1986) and Warren (1986), and states the form of all blending surfaces.

Theorem 1. *Let $\overline{V}$ be a vector of primary surfaces, and let $\overline{C}$ be a vector of disjoint clipping curves, such that each C_i is the intersection of V_i with an auxiliary surface U_i. Then, the set of all G^k-blending surfaces for $(\overline{V}, \overline{C})$ is included in the ideal*

$$\bigcap_{i=1}^{n} (g_i, h_i^{k+1}),$$

where g_i and h_i are the implicit equations of V_i and U_i, respectively. □

3. The parametric blending problem

Taking into account Theorem 1, the computation and analysis of blending surfaces can be approached by means of elimination theory techniques; for instance with Gröbner basis. Moreover, in Warren (1986), the author shows how to deal with the problem, for special cases, avoiding Gröbner basis computation.

[1] In this paper, whenever we say "derivatives up to order k", we mean order from 0 to k, understanding as usual that the zero order derivative is the rational function whose derivatives are considered.

Nevertheless, if one is interested in computing a parametric representation of a rational blending surface, the problem needs to be approached differently. Note that, even having a generic implicit expression of a single solution, one still would need to check the rationality and to apply parametrization algorithms (see Abhyankar and Bajaj, 1988, 1989; Schicho, 1998; Sendra and Winkler, 1991, 1997) in order to achieve a parametric solution for parametric inputs. In this paper we deal with this problem and we provide a method to generate all the parametric solutions without computing the implicit equations. Thus, we give a parametric counterpart version of Theorem 1.

More precisely, we will deal here with the problem of finding parametric blending surfaces for a tuple of rational primary surfaces, and a tuple of rational clipping curves. Furthermore, we will assume that we are given rational parametrizations of the primary surfaces such that under a suitable substitution of the parameters by univariate rational functions, one gets the clipping curves. Thus, our input will be a vector of rational surface parametrizations of the form

$$\overline{\mathcal{P}} = (\mathcal{P}_1(t, h), \ldots, \mathcal{P}_n(t, h)),$$

and a tuple of pairs of univariate rational functions

$$\overline{R} = ((M_1(t), N_1(t)), \ldots, (M_n(t), N_n(t))),$$

such that for $i = 1, \ldots, n$

$$\mathcal{P}_i(M_i(t), N_i(t))$$

parametrizes the ith clipping curve. Therefore, $\overline{\mathcal{P}}$, $\overline{R}$ and $(\mathcal{P}_i(M_i(t), N_i(t)))_{1\leq i\leq n}$ play the role of the primary surfaces, the auxiliary surfaces and the clipping curves, respectively.

We observe that, for every $s_0, \ldots, s_{n-1} \in \mathbb{K}$, where $s_i \neq s_j$ if $i \neq j$, one can reparametrize $\mathcal{P}_i(t, h)$ as

$$\mathcal{P}_i^\star(t, h) = \mathcal{P}_i(M_i(t), N_i(t) + h - s_{i-1}),$$

and therefore it holds that

$$\mathcal{P}_i^\star(t, s_{i-1}) = \mathcal{P}_i(M_i(t), N_i(t)).$$

Hence, one can always assume w.l.o.g. that the auxiliary tuple of a pair of univariate rational functions is of the form

$$((t, s_0), \ldots, (t, s_{n-1})).$$

This remark motivates the following definitions.

Definition 3. Let $(\overline{V}, \overline{C})$ be a blending data such that all primary surfaces and clipping curves are rational. Then, a *rational blending data* for $(\overline{V}, \overline{C})$ is a pair $(\overline{\mathcal{P}}, \bar{s})$ such that

(1) $\overline{\mathcal{P}} = (\mathcal{P}_1(t, h), \ldots, \mathcal{P}_n(t, h)) \in (\mathbb{K}(t, h)^3)^n$, and $\mathcal{P}_i(t, h)$ is a rational parametrization of the ith primary surface V_i.
(2) $\bar{s} = (s_0, \ldots, s_{n-1}) \in \mathbb{K}^n$ is a vector of n different field elements.
(3) For $i = 1, \ldots, n$, $\mathcal{P}_i(t, s_{i-1})$ parametrizes the ith clipping curve C_i.

Definition 4. Let $(\overline{\mathcal{P}}, \bar{s})$ be a rational blending data. Then, we say that a surface W is a *rational G^k-blending surface* for $(\overline{\mathcal{P}}, \bar{s})$ if W is rational and it has a rational parametrization $\mathcal{B}(t, h)$ such that for $i = 1, \dots, n$ all partial derivatives up to order k of the ith parametrization component of $\overline{\mathcal{P}}$ and of $\mathcal{B}(t, h)$ agree at (t, s_{i-1}). We say that $\mathcal{B}(t, h)$ is a *parametric solution for* $(\overline{\mathcal{P}}, \bar{s})$.

In this situation, the *parametric G^k-continuity blending problem* can be stated as follows:

Initial statement

- *Given* a rational blending data $(\overline{\mathcal{P}}, \bar{s})$.
- *Compute* a parametric representation of all rational G^k-blending surfaces for $(\overline{\mathcal{P}}, \bar{s})$; i.e. a rational parametrization $\mathcal{B}(t, h)$ of all rational G^k-blending surface for $(\overline{\mathcal{P}}, \bar{s})$, such that for $i = 1, \dots, n$ all partial derivatives up to order k of $\mathcal{P}_i(t, h)$ and of $\mathcal{B}(t, h)$ agree at (t, s_{i-1}).

In the following, we show that one can give a simpler, but equivalent, formulation of the problem.

Proposition 2. *$\mathcal{B}(t, h)$ is a parametric solution for $(\overline{\mathcal{P}}, \bar{s})$ if and only if*

$$\frac{\partial^j \mathcal{B}}{\partial^j h}(t, s_{i-1}) = \frac{\partial^j \mathcal{P}_i}{\partial^j h}(t, s_{i-1}) \qquad \text{for } j = 0, \dots, k, i = 1, \dots, n.$$

Proof. Clearly if $\mathcal{B}$ is a parametric solution, the condition is satisfied. Conversely, the condition for $j = 0$ implies that $\mathcal{B}(t, s_{i-1})$ parametrizes C_i. Thus, it only remains to prove that

$$\frac{\partial^{j_1+j_2} \mathcal{B}}{\partial^{j_1} h\, \partial^{j_2} t}(t, s_{i-1}) = \frac{\partial^{j_1+j_2} \mathcal{P}_i}{\partial^{j_1} h\, \partial^{j_2} t}(t, s_{i-1}), \qquad j_1 + j_2 = 1, \dots, k, i = 1, \dots, n.$$

However, since

$$\frac{\partial^{j_1} \mathcal{B}}{\partial^{j_1} h}(t, s_{i-1}) = \frac{\partial^{j_1} \mathcal{P}_i}{\partial^{j_1} h}(t, s_{i-1}), \qquad j_1 = 1, \dots, k, i = 1, \dots, n,$$

and taking into account that if $M(t, h) \in \mathbb{K}(t, h)$ then

$$\frac{\partial^{j_1+j_2} M}{\partial^{j_1} h\, \partial^{j_2} t}(t, s_{i-1}) = \frac{\partial^{j_2}}{\partial^{j_2} t}\left(\frac{\partial^{j_1} M}{\partial^{j_1} h}(t, s_{i-1})\right)$$

one concludes the proof. □

Therefore, the *parametric G^k-continuity blending problem* can be reformulated as follows:

Reduced (but equivalent) statement

- *Given* a rational blending data $\mathcal{S} = (\overline{\mathcal{P}}, \bar{s})$; i.e. the coordinates $\mathcal{P}_i(t, h)$ of $\overline{\mathcal{P}}$ are rational parametrizations of the primary surfaces, and $\mathcal{P}(t, s_{i-1})$ parametrizes the clipping curves.

- *Compute* all the parametric solutions; i.e. all rational surface parametrizations $\mathcal{B}(t,h)$ such that

$$\frac{\partial^j \mathcal{B}}{\partial^j h}(t, s_{i-1}) = \frac{\partial^j \mathcal{P}_i}{\partial^j h}(t, s_{i-1}) \qquad \text{for } j = 0, \ldots, k.$$

In the sequel, whenever we speak about the parametric G^k-continuity blending problem we will be considering the reduced version of it. Moreover, we will write "a parametric solution for $(\overline{\mathcal{P}}, \bar{s})$" meaning "a parametric solution to the parametric G^k-continuity blending problem for the rational blending data $(\overline{\mathcal{P}}, \bar{s})$".

4. Structure of the space of rational solutions

In this section we analyse the algebraic structure of the space of rational solutions for a rational blending data $\mathcal{S}$. We prove that the set of all parametric solutions for $\mathcal{S}$ can be directly related to a free module of rank 3. More precisely, we prove that any parametric solution can be expressed as the addition of a particular parametric solution and a generic linear combination of the basis of the module.

For this purpose, throughout this section we fix a rational blending data $(\overline{\mathcal{P}}, \bar{s})$, where $\overline{\mathcal{P}} = (\mathcal{P}_1(t,h), \ldots, \mathcal{P}_n(t,h))$ and $\bar{s} = (s_0, \ldots, s_{n-1})$ (note that $s_i \neq s_j$ if $i \neq j$). Also, we introduce the set

$$\mathcal{A}_{\bar{s}} = \left\{ \frac{A(t,h)}{B(t,h)} \in \mathbb{K}(t,h) \;\middle|\; B(t, s_{i-1}) \neq 0 \text{ for } i = 1, \ldots, n \right\}.$$

Note that $\mathcal{A}_{\bar{s}}$ is a subring of $\mathbb{K}(t,h)$. Furthermore, observe that if $A/B \in \mathbb{K}(t,h)$ and $B(t, s_{i-1}) = 0$, by Bézout's theorem (see e.g. Walker, 1950) the plane curve defined by $B(t,h)$ and the line $h = s_{i-1}$ have infinitely many common points, and therefore $(h - s_{i-1})$ divides $B(t,h)$. Conversely, if $(h - s_{i-1})$ divides $B(t,h)$, then $B(t, s_{i-1}) = 0$. Therefore, the commutative ring $\mathcal{A}_{\bar{s}}$ can be expressed as

$$\mathcal{A}_{\bar{s}} = \left\{ \frac{A(t,h)}{B(t,h)} \in \mathbb{K}(t,h) \;\middle|\; \gcd\left(\prod_{i=0}^{n-1} (h - s_i), B \right) = 1 \right\}.$$

Moreover, we consider the free $\mathcal{A}_{\bar{s}}$-module of rank 3 $(\mathcal{A}_{\bar{s}})^3$, and we denote it by $\mathbb{M}$:

$$\mathbb{M} = (\mathcal{A}_{\bar{s}})^3.$$

In this situation, one has the following theorem.

Theorem 2. *Let $\mathcal{B}(t,h)$ be a particular parametric solution for $(\overline{\mathcal{P}}, \bar{s})$. Then, the set of all the parametric solutions for $(\overline{\mathcal{P}}, \bar{s})$ can be expressed as*

$$\left\{ \mathcal{B}(t,h) + \mathcal{N}(t,h) \mid \mathcal{N} \in \mathbb{M}, \text{ and } \frac{\partial^j \mathcal{N}}{\partial^j h}(t, s_{i-1}) = 0,\; j = 0, \ldots, k,\; i = 1, \ldots, n \right\}.$$

Proof. Let Σ be the set of all the parametric solutions for $(\overline{\mathcal{P}}, \bar{s})$, and Ω the set in the statement of the theorem. Let $\mathcal{M}(t,h) = \mathcal{B}(t,h) + \mathcal{N}(t,h) \in \Omega$. Then one has that

$$\frac{\partial^j \mathcal{M}}{\partial^j h}(t, s_{i-1}) = \frac{\partial^j \mathcal{P}_i}{\partial^j h}(t, s_{i-1}), \qquad j = 0, \ldots, k, i = 0, \ldots, n.$$

Thus, by Proposition 2, $\mathcal{M}(t,h) \in \Sigma$. Conversely, let $\mathcal{R}(t,h) \in \Sigma$, and let

$$\mathcal{N}(t,h) = \mathcal{R}(t,h) - \mathcal{B}(t,h).$$

Then since $\mathcal{R}, \mathcal{B} \in \Sigma$, it holds that all partial derivatives w.r.t. h of $\mathcal{R}(t,h)$, and $\mathcal{B}(t,h)$ up to order k agree at the point (t, s_{i-1}), and therefore all partial derivatives w.r.t. h of $\mathcal{N}(t,h)$, up to order k, vanish at the point (t, s_{i-1}). Hence, $\mathcal{R}(t,h) = \mathcal{B}(t,h) + \mathcal{N}(t,h) \in \Omega$. □

The geometric interpretation of Theorem 2 is as follows. Any parametric solution can be expressed as the addition of a particular parametric solution and a parametrization of a variety in $\mathbb{K}^3$ of dimension less than or equal to 2, having the origin as a singularity of multiplicity at least $n(k+1)$. Note that we have not excluded zero dimensional varieties. In this case the parametrization to add to the particular solution is the origin, and consequently is not really a parametrization.

Also, one can interpret Theorem 2 in terms of systems of constraints. For this purpose, we consider the system of partial differential equations in $\mathbb{M}$:

$$\mathcal{E} = \left\{ \frac{\partial^j E}{\partial^j h}(t, s_{i-1}) = \frac{\partial^j \mathcal{P}_i}{\partial^j h}(t, s_{i-1}) \right\} \qquad \begin{array}{l} j = 0, \ldots, k, \\ i = 1, \ldots, n. \end{array}$$

Then, by Proposition 2, one has that the set of all the parametric solutions for $(\overline{\mathcal{P}}, \bar{s})$ is the set of all surface parametrizations (i.e. elements in $\mathbb{K}(t,h)^3 \backslash \mathbb{K}^3$) satisfying $\mathcal{E}$.

On the other hand, associated with $\mathcal{E}$, one can consider the homogeneous system of partial differential equations to $\mathcal{E}$, namely

$$\mathcal{E}_{\mathrm{H}} = \left\{ \frac{\partial^j E}{\partial^j h}(t, s_{i-1}) = 0 \right\} \qquad \begin{array}{l} j = 0, \ldots, k, \\ i = 1, \ldots, n. \end{array}$$

In this situation the elements $\mathcal{N}(t,h) \in \mathbb{M}$, introduced in Theorem 2 are the solutions of $\mathcal{E}_{\mathrm{H}}$ in $\mathbb{M}$. Therefore, Theorem 2 can be stated as follows:

Theorem 3. *A "general" parametric solution for $(\overline{\mathcal{P}}, \bar{s})$ can be expressed as the addition of a particular solution of the non-homogeneous system $\mathcal{E}$ and the "general" solution of the homogeneous system $\mathcal{E}_{\mathrm{H}}$.* □

In the following we investigate the algebraic structure of the set of solutions of the homogeneous system $\mathcal{E}_{\mathrm{H}}$. This study will allow us to be more precise with the meaning of "general solution". We start with the next lemma.

Lemma 1. *The set of solutions of $\mathcal{E}_{\mathrm{H}}$ in $\mathbb{M}$ is a submodule of $\mathbb{M}$ that we denote by $\mathbb{M}_{\mathrm{H}}$.*

Proof. Clearly $\mathbb{M}_{\mathrm{H}} \neq \emptyset$ since it contains the zero solution. First, we observe that if $\mathcal{N}_1, \mathcal{N}_2 \in \mathbb{M}_{\mathrm{H}}$ then $\mathcal{N}_1 + \mathcal{N}_2 \in \mathbb{M}_{\mathrm{H}}$. Now, let $R = A/B \in \mathcal{A}_{\bar{s}}$, and $\mathcal{N} \in \mathbb{M}_{\mathrm{H}}$. Since

$B(t, s_{i-1}) \neq 0$, and $\mathcal{N} \in \mathbb{M}$, $R(t,h)\mathcal{N}(t,h)$ is defined at (t, s_{i-1}), as well as its partial derivatives. Furthermore,

$$R(t, s_{i-1})\mathcal{N}(t, s_{i-1}) = 0 \qquad \text{for } i = 1, \ldots, n.$$

Moreover, since

$$\frac{\partial^j \mathcal{N}}{\partial^j h}(t, s_{i-1}) = 0, \qquad \text{for } j = 0, \ldots, k, i = 1, \ldots, n,$$

by Leibnitz's formula on the partial derivative of a product, one deduces that

$$\frac{\partial^j (R\mathcal{N})}{\partial^j h}(t, s_{i-1}) = \sum_{i=1}^{j} \binom{i}{j} \frac{\partial^i R}{\partial^i h}(t, s_{i-1}) \frac{\partial^{j-i} \mathcal{N}}{\partial^{j-i} h}(t, s_{i-1}) = 0$$

for $j = 1, \ldots, k$, and $i = 1, \ldots, n$. Therefore, $R\mathcal{N} \in \mathbb{M}_{\mathrm{H}}$, and $\mathbb{M}_{\mathrm{H}}$ is a submodule of $\mathbb{M}$. $\square$

Furthermore, one can be more precise and compute a basis of the submodule $\mathbb{M}_{\mathrm{H}}$.

Lemma 2. *Let* $A(h) = \prod_{i=0}^{n-1}(h - s_i)^{k+1}$. *Then,* $\{e_1, e_2, e_3\}$, *where*

$$e_1 = (A(h), 0, 0), \qquad e_2 = (0, A(h), 0), \qquad e_3 = (0, 0, A(h)),$$

is a basis of the submodule $\mathbb{M}_{\mathrm{H}}$.

Proof. Clearly $\{e_1, e_2, e_3\}$ is linearly independent. We now see that it generates the submodule. Let

$$\Sigma = \left\{ \sum_{i=1}^{3} R_i(t,h) e_i \;\middle|\; R_i \in \mathcal{A}_{\bar{s}} \right\}.$$

We have to prove that $\mathbb{M}_{\mathrm{H}} = \Sigma$. First we observe that e_i are clearly elements of $\mathbb{M}_{\mathrm{H}}$, since they are solutions of $\mathcal{E}_{\mathrm{H}}$. Let $R \in \Sigma$. Then, R can be written as

$$R = A(h)(R_1, R_2, R_3) \qquad \text{where } R_i \in \mathcal{A}_{\bar{s}}.$$

By Leibnitz's formula on the partial derivative of a product, one has that

$$\frac{\partial^j (R_\ell A)}{\partial^j h} = \sum_{i=0}^{j} \binom{i}{j} \frac{\partial^i R_\ell}{\partial^i h} \frac{\partial^{j-i} A}{\partial^{j-i} h}, \qquad \ell = 1, 2, 3.$$

Therefore, since partial derivatives w.r.t. h of A up to order k vanish at (t, s_{i-1}) one deduces that for $\ell = 1, 2, 3$

$$\frac{\partial^j (R_\ell A)}{\partial^j h}(t, s_{i-1}) = 0, \qquad j = 0, \ldots, k, i = 1, \ldots, n.$$

Thus, R is a solution of $\mathcal{E}_{\mathrm{H}}$, and hence $R \in \mathbb{M}_{\mathrm{H}}$. Conversely, let $F = (F_1, F_2, F_3) \in \mathbb{M}_{\mathrm{H}}$. We prove that F_ℓ, $\ell = 1, 2, 3$, can be written as

$$F_\ell = R_\ell(t,h) \prod_{i=0}^{n-1} (h - s_i)^{k+1}, \qquad \text{with } R_\ell \in \mathcal{A}_{\bar{s}}.$$

Let $F_\ell = N_\ell / M_\ell$ be the reduced form of F_ℓ (i.e. $\gcd(N_\ell, M_\ell) = 1$). Since $F \in \mathbb{M}_{\mathrm{H}}$, one has that N_ℓ vanishes at (t, s_{i-1}) for $i = 1, \ldots, n$. Thus, by Bézout's theorem, $C(t,h) := \prod_{i=0}^{n-1}(h - s_i)$ divides N_ℓ, and since $F_\ell \in \mathcal{A}_{\bar{s}}$, $\gcd(C, M_\ell) = 1$. Thus, F_ℓ can be written as

$$F_\ell = \frac{N_{\ell,0}}{M_\ell} C(t,h), \qquad \text{where } N_{\ell,0} \in \mathbb{K}(t,h), \text{ and } \gcd(N_{\ell,0}, M_\ell) = 1.$$

Now, since $F_\ell \in \mathbb{M}_{\mathrm{H}}$ it holds that $\frac{\partial F_\ell}{\partial h}(t, s_{i-1}) = 0$, for $i = 1, \ldots, n$. That is,

$$\frac{\partial \left(\frac{N_{\ell,0}}{M_\ell}\right)}{\partial h}(t, s_{i-1})\, C(t, s_{i-1}) + \frac{N_{\ell,0}}{M_\ell}(t, s_{i-1})\, \frac{\partial C}{\partial h}(t, s_{i-1}) = 0, \qquad i = 1, \ldots, n.$$

Taking into account that C vanishes at (t, s_{i-1}) but its partial derivative w.r.t. h does not, one gets that

$$\frac{N_{\ell,0}}{M_\ell}(t, s_{i-1}) = 0, \qquad i = 1, \ldots, n.$$

Thus, reasoning as before, C divides $N_{\ell,0}$ and $\gcd(C, M_\ell) = 1$. Therefore, F_ℓ can be written as

$$F_\ell = \frac{N_{\ell,1}}{M_\ell} C(t,h)^2, \qquad \text{where } N_{\ell,1} \in \mathbb{K}(t,h), \text{ and } \gcd(N_{\ell,1}, M_\ell) = 1.$$

The same reasoning can be done, using Leibnitz's formula, up to the k-th partial derivative. Finally, one gets that

$$F_\ell = \frac{N_{\ell,k}}{M_\ell} C(t,h)^{k+1}, \qquad \text{where } N_{\ell,k} \in \mathbb{K}(t,h), \text{ and } \gcd(N_{\ell,k}, M_\ell) = 1.$$

Therefore, we have proved that

$$F = \sum_{\ell=1}^{3} \frac{N_{\ell,k}}{M_\ell} e_\ell,$$

and hence $F \in \Sigma$. □

Now, we can be more precise on the meaning of "general" solution of $\mathcal{E}_{\mathrm{H}}$ saying that a *general solution* of $\mathcal{E}_{\mathrm{H}}$ is a generic linear combination in $\mathcal{A}_{\bar{s}}$ of the basis of the submodule $\mathbb{M}_{\mathrm{H}}$; i.e.

$$\prod_{i=0}^{n-1}(h - s_i)^{k+1}(R_1, R_2, R_3) \qquad \text{with } R_i \in \mathcal{A}_{\bar{s}}.$$

In this situation, Theorems 2 and 3 can be written as

Theorem 4. *Let $\mathcal{B}_{\mathrm{p}}(t,h)$ be a particular solution of the non-homogeneous system $\mathcal{E}$ and let $\mathcal{B}_{\mathrm{g}}(t,h)$ be the general solution of the homogeneous system $\mathcal{E}_{\mathrm{H}}$. Then all the parametric solutions for $(\overline{\mathcal{P}}, \bar{s})$ can be expressed as*

$$\mathcal{B}_{\mathrm{p}}(t,h) + \mathcal{B}_{\mathrm{g}}(t,h).$$

That is, all the parametric solutions for $(\overline{\mathcal{P}}, \bar{s})$ *are of the form*

$$\mathcal{B}_{\mathrm{p}}(t,h) + \prod_{i=0}^{n-1}(h-s_i)^{k+1}\left(\frac{N_1}{M_1}, \frac{N_2}{M_2}, \frac{N_3}{M_3}\right),$$

where $N_i, M_i \in \mathbb{K}[t,h]$ *and* $\gcd(\prod_{i=0}^{n-1}(h-s_i), M_i) = 1$. □

5. Determination of a particular rational solution of $\mathcal{E}$

In the previous section we have seen how the problem of computing all rational G^k blendings for several surfaces is reduced to the determination of a particular solution of the associated non-homogeneous system $\mathcal{E}$. There are several methods that approach this problem partially (see Filip, 1989; Hartmann, 2001a; Pérez-Díaz and Sendra, 2001). The approach in Pérez-Díaz and Sendra (2001) deals with n surfaces but only for $k = 1$ (i.e. for the case of G^1 geometric continuity), the algorithm in Hartmann (2001a) is given for $n = 2$ (i.e. for two primary surfaces) with G^k-continuity, and the method in Filip (1989) is also given for the case of two surfaces with G^1-continuity, and comments on the extension to G^k-continuity are done. In addition, these approaches provide families of solutions that depend on parameters. Nevertheless, this characteristic of the methods is not interesting in this context, since we indeed provide all parametric solutions.

These available procedures to compute particular solutions to the problem may be classified in two types: those where the particular solution is achieved by means of rational perturbations of the given primary parametrizations (this is the case of Hartmann, 2001a), and those where the perturbation is done on the given clipping parametrizations (this is the case of Pérez-Díaz and Sendra, 2001). The method in Filip (1989) perturbs, by means of Hermite polynomials, the clipping parametrization and vectors in the tangent spaces to the clipping curves. Thus, since tangent vectors are linear combinations of the partial derivatives of the clipping parametrizations, the approach in Filip (1989) can also be considered in the second type of methods.

In this section, we generalize our method in Pérez-Díaz and Sendra (2001) to arbitrary G^k-continuity. In fact, one may check that the method in Filip (1989) can be seen as a particular case of this generalization. This extension of the method might be done preserving also the capability of generating families of solutions, but for simplicity and because we only need to know a single solution, we do not develop this aspect here. Moreover, we show how the method in Hartmann (2001a) can also be extended to the case of n primary surfaces. We finish the section with a comparative analysis of the methods. Examples of these extended methods can be found in the last section of the paper. Furthermore, both methods have been implemented in Maple (see Section 5 and Appendix).

For this purpose, as we did in the previous section, we fix throughout this section a rational blending data $(\overline{\mathcal{P}}, \bar{s})$, where $\overline{\mathcal{P}} = (\mathcal{P}_1(t,h), \ldots, \mathcal{P}_n(t,h))$ and $\bar{s} = (s_0, \ldots, s_{n-1})$ (note that $s_i \neq s_j$ if $i \neq j$).

For simplicity in the derivation of the methods that we present, we shall suppose that at least one clipping curve, say C_1, is not planar. Note that this condition can be assumed without loss of generality, since this situation can always be achieved by means of a linear change of coordinates. Furthermore, we will denote the parametrization $\mathcal{P}_i(t, s_{i-1})$ of the clipping curve C_i as

$$\mathcal{Q}_i(t) = (q_{i,1}(t), q_{i,2}(t), q_{i,3}(t)) := \mathcal{P}_i(t, s_{i-1}).$$

Note that, since C_1 is not planar, then $q_{1,i}(t) \notin \mathbb{K}$; in particular $q_{1,1} \neq 0$, $q_{1,2} \neq 0$, $q_{1,3} \neq 0$.

Perturbing the clipping parametrizations

The basic idea of this new method is to construct, from the clipping parametrizations $\mathcal{Q}_i(t)$, a prototype of parametrized solution of the form

$$\mathcal{T}(t,h) = \overbrace{(A_{1,1}(h)q_{1,1}(t),\ A_{1,2}(h)q_{1,2}(t),\ A_{1,3}(h)q_{1,3}(t))}^{\text{Perturbation of } \mathcal{Q}_1(t)} + \overbrace{\sum_{i=1}^{n} A_i(h)\mathcal{Q}_i(t)}^{\substack{\text{Perturbation}\\ \text{of } \mathcal{Q}_i(t),\, i \geq 2}},$$

where $A_{i,j}(h), A_i(h)$ are polynomials. The polynomials $A_{1,j}(h)$ contain, initially, undetermined coefficients. Afterwards, we find explicit values for these undetermined coefficients that guarantee that $\mathcal{T}(t,h)$ is a particular solution of the problem.

We start with the following technical lemma that is the generalization of Lemma 3 in Pérez-Díaz and Sendra (2001) to the G^k-blending problem. It ensures (and describes) the existence of suitable interpolating polynomials $A_{i,j}, A_i$ guaranteeing G^k geometric continuity.

Lemma 3. *Let $\Lambda \in \mathbb{K}^{(k+1)n}$. Then, there exists a unique polynomial $A(h) \in \mathbb{K}[h]$ satisfying that*

(1) $\deg_h(A(h)) \leq (k+1)n - 1$.

(2) $\left(\frac{\partial^0 A}{\partial^0 h}(s_0), \ldots, \frac{\partial^0 A}{\partial^0 h}(s_{n-1}), \ldots, \frac{\partial^k A}{\partial^k h}(s_0), \ldots, \frac{\partial^k A}{\partial^k h}(s_{n-1})\right) = \Lambda.$

Proof. Let $\Lambda = (\lambda_{1,0}, \ldots, \lambda_{n,0}, \ldots, \lambda_{1,k}, \ldots, \lambda_{n,k})$, and let

$$A(h) = a_0 + a_1 h + \cdots + a_{(k+1)n-1} h^{(k+1)n-1},$$

where a_i are undetermined coefficients. If A satisfies (2), then for $i = 1, \ldots, n$, and $j = 0, \ldots, k$, one gets

$$j!a_j + (j+1)!s_{i-1}a_{(j+1)} + \cdots + \frac{((k+1)n-1)!\, s_{i-1}^{((k+1)n-1-j)}}{((k+1)n-1-j)!} a_{(k+1)n-1} = \lambda_{i,j}.$$

These conditions can be seen as a linear system of equations where a_i are the unknowns. By simple computation, one deduces that the determinant of the $(k+1)n \times (k+1)n$ matrix of this linear system is

$$k^n \prod_{i=1}^{k-1} (k-i)^{2(n+i-1)} \prod_{i \neq r, i=0}^{n-1} (s_i - s_r)^{(k+1)^2} \neq 0.$$

Thus, the system is solvable, and therefore, the result holds. □

Remark. It is easy to check that the lemma, with the assumption of uniqueness, can be extended to any degree greater than or equal to $(k+1)n$. However, we have stated the lemma in its simplest form since we are interested in finding a particular solution for $\mathcal{S}$ of small degree. Also, solving the corresponding linear system of equations appearing in the proof, one deduces that the polynomial $A(h)$ can be expressed as

$$A(h) = \sum_{i=1}^{n}\sum_{j=0}^{k}\lambda_{i,j}\sum_{m=0}^{k-j}\frac{1}{m!j!}\left[\frac{\partial^m}{\partial^m h}\left(\frac{(h-s_{i-1})^{k+1}}{\prod_{i=1}^{n}(h-s_{i-1})^{k+1}}\right)\right]_{s_{i-1}} \times \frac{\prod_{i=1}^{n}(h-s_{i-1})^{k+1}}{(h-s_{i-1})^{k+1-j-m}},$$

where $\Lambda = (\lambda_{1,0}, \ldots, \lambda_{n,0}, \ldots, \lambda_{1,k}, \ldots, \lambda_{n,k})$. □

Now, using Lemma 3, we proceed to construct the prototype parametrization $\mathcal{T}(t,h)$. This construction will be used for theoretical purposes in the proofs. Then (see Corollary 1) we will deduce a direct expression of the solution. The process consists of two different steps.

Theoretical construction of $\mathcal{T}$

Step 1. This step will generate the polynomial coefficients of the first clipping parametrization $\mathcal{Q}_1(t)$ in $\mathcal{T}(t,h)$. We take three different families of elements $\Lambda_{1,1}, \Lambda_{1,2}, \Lambda_{1,3} \in \mathbb{K}^{(k+1)n}$, where some of their components are left as undetermined coefficients. More precisely, we take

$$\Lambda_{1,j} = (1, 0, \ldots, 0, \overbrace{\lambda_{(j,1,1)}, \ldots, \lambda_{(j,n,1)}, \ldots, \lambda_{(j,1,k)}, \ldots, \lambda_{(j,n,k)}}^{\text{undetermined coefficients}}),$$

In these conditions, for $j = 1, 2, 3$, we apply Lemma 3 to $\Lambda_{1,j}$ to generate three different polynomials of degree less than or equal to $(k+1)n - 1$ that we denote by

$$A_{1,j}^{\Lambda_{1,j}}(h).$$

We introduce the index $\Lambda_{i,j}$ to emphasize that each of these polynomials depends on the undetermined coefficients in $\Lambda_{i,j}$. These polynomials will be the interpolating coefficient corresponding to the components of the first clipping curve parametrization $\mathcal{Q}_1$.

Step 2. In this step the polynomial coefficients of the remaining clipping parametrization $\mathcal{Q}_i(t)$, $i = 2, \ldots, n$, in $\mathcal{T}(t,h)$ are generated. For $i = 2\ldots, n$ we take elements $\Lambda_i \in \mathbb{K}^{(k+1)n}$ as follows:

$$\Lambda_i = (0, \ldots, 0, \overbrace{1}^{i\text{th}}, 0, \ldots, 0, 0, \ldots, 0, \ldots, 0, \ldots, 0).$$

Now, we apply Lemma 3 to each Λ_i to generate polynomials of degree less than or equal to than $(k+1)n - 1$ that we denote by

$$A_i(h).$$

Each of these polynomials does not depend on any undetermined coefficient (hence, we omit the corresponding index), and they will be the interpolating coefficient corresponding to the clipping parametrizations $\mathcal{Q}_i(t)$, $i = 2, \ldots, n$. Note that for all but the first, clipping parametrizations of all the components are perturbed equally.

In this situation, if $\Lambda = (\Lambda_{1,1}, \Lambda_{1,2}, \Lambda_{1,3})$, we introduce the pattern parametric solution for $(\overline{\mathcal{P}}, \bar{s})$ (i.e. the prototype parametrization) as

$$\mathcal{T}^{\Lambda}(t,h) = \overbrace{(A_{1,1}^{\Lambda_{1,1}}(h)q_{1,1}(t),\ A_{1,2}^{\Lambda_{1,2}}(h)q_{1,2}(t),\ A_{1,3}^{\Lambda_{1,3}}(h)q_{1,3}(t))}^{\text{Perturbation of the first clipping}} + \overbrace{\sum_{i=1}^{n} A_i(h)\mathcal{Q}_i(t)}^{\substack{\text{Perturbation}\\ \text{of the}\\ \text{remaining}\\ \text{clippings}}}.$$

This expression can be written in matrix form as follows. Let

$$\mathcal{Q}(t) = \begin{pmatrix} \mathcal{Q}_1 \\ \mathcal{Q}_2 \\ \vdots \\ \mathcal{Q}_n \end{pmatrix} = \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ q_{2,1} & q_{2,2} & q_{2,3} \\ \vdots & \vdots & \vdots \\ q_{n,1} & q_{n,2} & q_{n,3} \end{pmatrix},$$

$$\mathcal{H}^{\Lambda}(h) = \begin{pmatrix} \boxed{A_{1,1}^{\Lambda_{1,1}}} & \boxed{A_{1,2}^{\Lambda_{1,2}}} & \boxed{A_{1,3}^{\Lambda_{1,3}}} \\ A_2 & A_2 & A_2 \\ A_3 & A_3 & A_3 \\ \vdots & \vdots & \vdots \\ A_n & A_n & A_n \end{pmatrix}.$$

Then

$$\mathcal{T}^{\Lambda}(t,h) = (1,\ldots,1)[\mathcal{Q}(t) \circ \mathcal{H}^{\Lambda}(h)],$$

where $\circ$ denotes Hadamard's product, also often called the Schur product (see, e.g. Horn and Johnson, 1985); that is, if $A = (a_{i,j})_{1\le i,j\le r}$ and $B = (b_{i,j})_{1\le i,j\le r}$ then $A \circ B = (a_{i,j}\, b_{i,j})_{1\le i,j\le r}$.

Note that the undetermined coefficients are only at positions (1, 1), (1, 2) and (1, 3) of the matrix $\mathcal{H}^{\Lambda}$. In general, one can introduce the undetermined coefficients at different positions in the matrix, but it should happen that there is a polynomial with undetermined coefficients at each column of the matrix.

Taking into account the construction we have done, for $i = 1, \ldots, n$, it holds that (in the first matrix the non-zero row is the ith one)

$$\mathcal{H}^{\Lambda}(s_{i-1}) = \begin{pmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix},$$

$$\frac{\partial^j \mathcal{H}^{\Lambda}}{\partial^j h}(s_{i-1}) = \begin{pmatrix} \boxed{\lambda_{(1,i,j)}} & \boxed{\lambda_{(2,i,j)}} & \boxed{\lambda_{(3,i,j)}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix},$$

where for a given matrix A, $\partial^j A/\partial^j h$ denotes the matrix obtained by considering the j-partial derivatives of the entries of A.

Applying the above properties, it is clear that the parametrization $\mathcal{T}^{\Lambda}(t, h)$ satisfies the properties stated in the following lemma.

Lemma 4. *For $i = 1, \ldots, n$, it holds that*

(1) $\frac{\partial^j \mathcal{T}^{\Lambda}}{\partial^j t}(t, s_{i-1}) = \frac{\partial^j \mathcal{Q}_i}{\partial^j t}(t)$ *for* $j = 0, \ldots, k$.
(2) $\frac{\partial^j \mathcal{T}^{\Lambda}}{\partial^j h}(t, s_{i-1}) = (\lambda_{(1,i,j)} q_{1,1}(t),\ \lambda_{(2,i,j)} q_{1,2}(t),\ \lambda_{(3,i,j)} q_{1,3}(t))$ *for* $j = 1, \ldots, k$. □

Observe that if one takes $j = 0$ in Lemma 4(1), for almost all specializations Λ_0 of the undetermined coefficients one has that

$$\mathcal{T}^{\Lambda_0}(t, s_{i-1}) = \mathcal{Q}_i(t) = \mathcal{P}_i(t, s_{i-1}), \qquad i = 1, \ldots, n.$$

Thus, $\mathcal{T}^{\Lambda_0}(t, h)$ defines a rational surface containing the clipping curves. Therefore, the parametrization $\mathcal{T}^{\Lambda_0}(t, h)$ solves the blending problem with zero geometric continuity.

In order to achieve G^k-continuity, one can apply Proposition 2 and Lemma 4(2), and try to find algebraic conditions on the undetermined coefficients in Λ to get that

$$\frac{\partial^j \mathcal{T}^{\Lambda}}{\partial^j h}(t, s_{i-1}) = (\lambda_{(1,i,j)} q_{1,1}(t),\ \lambda_{(2,i,j)} q_{1,2}(t),\ \lambda_{(3,i,j)} q_{1,3}(t)) = \frac{\partial^j \mathcal{P}_i}{\partial^j h}(t, s_{i-1})$$

for $j = 1, \ldots, k$, and $i = 1, \ldots, n$.

In the next theorem, we see that the above conditions can always be satisfied and therefore a particular parametric solution for to the G^k-continuity blending problem is determined.

Theorem 5. *For $j = 1, \ldots, k$, and $i = 1, \ldots, n$, let*

$$\frac{\partial^j \mathcal{P}_i}{\partial^j h}(t, s_{i-1}) = (m_{(1,i,j)}(t), m_{(2,i,j)}(t), m_{(3,i,j)}(t)),$$

and let $\Lambda = (\Lambda_{1,1}, \Lambda_{1,2}, \Lambda_{1,3})$ be such that

$$\lambda_{(\ell,i,j)} = \frac{m_{(\ell,i,j)}(t)}{q_{1,\ell}(t)}, \qquad \ell = 1, 2, 3.$$

Then $\mathcal{T}^\Lambda(t, h)$ is a parametric solution for $(\overline{\mathcal{P}}, \bar{s})$.

Proof. Taking into account the comments done before Theorem 5, one just has to observe that the equations (where $j = 1, \ldots, k,\ \ i = 1, \ldots, n$)

$$(\lambda_{(1,i,j)} q_{1,1}(t),\ \lambda_{(2,i,j)} q_{1,2}(t),\ \lambda_{(3,i,j)} q_{1,3}(t)) = \frac{\partial^j \mathcal{P}_i}{\partial^j h}(t, s_{i-1}),$$

can always be solved in $\lambda_{(\ell,i,j)}$ because $q_{1,1} \neq 0,\ q_{1,2} \neq 0,\ q_{1,3} \neq 0$. Clearly the solution is the one in the statement of the theorem. □

We have described a theoretical construction of the particular rational solution $\mathcal{T}$. However, taking into account the explicit expression of the polynomials $A_{i,j}^{\Lambda_{i,j}}$, A_i obtained in the remark to Lemma 3, one can derive an explicit expression for it. In the next theorem we deal with this.

Corollary 1 (Direct Computation of $\mathcal{T}$). *Let $(m_{(1,i,j)}(t), m_{(2,i,j)}(t), m_{(3,i,j)}(t))$ be as in Theorem 5. Then, a parametric solution for $(\overline{\mathcal{P}}, \bar{s})$, is given by*

$$\mathcal{T}^\Lambda(t, h) = (A_{1,1}^{\Lambda_{1,1}}(h) q_{1,1}(t),\ A_{1,2}^{\Lambda_{1,2}}(h) q_{1,2}(t),\ A_{1,3}^{\Lambda_{1,3}}(h) q_{1,3}(t)) + \sum_{i=2}^{n} A_i(h) \mathcal{Q}_i(t),$$

where

$$A_{1,r}^{\Lambda_{1,r}}(h) = \sum_{\ell=0}^{k} \frac{1}{\ell!} \left[\frac{\partial^\ell}{\partial^\ell h} \left(\frac{(h - s_0)^{k+1}}{\prod_{i=1}^{n} (h - s_{i-1})^{k+1}} \right) \right]_{s_0} \frac{\prod_{i=1}^{n} (h - s_{i-1})^{k+1}}{(h - s_0)^{k+1-\ell}}$$
$$+ \sum_{i=1}^{n} \sum_{j=1}^{k} \frac{m_{(r,i,j)}(t)}{q_{1,r}(t)} \sum_{\ell=0}^{k-j} \frac{1}{\ell! j!} \left[\frac{\partial^\ell}{\partial^\ell h} \left(\frac{(h - s_{i-1})^{k+1}}{\prod_{i=1}^{n} (h - s_{i-1})^{k+1}} \right) \right]_{s_{i-1}}$$
$$\times \frac{\prod_{i=1}^{n} (h - s_{i-1})^{k+1}}{(h - s_{i-1})^{k+1-j-\ell}},$$

and

$$A_i(h) = \sum_{\ell=0}^{k} \frac{1}{\ell!} \left[\frac{\partial^\ell}{\partial^\ell h} \left(\frac{(h - s_{i-1})^{k+1}}{\prod_{i=1}^{n} (h - s_{i-1})^{k+1}} \right) \right]_{s_{i-1}} \frac{\prod_{i=1}^{n} (h - s_{i-1})^{k+1}}{(h - s_{i-1})^{k+1-\ell}}. \quad \square$$

Theorem 5 and Corollary 1 provide the following algorithm to compute a parametric solution for the blending data $(\overline{\mathcal{P}}, \bar{s})$. The input of the algorithm is as it is described in the statement of the problem (see Section 3).

Algorithm 1. Given a rational blending data $(\overline{\mathcal{P}}, \bar{s})$, the algorithm computes a parametric solution for $(\overline{\mathcal{P}}, \bar{s})$.

(1) For $j = 1, \ldots, k$ and for $i = 1, \ldots, n$ compute $\frac{\partial^j \mathcal{P}_i}{\partial^j h}(t, s_{i-1})$.

(2)

$$\mathcal{T}^{\Lambda}(t,h) = \sum_{i=1}^{n}\sum_{\ell=0}^{k}\frac{1}{\ell!}\left[\frac{\partial^\ell}{\partial^\ell h}\left(\frac{(h-s_{i-1})^{k+1}}{\prod_{i=1}^{n}(h-s_{i-1})^{k+1}}\right)\right]_{s_{i-1}} \times \frac{\prod_{i=1}^{n}(h-s_{i-1})^{k+1}}{(h-s_{i-1})^{k+1-\ell}}\mathcal{Q}_i(t) + \sum_{i=1}^{n}\sum_{j=1}^{k}\frac{\partial^j \mathcal{P}_i}{\partial^j h}(t,s_{i-1}) \times \sum_{\ell=0}^{k-j}\frac{1}{\ell!j!}\left[\frac{\partial^\ell}{\partial^\ell h}\left(\frac{(h-s_{i-1})^{k+1}}{\prod_{i=1}^{n}(h-s_{i-1})^{k+1}}\right)\right]_{s_{i-1}} \times \frac{\prod_{i=1}^{n}(h-s_{i-1})^{k+1}}{(h-s_{i-1})^{k+1-j-\ell}}.$$

(3) Return $\mathcal{T}^{\Lambda}(t,h)$.

Perturbing the primary parametrizations

In Hartmann (2001a), Hartmann provides a method to generate a family of parametric solutions with G^k-continuity for a rational blending data of the form $((\mathcal{P}_1, \mathcal{P}_2), (s_0, s_1))$ (more precisely $s_0 = 0, s_1 = 1$), and therefore only two primary surfaces are considered. In Hartmann (2001a), the perturbation is done on the primary parametrizations. So, the basic idea of Hartmann's method is to generate a prototype of parametrized solution of the form

$$\mathcal{T}(t,h) = \overbrace{f_1(h)\mathcal{P}_1(t,h) + f_2(h)\mathcal{P}_2(t,h)}^{\substack{\text{Uniform perturbation of the} \\ \text{primary parametrizations } \mathcal{P}_i(t,h)}},$$

where $f_i(h)$ are in general rational functions.

More precisely, Hartmann gives the following family of particular solutions that depends on a parameter u (the *balance* parameter).

Theorem 6. *Let $\mathcal{S} = ((\mathcal{P}_1, \mathcal{P}_2), (s_0, s_1))$ be a rational blending data, let $u \in \mathbb{K}\setminus\{0,1\}$, and let*

$$f(h) = \frac{u(s_1-h)^{k+1}}{u(s_1-h)^{k+1} + (1-u)(h-s_0)^{k+1}}.$$

Then, the parametrization

$$\mathcal{T}(t,h) = f(h)\mathcal{P}_1(t,h) + (1-f(h))\mathcal{P}_2(t,h)$$

is a parametric solution for $\mathcal{S}$. □

Remark. Let $\mathbb{K} = \mathbb{R}$ be the field of real numbers, and let $s_0, s_1 \in \mathbb{R}$, with $s_0 < s_1$. Then, for $u \in (0, 1)$ the rational function $f(h)$ in Theorem 6 is a C^∞-continuous rational function in $[s_0, s_1]$. □

Hartmann's method can be easily generalized to $n \geq 2$ primary surfaces, thus a solution of the type

$$\mathcal{T}(t,h) = \overbrace{f_1(h)\mathcal{P}_1(t,h) + \cdots + f_n(h)\mathcal{P}_n(t,h)}^{\substack{\text{Uniform perturbation of the}\\ \text{primary parametrizations } \mathcal{P}_i(t,h)}},$$

is generated. More precisely, one gets the following theorem.

Theorem 7. *Let $\mathcal{S} = ((\mathcal{P}_1, \ldots, \mathcal{P}_n), (s_0, \ldots, s_{n-1}))$ be a rational blending data, let $u_1, \ldots, u_n \in \mathbb{K}\setminus\{0, 1\}$, and for $i = 1, \ldots, n$ let*

$$f_i(h) = \frac{u_i \prod_{j=1}^{i-2}(h - s_{j-1})^{k+1} \prod_{j=i-1, j\neq i}^{n}(s_{j-1} - h)^{k+1}}{u_i \prod_{j=1}^{i-2}(h - s_{j-1})^{k+1} \prod_{j=i-1, j\neq i}^{n}(s_{j-1} - h)^{k+1} + (1 - u_i)(h - s_{i-1})^{k+1}}.$$

Then $\mathcal{T}(t,h) = f_1(h)\mathcal{P}_1(t,h) + \cdots + f_n(h)\mathcal{P}_n(t,h)$, is a parametric solution for the blending data $\mathcal{S}$.

Proof. For $i = 1, \ldots, n$, the functions $f_i(h)$ satisfy that:

$$f_i(s_{i-1}) = 1, \qquad \text{and} \qquad f_i(s_{\ell-1}) = 0 \qquad \text{for } \ell \in \{1, \ldots, n\}\setminus\{i\}.$$

Furthermore, for $j = 1, \ldots, k$ and $r = 1, \ldots, n$ one has that $\frac{\partial^j f_r}{\partial^j h}(s_{i-1}) = 0$. Thus, the parametrization $\mathcal{T}(t,h)$ satisfies that

$$\frac{\partial^j \mathcal{T}}{\partial^j h}(t, s_{i-1}) = \frac{\partial^j \mathcal{P}_i}{\partial^j h}(t, s_{i-1}), \qquad j = 0, \ldots, k, \text{ and } i = 1, \ldots, n.$$

Therefore, taking into account Proposition 2, $\mathcal{T}(t,h)$ is a parametric solution for $\mathcal{S}$. □

Remark. Note that for $n = 2$, taking in Theorem 7

$$f_1(h) = \frac{u_1(s_1 - h)^{k+1}}{u_1(s_1 - h)^{k+1} + (1 - u_1)(h - s_0)^{k+1}},$$

$$f_2(h) = \frac{u_2(s_0 - h)^{k+1}}{u_2(s_0 - h)^{k+1} + (1 - u_2)(h - s_1)^{k+1}}$$

with $u_2 = 1 - u_1$, we get Theorem 6. □

We have already analysed, after Theorem 6, the continuity of the functions f_i when $n = 2$. In the next proposition, we study the continuity for arbitrary n.

Proposition 3. *Let $\mathbb{K} = \mathbb{R}$ be the field of real numbers, and $s_0, \ldots, s_{n-1} \in \mathbb{R}$, with $s_0 < \cdots < s_{n-1}$. Then if $u_i \in (0, 1)$, the rational functions $f_i(h)$, for $i = 2, \ldots, n$, in Theorem 7 are C^∞-continuous in $[s_{i-2}, s_{i-1}]$, and $f_1(h)$ is C^∞-continuous in $[s_0, s_1]$.*

Proof. Let the denominator of $f_i(h)$ vanish at $a \in [s_{i-2}, s_{i-1}]$. Then,

$$u_i \prod_{j=1}^{i-2}(a - s_{j-1})^{k+1} \prod_{j=i-1, j\neq i}^{n} (s_{j-1} - a)^{k+1} + (1 - u_i)(a - s_{i-1})^{k+1} = 0.$$

Note that from the above equality it is clear that $a \neq s_{i-1}$. Thus,

$$u_i = \frac{1}{1 - \left(\frac{\prod_{j=1}^{i-2}(a-s_{j-1}) \prod_{j=i-1, j\neq i}^{n}(s_{j-1}-a)}{a-s_{i-1}}\right)^{k+1}}.$$

Therefore, since $u_i < 1$ one deduces that

$$0 < -\left(\frac{\prod_{j=1}^{i-2}(a - s_{j-1}) \prod_{j=i-1, j\neq i}^{n}(s_{j-1} - a)}{a - s_{i-1}}\right)^{k+1},$$

which is impossible because $a \in [s_{i-2}, s_{i-1}]$. □

Theorem 7 provides the following algorithm to compute a parametric solution for the blending data $(\overline{\mathcal{P}}, \bar{s})$. The input of the algorithm is as it is described in the statement of the problem (see Section 3).

Algorithm 2. Given a rational blending data $(\overline{\mathcal{P}}, \bar{s})$, the algorithm computes a parametric solution for $(\overline{\mathcal{P}}, \bar{s})$.

(1) For $i = 1, \ldots, n$ take $u_i \in \mathbb{K}\setminus\{0, 1\}$ and compute

$$f_i(h) := \frac{u_i \prod_{j=1}^{i-2}(h - s_{j-1})^{k+1} \prod_{j=i-1, j\neq i}^{n}(s_{j-1} - h)^{k+1}}{u_i \prod_{j=1}^{i-2}(h - s_{j-1})^{k+1} \prod_{j=i-1, j\neq i}^{n}(s_{j-1} - h)^{k+1} + (1 - u_i)(h - s_{i-1})^{k+1}}.$$

(2) $\mathcal{T}(t, h) := f_1(h)\mathcal{P}_1(t, h) + \cdots + f_n(h)\mathcal{P}_n(t, h)$.
(3) Return $\mathcal{T}(t, h)$.

Comparison of methods

We finish this section with a comparative discussion of the two methods for computing particular solutions. We base our discussion on four different aspects:

1. Algebraic manipulation required in the algorithms.
2. Upper bounds of the degrees of the output parametrizations.
3. Capability of the methods to provide polynomial parametrizations as outputs (for more details on the polynomiality see next section).
4. Actual computing times in the implementation.

Concerning algebraic manipulations required to derive the output, extension of Hartmann's method is much better since it only involves basic rational function arithmetic. Thus, it can be considered as a very direct approach. In the case of Algorithm 1, evaluations and derivative computations are required, and therefore it is not as direct. Nevertheless, in both cases (clearly for Algorithm 2) the complexity is polynomial, and empirical analysis shows that both are quite efficient.

In order to study the degree of the outputs, we first recall that the degree of a rational function $R \in \mathbb{K}(t_1, \ldots, t_\ell)$ w.r.t. t_i is defined as the maximum of the degrees of the numerator and denominator of R (where R is given in reduced form) w.r.t. t_i. And, we define the degree of a rational parametrization as the maximum of the degrees of its rational components. Therefore, if $\mathcal{P}(t, h) = (p_1(t, h), p_2(t, h), p_3(t, h))$ then

$$\deg_t(\mathcal{P}(t, h)) = \max\{\deg_t(p_i(t, h)) \mid i = 1, 2, 3\}.$$

Similarly one defines $\deg_h(\mathcal{P}(t, h))$. Moreover, the total degree of the parametrization $\mathcal{P}(t, h)$ is defined as

$$\mathrm{totaldeg}(\mathcal{P}(t, h)) = \max\{\mathrm{totaldeg}(p_i(t, h)) \mid i = 1, 2, 3\},$$

where $\mathrm{totaldeg}(p_i(t, h))$ denotes the total degree of the rational function $p_i(t, h)$; that is, the maximum of the total degrees of the numerator and denominator of $p_i(t, h)$ in reduced form.

In these conditions, let $\mathcal{S} = (\overline{\mathcal{P}}, \bar{s}) = ((\mathcal{P}_1, \ldots, \mathcal{P}_n), (s_0, \ldots, s_{n-1}))$ be a rational blending data, and let

$$\alpha = \max\{\deg_t(\mathcal{P}_i(t, h)) \mid i = 1, \ldots, n\},$$
$$\beta = \max\{\deg_h(\mathcal{P}_i(t, h)) \mid i = 1, \ldots, n\},$$
$$\gamma = \max\{\mathrm{totaldeg}(\mathcal{P}_i(t, h)) \mid i = 1, \ldots, n\}.$$

Then, a simple analysis of the algorithms shows the following upper bounds for the degrees:

- *Algorithm* 1. Let $\mathcal{T}(t, h)$ be the output of Algorithm 1 performed on $\mathcal{S}$. Then,

  (i) $\deg_t(\mathcal{T}(t, h)) \leq \alpha n(k + 1)$.
  (ii) $\deg_h(\mathcal{T}(t, h)) \leq n(k + 1) - 1$.
  (iii) $\mathrm{totaldeg}(\mathcal{T}(t, h)) = \mathcal{O}(\gamma nk)$.

- *Algorithm* 2. Let $\mathcal{T}(t, h)$ be the output of Algorithm 2 performed on $\mathcal{S}$. Then,

  (i) $\deg_t(\mathcal{T}(t, h)) \leq n\alpha$.
  (ii) $\deg_h(\mathcal{T}(t, h)) \leq n((n - 1)(k + 1) + \beta)$.
  (iii) $\mathrm{totaldeg}(\mathcal{T}(t, h)) = \mathcal{O}(n^2k + \gamma n)$.

Comparing Algorithms 1 and 2 in terms of the polynomiality of the output, one sees that Algorithm 1 is much better than Algorithm 2. Algorithm 1 outputs a parametric polynomial solution for $(\overline{\mathcal{P}}, \bar{s})$ if any exists (see Corollary 2). However, Algorithm 2 is not optimal in this sense. For more details on the polynomiality see the next section.

Algorithms 1 and 2 have been implemented in Maple. In Table 1 we illustrate the performance of the implementations, showing times for the parametrizations appearing in the Appendix. In the table we also show:

Table 1
Performance of the implementation

Input	n	k	D_p	D_c	Algorithm 1	Algorithm 2
I	2	2	[5, 18]	[2,2]	0.149	0.295
II	2	8	[2, 2]	[2, 2]	0.488	0.226
III	2	5	[2, 6]	[2, 2]	0.230	0.035
IV	2	4	[4, 3]	[2, 3]	0.510	0.045
V	3	3	[5, 2, 4]	[4, 2, 2]	0.580	0.760
VI	3	1	[2, 2, 3]	[2, 2, 3]	0.130	0.120
VII	4	2	[2, 2, 2, 2]	[2, 2, 2, 1]	0.300	0.370
VIII	4	1	[3, 2, 2, 1]	[2, 1, 1, 1]	0.215	0.215
IX	5	2	[2, 2, 3, 3, 2]	[2, 2, 1, 2, 2]	0.425	1.700
X	5	1	[2, 2, 2, 1, 2]	[2, 2, 2, 1, 1]	0.270	1.150
XI	6	3	[2, 2, 1, 1, 1, 2]	[1, 1, 1, 1, 1, 1]	1.115	4.965

n = number of primary surfaces.
k = order of geometry continuity.
D_P = list with the total degrees of the parametrizations of the primary surfaces.
D_C = list with the degrees of the parametrizations of the clipping curves.

1. The degree of the parametrizations of the input primary surfaces $\mathcal{P}_i(t, h)$,
2. the number n of primary surfaces involved in the blending,
3. the degree of the parametrizations of the input clipping curves $\mathcal{Q}_i(t)$ and
4. the order k of geometry continuity.

Actual computing times are measured on a PC Pentium III Processor 128 MB of RAM, and times are given in seconds of CPU. We remark that the outputs provided by Algorithm 2 are in general more complicated, in the sense of density, than the outputs given by Algorithm 1 (see Section 7).

The following Table 2 summarizes the comparative analysis of the methods in terms of degrees, required algebraic manipulations, polynomiality, computing times and density of the output.

Table 2
Comparative analysis of the Method

Characteristic	Algorithm 1	Algorithm 2
Degree in t		Better
Degree in h	Better	
Total degree	$\mathcal{O}(\gamma nk)$	$\mathcal{O}(n^2k + \gamma n)$
Polynomiality of the output (see Section 6 for details)	Better	
Required algebraic manipulations		Better
Actual computing times	Equivalent	Equivalent
Density of the output	Better	

6. Structure of the space of polynomial solutions

The generation of polynomial blendings (i.e. polynomial parametrizations that are blendings) is important in applications. For instance one avoids the unstable numerical behaviour of the denominators when tracing the surface.

In this section, we prove that the set of all polynomial parametric solutions for a parametric blending problem can also be expressed in terms of a free module of rank 3; in this case over a bivariate polynomial ring. Moreover we state an algorithmic criterion to decide whether there exist parametric polynomial solutions and we prove that the extension of the method in Pérez-Díaz and Sendra (2001) always reaches a polynomial parametrization if any exists.

For this purpose, throughout this section we fix a rational blending data $(\overline{\mathcal{P}}, \bar{s})$, where $\overline{\mathcal{P}} = (\mathcal{P}_1(t,h), \ldots, \mathcal{P}_n(t,h))$ and $\bar{s} = (s_0, \ldots, s_{n-1})$ (note that $s_i \neq s_j$ if $i \neq j$). In this situation, we consider the free $\mathbb{K}[t,h]$-module of rank 3 $(\mathbb{K}[t,h])^3$, and denote it by

$$\mathbb{M}^{\mathrm{Pol}} = (\mathbb{K}[t,h])^3.$$

Also, we denote by

$$\mathcal{E}^{\mathrm{Pol}} \qquad \text{and} \qquad \mathcal{E}_{\mathrm{H}}^{\mathrm{Pol}}$$

the systems (non-homogenous and homogeneous, respectively) introduced in Section 4, but now over $\mathbb{M}^{\mathrm{Pol}}$ instead of over $\mathbb{M}$. The following lemmas are stated similarly as Lemmas 1 and 2.

Lemma 5. *The set of solutions of $\mathcal{E}_{\mathrm{H}}^{\mathrm{Pol}}$ in $\mathbb{M}^{\mathrm{Pol}}$ is a submodule of $\mathbb{M}^{\mathrm{Pol}}$ that we denote by $\mathbb{M}_{\mathrm{H}}^{\mathrm{Pol}}$.* □

Lemma 6. *Let $A(h) = \prod_{i=0}^{n-1}(h - s_i)^{k+1}$. Then, $\{e_1, e_2, e_3\}$, where*

$$e_1 = (A(h), 0, 0), \qquad e_2 = (0, A(h), 0), \qquad e_3 = (0, 0, A(h)),$$

is a basis of the submodule $\mathbb{M}_{\mathrm{H}}^{\mathrm{Pol}}$. □

Similarly, as we did in Section 4, we introduce the notion of "general" solution of $\mathcal{E}_{\mathrm{H}}^{\mathrm{Pol}}$ saying that it is a generic linear combination in $\mathbb{K}[t,h]$ of the basis of the submodule $\mathbb{M}_{\mathrm{H}}^{\mathrm{Pol}}$; i.e.

$$\prod_{i=0}^{n-1}(h - s_i)^{k+1}(R_1, R_2, R_3) \qquad \text{with } R_i \in \mathbb{K}[t,h].$$

In this situation, one may state the analogous result to Theorem 4 for the polynomial case.

Theorem 8. *Let $\mathcal{B}_{\mathrm{p}}^{\mathrm{Pol}}(t,h)$ be a particular polynomial solution of the non-homogeneous system $\mathcal{E}^{\mathrm{Pol}}$ and let $\mathcal{B}_{\mathrm{g}}^{\mathrm{Pol}}(t,h)$ be a general solution of $\mathcal{E}_{\mathrm{H}}^{\mathrm{Pol}}$. Then all the parametric polynomial solutions for $(\overline{\mathcal{P}}, \bar{s})$ can be expressed as*

$$\mathcal{B}_{\mathrm{p}}^{\mathrm{Pol}}(t,h) + \mathcal{B}_{\mathrm{g}}^{\mathrm{Pol}}(t,h).$$

That is, all the parametric polynomial solutions for $(\overline{\mathcal{P}}, \bar{s})$ *are of the form*

$$\mathcal{B}_{\mathrm{p}}^{\mathrm{Pol}}(t, h) + \prod_{i=0}^{n-1}(h - s_i)^{k+1}(R_1, R_2, R_3),$$

where $R_i \in \mathbb{K}[t, h]$.

Proof. Let $A(h) = \prod_{i=0}^{n-1}(h - s_i)^{k+1}$. Let Σ^{Pol} be the set of all the polynomial parametric solutions for $(\overline{\mathcal{P}}, \bar{s})$, and

$$\Omega^{\mathrm{Pol}} = \{\mathcal{B}_{\mathrm{p}}^{\mathrm{Pol}}(t, h) + A(h)(R_1, R_2, R_3) \mid R_i \in \mathbb{K}[t, h]\}.$$

By Theorem 4, $\Omega^{\mathrm{Pol}} \subset \Sigma^{\mathrm{Pol}}$. Now, let $\mathcal{M}(t, h) \in \Sigma^{\mathrm{Pol}}$. Then, by Theorem 4, one has that $\mathcal{M}(t, h)$ can be expressed as

$$\mathcal{M}(t, h) = \mathcal{B}_{\mathrm{p}}^{\mathrm{Pol}}(t, h) + A(h)\left(\frac{N_1}{M_1}, \frac{N_2}{M_2}, \frac{N_3}{M_3}\right),$$

where $N_i, M_i \in \mathbb{K}[t, h]$ and $\gcd(\prod_{i=0}^{n-1}(h - s_i), M_i) = 1$. Since $\mathcal{M}(t, h)$, and $\mathcal{B}_{p}^{\mathrm{Pol}}(t, h)$ are polynomial parametrizations, it holds that

$$A(h)\left(\frac{N_1}{M_1}, \frac{N_2}{M_2}, \frac{N_3}{M_3}\right) \in \mathbb{K}[t, h]^3.$$

Furthermore, since $\gcd(\prod_{i=0}^{n-1}(h - s_i), M_i) = 1$, one deduces that M_i divides N_i, and therefore

$$\left(\frac{N_1}{M_1}, \frac{N_2}{M_2}, \frac{N_3}{M_3}\right) \in \mathbb{K}[t, h]^3.$$

Thus, $\mathcal{M}(t, h) \in \Omega^{\mathrm{Pol}}$. □

In Theorem 8, we have seen the expression of all polynomial solutions for a rational blending data, if any exists. However, we still do not have a criterion for deciding the existence of polynomial solutions. In the next theorem we characterize the existence of polynomial blending by means of the clipping curves. For this purpose, we apply the ideas in Algorithm 1. More precisely, we state the following theorem.

Theorem 9. *The following statements are equivalent:*

(1) *There exists a parametric polynomial solution for* $(\overline{\mathcal{P}}, \bar{s})$.

(2) *There exist infinitely many parametric polynomial solutions for* $(\overline{\mathcal{P}}, \bar{s})$.

(3) *The rational functions*

$$\frac{\partial^j \mathcal{P}_i}{\partial^j h}(t, s_{i-1}) \qquad \text{for } j = 0, \ldots, k, i = 1, \ldots, n,$$

are polynomial.

Proof. (1) implies (2) follows from Theorem 8, and (2) implies (3) follows from Proposition 2. In order to prove that (3) implies (1), we consider the output parametrization $\mathcal{T}^{\Lambda}(t,h)$ given by Algorithm 1. It is of the form

$$\mathcal{T}^{\Lambda}(t,h) = (A_{1,1}^{\Lambda_{1,1}}(h)q_{1,1}(t),\ A_{1,2}^{\Lambda_{1,2}}(h)q_{1,2}(t),\ A_{1,3}^{\Lambda_{1,3}}(h)q_{1,3}(t)) + \sum_{i=2}^{n} A_i(h)\mathcal{Q}_i(t).$$

$\sum_{i=2}^{n} A_i(h)\mathcal{Q}_i(t)$ is polynomial because (see Corollary 1 for the expression of A_i)

$$(h-s_{i-1})^{k+1-\ell} \text{ divides } \prod_{i=1}^{n}(h-s_{i-1})^{k+1}.$$

Furthermore, for $r = 1,2,3$ one has that $A_{1,r}^{\Lambda_{1,r}}(h)q_{1,r}(t)$ are also polynomial (see Corollary 1 for the expression of $A_{1,r}^{\Lambda_{1,r}}$) because

$$(h-s_{i-1})^{k+1-j-\ell} \text{ divides } \prod_{i=1}^{n}(h-s_{i-1})^{k+1},$$

and the components of

$$\frac{\partial^j \mathcal{P}_i}{\partial^j h}(t,s_{i-1}) = (m_{(1,i,j)}(t), m_{(2,i,j)}(t), m_{(3,i,j)}(t))$$
$$\text{for } j=0,\ldots,k,\ i=1,\ldots,n$$

are polynomials. Therefore, $\mathcal{T}^{\Lambda}(t,h)$ is a parametric polynomial solution for $(\overline{\mathcal{P}},\bar{s})$. □

Comparing Algorithms 1 and 2 in terms of the polynomiality of the output, one sees that although in Hartmann's method (Algorithm 2) the coefficients f_i can be taken as polynomials (see Hartmann, 2001a), the polynomiality of the output depends on the polynomiality of the primary parametrizations. However, a direct consequence of the proof of (3) implies (1) in Theorem 9 shows that Algorithm 1 is optimal in this sense. More precisely, one has the following corollary.

Corollary 2. *Algorithm 1 outputs a parametric polynomial solution for $(\overline{\mathcal{P}},\bar{s})$, if any exists.* □

In the following example, we show that Algorithm 2 does not have the property of Algorithm 1 described in the previous corollary.

Example 1. We consider the problem of blending with G^1 geometric continuity of two surfaces. More precisely, let V_1 and V_2 be the primary surfaces parametrized by

$$\mathcal{P}_1(t,h) = \left(\frac{6t^2h-3t^2h^2-5h^2-30h-45}{t^2h^2+h^2+6h+9},\ \frac{(t^2h^2-h^2-6h-9)t}{t^2h^2+h^2+6h+9},\ \frac{2}{3}t^2+\frac{4}{3}\right),$$

$$\mathcal{P}_2(t,h) = \left(\frac{3(t^4h^2-2t^4h+4t^3h-10t^2h+t^4-4t^3+8t^2-20t+25+6t^2h^2+24th)}{5(th-t+2)^2},\right.$$
$$\left.\frac{(t^2h-t^2+2t+1)(h^2-1)}{(th-t+2)(h^2+1)},\ \frac{2(t^2h-t^2+2t+1)h}{(th-t+2)(h^2+1)}\right).$$

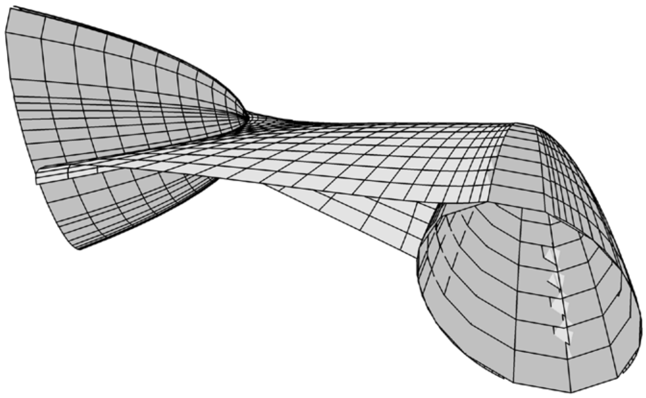

Fig. 2. Primary surfaces and blending surface generated by Algorithm 1.

Now, let

$$\mathcal{Q}_1(t) = \mathcal{P}_1(t,0) = (-5, -t, \tfrac{2}{3}t^2 + \tfrac{4}{3}),$$
$$\mathcal{Q}_2(t) = \mathcal{P}_2(t,1) = (\tfrac{3}{5}t^2 + \tfrac{3}{5}t + \tfrac{15}{4}, 0, t + \tfrac{1}{2}),$$

be the parametrization of the clipping curves C_i. Thus, we consider the rational blending data $\mathcal{S} = ((\mathcal{P}_1, \mathcal{P}_2), (s_0, s_1))$ where $s_0 = 0$, and $s_1 = 1$. We observe that

$$\frac{\partial \mathcal{P}_1}{\partial h}(t,0) = \left(\tfrac{2}{3}t^2, 0, 0\right),$$
$$\frac{\partial \mathcal{P}_2}{\partial h}(t,1) = \left(\tfrac{3}{5}t^3 + \tfrac{3}{10}t^2 + \tfrac{18}{15}t - \tfrac{3}{20}(4t^2 + 4t + 25)t, t + \tfrac{1}{2}, \tfrac{1}{2}t^2 - \tfrac{1}{4}(2t+1)t\right).$$

Therefore, the clipping curves satisfy condition (3) in Theorem 9, and therefore there exist polynomial solutions. In the following we illustrate how Algorithm 1 determines a polynomial solution, but Algorithm 2 (even taking the rational functions f_i in Theorem 7 to be polynomial) does not reach a polynomial parametrization.

Algorithm 1 outputs the polynomial parametrization (see Fig. 2).

$$\begin{aligned}\mathcal{T}(t,h) = \Big(&\tfrac{175}{2}h^3 - 5 - \tfrac{525}{4}h^4 + \tfrac{105}{2}h^5 + \tfrac{191}{48}t^2h^3 + \tfrac{33}{5}h^3t \\ &- \tfrac{799}{120}h^4t^2 - \tfrac{201}{20}h^4t + \tfrac{2041}{720}h^5t^2 + \tfrac{81}{20}h^5t \\ &+ \tfrac{2}{3}t^2h - \tfrac{2}{9}t^2h^2 + \tfrac{3}{20}h^5t^3 - \tfrac{3}{10}t^3h^4 + \tfrac{3}{20}t^3h^3, \tfrac{21}{4}h^3t \\ &- \tfrac{13}{2}h^4t + \tfrac{9}{4}h^5t - t - \tfrac{2}{9}h^5t^3 + \tfrac{2}{3}t^3h^4 - \tfrac{2}{3}t^3h^3 + \tfrac{2}{9}t^3h^2 \\ &+ 4h^4 - \tfrac{7}{4}h^5 - \tfrac{9}{4}h^3, -\tfrac{157}{24}t^2h^3 - \tfrac{103}{12}h^3 + \tfrac{2}{3}t^2 + \tfrac{4}{3} \\ &+ \tfrac{39}{4}h^4t^2 + 13h^4 - \tfrac{31}{8}h^5t^2 - \tfrac{21}{4}h^5 + \tfrac{21}{2}h^3t - \tfrac{63}{4}h^4t \\ &+ \tfrac{25}{4}h^5t\Big).\end{aligned}$$

Algorithm 2. We compute a polynomial function $f(h)$, satisfying that,

$$f(s_0) = 1, \qquad f(s_1) = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial h}(s_0) = \frac{\partial f}{\partial h}(s_1) = 0.$$

For instance,

$$f(h) = 1 - 3h^2 + 2h^3.$$

Then, Algorithm 2 outputs the following parametrization that is not polynomial

$$\mathcal{T}(t,h) = f(h)\mathcal{P}_1(t,h) + (1 - f(h))\mathcal{P}_2(t,h) = \left(\frac{D_1}{C_1}, \frac{D_2}{C_2}, \frac{D_3}{C_3}\right)$$

$$\begin{aligned}
D_1 = & -\tfrac{1}{5}(-900t + 600h + 4820h^2t + 300th - 420t^2h - 4544h^3t + 852t^3h^4 \\
& - 624t^3h^3 + 2620t^2h^3 + 144t^3h^2 - 1313t^2h^2 + 225t^2 - 4625h^2 \\
& + 1575h^4 - 352h^5t^2 + 1228h^5t - 1340h^4t^2 - 1440h^4t + 350h^5 + 900 \\
& - 6t^4h^2 - 30t^4h + 120t^3h + 6h^7t^6 - 21h^6t^6 + 24h^6t^5 + 24h^5t^6 \\
& - 60h^5t^5 - 9t^6h^4 + 192t^4h^3 + 420h^5t^4 - 396h^4t^4 + 72h^7t^4 - 264h^6t^4 \\
& + 288h^6t^3 + 86h^7t^2 + 227h^6t^2 + 344h^6t + 36t^5h^4 - 792h^5t^3),
\end{aligned}$$

$$\begin{aligned}
D_2 = & -18t - 20h^2t - 12th - 3t^2h - 12h^3t - 10t^3h^4 - 4t^2h^3 + 2t^3h^2 \\
& + 4t^2h^2 + 9t^2 - 27h^2 + 36h^4 + 55h^5t^2 - 62h^4t^2 + 106h^4t \\
& + 2h^5 - t^4h^2 - 2h^7 - 9h^6 + t^4h^3 - 9h^5t^4 + 5h^4t^4 + 4h^6t^4 - 16h^7t^2 \\
& + 21h^6t^2 - 36h^6t - 4h^8t^2 - 8h^7t + 8h^5t^3,
\end{aligned}$$

$$\begin{aligned}
D_3 = & \tfrac{2}{3}(-2t + 4h^2t + 2th + 10h^3t + 5t^3h^4 - 4t^3h^3 - 5t^2h^3 + 2t^3h^2 \\
& - 4t^2h^2 + 2t^2 - 8h^2 - t^3 - 18h^4 + 17h^3 - 2h^5t^2 - 10h^5t + 9h^4t^2 \\
& - 2h^4t + 4 + 8h^5 + t^3h + 2h^6t^3 + 4h^6t - 5h^5t^3),
\end{aligned}$$

$$\begin{aligned}
C_1 &= (t^2h^2 + h^2 + 6h + 9)(th - t + 2)^2, \\
C_2 &= (t^2h^2 + h^2 + 6h + 9)(th - t + 2)(h^2 + 1), \\
C_3 &= (th - t + 2)(h^2 + 1).
\end{aligned}$$

Finally, if $\mathcal{T}(t,h)$ is the polynomial parametric solution obtained by Algorithm 1, we have that all the polynomial parametric solutions for $(\overline{\mathcal{P}}, \bar{s})$, are

$$\mathcal{T}(t,h) + h^2(h-1)^2(R_1,\ R_2,\ R_3),$$

where $R_i \in \mathbb{K}[t,h]$. □

7. Computation of all parametric blending solutions

Combining the results presented in the previous section, one can derive an algorithm for computing all parametric solutions for a given rational blending data $(\overline{\mathcal{P}}, \bar{s})$. Furthermore, we also present an algorithm that decides whether the blending data $(\overline{\mathcal{P}}, \bar{s})$ has a parametric polynomial solution, and in the affirmative case computes all the polynomial solutions for the rational blending data.

Algorithm (General Rational Solution). Given a rational blending data $(\overline{\mathcal{P}}, \bar{s})$, the algorithm computes all the parametric solutions for $(\overline{\mathcal{P}}, \bar{s})$.

(1) Compute a particular parametric solution for $(\overline{\mathcal{P}}, \bar{s})$ (apply any of the algorithms described in the previous section). Let $\mathcal{B}_{\mathrm{p}}(t, h)$ be the output parametrization.
(2) Let $\mathcal{B}_{\mathrm{g}}(t, h)$ be the general solution of the homogeneous system $\mathcal{E}_{\mathrm{H}}$, that is

$$\mathcal{B}_{\mathrm{g}}(t, h) := \prod_{i=0}^{n-1}(h - s_i)^{k+1}\left(\frac{N_1}{M_1}, \frac{N_2}{M_2}, \frac{N_3}{M_3}\right),$$

where $N_i, M_i \in \mathbb{K}[t, h]$ and $\gcd(\prod_{i=0}^{n-1}(h - s_i), M_i) = 1$.
(3) Return $\mathcal{B}_{\mathrm{p}}(t, h) + \mathcal{B}_{\mathrm{g}}(t, h)$.

Algorithm (General Polynomial Solution). Given a rational blending data $(\overline{\mathcal{P}}, \bar{s})$, the algorithm decides whether there exists a parametric polynomial solution for $(\overline{\mathcal{P}}, \bar{s})$, and in the affirmative case computes all the parametric polynomial solutions for the blending data.

(1) If the rational functions

$$\frac{\partial^j \mathcal{P}_i}{\partial^j h}(t, s_{i-1}) \qquad \text{for } j = 0, \ldots, k, i = 1, \ldots, n,$$

are not all polynomial, then return "`There is not a polynomial solution`". Else, apply Algorithm 1 to compute a particular polynomial parametric solution for $(\overline{\mathcal{P}}, \bar{s})$. Let $\mathcal{B}_{\mathrm{p}}^{\mathrm{Pol}}(t, h)$ be the output parametrization.
(2) Let $\mathcal{B}_{\mathrm{g}}^{\mathrm{Pol}}(t, h)$ be the general solution of the homogeneous system $\mathcal{E}_{\mathrm{H}}^{\mathrm{Pol}}$, that is

$$\mathcal{B}_{\mathrm{g}}^{\mathrm{Pol}}(t, h) := \prod_{i=0}^{n-1}(h - s_i)^{k+1}(R_1,\ R_2,\ R_3), \qquad \text{where } R_i \in \mathbb{K}[t, h].$$

(3) Return $\mathcal{B}_{\mathrm{p}}^{\mathrm{Pol}}(t, h) + \mathcal{B}_{\mathrm{g}}^{\mathrm{Pol}}(t, h)$.

We illustrate Algorithm General Rational Solution by some examples, where the two possible algorithms in Step 1 are considered. For Algorithm General Polynomial Solution, see Example 1.

Example 2. We consider the typical example of blending two cylinders. Let V_1 and V_2 be the cylinders parametrized by

$$\mathcal{P}_1(t, h) = \left(\frac{t^2 - 1}{t^2 + 1}, \frac{8t^2 + 8 - 15t^2h - 15h + 18th + 4t^2h^3 + 4h^3}{4(t^2 + 1)}, \frac{2t}{t^2 + 1}\right),$$

$$\mathcal{P}_2(t, h) = \left(\frac{t^2 - 1}{t^2 + 1}, \frac{2t}{t^2 + 1}, \frac{2(-t^2 - 1 + 2t - 2th + 2t^2h + 2h)}{t^2 + 1}\right)$$

and let

$$\mathcal{Q}_1(t) = \mathcal{P}_1(t, 0) = \left(\frac{t^2 - 1}{t^2 + 1},\ 2,\ \frac{2t}{t^2 + 1}\right),$$

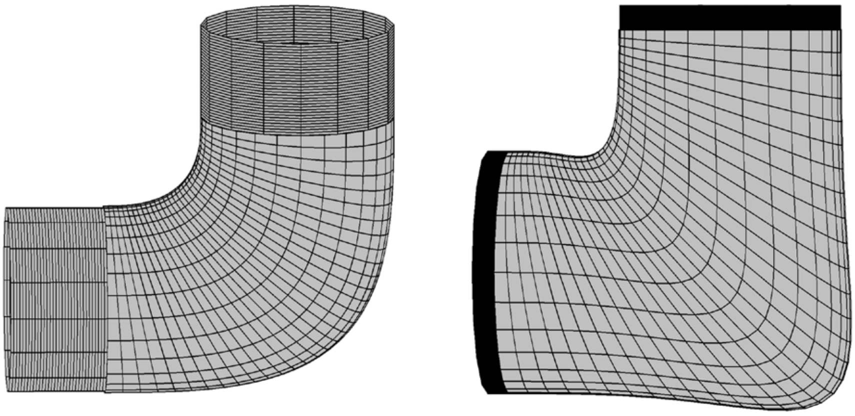

Fig. 3. Blending surface by Algorithm 1 (left), and by Algorithm 2 (right).

$$\mathcal{Q}_2(t) = \mathcal{P}_2(t, 1) = \left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}, 2\right),$$

be the parametrization of the clipping curves C_i. Thus, we consider the rational blending data $\mathcal{S} = ((\mathcal{P}_1, \mathcal{P}_2), (s_0, s_1))$ where $s_0 = 0$, and $s_1 = 1$.

We apply Algorithm General Rational Solution to compute all parametric solutions for $\mathcal{S}$ with G^2-geometric continuity. In Step 1, we compute a particular solution. For this purpose, we may choose either Algorithm 1 or 2.

Algorithm 1. We compute $\frac{\partial^j \mathcal{P}_i}{\partial^j h}(t, s_{i-1})$ for $i = 1, 2$, and $j = 1, 2$:

$$\begin{aligned}
\frac{\partial \mathcal{P}_1}{\partial h}(t, 0) &= \left(0, \frac{-3(5t^2+5-6t)}{4(t^2+1)}, 0\right),\\
\frac{\partial \mathcal{P}_2}{\partial h}(t, 1) &= \left(0, 0, \frac{4(-t+t^2+1)}{t^2+1}\right),\\
\frac{\partial^2 \mathcal{P}_1}{\partial^2 h}(t, 0) &= \frac{\partial^2 \mathcal{P}_2}{\partial^2 h}(t, 1) = (0, 0, 0).
\end{aligned}$$

Thus, the particular solution generated by Algorithm 1 is

$$\mathcal{T}(t, h) = \left(-\frac{t^2-1}{t^2+1}, \frac{8t^2-15t^2h-3t^2h^5+10t^2h^3+18th-28h^3t-6th^5}{4(t^2+1)} + \frac{24h^4t+8-15h-3h^5+10h^3}{4(t^2+1)}, \frac{2(t-2h^3t+2t^2h^3+2h^3+h^4t-t^2h^4-h^4)}{t^2+1}\right).$$

In Fig. 3 (left) we plot together the cylinders V_1, V_2, and the surface parametrized by $\mathcal{T}(t, h)$.

Algorithm 2. We compute the rational function, with $u = 1/2$,

$$f(h) = \frac{u(1-h)^3}{u(1-h)^3 + (1-u)h^3}.$$

Thus, the particular solution generated by Algorithm 2 is

$$\mathcal{T}(t,h) = \left(\frac{t^2-1}{t^2+1}, \frac{8 + 62h^3t + 18th - 39t^2h - 18h^4t - 54h^2t - 4h^6 - 39h}{4(1-3h+3h^2)(t^2+1)} \right.$$
$$+ \frac{8t^2 + 69h^2 + 3h^4 69h^2t^2 - 49h^3 + 12h^5 - 49t^2h^3 + 3t^2h^4 + 12t^2h^5 - 4h^6t^2}{(1-3h+3h^2)(t^2+1)},$$
$$\left. 2\frac{-3th + 3h^2t + h^3t + t - t^2h^3 - h^3 - 2h^4t + 2t^2h^4 + 2h^4}{(1-3h+3h^2)(t^2+1)} \right).$$

In Fig. 3 (right) we plot together the cylinders V_1, V_2, and the surface parametrized by this particular solution $\mathcal{T}(t,h)$.

Finally, if $\mathcal{T}(t,h)$ is any of the parametric solutions obtained above, we have that all the parametric solutions for $(\overline{\mathcal{P}}, \bar{s})$, are

$$\mathcal{T}(t,h) + h^3(h-1)^3 \left(\frac{N_1}{M_1}, \frac{N_2}{M_2}, \frac{N_3}{M_3}\right),$$

where $N_i, M_i \in \mathbb{K}[t,h]$ and $\gcd(h(h-1), M_i) = 1$. $\square$

Example 3. In this example we apply Algorithm General Rational Solution to obtain all the parametric solutions for three surfaces with G^1-continuity. Let V_1 be the sphere $x^2 + y^2 + (z-1)^2 - 1$, V_2 be the cylinder $x^2 + y^2 - 4$, and V_3 be the sphere $x^2 + y^2 + (z+1)^2 - 1$. We consider the following parametrizations of V_1, V_2, V_3:

$$\mathcal{P}_1(t,h) = \left(\frac{h^2+t^2-1}{h^2+t^2+1}, \frac{2t}{h^2+t^2+1}, \frac{2h}{h^2+t^2+1} + 1\right),$$
$$\mathcal{P}_2(t,h) = \left(2\frac{t^2-1}{t^2+1}, \frac{4t}{t^2+1}, h-1\right),$$
$$\mathcal{P}_3(t,h) = \left(\frac{h^2-4h+3+t^2}{h^2-4h+5+t^2}, \frac{2t}{h^2-4h+5+t^2}, -2\frac{-5h+7+h^2+t^2}{5-4h+h^2+t^2}\right)$$

and the parametrization of the clipping curves C_i:

$$\mathcal{Q}_1(t) = \mathcal{P}_1(t,0) = \left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}, 1\right),$$
$$\mathcal{Q}_2(t) = \mathcal{P}_2(t,1) = \left(2\frac{t^2-1}{t^2+1}, \frac{4t}{t^2+1}, 0\right),$$
$$\mathcal{Q}_3(t) = \mathcal{P}_3(t,2) = \left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}, -2\right).$$

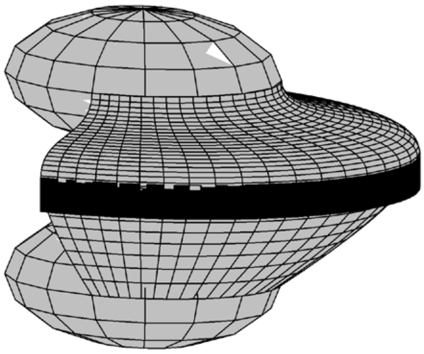

Fig. 4. Spheres, cylinder (black colour) and blending surface generated by Algorithm 1.

Thus, we consider the rational blending data

$$\mathcal{S} = ((\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3), (0, 1, 2)).$$

We apply Algorithm General Rational Solution to compute all parametric solutions for $\mathcal{S}$ with G^1-geometric continuity. In Step 1, we compute a particular solution. For this purpose, we may choose either Algorithm 1 or 2.

Algorithm 1. We compute $\frac{\partial^j \mathcal{P}_i}{\partial^j h}(t, s_{i-1})$ for $i = 1, 2, 3$ and $j = 1$:

$$\frac{\partial \mathcal{P}_1}{\partial h}(t, 0) = \left(0, 0, \frac{2}{t^2+1}\right), \qquad \frac{\partial \mathcal{P}_2}{\partial h}(t, 1) = (0, 0, 1),$$
$$\frac{\partial \mathcal{P}_3}{\partial h}(t, 2) = \left(0, 0, \frac{2}{t^2+1}\right).$$

In this situation the parametric solution for the rational blending data $\mathcal{S}$, provided by Algorithm 1 is

$$\mathcal{T}(t, h) = \left(\frac{(t^2-1)(1+4h^2+h^4-4h^3)}{t^2+1}, \frac{2t(1+4h^2-4h^3+h^4)}{t^2+1}, \right.$$
$$\left. \frac{4t^2+4+8h-53h^2t^2-81h^2+99t^2h^3+135h^3-63t^2h^4-83h^4+13t^2h^5+17h^5}{4(t^2+1)}\right).$$

In Fig. 4 we plot together the spheres V_1, V_3, part of the cylinder V_2 and part of the blending surface parametrized by $\mathcal{T}(t, h)$.

Algorithm 2. We compute the rational functions with $u_1 = 1/2$, $u_2 = 3/4$, $u_3 = 2/3$,

$$f_1(h) = \frac{u_1(1-h)^2(2-h)^2}{u_1(1-h)^2(2-h)^2 + (1-u_1)h^2},$$
$$f_2(h) = \frac{u_2(-h)^2(2-h)^2}{u_2(-h)^2(2-h)^2 + (1-u_2)(h-1)^2},$$
$$f_3(h) = \frac{u_3(1-h)^2(h)^2}{u_3(1-h)^2(h)^2 + (1-u_3)(h-2)^2}.$$

Thus, the particular solution generated by Algorithm 2 is

$$\mathcal{T}(t,h) = \left(\frac{D_1}{D}, \frac{D_2}{D}, \frac{D_3}{D}\right).$$

$$\begin{aligned}
D ={}& (4 - 12h + 14h^2 - 6h^3 + h^4)(h^2 + t^2 + 1)(13h^2 - 12h^3 + 3h^4 \\
&+ 1 - 2h)(t^2 + 1)(3h^2 - 4h^3 + 2h^4 + 4 - 4h)(5 - 4h + h^2 + t^2)
\end{aligned}$$

$$\begin{aligned}
D_1 ={}& -80 + 96t^2h + 8884t^6h^8 - 11\,968t^6h^7 + 7335t^6h^4 - 10\,756t^6h^5 \\
&+ 12\,442h^6t^6 - 100\,254h^6 + 840h^2t^6 - 3352t^6h^3 - 96t^6h - 288t^6h^{11} \\
&- 4594t^6h^9 + 1519h^{10}t^6 - 384h^{15}t^2 - 672t^4h^{13} + 24h^{16}t^2 + 48t^4h^{14} \\
&+ 24t^6h^{12} + 4310t^4h^{12} - 12\,158h^{13}t^2 + 2791h^{14}t^2 - 16\,704t^4h^{11} \\
&- 82\,442t^4h^9 + 43\,739h^{10}t^4 - 73\,764t^2h^{11} + 35\,450t^2h^{12} - 544t^4h \\
&+ 4616h^2t^4 - 20\,312t^4h^3 + 51\,763t^4h^4 - 89\,824t^4h^5 + 119\,904h^6t^4 \\
&- 106\,844t^2h^7 + 138\,671t^2h^8 - 131\,120t^4h^7 + 115\,179h^{10}t^2 \\
&- 141\,034h^9t^2 + 80t^4 + 117\,176t^4h^8 - 27\,121h^{10} + 54\,702h^9 + 544h \\
&- 16t^2 - 4456h^2 - 2508h^{12} + 394h^{13} - 29h^{14} + 9900h^{11} - 47\,571h^4 \\
&- 680h^2t^2 + 16t^6 + 19\,160h^3 + 79\,376h^5 + 2200t^2h^3 - 71t^2h^4 \\
&- 18\,124t^2h^5 - 84\,245h^8 + 102\,164h^7 + 58\,664h^6t^2
\end{aligned}$$

$$\begin{aligned}
D_2 ={}& 2t(80 - 640t^2h + 131\,873h^6 + 24t^4h^{12} - 504h^{13}t^2 + 36h^{14}t^2 - 288t^4h^{11} \\
&- 4594t^4h^9 + 1519h^{10}t^4 - 12\,906t^2h^{11} + 3265t^2h^{12} - 96t^4h - 192h^{15} \\
&+ 12h^{16} + 840h^2t^4 - 3352t^4h^3 + 7335t^4h^4 - 10\,756t^4h^5 + 12\,442h^6t^4 \\
&- 121\,540t^2h^7 + 102\,346t^2h^8 - 11\,968t^4h^7 + 34\,834h^{10}t^2 - 68\,420h^9t^2 \\
&+ 16t^4 + 8884t^4h^8 + 83\,355h^{10} - 122\,770h^9 - 544h + 96t^2 + 4568h^2 \\
&+ 20\,077h^{12} - 6444h^{13} + 1422h^{14} - 46\,242h^{11} + 51\,547h^4 + 5408h^2t^2 \\
&- 19\,960h^3 - 93\,032h^5 - 23\,312t^2h^3 + 57\,346t^2h^4 - 94\,572t^2h^5 \\
&+ 150\,774h^8 - 154\,500h^7 + 118\,587h^6t^2)
\end{aligned}$$

$$\begin{aligned}
D_3 ={}& (h - 1)(-80 + 512t^2h + 77453h^6 + 6t^4h^{12} - 180h^{13}t^2 + 12h^{14}t^2 \\
&- 78t^4h^{11} - 1475t^4h^9 + 447h^{10}t^4 - 5494t^2h^{11} + 1266t^2h^{12} + 80t^4h \\
&- 102h^{15} + 6h^{16} - 160h^2t^4 - 785t^4h^4 - 1923t^4h^5 + 2803h^6t^4 \\
&- 43\,922t^2h^7 + 39\,686t^2h^8 - 2965t^4h^7 + 12\,950h^{10}t^2 - 48t^4h^3 \\
&+ 1005t^4h^4 - 2579t^4h^5 + 3981h^6t^4 - 57\,566t^2h^7 + 52\,054t^2h^8 \\
&- 4235t^4h^7 + 16\,246h^{10}t^2 - 34\,182h^9t^2 - 16t^4 + 3075t^4h^8 + 66\,858h^{10} \\
&- 97\,368h^9 + 304h - 96t^2 - 256h^2 + 14\,113h^{12} - 4091h^{13} \\
&+ 819h^{14} - 35\,429h^{11} + 18\,113h^4 - 1120h^2t^2 - 3616h^3 - 44\,985h^5 \\
&- 432t^2h^3 + 8894t^2h^4 - 25\,940t^2h^5 + 112\,098h^8 - 103\,825h^7 \\
&+ 46\,038h^6t^2).
\end{aligned}$$

Finally, if $\mathcal{T}(t, h)$ is one of the parametric solutions obtained above, we have that all the parametric solutions for $(\overline{\mathcal{P}}, \bar{s})$, are

$$\mathcal{T}(t, h) + h^2(h-1)^2(h-2)^2\left(\frac{N_1}{M_1}, \frac{N_2}{M_2}, \frac{N_3}{M_3}\right),$$

where $N_i, M_i \in \mathbb{K}[t, h]$ and $\gcd(h(h-1)(h-2), M_i) = 1$. □

The previous examples deal with blending involving quadrics. In the following example we treat a blending where the primary surfaces are not so simple. Further examples of this type have been considered in the implementation analysis (see Appendix).

Example 4. In this example we apply Algorithm General Rational Solution to obtain all the parametric solutions for two surfaces with G^2-continuity. Let V_1 and V_2 be the primary surfaces defined by the parametrizations $\mathcal{P}_1$ and $\mathcal{P}_2$ considered in Input I (see Appendix), respectively. The parametrizations of the clipping curves C_i are given by

$$\mathcal{Q}_1(t) = \mathcal{P}_1(t, 0), \qquad \mathcal{Q}_2(t) = \mathcal{P}_2(t, 1).$$

Thus, we have the rational blending data

$$\mathcal{S} = ((\mathcal{P}_1, \mathcal{P}_2), (0, 1)).$$

We apply Algorithm General Rational Solution to compute all parametric solutions for $\mathcal{S}$ with G^2-geometric continuity. In Step 1, we compute a particular solution. For this purpose, we may choose either Algorithm 1 or 2.

Algorithm 1. We compute $\frac{\partial^j \mathcal{P}_i}{\partial^j h}(t, s_{i-1})$ for $i = 1, 2$ and $j = 1, 2$. The parametric solution for the rational blending data $\mathcal{S}$, provided by Algorithm 1 is

$$\begin{aligned}\mathcal{T}(t, h) = \Big(&-\tfrac{1}{89}(1333h^3t^2 + 1333h^3t^4 - 2310h^4t^2 - 2310h^4t^4 + 1155h^5t^2\\ &+ 1155h^5t^4 - 89t^2 - 89t^4 - 2670h^3 + 4628h^4 - 2314h^5 + 178\\ &- 267h^2t^2 - 267h^2t^4 + 534h^2)/((t^2+1)(2+t^2)),\ -2/89t\\ &\times (2668h^3 + 1333h^3t^2 - 4624h^4 - 2310h^4t^2 + 2312h^5 + 1155h^5t^2\\ &- 178 - 89t^2 - 534h^2 - 267h^2t^2)/((t^2+1)(2+t^2)),\ -\tfrac{980}{89}h^3\\ &+ 1 + \tfrac{1337}{89}h^4 - 535/89h^5 - 2h + 3h^2\Big).\end{aligned}$$

In Fig. 5 we plot together the primary surfaces V_1, V_2 (left) and part of the blending surface parametrized by $\mathcal{T}(t, h)$ (right).

Algorithm 2. We compute the rational function, with $u = 1/2$,

$$f(h) = \frac{u(1-h)^3}{u(1-h)^3 + (1-u)h^3}.$$

Thus, the particular solution generated by Algorithm 2 is

$$\mathcal{T}(t, h) = \left(\frac{D_1}{D}, \frac{D_2}{D}, \frac{D_3}{D}\right),$$

$$D = (3h^2 - 3h + 1)(4 - 12h + 102h^2 - 6h^3 + h^4)(h^2 + t^2 + 1)$$

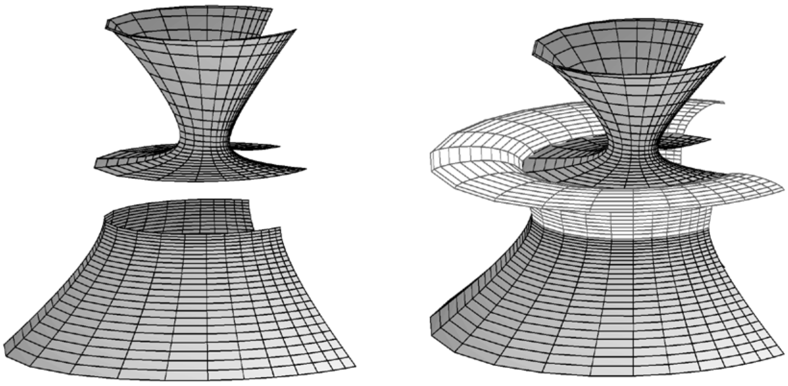

Fig. 5. Primary surfaces and blending surface generated by Algorithm 1.

$$\times (12h^2 - 4h^3 + h^4 + 8 - 16h)(t^2 + 1)(90h^2 - 2h^3 + h^4 + 356 - 356h)$$
$$\times (5 - 4h + h^2 + t^2),$$

$$D_1 = 558\,208h - 672\,004h^{17}t^2 + 88\,120h^{18}t^2 - 9108h^{19}t^2 + 759h^{20}t^2 + 148\,387h^{16}t^4 - 15\,641h^{17}t^4 + 1380h^{18}t^4 - 1085\,998h^{15}t^4 + 17\,938\,432h^3 - 55\,806\,576h^4 - 4024\,384h^2 + 137\,050\,192h^5 - 277\,070\,208h^6 + 11\,392t^6 - 11\,392t^2 - 622\,447\,132h^8 + 458\,753\,968h^7 + 17\,190\,810h^{15} - 3969\,852h^{16} - 56\,950\,240h^{14} - 642\,977\,632h^{10} + 146\,858\,101h^{13} + 9186h^{19} + 486\,967\,632h^{11} - 298\,989\,869h^{12} + 696\,304\,716h^9 + 56\,960t^4 + 2532\,176h^5t^2 + 709\,184h^2t^6 - 2888\,192h^3t^6 + 7915\,440h^4t^6 + 40\,604\,604h^8t^6 + 29\,706\,848h^6t^6 - 40\,192\,720h^7t^6 - 16\,052\,936h^{15}t^2 + 3813\,398h^{16}t^2 + 5819\,380h^{14}t^4 + 16\,559\,560h^{10}t^6 - 23\,115\,337h^{13}t^4 - 6647\,704h^{11}t^6 + 1945\,661h^{12}t^6 - 30\,250\,812h^9t^6 - 17\,001\,168h^5t^6 - 102\,528ht^6 + 102\,528ht^2 - 663\,616h^2t^2 + 2432\,512h^3t^2 - 4657\,200h^4t^2 + 50\,867\,406h^{14}t^2 + 285\,010\,256h^{10}t^4 - 159\,978\,384h^{11}t^4 + 69\,438\,241h^{12}t^4 - 122\,641\,835h^{13}t^2 + 225\,792\,143h^{12}t^2 - 394\,057\,916h^9t^4 - 316\,691\,688h^{11}t^2 + 153\,528\,244h^8t^2 - 122\,443\,984h^5t^4 - 558\,208ht^4 + 3978\,816h^2t^4 - 17\,482\,752h^3t^4 + 52\,548\,336h^4t^4 + 423\,207\,596h^8t^4 + 335\,831\,624h^{10}t^2 + 232\,369\,792h^6t^4 - 353\,793\,328h^7t^4 - 265\,683\,028h^9t^2 + 14\,078\,112h^6t^2 - 61\,984\,176h^7t^2 - 56\,960 - 41h^{21}t^2 - 74h^{19}t^4 + 41h^{21} - 2h^{22} + 2h^{22}t^2 + 4h^{20}t^4 + 619h^{16}t^6 - 33h^{17}t^6 + 2h^{18}t^6 - 6500h^{15}t^6 + 59\,646h^{14}t^6 - 407\,457h^{13}t^6 + 688\,530h^{17} - 89\,566h^{18} - 763h^{20},$$

$$D_2 = 2t(-558\,208h - 15\,676h^{17}t^2 + 1382h^{18}t^2 - 74h^{19}t^2 + 4h^{20}t^2 + 619h^{16}t^4 - 33h^{17}t^4 + 2h^{18}t^4 - 6500h^{15}t^4 - 17\,938\,432h^3 + 55\,806\,576h^4$$

$$
\begin{aligned}
&+ 4024\,384h^2 - 136\,993\,104h^5 + 276\,681\,856h^6 + 68\,352t^2\\
&+ 620\,201\,116h^8 - 457\,550\,960h^7 - 17\,168\,492h^{15} + 3966\,156h^{16}\\
&+ 56\,851\,320h^{14} + 640\,442\,016h^{10} - 146\,526\,653h^{13} - 485\,280\,200h^{11}\\
&+ 298\,137\,101h^{12} - 693\,481\,036h^9 + 11\,392t^4 + 56\,960 - 139\,456\,672h^5t^2\\
&-1092\,868h^{15}t^2 + 149\,034h^{16}t^2 + 59\,646h^{14}t^4 - 407\,457h^{13}t^4\\
&- 660\,736ht^2 + 4688\,000h^2t^2 - 20\,370\,944h^3t^2 + 60\,463\,776h^4t^2\\
&+ 5882\,010h^{14}t^2 + 16\,559\,560h^{10}t^4 - 6647\,704h^{11}t^4 + 1945\,661h^{12}t^4\\
&- 23\,538\,082h^{13}t^2 + 71438\,046h^{12}t^2 - 30\,250\,812h^9t^4\\
&- 166\,763\,584h^{11}t^2 + 464\,119\,528h^8t^2 - 17\,001\,168h^5t^4 - 102\,528ht^4\\
&+ 709\,184h^2t^4 - 2888\,192h^3t^4 + 7915\,440h^4t^4 + 40\,604\,604h^8t^4\\
&- 9184h^{19} + 763h^{20} + 301\,822\,296h^{10}t^2 + 29\,706\,848h^6t^4\\
&- 40\,192\,720h^7t^4 - 424\,641\,528h^9t^2 + 89\,534h^{18} + 262\,146\,016h^6t^2\\
&- 394\,174\,912h^7t^2 - 41h^{21} + 2h^{22} - 688\,106h^{17}),
\end{aligned}
$$

$$
\begin{aligned}
D_3 = {}& (h-1)(615\,168h + 1318h^{17}t^2 - 70h^{18}t^2 + 4h^{19}t^2 - 31h^{16}t^4 + 2h^{17}t^4\\
&+ 591h^{15}t^4 + 21\,119\,808h^3 - 67\,214\,384h^4 - 4525\,632h^2 + 157\,616\,288h^5\\
&- 285\,436\,320h^6 - 68\,352t^2 - 480\,514\,492h^8 + 411\,357\,856h^7\\
&+ 3462\,115h^{15} - 618\,298h^{16} - 14\,533\,483h^{14} - 360\,139\,344h^{10}\\
&+ 46\,664\,923h^{13} + 229\,018\,510h^{11} - 116\,459\,441h^{12} + 459\,601\,108h^9\\
&- 11\,392t^4 + 158\,912\,512h^5t^2 + 136\,528h^{15}t^2 - 14\,464h^{16}t^2 - 5957h^{14}t^4\\
&+ 54\,575h^{13}t^4 + 729\,088ht^2 - 5280\,384h^2t^2 + 24\,056\,448h^3t^2\\
&- 73\,070\,880h^4t^2 - 977\,452h^{14}t^2 + 727h^{19} - 39h^{20} - 5467\,704h^{10}t^4\\
&+ 1660\,422h^{11}t^4 - 361\,681h^{12}t^4 + 5095\,654h^{13}t^2 - 19\,751\,782h^{12}t^2\\
&+ 13\,077\,764h^9t^4 + 58\,010\,404h^{11}t^2 - 306\,776\,328h^8t^2 + 18\,909\,792h^5t^4\\
&+ 113\,920ht^4 - 800\,320h^2t^4 + 3437\,888h^3t^4 - 9615\,984h^4t^4\\
&- 22\,745\,660h^8t^4 - 130\,678\,872h^{10}t^2 - 27\,163\,328h^6t^4 + 28\,925\,024h^7t^4\\
&+ 227\,463\,288h^9t^2 - 258\,102\,624h^6t^2 + 320\,347\,648h^7t^2 - 56\,960 + 2h^{21}\\
&+ 82\,087h^{17} - 8515h^{18}).
\end{aligned}
$$

Finally, if $\mathcal{T}(t,h)$ is one of the parametric solutions obtained above, we have that all the parametric solutions for $(\overline{\mathcal{P}}, \bar{s})$, are

$$\mathcal{T}(t,h) + h^3(h-1)^3\left(\frac{N_1}{M_1}, \frac{N_2}{M_2}, \frac{N_3}{M_3}\right),$$

where $N_i, M_i \in \mathbb{K}[t,h]$ and $\gcd(h(h-1),\ M_i) = 1$. □

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Appendix. Parametrizations in Section 5

In this section we show the data parametrizations used in Section 5. We assume that the clipping curves C_i are given by the parametrizations of the form

$$\mathcal{Q}_i(t) = \mathcal{P}_i(t, i-1), \qquad \text{for } i = 1, \ldots, n,$$

where $\mathcal{P}_i(t, h)$ are the parametrizations of the primary surfaces V_i.

Input I:

$$\mathcal{P}_1 = \left(\frac{-(t^2-1)(-1-3h^2+2h^3)}{t^2+1}, \right. \\ \left. \frac{-2t(-1-3h^2+2h^3)}{t^2+1},\ 1-2h+3h^2-2h^3 \right),$$

$$\begin{aligned}
\mathcal{P}_2 = {} & (2(193\,664h + 1082\,464h^3 - 1874\,632h^4 - 5696t^2 - 598\,112h^2 \\
& - 981\,968h^5t^2 + 2694\,992h^5 - 3183\,560h^6 - 28\,480 + 5696t^6 \\
& + 96\,864h^2t^6 - 171\,104h^3t^6 + 333\,928h^4t^6 + 22\,702h^8t^6 + 299\,352h^6t^6 \\
& - 111\,200h^7t^6 - 32h^{15}t^2 + 2h^{16}t^2 + 4h^{14}t^4 + 399h^{10}t^6 - 56h^{13}t^4 \\
& - 24h^{11}t^6 + 2h^{12}t^6 - 3110h^9t^6 - 431\,408h^5t^6 - 34\,176ht^6 + 34\,176ht^2 \\
& - 62\,560h^2t^2 - 57\,760h^3t^2 + 384\,792h^4t^2 - 15\,324h^{12} + 1162h^{13} - 83h^{14} \\
& + 123\,464h^{11} + 505h^{14}t^2 + 59\,839h^{10}t^4 - 7936h^{11}t^4 + 904h^{12}t^4 \\
& - 4830h^{13}t^2 + 37\,106h^{12}t^2 - 328\,758h^9t^4 - 217\,408h^{11}t^2 + 3371\,762h^8t^2 \\
& - 3834\,960h^5t^4 - 193\,664ht^4 + 632\,416h^2t^4 - 1311\,328h^3t^4 \\
& + 2547\,784h^4t^4 + 1157\,014h^8t^4 + 849\,849h^{10}t^2 + 3890\,248h^6t^4 \\
& - 2616\,224h^7t^4 - 563\,351h^{10} + 28\,480t^4 - 2124\,650h^9t^2 + 2150\,440h^6t^2 \\
& - 3373\,728h^7t^2 + 3313\,056h^7 - 2726\,550h^8 + 1549\,606h^9)/((4 - 12h \\
& + 102h^2 - 6h^3 + h^4)(h^2 + t^2 + 1)(12h^2 - 4h^3 + h^4 + 8 - 16h)(t^2 + 1) \\
& \times (90h^2 - 2h^3 + h^4 + 356 - 356h)(5 - 4h + h^2 + t^2)), 4t(-19\,3664h \\
& -1276\,640h^3 + 2476\,136h^4 + 34\,176t^2 + 626\,656h^2 - 4112\,704h^5t^2 \\
& - 3818\,000h^5 + 4595\,400h^6 + 28\,480 - 227\,840ht^2 + 723\,520h^2t^2 \\
& -1447\,744h^3t^2 + 2787\,280h^4t^2 + 26\,483h^{12} - 3010h^{13} + 295h^{14} \\
& - 172\,924h^{11} + 3h^{14}t^2 + 399h^{10}t^4 - 24h^{11}t^4 + 2h^{12}t^4 - 42h^{13}t^2 \\
& + 721h^{12}t^2 - 3110h^9t^4 - 6468h^{11}t^2 + 1110\,968h^8t^2 - 431\,408h^5t^4 \\
& - 34\,176ht^4 + 96\,864h^2t^4 - 171\,104h^3t^4 + 333\,928h^4t^4 + 22\,702h^8t^4 \\
& + 52\,594h^{10}t^2 + 299\,352h^6t^4 - 111\,200h^7t^4 - 304\,796h^9t^2 + 729\,075h^{10} \\
& +5696t^4 + 4023\,200h^6t^2 - 2601\,184h^7t^2 - 4580\,864h^7 + 3570\,266h^8 \\
& - 1975\,990h^9 - 16h^{15} + h^{16})/((4 - 12h + 102h^2 - 6h^3 + h^4) \\
& \times (h^2 + t^2 + 1)(12h^2 - 4h^3 + h^4 + 8 - 16h)(t^2 + 1)(90h^2 - 2h^3 \\
& + h^4 + 356 - 356h)(5 - 4h + h^2 + t^2)), (h - 1)(216\,448h - 212\,992h^3
\end{aligned}$$

$$
\begin{aligned}
&+1536\,928h^4 - 68\,352t^2 - 233\,248h^2 - 1816\,864h^5t^2 - 3072\,928h^5\\
&+3995\,132h^6 + 364\,544ht^2 - 854\,080h^2t^2 + 1125\,632h^3t^2 - 80\,128h^4t^2\\
&+27\,629h^{12} - 3250h^{13} + 313h^{14} - 174\,205h^{11} + 2h^{14}t^2 + 251h^{10}t^4\\
&-13h^{11}t^4 + h^{12}t^4 - 30h^{13}t^2 + 564h^{12}t^2 - 2108h^9t^4 - 5346h^{11}t^2\\
&+919\,196h^8t^2 - 260\,736h^5t^4 + 56\,960ht^4 - 119\,584h^2t^4 + 130\,816h^3t^4\\
&+59\,616h^4t^4 + 16\,898h^8t^4 + 44\,370h^{10}t^2 + 224\,188h^6t^4 - 86\,976h^7t^4\\
&-258\,792h^9t^2 + 717\,379h^{10} - 11\,392t^4 + 2671\,320h^6t^2 - 2010\,352h^7t^2\\
&-4156\,080h^7 + 3351\,418h^8 - 1903\,884h^9 - 56\,960 - 17h^{15} + h^{16})\\
&\times((4 - 12h + 102h^2 - 6h^3 + h^4) \times (h^2 + t^2 + 1)(12h^2 - 4h^3 + h^4\\
&+8 - 16h)(90h^2 - 2h^3 + h^4 + 356 - 356h)(5 - 4h + h^2 + t^2))^{-1}).
\end{aligned}
$$

Input II:

$$
\mathcal{P}_1 = \left(h+1,\ \frac{2t}{t^2+1},\ \frac{t^2-1}{t^2+1}\right), \qquad \mathcal{P}_2 = \left(\frac{4t}{t^2+1},\ 2\frac{t^2-1}{t^2+1},\ h+2\right).
$$

Input III:

$$
\begin{aligned}
\mathcal{P}_1 &= (-2t,\ 10h-5,\ 6h - 6h^2 + 3/2 - 6/25t^2),\\
\mathcal{P}_2 &= (4 - 3t + 8th - 8th^2,\ 10h - 5,\ 6h - 6h^2 + 27/50 + 36/25t - 96/25th\\
&\quad + 96/25th^2 - 27/50t^2 + 72/25t^2h - 168/25t^2h^2 + 192/25t^2h^3\\
&\quad - 96/25t^2h^4).
\end{aligned}
$$

Input IV:

$$
\begin{aligned}
\mathcal{P}_1 = \Bigg(&\frac{15 + 35t - 98h + 74th + 15h^2 + 41t^2}{9(1 + 10t - 8h)},\\
&\frac{-2(31 + 19t - 44h)}{47t - 19t^3h + 72h - 17t^2h^2 - 55h^2 + 72h^3},\\
&\frac{42 + 18t - 42h - 27th + 2h^2 + 74t^2}{90 - 81t + 67h - 85th^2 - 84h^2 - 42t^2}\Bigg),\\
\mathcal{P}_2 = \Bigg(&1,\ -\frac{90 - 10th - 95h^2 - 80h^3 - 90t^2 + 19t^2h}{-40 - 45t + 91h - 7th + 30h^2 + 37t^2},\\
&-\frac{42 + 39t - 20h}{-31t + 71t^3 + 47th + 58th^2 + 30h^3 + 28t^2}\Bigg).
\end{aligned}
$$

Input V:

$$
\begin{aligned}
\mathcal{P}_1 = \Bigg(&\frac{5 - 88h - 43th^2 - 73h^3 + 25t^2 + 4t^2h}{40 - 78t^3 + 62t^4 + 11h^4 + 88t^3h + th^3},\\
&\frac{30 + 81t - 5h - 28th + 4h^2 - 11t^2}{10 + 57t - 82h},
\end{aligned}
$$

$$-14/73t + 35/73h^2 + 14/73h^3 + 9/73t^2h + 51/73t^4h + t^2h^3\Bigg),$$

$$\mathcal{P}_2 = \left(\frac{32 - 37t + 93h + 58th + 90h^2 + 53t^2}{69 + 84t - 46h},\right.$$

$$\frac{-59}{56 + 83t + 91h - 92th + 93h^2 - 91t^2},$$

$$\left. 18 - 10/3t + 77/3h + 21th + 30h^2 - 61/3t^2\right),$$

$$\mathcal{P}_3 = \left(\frac{68 - 65t + 43h + 6th + 39t^2}{th(20 + 93h^2)}, \frac{-h(44 + 80t + 5h)}{34t},\right.$$

$$\left.\frac{-3h(-21h + 8t^2)}{35t - 67h^3 + 19t^2h}\right).$$

Input VI:

$$\mathcal{P}_1 = \left(h + 1, \frac{2t}{t^2 + 1}, \frac{t^2 - 1}{t^2 + 1}\right), \qquad \mathcal{P}_2 = \left(\frac{4t}{t^2 + 1}, 2\frac{t^2 - 1}{t^2 + 1}, h + 2\right),$$

$$\mathcal{P}_3 = \left(-\frac{-54 - 56t - 93h + 67th - 94h^2 + 84t^2}{39 - 89h - 22th + 77th^2}, h,\right.$$

$$\left. -\frac{76 - 63t + 63h + 69th - 89h^2 + 99t^2}{43 - 8t - 96h + 89t^3 + 58th + 81h^2}\right).$$

Input VII:

$$\mathcal{P}_1 = \left(h + 3, \frac{2t}{t^2 + 1}, \frac{t^2 - 1}{t^2 + 1}\right), \qquad \mathcal{P}_2 = \left(h - 4, \frac{2t}{t^2 + 1}, \frac{t^2 - 1}{t^2 + 1}\right),$$

$$\mathcal{P}_3 = \left(h - 1, 6t + h + t^2 + 1, \frac{h - 3th - 5t^2 + 12t + 1}{h + t + 1}\right),$$

$$\mathcal{P}_4 = \left(\frac{-1024t}{5th + 1}, -2t + th + h + 1, h - 2\right).$$

Input VIII:

$$\mathcal{P}_1 = \left(-\frac{-2 + 4ht + t^2 + 1}{6 + 7t + 8h}, \frac{5t + 4h - 2t^2 + 8t}{-1 + 4t - 8h}, 5 + 2th^2 + 7t^2\right),$$

$$\mathcal{P}_2 = (1, t + h, ht - 2), \qquad \mathcal{P}_3 = (-h, -5 + 9t - 6h + 4th + 6t, -6 + 3t),$$

$$\mathcal{P}_4 = \left(-\frac{-35t - 2h + 1}{1 + 7t - 6h}, 83t - h, 35 - h - t\right).$$

Input IX:

$$\mathcal{P}_1 = (th - t - 2,\ t^2 + h^2 - 3,\ 0), \qquad \mathcal{P}_2 = (1,\ t - t^2 + 1,\ 4 + t - 16h - th),$$

$$\mathcal{P}_3 = (t + h,\ h^2 t - t + 2,\ 2), \qquad \mathcal{P}_5 = \left(h,\ \frac{2t^2}{t^2+1},\ \frac{t^2-1}{t^2+1}\right),$$

$$\mathcal{P}_4 = \left(\frac{1}{7 + 2t + h},\ t + h,\ -(-6 - 2t + 8h - 2th^2 + 9t^2 h - 52t^2 h)\right).$$

Input X:

$$\mathcal{P}_1 = \left(h + 3,\ \frac{2t}{t^2+1},\ \frac{t^2-1}{t^2+1}\right), \qquad \mathcal{P}_2 = \left(\frac{1}{th + 1 - h^2},\ t - t^2 + 1,\ \frac{1}{t - h}\right),$$

$$\mathcal{P}_3 = \left(\frac{4t}{t^2+1},\ 2\frac{t^2-1}{t^2+1},\ h + 2\right),$$

$$\mathcal{P}_4 = \left(-\frac{-3t - 2h + 1}{1 + 7t - 6h},\ -3t - h,\ -5 - h - t\right)$$

$$\mathcal{P}_5 = (-2t,\ 10h - 5,\ h - 6h^2 + 3/2 - 6/2t).$$

Input XI:

$$\mathcal{P}_1 = \left(1 + 10t - 8h,\ \frac{1}{3 + t - 4h},\ -(2 + t - 2h - 2th)\right),$$

$$\mathcal{P}_2 = \left(0,\ \frac{-th}{-4 - 4t + 9h - t},\ -(2 + 1t - 2h)\right), \qquad \mathcal{P}_3 = (t,\ h,\ t + h),$$

$$\mathcal{P}_4 = \left(\frac{1}{h},\ t,\ \frac{h}{t}\right), \qquad \mathcal{P}_5 = (1,\ h,\ t), \qquad \mathcal{P}_6 = (h,\ t,\ h^2 - t + 1).$$

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