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Parametrization of Approximate Algebraic Curves by Lines ⋆

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Abstract

It is well known that irreducible algebraic plane curves having a singularity of maximum multiplicity are rational and can be parametrized by lines. In this paper, given a tolerance $\epsilon > 0$ and an ϵ–irreducible algebraic plane curve $\mathcal{C}$ of degree d having an ϵ-singularity of multiplicity $d-1$, we provide an algorithm that computes a proper parametrization of a rational curve that is exactly parametrizable by lines. Furthermore, the error analysis shows that under certain initial conditions that ensures that points are projectively well defined, the output curve lies within the offset region of $\mathcal{C}$ at distance at most $2\sqrt{2}\epsilon^{1/(2d)}\exp(2)$.

Key words: Approximate Algebraic Curves, Rational Parametrization, Hibrid Symbolic-Numeric Methods.

1 Introduction

Over the past several years, many authors have approached computer algebra problems by means of symbolic-numeric techniques. For instance, among others, methods for computing greatest common divisors of approximate polynomials (see (6), (9), (15), (29)), for determining functional decomposition (see (11)), for testing primality (see (20)), for finding zeros of multivariate systems (see (9), (16), (18)), for factoring approximate polynomials (see (10),(21), (30),

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(31)), or for numerical computation of Gröbner basis (see (28), (36)) have been developed.

Similarly, hybrid (i.e. symbolic and numeric) methods for the algorithmic treatment of algebraic curves and surfaces have been presented. For instance, computation of singularities have been treated in (3), (5), (13), (22), (26), implicitization methods have been proposed in (12) and (14), and the numerical condition of implicitly given algebraic curves and surfaces have been analyzed (see (17)). Also, piecewise parametrizations are provided (see (10), (23), (19)) by means of combination of both algebraic and numerical techniques for solving differential equations and rational B-spline manipulations.

However, although many authors have addressed the problem of globally and symbolically parametrizing algebraic curves and surfaces (see, (1), (24), (25), (32), (33), (34)), only few results have been achieved for the case of approximate algebraic varieties. The statement of the problem for the approximate case is slightly different than the classical symbolic parametrization question. Intuitively speaking, one is given an irreducible affine algebraic plane curve $\mathcal{C}$, that may or not be rational, and a tolerance $\epsilon > 0$, and the problem consists in computing a rational curve $\overline{\mathcal{C}}$, and its parametrization, such that almost all points of the rational curve $\overline{\mathcal{C}}$ are in the "*vicinity*" of $\mathcal{C}$. The notion of vicinity may be introduced as the offset region limited by the external and internal offset to $\mathcal{C}$ at distance ϵ (see Section 4 for more details, and (2) for basic concept on offsets), and therefore the problem consists in finding, if it is possible, a rational curve $\overline{\mathcal{C}}$ lying within the offset region of $\mathcal{C}$. For instance, let us suppose that we are given a tolerance $\epsilon = 0.001$, and that we are given the quartic $\mathcal{C}$ defined by

$$16.001 + 24.001x + 8y - 2y^2 + 12yx + 14.001x^2 + 2y^2x + x^2y + x^4 - y^3 + 6.001x^3.$$

Note that $\mathcal{C}$ has genus 3, and therefore the input curve is not rational. Our method provides as an answer the quartic $\overline{\mathcal{C}}$ defined by

$$16.008 + 24.012x + 8y - 2y^2 + 12yx + 14.006x^2 + 2y^2x + x^2y + x^4 - y^3 + 6.001x^3.$$

Now, it is easy to check that the new curve $\overline{\mathcal{C}}$ has an affine triple point at $(-2, -2)$, and hence it is rational. Furthermore, it can be parametrized by

$$\mathcal{P}(t) = (t^3 - 0.001 - t - 2t^2, t^4 + 1.999t - t^2 - 2t^3 - 2).$$

In Fig. 1 one may check that $\mathcal{C}$ and $\overline{\mathcal{C}}$ are close (see Example 2 in Section 3 for more details).

The notion of vicinity is geometric and in general may be difficult to deduce it directly from the coefficients of the implicit equations; in the sense that two implicit equations f_1 and f_2 may satisfy that $||f_1 - f_2||$ is small, and however

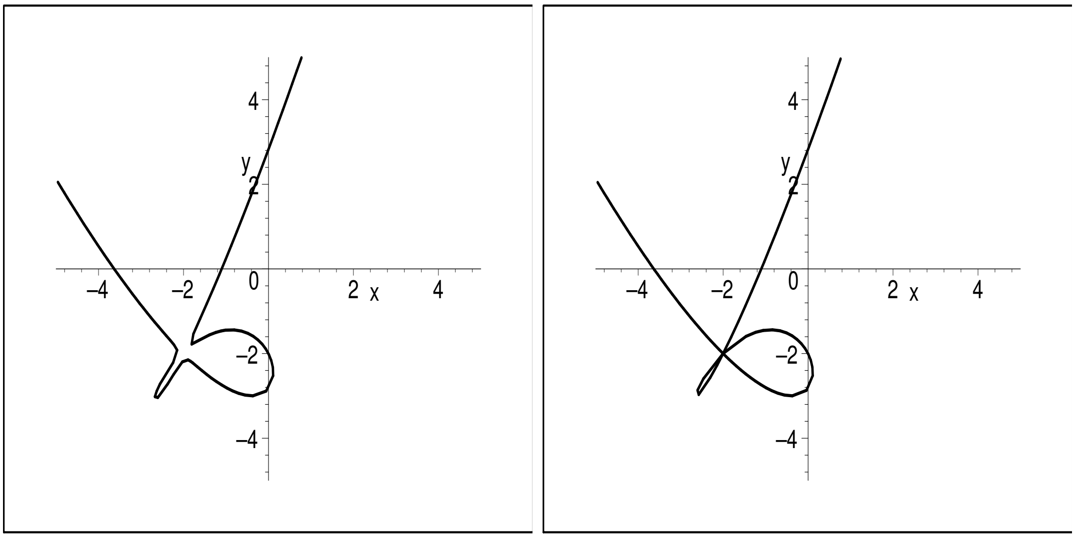


Fig. 1. Curve $\mathcal{C}$ (left), curve $\overline{\mathcal{C}}$ (right)

they may define algebraic curves that are not close; i.e none of them lie in the vicinity of the other. For example, if we consider the line $f_1 = x + y$ and the conic $f_2 = x + y + \frac{1}{1000}x^2 + \frac{1}{1000}y^2 - \frac{1}{1000}$, we have that $||f_1 - f_2||_\infty = \frac{1}{1000}$. Nevertheless, the curves defined by f_1 and f_2 are not close.

The problem of relating the tolerance with the vicinity notion, may be approached either analyzing locally the condition number of the implicit equations (see (17)) or studying whether for almost every point P on the original curve, there exists a point Q on the output curve such that the euclidean distance of P and Q is significantly smaller than the tolerance. In this paper our error analysis will be based on the second approach. From this fact, and using (17), one may derive upper bounds for the distance of the offset region.

In (4), the problem described above is studied for the case of approximate irreducible conics, rational cubics and quadrics, and the error analysis for the conic case is presented. In this paper, although we do not give an answer for the general case, we extend the results in (4) by showing how to solve the question for the special case of curves parametrizable by lines. More precisely, we provide an algorithm that parametrizes approximate irreducible algebraic curves of degree d having an ϵ–singularity of multiplicity $d-1$ (see Section 2). We illustrate the results by some examples (see Section 3), and we analyze the numerical error showing that the output rational curve lies within the offset region of the input perturbated curve at distance at most $2\sqrt{2}\epsilon^{1/(2d)}\exp(2)$ (see Section 4).

2 Numerical Parametrization by Lines.

It is well known that irreducible algebraic curves having a singularity of maximum multiplicity are rational, and that they can be parametrized by lines. Examples of curves parametrizable by lines are irreducible conics, irreducible cubics with a double point, irreducible quartics with a triple point, etc. In this section, we show that this property is also true if one considers approximate irreducible algebraic curves that "almost" have a singularity of maximum multiplicity.

Before describing the method for the approximate case, and for reasons of completeness, we briefly recall here the algorithmic approach for symbolically parametrize curves having a singularity of maximum multiplicity. The geometric idea for these type of curves is to consider a pencil of lines passing through the singular point if the curve has degree bigger than 2, or through a simple point if the curve is a conic. In this situation, all but finitely many lines in the pencil intersect the original curve exactly at two different points: the base point of the pencil and a free point on the curve. The free intersection point depends rationally on the parameter defining the line, and it yields a rational parametrization of the curve. More precisely, the symbolic algorithm for parametrizing curves by lines (where the trivial case of lines is excluded) can be outlined as follows (see (33), (34) for details):

Symbolic Parametrization By Lines

- Given an irreducible polynomial $f(x, y) \in \mathbb{K}[x, y]$ ($\mathbb{K}$ is an algebraically closed field of characteristic zero), defining an irreducible affine algebraic plane curve $\mathcal{C}$ of degree $d > 1$, with a $(d-1)$–fold point if $d \geq 3$.
- Compute a rational parametrization $\mathcal{P}(t) = (p_1(t), p_2(t))$ of $\mathcal{C}$.

1. If $d = 2$ take a point P on $\mathcal{C}$, else determine the $(d-1)$–fold point P of $\mathcal{C}$.
2. If P is at infinity, consider a linear change of variables such that P is transformed into an affine point. Let $P = (a, b)$.
3. Compute

$$A(x,y,t) = \frac{\dfrac{\partial^{(d-1)} f}{\partial^{(d-1)} x}\dfrac{1}{(d-1)!} + \dfrac{\partial^{(d-1)} f}{\partial^{(d-2)} x \partial y}\dfrac{t}{(d-2)!} + \cdots + \dfrac{t^{(d-1)}}{(d-1)!}\dfrac{\partial^{(d-1)} f}{\partial^{(d-1)} y}}{\dfrac{\partial^{d} f}{\partial^{d} x}\dfrac{1}{d!} + \dfrac{\partial^{d} f}{\partial^{(d-1)} x \partial y}\dfrac{t}{(d-1)!} + \cdots + \dfrac{t^{d}}{d!}\dfrac{\partial^{d} f}{\partial^{d} y}}.$$

   and return

$$\mathcal{P}(t) = (-A(P,t) + a, -tA(P,t) + b).$$

Remark. The parametrization can also be obtained as

$$\mathcal{P}(t) = \left(\frac{-g_{d-1}(1,t)}{g_d(1,t)} + a, \frac{-tg_{d-1}(1,t)}{g_d(1,t)} + b \right),$$

where $g_d(x, y)$ and $g_{d-1}(x, y)$ are the homogeneous components of $g(x, y) = f(x+a, y+b)$ of degree d and $d-1$, respectively. Observe that both components of $\mathcal{P}(t)$ have the same denominator.

Now, we proceed to describe the method to parametrize by lines approximate algebraic curves. For this purpose, we distinguish between the conic case and the general case. The main difference between these two cases is that in the case of conics, if the approximate curve is irreducible, the rationality is preserved. As we will see, the results obtained for conics are similar to those presented in (4). Afterwards, the ideas for the 2 degree case will be generalized to any degree an therefore results in (4) will be extended. Throughout this section, we fix a tolerance $\epsilon > 0$ and we will use the polynomial ∞–norm; i.e if $p(x, y) = \sum_{i,j \in I} a_{i,j} x^i y^j \in \mathbb{C}[x, y]$ then $||p(x, y)||$ is defined as $\max\{|a_{i,j}| \ / \ i, j \in I\}$. In particular if $p(x, y)$ is a constant coefficient $||p(x, y)||$ will denote its module.

Parametrization of Approximate Conics

Let $\mathcal{C}$ be a conic defined by an ϵ–irreducible (over $\mathbb{C}$) polynomial $f(x, y) \in \mathbb{C}[x, y]$; that is $f(x, y)$ can not be expressed as $f(x, y) = g(x, y)h(x, y) + \mathcal{E}(x, y)$ where $g, h, \mathcal{E} \in \mathbb{C}[x, y]$ and $\|\mathcal{E}(x, y)\| < \epsilon \|f(x, y)\|$ (see for instance (10)). In particular, this implies that $f(x, y)$ is irreducible and therefore $\mathcal{C}$ is rational. Thus, one may try to apply the symbolic parametrization algorithm to $\mathcal{C}$. In order to do that one has to compute a simple point on $\mathcal{C}$. Furthermore, one may check whether the simple point can be taken over $\mathbb{R}$ and, if possible, compute it. This can be done either symbolically, for instance introducing algebraic numbers with the techniques presented in (35), or numerically by root finding methods. If one works symbolically then the direct application of the algorithm will provide an exact answer. Let us assume that the simple point is approximated. For this purpose, we introduce the notion of ϵ–point.

Definition 1. *We say that $\overline{P} = (\overline{a}, \overline{b}) \in \mathbb{C}^2$ is an ϵ–affine point of an algebraic plane curve $\mathcal{C}$ defined by a polynomial $f(x, y) \in \mathbb{C}[x, y]$ if it holds that*

$$\frac{|f(\overline{P})|}{\|f(x, y)\|} < \epsilon;$$

that is, $\overline{P}$ is a simple point on $\mathcal{C}$ computed under fixed precision $\epsilon \|f(x, y)\|$. □

Note that we required the relative error w.r.t $\|f(x, y)\|$ because for any non-zero complex number λ the polynomial $\lambda f(x, y)$ also defines $\mathcal{C}$.
In this situation, let $\overline{P} = (\overline{a}, \overline{b})$ be an ϵ–affine point of $\mathcal{C}$, and let us consider

the conic $\overline{\mathcal{C}}$ defined by the polynomial

$$\overline{f}(x,y) = f(x,y) - f(\,\overline{P}\,).$$

Now, $\overline{P}$ is really a point on $\overline{\mathcal{C}}$. Furthermore, $\overline{\mathcal{C}}$ is irreducible. Indeed, if $\overline{f}$ factors as $\overline{f} = \overline{g}\overline{h}$ then $f = \overline{g}\overline{h} + f(\,\overline{P}\,)$ and $|f(\,\overline{P}\,)| < \epsilon\|f(x,y)\|$, that is f is not ϵ–irreducible, which is impossible. Therefore, we have constructed a rational conic, namely $\overline{\mathcal{C}}$ on which we know a simple point, namely $\overline{P}$. Hence, we may directly apply the symbolic algorithm to $\mathcal{C}$ to get the rational parametrization

$$\overline{\mathcal{P}}(t) = \left(-A(\overline{P},t) + \overline{a}, -tA(\overline{P},t) + \overline{b}\right),$$

where

$$A(x,y,t) = \frac{\dfrac{\partial f}{\partial x} + t\dfrac{\partial f}{\partial y}}{\dfrac{\partial^2 f}{\partial^2 x}\dfrac{1}{2!} + t\dfrac{\partial^2 f}{\partial x \partial y} + \dfrac{t^2}{2!}\dfrac{\partial^2 f}{\partial^2 y}}.$$

Parametrization of Approximate Curves

In this subsection we deal with approximate curves of degree bigger than 2. In this case, the main difficulty is that the given approximate algebraic curve is, in general, non–rational even though it might correspond to the perturbation of a rational curve. The idea to solve the problem is to generalize the construction done for conics. For this purpose, we observe that the output curve in the 2-degree case is the original polynomial minus its Taylor expansion up to order 1 at the ϵ–point, i.e. the evaluation of the polynomial at the point. We will see that for curves of degree d having "almost" a singularity of multiplicity $d-1$ one may subtract to the original polynomial its Taylor expansion up to order $d-1$ at the quasi singularity to get a rational curve close to the given one.

To be more precise, we first introduce the notion of ϵ–singularity.

Definition 2. *We say that* $\overline{P} = (\overline{a}, \overline{b}) \in \mathbb{C}^2$ *is an ϵ–affine singularity of multiplicity r of an algebraic plane curve defined by a polynomial* $f(x,y) \in \mathbb{C}[x,y]$ *if, for* $0 \leq i+j \leq r-1$, *it holds that*

$$\frac{\left\| \dfrac{\partial^{i+j} f}{\partial^i x \partial^j y}(\overline{P}) \right\|}{\|f(x,y)\|} < \epsilon.$$

□

Note that an ϵ–singularity of multiplicity 1 is an ϵ–point on the curve. Similarly, one may introduce the corresponding notion for ϵ–singularities at infinity. However, here we will work only with ϵ–affine singularities taking into account that the user can always prepare the input, by means of a suitable linear change of coordinates, in order to be in the affine case. Alternatively, one may also use the method described in (9).

In this situation, we denote by $\mathcal{L}_\epsilon^d$ the set of all ϵ–irreducible (over $\mathbb{C}$) real algebraic curves of degree d having an ϵ–singularity of multiplicity $d-1$, that we assume is real. In the previous subsection we have seen how to parametrize by lines elements in $\mathcal{L}_\epsilon^2$. In the following, we assume that $d > 2$ and we show that also elements in $\mathcal{L}_\epsilon^d$ can be parametrized by lines.

In order to check whether a given curve $\mathcal{C}$ of degree d, defined by a polynomial $f(x, y)$, belongs to $\mathcal{L}_\epsilon^d$, one has to check the ϵ–irreducibility of $f(x, y)$ as well as the existence of an ϵ–singularity of multiplicity $d-1$. For this purpose, to analyze the ϵ–irreducibility, one may use any of the existing algorithms (e.g. (10), (20), (21), (31)). The algorithm given in (10) has polynomial complexity. However, although the algorithm given in (20) has exponential complexity, in practice has very good performance. Furthermore, algorithms in (21), (31) provide improvements to the methods described in (20).

For checking the existence and computation of ϵ-singularities of multiplicity $d-1$ one has to solve the system of algebraic equations

$$\frac{\partial^{i+j} f}{\partial^i x \partial^j y}(x, y) = 0, \qquad i + j = 0, \ldots, d-2,$$

under fixed precision $\epsilon \cdot \|f(x, y)\|$, by applying root finding techniques (see (9), (22), (26), (27)). Nevertheless, one may accelerate the computation by reducing the number of equations and degrees involved in the system. More precisely, for some i_0, j_0, i_1, j_1, such that $i_0+j_0 = i_1+j_1 = d-2$, one computes the solutions of the system

$$\frac{\partial^{i_0+j_0} f}{\partial^{i_0} x \partial^{j_0} y}(x, y) = \frac{\partial^{i_1+j_1} f}{\partial^{i_1} x \partial^{j_1} y}(x, y) = 0,$$

under fixed precision $\epsilon\|f(x, y)\|$. Note that the two equations involved are quadratic. For this purpose, one may use well known methods (see for instance (9), (22), (26), (27)). Once these solutions have been approximated, one may proceed as follows: if any of the roots obtained above, say $\overline{P}$, satisfies that

$$\left\| \frac{\partial^{i+j} f}{\partial^i x \partial^j y}(\overline{P}) \right\| \leq \epsilon\|f(x, y)\|, \quad i + j = 0, \ldots d-3,$$

then $\overline{P}$ is an ϵ–singularity of multiplicity $d-1$; otherwise, $\mathcal{C}$ does not have ϵ–singularities of multiplicity $d-1$.

As an example (see Example 3 in Section 3), let $\epsilon = 0.001$, and let $\mathcal{C}$ be the real ϵ–irreducible quartic defined by

$$f(x, y) = x^4+2y^4+1.001x^3+3x^2y-y^2x-3y^3+0.00001y^2-0.001x-0.001y-0.001.$$

Applying the process described above one gets that $\mathcal{C}$ has a 3-fold ϵ–singularity at $\overline{P} = (-.1248595915\,10^{-6}, .1249844199\,10^{-6})$. In Fig. 2 appears the plot of the real part of $\mathcal{C}$, and one sees that $\overline{P}$ is "almost" a triple point of the curve.

Alternatively to the approach described above one may use the techniques presented in (5) in combination with the Gap Theorem (see (8)), and the Test Criterion.

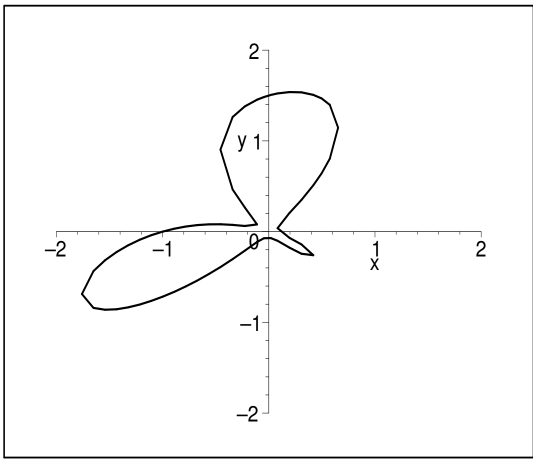


Fig. 2. Real part of the curve $\mathcal{C}$

Now, in order to parametrize the approximate algebraic curve $\mathcal{C} \in \mathcal{L}_{\epsilon}^{d}$ we consider a pencil of lines $\mathcal{H}_t$ passing through the ϵ–singularity $\overline{P} = (\overline{a}, \overline{b})$ of multiplicity $d-1$. That is, $\mathcal{H}_t$ is defined by the polynomial

$$\mathcal{H}_t(x, y, t) = y - tx - \overline{b} + \overline{a}t.$$

If $\overline{P}$ had been really a singularity, then the above symbolic algorithm would have output the parametrization $(\overline{p}_1(t), \overline{p}_2(t)) \in \mathbb{R}(t)^2$, where $\overline{p}_1(t)$ is the root in $\mathbb{R}(t)$ of the polynomial

$$\frac{f(x, tx + \overline{b} - \overline{a}t)}{(x - \overline{a})^{d-1}}$$

and $\overline{p}_2(t) = t\overline{p}_1(t) + \overline{b} - t\overline{a}$. However, in our case $\overline{P}$ is not a singularity but an ϵ–singularity. Then, the idea consists in computing the root in $\mathbb{R}(t)$ of the quotient of $f(x, tx + \overline{b} - \overline{a}t)$ and $(x - \overline{a})^{d-1}$ w.r.t. x (note that $\deg_x(f(x, tx + \overline{b} - \overline{a}t)) = d$, and therefore the quotient has degree 1 in x), say $\overline{p}_1(t)$, to finally consider $\overline{\mathcal{P}}(t) = (\overline{p}_1(t), t\overline{p}_1(t) + \overline{b} - t\overline{a})$ as approximate parametrization of $\mathcal{C}$. In the next lemma we prove that $\overline{\mathcal{P}}(t)$ is really a rational parametrization, and in Section 4, we will see that the error analysis shows that this construction generates a rational curve close to the original one.

Lemma 1. *Let $f(x, y)$ be the implicit equation of a curve $\mathcal{C} \in \mathcal{L}_{\epsilon}^{d}$ and let $\overline{P} = (\overline{a}, \overline{b})$ be the ϵ–singularity of multiplicity $d-1$ of $\mathcal{C}$. Let $\overline{p}_1(t)$ be the root* *in* $\mathbb{R}(t)$ *of the quotient of* $f(x, tx+\overline{b}-\overline{a}t)$ *and* $(x-\overline{a})^{d-1}$*, and let* $\overline{p}_2(t) = t\overline{p}_1(t)+\overline{b}-t\overline{a}$*. Then* $\overline{P}(t) = (\overline{p}_1(t), \overline{p}_2(t))$ *is a rational parametrization.*

Proof. To prove the lemma one has to show that at least one of the components of $\overline{P}(t)$ is not a constant. Let $g(x,t) = f(x, tx+\overline{b}-\overline{a}t)$. We see that $\overline{p}_1(t) \neq \overline{a}$. Indeed, if $\overline{p}_1(t) = \overline{a}$, since $\overline{p_1}(t)$ is the root of quotient of $g(x,t)$ and $(x-\overline{a})^{d-1}$, one has that $g(x,t) = \lambda(x-\overline{a})^d + R(t)$, where $\lambda \in \mathbb{R}^\star$, and $R(t) \in \mathbb{R}(t)$. Moreover, since $R(t)$ is the remainder and $(x-\overline{a})^{d-1}$ is monic in x, one has that $R(t)$ is a polynomial. Let us say that $R(t) = a_s t^s + \cdots + a_0$, with $a_s \neq 0$. Thus,

$$f(x,y) = g\left(x, \frac{y-\overline{b}}{x-\overline{a}}\right) =$$

$$\lambda(x-\overline{a})^d + \frac{a_s(y-\overline{b})^s + a_{s-1}(y-\overline{b})^{s-1}(x-\overline{a}) + \cdots + a_0(x-\overline{a})^s}{(x-\overline{a})^s}.$$

However, if $s>0$ this implies that $(x-\overline{a})$ divides $a_s(y-\overline{b})^s$ which is impossible because $a_s \neq 0$. Hence $s=0$; i.e. $R(t)$ is a constant μ. That is, $f(x,y) = \lambda(x-\overline{a})^d + \mu$. Therefore, since $f(x,y)$ is a univariate of polynomial of degree bigger than 1, it is reducible and hence it is not ϵ-irreducible which is impossible. □

Lemma 2. *The parametrization* $\overline{P}(t) = (\overline{p}_1(t), \overline{p}_2(t))$ *in Lemma 1 is proper.*

Proof. Note that $t = \dfrac{\overline{p}_2-\overline{b}}{\overline{p}_1-\overline{a}}$. Thus, $\overline{\mathcal{P}}(t)$ is proper and its inverse is $\dfrac{y-\overline{b}}{x-\overline{a}}$. □

In the next lemma, for $P \in \mathbb{R}^2$ and $\delta > 0$, we denote by $D(P,\delta)$ the Euclidean disk

$$D(P,\delta) = \{(x,y) \in \mathbb{R}^2 \quad | \quad \|(x,y)-P\|_2 \leq \delta\}.$$

Lemma 3. *Let* $\mathcal{C}$ *be an affine algebraic curve, defined by a polynomial* $f(x,y) \in \mathbb{R}[x,y]$*, having a real* ϵ*–singularity* $\overline{P}$ *of multiplicity* r*. Then, there exists* $\delta > 0$ *such that any point* $Q \in D(\overline{P},\delta)$ *is also an* ϵ*–singularity of multiplicity* r *of* $\mathcal{C}$*.*

Proof. We denote by $f_{i,j}$ the partial derivative $\frac{\partial^{i+j} f}{\partial^i x \partial^j y}$. Since $\overline{P}$ is an ϵ–singularity of multiplicity r, for $i+j = 1,\ldots,r-1$, it holds that $|f_{i,j}(\overline{P})| < \epsilon\|f(x,y)\|$. Let us denote $|f_{i,j}(\overline{P})| = \epsilon_{i,j}$ for $i+j = 1,\ldots,r-1$. Then, for each $\epsilon_{i,j}$ there exist $\lambda_{i,j} > 0$ such that

$$\epsilon_{i,j} = \epsilon\|f(x,y)\| - \lambda_{i,j} < \epsilon\|f(x,y)\|.$$

We consider $\lambda = \min\{\lambda_{i,j},\ \ i+j = 1,\ldots,r-1\}$ (note that $\lambda > 0$). On the other hand, since all partial derivatives are continuous, let M bound all partial derivatives up to order r in the compact set $D(\overline{P},\epsilon)$, and let δ be strictly smaller than $\min\{\lambda/(2M), \epsilon\}$; note that $M > 0$ since otherwise it would imply that $\mathcal{C}$ contains a disk of points which is impossible. Now, take $Q \in D(\overline{P}, \delta)$. Then, by applying the Mean Value Theorem, we have that for $i + j = 1, \ldots, r-1$

$$|f_{i,j}(Q)| \leq |f_{i,j}(\overline{P})| + |f_{i,j}(\overline{P}) - f_{i,j}(Q)| \leq \epsilon_{i,j} + |\nabla(f_{i,j}(\xi_{i,j})) \cdot (\overline{P} - Q)^T|,$$

where $\xi_{i,j}$ is on the segment joining Q and $\overline{P}$. Then, one concludes that

$$|f_{i,j}(Q)| \leq \epsilon\|f(x,y)\| - \lambda_{i,j} + 2\delta M \leq \epsilon\|f(x,y)\| - \lambda + 2\delta M < \epsilon\|f(x,y)\|.$$

Therefore, Q is an ϵ–singularity of multiplicity r of $\mathcal{C}$. □

Now, let $\mathcal{C} \in \mathcal{L}_\epsilon^d$ be defined by the polynomial $f(x,y)$. Then by Lemma 3, one deduces that $\mathcal{C}$ has infinitely many $(d-1)$–fold ϵ–singularities. For our purposes, we are interested in choosing the singularity appropriately. More precisely, we say that $\overline{P} = (\overline{a}, \overline{b})$ is a *proper* $(d-1)$*–fold* ϵ*–singularity of* $\mathcal{C}$ if the polynomial

$$\sum_{j_1+j_2=d-1}^{d} \frac{\partial^{j_1+j_2} f}{\partial^{j_1}x\partial^{j_2}y}(\,\overline{P}\,)(x-\overline{a})^{j_1}(y-\overline{b})^{j_2}\frac{1}{j_1!j_2!},$$

is irreducible over $\mathbb{C}$. Note that this is always possible because an small perturbation of the coefficients of a polynomial transforms it onto an irreducible polynomial.

The following theorem shows that the implicit equation of the rational curve defined by the parametrization generated by the above process can be obtained also, as in the conic case, by Taylor expansions at the ϵ–singularity. In fact, the theorem includes as a particular case the result for conics. This result will avoid quotient computations and will be used to analyze the error.

Theorem 1. *Let $f(x,y)$ be the implicit equation of a curve $\mathcal{C} \in \mathcal{L}_\epsilon^d$ and let $\overline{P} = (\overline{a}, \overline{b})$ be a proper ϵ–singularity of multiplicity $d-1$ of $\mathcal{C}$. Let $\overline{p}_1(t)$ be the root in $\mathbb{R}(t)$ of the quotient of $f(x, tx + \overline{b} - \overline{a}t)$ and $(x-\overline{a})^{d-1}$, and let $\overline{p}_2(t) = t\overline{p}_1(t) + \overline{b} - t\overline{a}$. Then the implicit equation of the rational curve $\overline{\mathcal{C}}$ defined by the parametrization $\overline{P}(t) = (\overline{p}_1(t), \overline{p}_2(t))$ is*

$$\overline{f}(x,y) = f(x,y) - T(x,y)$$

where $T(x,y)$ is the Taylor expansion up to order $d-1$ of $f(x,y)$ at $\overline{P}$.

Proof. Let

$$f(x,y) = f(\,\overline{P}\,) + \sum_{j_1+j_2=1}^{d} \frac{\partial^{j_1+j_2} f}{\partial^{j_1}x\partial^{j_2}y}(\,\overline{P}\,)(x-\overline{a})^{j_1}(y-\overline{b})^{j_2}\frac{1}{j_1!j_2!}$$

be the Taylor expansion of $f(x,y)$ at $\overline{P}$. Thus,

$$f(x, tx+\bar{b}-t\bar{a}) = f(\,\overline{P}\,) + \sum_{j_1+j_2=1}^{d} \frac{\partial^{j_1+j_2} f}{\partial^{j_1} x \partial^{j_2} y}(\,\overline{P}\,)(x-\bar{a})^{j_1+j_2} t^{j_2} \frac{1}{j_1! j_2!} =$$

$$(x-\bar{a})^{d-1} \left(\sum_{j_1+j_2=d-1}^{d} \frac{\partial^{j_1+j_2} f}{\partial^{j_1} x \partial^{j_2} y}(\,\overline{P}\,)(x-\bar{a})^{j_1+j_2-d+1} t^{j_2} \frac{1}{j_1! j_2!} \right) +$$

$$\left(f(\,\overline{P}\,) + \sum_{j_1+j_2=1}^{d-2} \frac{\partial^{j_1+j_2} f}{\partial^{j_1} x \partial^{j_2} y}(\,\overline{P}\,)(x-\bar{a})^{j_1+j_2} t^{j_2} \frac{1}{j_1! j_2!} \right) = (x-\bar{a})^{d-1} M(x,t) + N(x,t)$$

where

$$N(x,t) = T(x, tx+\bar{b}-t\bar{a}), \quad M(x,t) = \frac{S(x, tx+\bar{b}-t\bar{a})}{(x-\bar{a})^{d-1}},$$

and $S(x,y)$ is the Taylor expansion from order $d-1$ up to order d at $\overline{P}$. We observe that $\deg_x(M) = 1$, and $\deg_x(N) \leq d-2$. On the other hand, let $U(x,t)$ and $V(x,t)$ be the quotient and the remainder of $f(x, tx+\bar{b}-t\bar{a})$ and $(x-\bar{a})^{d-1}$ w.r.t. x, respectively. Then,

$$f(x, tx+\bar{b}-t\bar{a}) = (x-\bar{a})^{d-1} U(x,t) + V(x,t)$$

with $\deg_x(V) \leq d-2$. Therefore,

$$(x-\bar{a})^{d-1}(M(x,t) - U(x,t)) = V(x,t) - N(x,t).$$

Thus, since the degree w.r.t. x of $V-N$ is smaller or equal $d-2$, and $(x-\bar{a})^{d-1}$ divides $V-N$, one gets that $M=U$ and $V=N$. In this situation,

$$\overline{f}(\overline{\mathcal{P}}(t)) = f(\overline{\mathcal{P}}(t)) - T(\overline{\mathcal{P}}(t)) = f(\overline{p}_1(t), t\overline{p}_1(t)+\bar{b}-t\bar{a}) - T(\overline{\mathcal{P}}(t)) =$$

$$= (\overline{p_1}(t)-\bar{a})^{d-1} U(\overline{p_1}(t), t) + N(\overline{p}_1(t), t) - T(\overline{\mathcal{P}}(t)) = T(\overline{\mathcal{P}}(t)) - T(\overline{\mathcal{P}}(t)) = 0.$$

Moreover, since $\overline{P}$ is a proper ϵ–singularity of multiplicity $d-1$ of $\mathcal{C}$, one has that $\overline{f}$ is irreducible, and thus $\overline{\mathcal{P}}(t)$ parametrizes $\overline{\mathcal{C}}$. □

This result can be applied to derive a similar algorithm for parametrizing approximate algebraic curves by lines similar to the symbolic algorithm.

Numerical Parametrization By Lines

- Given the defining polynomial $f(x,y)$ of $\mathcal{C} \in \mathcal{L}_{\epsilon}^{d}$, $d \geq 2$.
- Compute a rational parametrization $\overline{\mathcal{P}}(t)$ of a rational curve $\overline{\mathcal{C}}$ close to $\mathcal{C}$.

1. If $d = 2$ compute an affine ϵ–point $\overline{P}$ of $\mathcal{C}$, else compute a proper ϵ–singularity $\overline{P}$ of $\mathcal{C}$ of multiplicity $d-1$.
2. Compute $\overline{f}(x,y) = f(x,y) - T(x,y)$ where $T(x,y)$ is the Taylor expansion of $f(x,y)$ up to order $d-1$ at $\overline{P}$.
3. Apply step 3 of the symbolic algorithm to $\overline{f}$ and $\overline{P}$.

3 Examples

In this section, we illustrate the numerical parametrization algorithm developed in Section 2 by some examples where one can check that the output rational curve $\overline{\mathcal{C}}$ is close to the original curve $\mathcal{C}$. This behavior will be clarified in the error analysis section.

We give an example in detail, where we explain how the algorithm is performed, and we summarize seven other examples in different tables. In these tables we show the input curve $\mathcal{C}$, the tolerance ϵ considered, the ϵ–singularity, the output curve $\overline{\mathcal{C}}$, the output parametrization $\overline{\mathcal{P}}(t)$ defining the curve $\overline{\mathcal{C}}$, and a figure representing $\mathcal{C}$ and $\overline{\mathcal{C}}$.

Example 1. We consider $\epsilon = 0.001$ and the curve $\mathcal{C}$ of degree 6 defined by the polynomial

$$f(x, y) = y^6 + x^6 + 2.yx^4 - 2.y^4x + 10^{-3}x + .10^{-3}y + 2 \cdot 10^{-3} + 10^{-3}x^4.$$

First of all, by applying the algorithm developed in (10), we observe that the polynomial $f(x, y)$ is ϵ–irreducible. Now, we apply the first step of the Algorithm Numerical Parametrization by Lines, and we compute the ϵ–singularity. For this purpose, we determine the solutions of the system (see (9), (27))

$$\frac{\partial^4 f}{\partial^4 x}(x, y) = \frac{\partial^4 f}{\partial^4 y}(x, y) = 0,$$

under fixed precision $\epsilon\|f(x, y)\| = 0.002$. We get four solutions

$$\overline{P}_1 = (-.06650062380 + .1157587268I, .06683312414 + .1154704132I),$$

$$\overline{P}_2 = (-.06650062380 - .1157587268I, .06683312414 - .1154704132I),$$

$$\overline{P}_3 = (.1875000000 \cdot 10^{-5}, -.50000002 \cdot 10^{-3}), \quad \overline{P}_4 = (.1329993725, -.1331662483).$$

Only the root $\overline{P}_3$, satisfies that

$$\left\| \frac{\partial^{i+j} f}{\partial^i x \partial^j y}(\overline{P}_3) \right\| \leq 0.002, \quad i + j = 0, \ldots 3.$$

Then $\overline{P} = \overline{P}_3 = (.1875000000 \cdot 10^{-5}, -.50000002 \cdot 10^{-3})$ is an ϵ–singularity of multiplicity 5, and therefore $\mathcal{C} \in \mathcal{L}^6_{0.001}$.

Applying the second step of the Algorithm Numerical Parametrization by

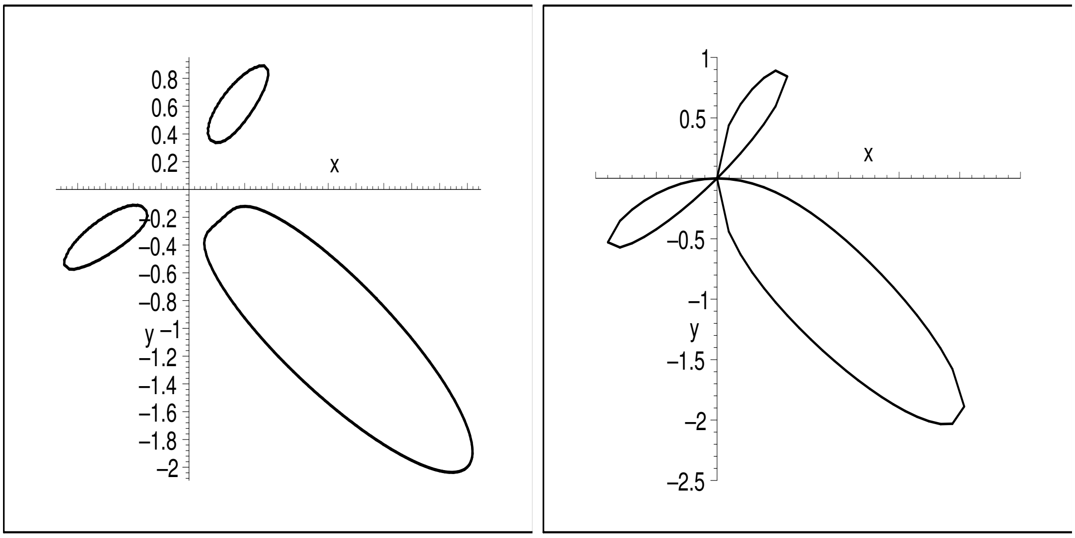


Fig. 3. Input curve $\mathcal{C}$ (left), output curve $\overline{\mathcal{C}}$ (right)

Lines, we compute

$$\overline{f}(x,y) = f(x,y) - T(x,y),$$

where $T(x,y)$ is the Taylor expansion of $f(x,y)$ up to order 5 at $\overline{P}$,

$T(x,y) = .001000000000x + .0010000000000y + .1000000173 \cdot 10^{-8}yx + .1300000000 \cdot 10^{-10}x^4 + .7500000034 \cdot 10^{-8}x^3 - .2499999700 \cdot 10^{-8}y^3 + .4000000160 \cdot 10^{-2}xy^3 + .1500000000 \cdot 10^{-4}x^3y - .2109375027 \cdot 10^{-13}x^2 + .3000000000 \cdot 10^{-12}y^4 - .2812500001 \cdot 10^{-11}y^2 - .4218750000 \cdot 10^{-10}yx^2 + .3000000240 \cdot 10^{-5}y^2x + .2000000000 \cdot 10^{-2}$.

One gets the curve $\overline{\mathcal{C}}$ defined by

$\overline{f}(x,y) = -.1250000464 \cdot 10^{-12}x + .1125000100 \cdot 10^{-14}y + .9999999873 \cdot 10^{-3}x^4 + 2.yx^4 - 2.y^4x - .1000000173 \cdot 10^{-8}yx + y^6 + x^6 - .7500000036 \cdot 10^{-8}x^3 + .2499999700 \cdot 10^{-8}y^3 + .2109375029 \cdot 10^{-13}x^2 - .3000000180 \cdot 10^{-12}y^4 + .2812500000 \cdot 10^{-11}y^2 - .1500000000 \cdot 10^{-4}x^3y - .4000000160 \cdot 10^{-2}xy^3 - .3000000240 \cdot 10^{-5}y^2x + .4218750000 \cdot 10^{-10}yx^2 + .1562500311 \cdot 10^{-18}$.

Now, we apply step 3 of the symbolic algorithm to $\overline{f}$ and $\overline{P}$. Thus, we compute

$$A(x,y,t) = \frac{\dfrac{\partial^5 \overline{f}}{\partial^5 x}\dfrac{1}{5!} + \dfrac{\partial^5 \overline{f}}{\partial^4 x \partial y}\dfrac{t}{4!} + \cdots + \dfrac{t^5}{5!}\dfrac{\partial^5 \overline{f}}{\partial^5 y}}{\dfrac{\partial^6 \overline{f}}{\partial^6 x}\dfrac{1}{6!} + \dfrac{\partial^6 \overline{f}}{\partial^5 x \partial y}\dfrac{t}{5!} + \cdots + \dfrac{t^6}{6!}\dfrac{\partial^6 \overline{f}}{\partial^6 y}} =$$

$$\frac{6x + 2.000000000t - 2.000000000t^4 + 6yt^5}{1 + t^6}.$$

and we return

$$\mathcal{P}(t) = \Big(-A(\overline{P}, t) + .1875000000 \cdot 10^{-5}, -tA(\overline{P}, t) - .50000002 \cdot 10^{-3}\Big) = (\overline{p}_1(t), \overline{p}_2(t)),$$

where

$$\bar{p}_1(t) = \frac{-2.000000000t + .3000000120 \cdot 10^{-2}t^5 + .1875000000 \cdot 10^{-5}t^6}{1+t^6} +$$

$$\frac{2.000000000t^4 - .9375000000 \cdot 10^{-5}}{1+t^6},$$

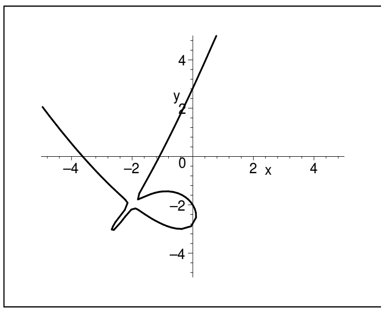
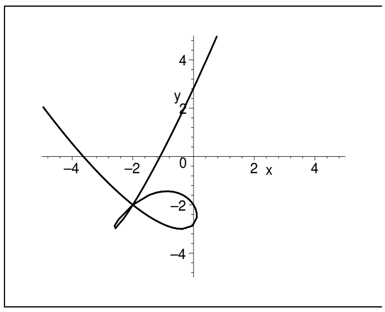
and

$$\bar{p}_2(t) = \frac{-.4887500200 \cdot 10^{-3} - 2.000000000t^4 - .3000000120 \cdot 10^{-2}t^5}{1+t^6} +$$

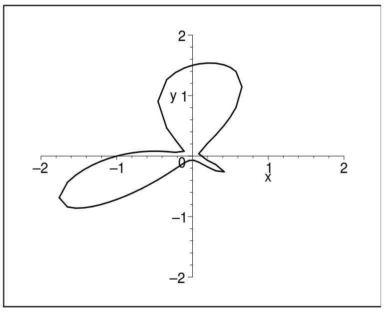
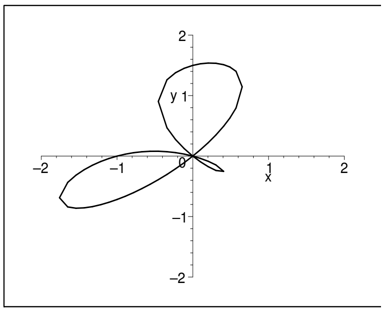
$$\frac{2.000000000t - .5000000200 \cdot 10^{-3}t^6}{1+t^6}.$$

See Fig. 3 to compare the input curve and the rational output curve. □

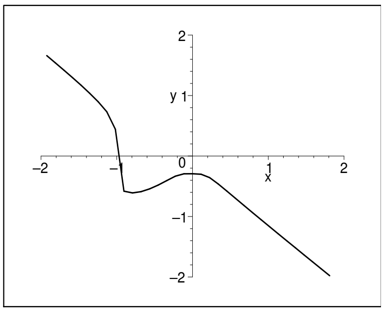
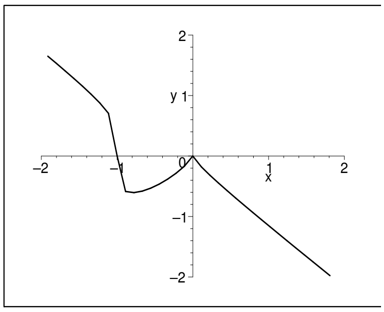
Example 2.

Input Curve $\mathcal{C}$	$16.001 + 24.001x + 8y - 2y^2 + 12yx + 14.001x^2 + 2y^2x + x^2y + x^4 - y^3 + 6.001x^3$
Tolerance ϵ	0.001
ϵ–Singularity	$(-2,\ -2)$
Output Curve $\overline{\mathcal{C}}$	$16.008 + 24.012x + 8y - 2y^2 + 12yx + 14.006x^2 + 2y^2x + x^2y + x^4 - y^3 + 6.001x^3$
Parametrization $\overline{\mathcal{P}}(t) = (\overline{p}_1(t), \overline{p}_2(t))$	$\bar{p}_1 = t^3 - 0.001 - t - 2t^2, \quad \bar{p}_2 = t^4 + 1.999t - t^2 - 2t^3 - 2.$
Figures Curve $\mathcal{C}$(left) Curve $\overline{\mathcal{C}}$(right)	 

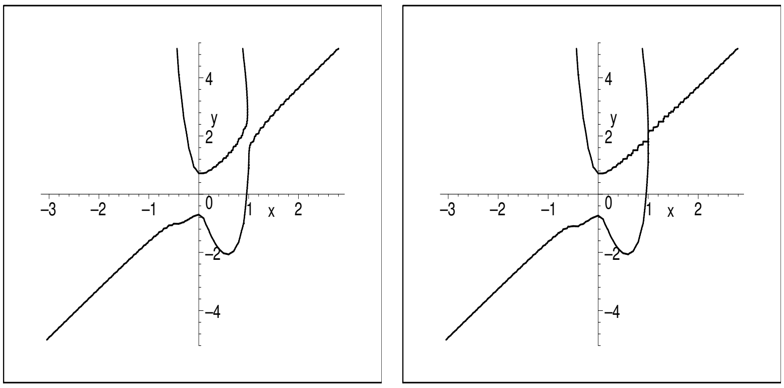
Example 3.

Input Curve $\mathcal{C}$	$x^4+2y^4+1.001x^3+3x^2y-y^2x-3y^3+0.00001y^2$ $-0.001x-0.001y-0.001$
Tolerance ϵ	0.001
ϵ–Singularity	$(-.1248595915\,10^{-6},\ .1249844199\,10^{-6})$
Output Curve $\overline{\mathcal{C}}$	$x^4+2.y^4+1.001x^3+3.x^2y-y^2x-3.y^3+10^{-6}y^2-$ $.6243761996\cdot10^{-13}x-.6260915576\cdot10^{-13}y+$ $.9744187291\cdot10^{-23}-.3522924910\cdot10^{-16}x^2+$ $.9991263887\cdot10^{-6}xy$
Parametrization $\overline{\mathcal{P}}(t)=(\overline{p}_1(t),\overline{p}_2(t))$	$\bar{p}_1=-.487671\cdot\frac{2.0526-2.05055t^2+6.15167t+.512063\cdot10^{-6}t^4-6.15167t^3}{1.+2.t^4}$, $\bar{p}_2=.487671\cdot\frac{-2.05260t+2.05055t^3-6.15167t^2+6.15167t^4+.256287\cdot10^{-6}}{1.+2.t^4}$.
Figures Curve $\mathcal{C}$(left) Curve $\overline{\mathcal{C}}$(right)	 

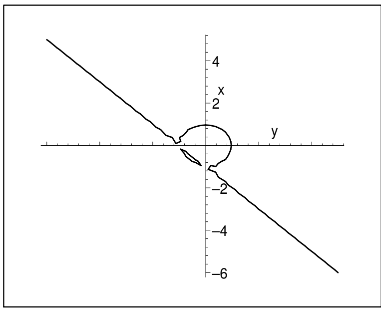
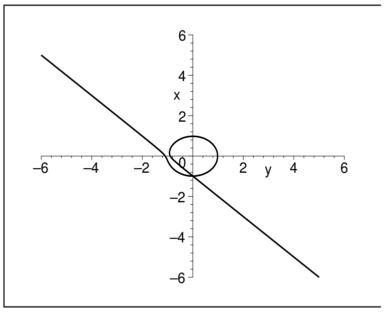
Example 4.

Input Curve $\mathcal{C}$	$y^5+x^5+x^4+.001x+.001y+.002+.001x^2+.005y^2+.001x^3$
Tolerance ϵ	0.01
ϵ–Singularity	$(-0.0002501, 0)$
Output Curve $\overline{\mathcal{C}}$	$y^5+x^5+x^4+.6255863298\cdot10^{-10}x+.9999998183\cdot10^{-3}x^3+$ $.3912115701\cdot10^{-14}+.3751562603\cdot10^{-6}x^2$
Parametrization $\overline{\mathcal{P}}(t)=(\overline{p}_1(t),\overline{p}_2(t))$	$\bar{p}_1=-41902244\cdot10^{-6}\cdot\frac{2384119+597t^5}{1+t^5}$, $\bar{p}_2=-.9987492180\frac{t}{1+t^5}$.
Figures Curve $\mathcal{C}$(left) Curve $\overline{\mathcal{C}}$(right)	 

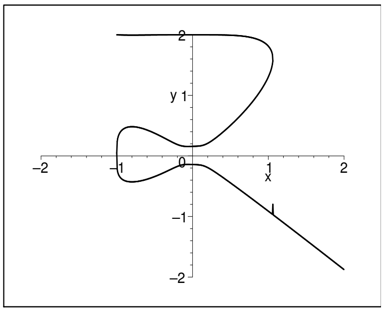
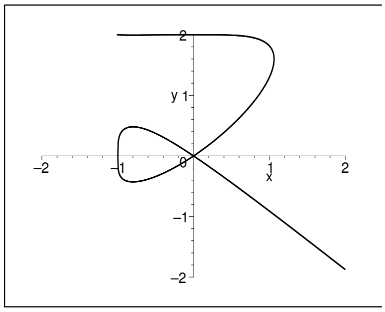
Example 5.

Input Curve $\mathcal{C}$	$-10.x + 2.y + xy^4 + 862.x^4y - 359.x^3y^2 + 3.099.- 859.967x^3y + 39.x^2y^3 + 299.011x^2y^2 + 52.x^2y - 3.xy^3 + 5.xy^2 - 7.901xy + 687.x^4 - 642.x^5 - 67.989x^3 + 14.x^2 - 9.989y^4 + y^5 - 4.y^3 - y^2$
Tolerance ϵ	0.1
ϵ–Singularity	$(.999067678, 1.99734)$
Output Curve $\overline{\mathcal{C}}$	$10.12701492x + 1.548607302y + xy^4 + 862.x^4y - 359.x^3y^2 - 859.9670000x^3y + 39.x^2y^3 + 299.0110000x^2y^2 + 52.18519488x^2y - 3.xy^3 + 4.626307400xy^2 - 7.063248589xy - 642.x^5 - 67.98172465x^3 + 13.33333837x^2 - 9.989000000y^4 + y^5 - 3.999974822y^3 - .9012712980y^2 + 687.x^4 + 3.247948193$
Parametrization $\overline{\mathcal{P}}(t) = (\overline{p}_1(t), \overline{p}_2(t))$	$\bar{p}_1 = .22545229 \cdot \frac{.69592866\cdot 10^3 - .128422685\cdot 10^4 t + 0.0102.t^4 + 4.4313t^5}{t^5+t^4+862.t+39.t^3-642.-359.t^2} + .22545229 \cdot \frac{.81893515\cdot 10^3 t^2 - .19495476\cdot 10^3 t^3}{t^5+t^4+862.t+39.t^3-642.-359.t^2},$ $\bar{p}_2 = .22545229 \cdot \frac{.111775629\cdot 10^5 t - .82845609\cdot 10^4 t^2 + 4.4380666t^5}{t^5+t^4+862.t+39.t^3-642.-359.t^2} + .22545229 \cdot \frac{.27553162\cdot 10^4 t^3 - .35891982\cdot 10^3 t^4 - .56876434\cdot 10^4}{t^5+t^4+862.t+39.t^3-642.-359.t^2}.$
Figures Curve $\mathcal{C}$(left) Curve $\overline{\mathcal{C}}$(right)	

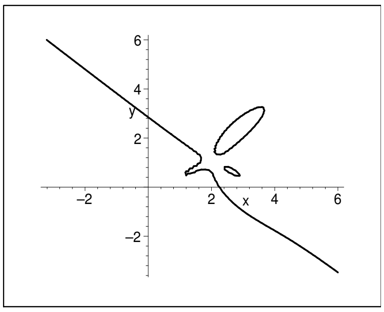
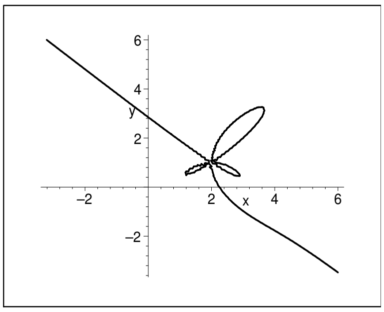
Example 6.

Input Curve $\mathcal{C}$	$x^3+x^2y+x^2+xy^2+y^3+y^2-$ $.999990x-.999980y-.9999600$
Tolerance ϵ	0.01
ϵ–Singularity	$(-.99000000, 0)$
Output Curve $\overline{\mathcal{C}}$	$x^3+x^2y+x^2+xy^2+y^3+y^2-$ $.9603000x-.9801000y-.9604980,$
Parametrization $\overline{\mathcal{P}}(t)=(\overline{p}_1(t),\overline{p}_2(t))$	$\bar{p}_1=\frac{0.99t+0.98-t^2-0.99t^3}{1+t+t^2+t^3}, \qquad \bar{p}_2=\frac{t(1.98t+1.97-0.01t^2)}{1+t+t^2+t^3}.$
Figures Curve $\mathcal{C}$(left) Curve $\overline{\mathcal{C}}$(right)	 

Example 7.

Input Curve $\mathcal{C}$	$y^5+x^5+x^4-2.y^4+10^{-3}x+10^{-3}y+10^{-3}+$ $10^{-3}x^2+10^{-3}x^3+2\cdot 10^{-3}y^2x+10^{-3}y^3$
Tolerance ϵ	0.01
ϵ–Singularity	$(-.2501564001\cdot 10^{-3}, .1250195\cdot 10^{-3})$
Output Curve $\overline{\mathcal{C}}$	$.6255863298\cdot 10^{-10}x+.1562864926\cdot 10^{-10}y+$ $y^5+x^5+x^4-2.y^4+.9999998183\cdot 10^{-3}x^3+$ $.3751562603\cdot 10^{-6}x^2+.9999997015\cdot 10^{-3}y^3-$ $.1875194239\cdot 10^{-6}y^2+.3423651857\cdot 10^{-14}$
Parametrization $\overline{\mathcal{P}}(t)=(\overline{p}_1(t),\overline{p}_2(t))$	$\bar{p}_1=-.114881528\cdot\frac{8.695909548-.1740379799\cdot 10^2t^4+.2177516307\cdot 10^{-2}t^5}{1+t^5}$ $\bar{p}_2=.2297630556\cdot\frac{8.702443119t^5-4.346866016t+.544123596\cdot 10^{-3}}{1+t^5}.$
Figures Curve $\mathcal{C}$(left) Curve $\overline{\mathcal{C}}$(right)	 

Example 8.

Input Curve $\mathcal{C}$	$291.9690000x - 17.00300000y - 100.9940000y^2 + 20.y^4x - 511.9760000x^2 + x^7 - 14.x^6 + 82.x^5 - 259.9990000x^4 + 479.9920000x^3 + 29.y^5 - 74.99900000y^4 - 40.y^3x + 40.y^2x - 160.x^2y + 140.xy + 2.x^5y - 20.x^4y + 80.x^3y + y^7 - 7.y^6 + 114.9960000y^3 - 72.98400000 - 4.y^5x.$
Tolerance ϵ	0.001
ϵ–Singularity	$(2, 1)$
Output Curve $\overline{\mathcal{C}}$	$-73. + 292.x - 17.y - 101.y^2 - 512.x^2 + x^7 - 14.x^6 - 260.x^4 + 480.x^3 + 29.y^5 - 75.y^4 - 40.y^3x - 160.x^2y + 140.xy + 2.x^5y - 20.x^4y + 80.x^3y + y^7 - 7.y^6 + 115.y^3 - 4.y^5x + 20.y^4x + 82.x^5 + 40.y^2x.$
Parametrization $\overline{\mathcal{P}}(t) = (\overline{p}_1(t), \overline{p}_2(t))$	$\bar{p}_1 = \frac{2(t^7+1+2t^5-t)}{t^7+1}, \qquad \bar{p}_2 = \frac{4t^6-2t^2+t^7+1.}{t^7+1}.$
Figures Curve $\mathcal{C}$(left) Curve $\overline{\mathcal{C}}$(right)	 

4 Error Analysis

Examples in Section 3 show that, in practice, the output curve of our algorithm is quite close to the input one. In this section we analyze how far these two affine curves are.

To be more precise let $\mathcal{C} \in \mathcal{L}_\epsilon^d$ be defined by $f(x, y)$. In addition, we will denote by

$$\overline{\mathcal{P}}(t) = \left(\frac{\overline{p}_1(t)}{\overline{q}(t)}, \frac{\overline{p}_2(t)}{\overline{q}(t)} \right),$$

where $\gcd(\overline{p}_i, \overline{q}) = 1$, the generated parametrization of the output curve $\overline{\mathcal{C}}$. Moreover, since we will measure distances, we may assume that the $\epsilon-$singularity of $\mathcal{C}$ is the origin, otherwise one can apply a translation such that it is moved to the origin and distances are preserved. Also we assume that $\|f(x,y)\| = 1$, otherwise we consider $\frac{f(x,y)}{\|f(x,y)\|}$. If one does not normalize the input polynomial $f(x,y)$, a similar treatment with relative errors can be done.

In this situation, the general strategy we will follow is to show that almost any affine real point on $\overline{\mathcal{C}}$ is at small distance of an affine real point on $\mathcal{C}$. For this purpose, we observe that $\overline{\mathcal{P}}(t)$ is an exact parametrization of $\overline{\mathcal{C}}$ obtained by lines, and therefore all affine real points on $\overline{\mathcal{C}}$ are obtained as the intersection of a line of the form $y = tx$, for t real, with $\overline{\mathcal{C}}$. Then, if one intersects the curve $\mathcal{C}$ with the same line one gets d points on $\mathcal{C}$, counted properly, and we show that at least one of these intersection points on $\mathcal{C}$ is close to the initial point on $\overline{\mathcal{C}}$. Also, we observe that it is enough to reason with slope parameter values of t in the interval $[-1, 1]$ because if $|t| > 1$ one may apply a similar strategy intersecting with lines of the form $x = ty$. Therefore, let $t_0 \in \mathbb{R}$ be such that $|t_0| \leq 1$ and $\overline{q}(t_0) \neq 0$. Then, the corresponding point $\overline{Q}$ on $\overline{\mathcal{C}}$ is $\overline{Q} = \overline{\mathcal{P}}(t_0)$. Let us expressed $\overline{Q}$ as

$$\overline{Q} = \left(\overline{a}, \overline{b}\right) = \left(\frac{\overline{a}_1}{\overline{c}}, \frac{\overline{b}_1}{\overline{c}}\right)$$

where $\overline{a}_1 = \overline{p}_1(t_0), \overline{a}_2 = \overline{p}_2(t_0)$ and $\overline{c} = \overline{q}(t_0)$. Observe that, since we are cutting with the line $y = t_0 x$, it holds that $\overline{b} = t_0 \overline{a}$. Thus, if we write the affine point $\overline{Q}$ projectively one has that $(\overline{a}_1 : t_0 \overline{a}_1 : \overline{c})$. Now, observe that if $|\overline{a}_1|$ and $|\overline{c}|$ are simultaneously very small, i.e very close to ϵ, this point is not well defined as an element in $\mathbb{P}^2(\mathbb{R})$. For this reason, we will assume that either $|\overline{a}_1|$ or $|\overline{c}|$ is bigger than a certain bound that depends on the tolerance. In fact, for our error analysis, we fix that

$$|\overline{a}_1| > \epsilon^{1/d} \quad \text{or} \quad |\overline{c}| > \epsilon^{1/d}.$$

Furthermore, we observe that the defining polynomials of $\overline{\mathcal{C}}$ and $\mathcal{C}$ have the same homogeneous form of maximum degree, and hence both curves have the same points at infinity.

Now, let $Q = (a, b)$ be any affine point in $\mathcal{C} \cap \{y = t_0 x\}$; note that here it also holds that $b = t_0 a$. We want to compute the Euclidean distance between $\overline{Q}$ and Q. In order to do that, we observe that

$$\|\overline{Q} - Q\|_2 = \sqrt{(\,\overline{a} - a)^2 + (\,\overline{b} - b)^2} = \sqrt{(\,\overline{a} - a)^2(1 + t_0^2)} \leq \sqrt{2}|\,\overline{a} - a|.$$

Therefore, we focus on the problem of computing a good bound for $|\,\overline{a} - a|$. For this purpose we first prove two different lemmas that will be used as general strategies in our reasonings.

Lemma 2. *It holds that*

$$|\overline{a} - a| \leq \epsilon \cdot C,$$

where

$$C = \frac{\displaystyle\sum_{j_1+j_2=0}^{d-2} |\overline{a}|^{j_1+j_2} |t_0|^{j_2} \frac{1}{j_1! j_2!}}{|\overline{a}|^{d-1} |\overline{c}|}.$$

Proof. First of all, we note that $\overline{a}$ is a root of the univariate polynomial $\overline{f}(x, t_0 x) = x^{d-1}(\overline{c}x - \overline{a}_1)$, and that a is a root of the univariate polynomial

$$f(x, t_0 x) = x^{d-1}(\overline{c}x - \overline{a}_1) + \sum_{j_1+j_2=0}^{d-2} \frac{\partial^{j_1+j_2} f}{\partial^{j_1} x \partial^{j_2} y}(0,0) x^{j_1} (t_0 x)^{j_2} \frac{1}{j_1! j_2!},$$

Since $(0,0)$ is the $(d-1)$-fold ϵ–singularity of $\overline{\mathcal{C}}$ it holds that

$$\|f(x, t_0 x) - \overline{f}(x, t_0 x)\| = \max_{j_1+j_2=0,\ldots d-2} \left\{ \left| \frac{\partial^{j_1+j_2} f}{\partial^{j_1} x \partial^{j_2} y}(0,0) \right| |t_0|^{j_2} \frac{1}{j_1! j_2!} \right\} \leq$$

$$\max_{j_1+j_2=0,\ldots d-2} \left\{ \left| \frac{\partial^{j_1+j_2} f}{\partial^{j_1} x \partial^{j_2} y}(0,0) \right| \right\} < \epsilon \|f(x,y)\| = \epsilon,$$

and thus $\overline{f}(x, t_0 x)$ can be written as

$$\overline{f}(x, t_0 x) = f(x, t_0 x) + R(x) \text{ where } R \in \mathbb{R}[x] \text{ and } \|R(x)\| < \epsilon.$$

Therefore, by applying standard numerical techniques to measure $|\overline{a} - a|$ by means of the condition number (see for instance (7), pg. 303), one deduces that

$$|\overline{a} - a| \leq \epsilon \cdot C,$$

where

$$C = \frac{\displaystyle\sum_{j_1+j_2=0}^{d-2} |\overline{a}|^{j_1+j_2} |t_0|^{j_2} \frac{1}{j_1! j_2!}}{\left| \dfrac{\partial \overline{f}}{\partial x}(\overline{a}, t_0 \overline{a}) \right|} = \frac{\displaystyle\sum_{j_1+j_2=0}^{d-2} |\overline{a}|^{j_1+j_2} |t_0|^{j_2} \frac{1}{j_1! j_2!}}{|\overline{a}|^{d-1} |\overline{c}|}.$$

□

Lemma 3. *Let*

$$h(x) = c \prod_{i=1}^{n} (x - c_i) \in \mathbb{C}[x] \text{ with } \deg(h) = n,$$

and let $\lambda \in \mathbb{C}$ be such that $|h(\lambda)| \leq \delta$. Then, there exists a root c_{i_0} of $h(x)$ such that

$$|\lambda - c_{i_0}| \leq \left(\frac{\delta}{|c|} \right)^{\frac{1}{n}}.$$

Proof. Let us assume that for $i = 1, \ldots, n$, $|\lambda - c_i| > \left(\frac{\delta}{|c|}\right)^{\frac{1}{n}}$. Then,

$$|h(\lambda)| = |c| \prod_{i=1}^{n} |\lambda - c_i| > \delta,$$

which contradicts that $|h(\lambda)| \leq \delta$. □

Now, we proceed to analyze $|\overline{a} - a|$ by using the previous lemmas. For this purpose, we distinguish different cases depending on the values of $|\overline{a}_1|$ and $|\overline{c}|$:

Lemma 4. *Let $|\overline{c}| \geq 1$. Then, it holds that:*

1. *If $|\overline{a}| > 1$, then $|\overline{a} - a| \leq \epsilon \cdot \exp(2)$.*
2. *If $|\overline{a}| \leq 1$, then $|\overline{a} - a| \leq (\epsilon \cdot \exp(2))^{\frac{1}{d}}$.*

Proof.

1. If $|\overline{a}| > 1$, we have that the constant C in Lemma 2 can be bounded as

$$C = \frac{\sum_{j_1+j_2=0}^{d-2} |\overline{a}|^{j_1+j_2} |t_0|^{j_2} \frac{1}{j_1! j_2!}}{|\overline{a}|^{d-1}|\overline{c}|} = \frac{\sum_{k=0}^{d-2} \frac{(|\overline{a}|+|\overline{a}||t_0|)^k}{k!}}{|\overline{a}|^{d-1}|\overline{c}|} \leq$$

$$\sum_{k=0}^{d-2} \frac{(1+|t_0|)^k}{k!|\overline{a}|^{d-1-k}} \leq \sum_{k=0}^{d-2} \frac{(1+|t_0|)^k}{k!} \leq \exp(1+|t_0|) \leq \exp(2).$$

Therefore, by Lemma 2 we deduce that

$$|\overline{a} - a| \leq \epsilon \cdot \exp(2).$$

2. If $|\overline{a}| \leq 1$, we have that

$$|f(\overline{a}, \overline{a}t_0)| = \left| \overline{f}(\overline{a}, \overline{a}t_0) + \sum_{j_1+j_2=0}^{d-2} \frac{\partial^{j_1+j_2} f}{\partial^{j_1} x \partial^{j_2} y}(0,0) \overline{a}^{j_1} (t_0 \overline{a})^{j_2} \frac{1}{j_1! j_2!} \right| =$$

$$= \left| \sum_{j_1+j_2=0}^{d-2} \frac{\partial^{j_1+j_2} f}{\partial^{j_1} x \partial^{j_2} y}(0,0) \overline{a}^{j_1} (t_0 \overline{a})^{j_2} \frac{1}{j_1! j_2!} \right| \leq$$

$$\sum_{j_1+j_2=0}^{d-2} \left| \frac{\partial^{j_1+j_2} f}{\partial^{j_1} x \partial^{j_2} y}(0,0) \right| |\overline{a}|^{j_1} |t_0|^{j_2} |\overline{a}|^{j_2} \frac{1}{j_1! j_2!} \leq \epsilon \cdot \exp(|\overline{a}|(1+|t_0|)) \leq \epsilon \cdot \exp(2).$$

In this situation, by Lemma 3 we deduce that there exists a root of the univariate polynomial $f(x, t_0 x)$, that we can assume w.l.o.g. that is a, such that

$$|\overline{a} - a| \leq \left(\frac{\epsilon \cdot \exp(2)}{|\overline{c}|} \right)^{\frac{1}{d}} \leq (\epsilon \cdot \exp(2))^{\frac{1}{d}}.$$

□

Lemma 5. *Let $|\overline{c}| < 1$ and $|\overline{a}_1| \geq 1$. Then, it holds that $|\overline{a} - a| \leq \epsilon \cdot \exp(2)$.*

Proof. Since $|\overline{c}| < 1$ and $|\overline{a}_1| \geq 1$, we have that the constant C in Lemma 2 can be bounded as

$$C = \frac{\sum_{j_1+j_2=0}^{d-2} |\overline{a}|^{j_1+j_2} |t_0|^{j_2} \frac{1}{j_1! j_2!}}{|\overline{a}|^{d-1}|\overline{c}|} = \frac{\sum_{k=0}^{d-2} \frac{(|\overline{a}_1|+|\overline{a}_1||t_0|)^k |\overline{c}|^{(d-2-k)}}{k!}}{|\overline{a}_1|^{d-1}} \leq$$

$$\sum_{k=0}^{d-2} \frac{(1+|t_0|)^k}{k!|\overline{a}_1|^{d-1-k}} \leq \sum_{k=0}^{d-2} \frac{(1+|t_0|)^k}{k!} \leq \exp(1+|t_0|) \leq \exp(2).$$

Therefore, by Lemma 2 we deduce that

$$|\overline{a} - a| \leq \epsilon \cdot \exp(2).$$

□

Finally, it only remains to analyze the case where $|\overline{c}| < 1$ and $|\overline{a}_1| < 1$. In order to do that, we recall that we have assumed that either $|\overline{a}_1|$ or $|\overline{c}|$ is bigger than $\epsilon^{1/d}$. In the next lemma, we study these cases.

Lemma 6. *It holds that:*

1. *If $|\overline{c}| < 1$ and $\epsilon^{\frac{1}{d}} < |\overline{a}_1| < 1$, then $|\overline{a} - a| \leq \epsilon^{\frac{1}{d}} \cdot \exp(2)$.*
2. *If $|\overline{a}_1| < 1$ and $\epsilon^{\frac{1}{d}} < |\overline{c}| < 1$, then $|\overline{a} - a| \leq (\epsilon^{1/2} \cdot \exp(2))^{\frac{1}{d}}$.*

Proof.

1. If $|\overline{c}| < 1$ and $|\overline{a}_1| > \epsilon^{1/d}$, we have that the constant C in Lemma 2 can be bounded as

$$C = \frac{\sum_{j_1+j_2=0}^{d-2} |\overline{a}|^{j_1+j_2} |t_0|^{j_2} \frac{1}{j_1! j_2!}}{|\overline{a}|^{d-1}|\overline{c}|} = \frac{\sum_{j_1+j_2=0}^{d-2} |\overline{a}_1|^{j_1+j_2-d+1} |t_0|^{j_2} \frac{1}{j_1! j_2!}}{|\overline{c}|^{j_1+j_2-d+2}} =$$

$$\frac{\sum_{j_1+j_2=0}^{d-2} |\overline{c}|^{d-j_1-j_2-2} |t_0|^{j_2} \frac{1}{j_1! j_2!}}{|\overline{a}_1|^{d-j_1-j_2-1}} \leq$$

$$\frac{\sum_{j_1+j_2=0}^{d-2} |t_0|^{j_2} \frac{1}{j_1! j_2!}}{|\overline{a}_1|^{d-1}} \leq \frac{\exp(2)}{|\overline{a}_1|^{d-1}} \leq \exp(2) \cdot \epsilon^{-1+1/d}$$

   Therefore, by Lemma 2 we deduce that

$$|\overline{a} - a| \leq \epsilon^{\frac{1}{d}} \cdot \exp(2).$$

2. Let $\epsilon^{1/d} < |\overline{c}| < 1$ and $|\overline{a}_1| < 1$. First we assume that $|\overline{a}_1| \leq \epsilon^{1/d}$. Otherwise we would reason as in (1). Thus, one has that $|\overline{a}_1| \leq \epsilon^{1/d} < |\overline{c}| < 1$. In these

conditions, we deduce that

$$|f(\overline{a},\overline{a}t_0)| = \left|\overline{f}(\overline{a},\overline{a}t_0) + \sum_{j_1+j_2=0}^{d-2} \frac{\partial^{j_1+j_2} f}{\partial^{j_1}x\partial^{j_2}y}(0,0)\overline{a}^{j_1}(t_0\overline{a})^{j_2}\frac{1}{j_1!j_2!}\right| =$$

$$= \left|\sum_{j_1+j_2=0}^{d-2} \frac{\partial^{j_1+j_2} f}{\partial^{j_1}x\partial^{j_2}y}(0,0)\overline{a}^{j_1}(t_0\overline{a})^{j_2}\frac{1}{j_1!j_2!}\right| \leq$$

$$\sum_{j_1+j_2=0}^{d-2} \left|\frac{\partial^{j_1+j_2} f}{\partial^{j_1}x\partial^{j_2}y}(0,0)\right| |\overline{a}|^{j_1}|t_0|^{j_2}|\overline{a}|^{j_2}\frac{1}{j_1!j_2!} \leq \epsilon\cdot\exp(|\overline{a}|(1+|t_0|)) \leq \epsilon\cdot\exp(2).$$

Now, by Lemma 3 we deduce that there exists a root of the univariate polynomial $f(x,t_0x)$, that we can assume w.l.o.g. that is a, such that

$$|\overline{a}-a| \leq \left(\frac{\epsilon\cdot\exp(2)}{|\overline{c}|}\right)^{\frac{1}{d}} = (\epsilon\cdot\exp(2))^{\frac{1}{d}}\frac{1}{|\overline{c}|^{1/d}} \leq (\epsilon\cdot\exp(2))^{\frac{1}{d}}\frac{1}{\epsilon^{1/d^2}} \leq$$

$$(\epsilon\cdot\exp(2))^{\frac{1}{d}}\frac{1}{\epsilon^{\frac{1}{2d}}} = (\epsilon^{1/2}\cdot\exp(2))^{\frac{1}{d}}.$$

□

From the previous lemmas, one deduces the following theorem.

Theorem 2. *For almost all affine real point $\overline{Q}\in\overline{\mathcal{C}}$ there exists an affine real point $Q\in\mathcal{C}$ such that*

$$\|\overline{Q}-Q\|_2 \leq \sqrt{2}\epsilon^{\frac{1}{2d}}\exp(2).$$

Proof. Applying Lemmas 4, 5 and 6 one deduces that

$$\|\overline{Q}-Q\|_2 = \sqrt{(\,\overline{a}-a)^2+(\,\overline{b}-b)^2} = \sqrt{(\,\overline{a}-a)^2(1+t_0^2)} \leq$$

$$\sqrt{2}|\,\overline{a}-a| \leq \sqrt{2}\epsilon^{\frac{1}{2d}}\exp(2).$$

□

Now, let $\overline{Q}=(\overline{a},\overline{b})$ be a regular point on $\overline{\mathcal{C}}$ such that there exists $Q=(a,b)\in\mathcal{C}$ with $\|\overline{Q}-Q\|_2\leq\sqrt{2}\epsilon^{\frac{1}{2d}}\exp(2)$ (see Theorem 2). In this situation, we consider the tangent line to $\overline{\mathcal{C}}$ at $\overline{Q}$; i.e $T(x,y)=n_x(x-\overline{a})+n_y(y-\overline{b})$, where (n_x,n_y) is the unitary normal vector to $\overline{\mathcal{C}}$ at $\overline{Q}$. Then, we bound the value $\|T(Q)\|$:

$$\|T(Q)\| \leq \|n_x\|\cdot|a-\overline{a}|+\|n_y\|\cdot|b-\overline{b}| \leq \|\overline{Q}-Q\|_2(\|n_x\|+\|n_y\|) \leq 2\sqrt{2}\epsilon^{\frac{1}{2d}}\exp(2).$$

Therefore, reasoning as in Subsection 2.2 of (17) one deduces the following theorem.

Theorem 3. *$\mathcal{C}$ is contained in the offset region of $\overline{\mathcal{C}}$ at distance $2\sqrt{2}\epsilon^{\frac{1}{2d}}\exp(2)$.*
□

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