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A First Approach Towards Normal Parametrizations of Algebraic Surfaces*

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Abstract

In this paper we analyze the problem of deciding the normality (i.e. the surjectivity) of a rational parametrization of a surface $\mathcal{S}$. The problem can be approached by means of elimination theory techniques, providing a proper close subset $\mathcal{Z} \subset \mathcal{S}$ where surjectivity needs to be analyzed. In general, these direct approaches are unfeasible because $\mathcal{Z}$ is very complicated and its elements computationally hard to manipulate. Motivated by this fact, we study ad hoc computational alternative methods that simplifies $\mathcal{Z}$. For this goal, we introduce the notion of pseudo-normality, a concept that provides necessary conditions for a parametrization for being normal. Also, we provide an algorithm for deciding the pseudo-normality. Finally, we state necessary and sufficient conditions on a pseudo-normal parametrization to be normal. As a consequence, certain types of parametrizations are shown to be always normal. For instance, pseudo-normal polynomial parametrizations are normal. Moreover, for certain class of parametrizations, we derive an algorithm for deciding the normality.

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1 Introduction

Let $\mathcal{Q}(\bar{t})$, where $\bar{t} = (t_1, \ldots, t_r)$, be a rational parametrization of an r-dimensional variety $\mathcal{H}$ in $\mathbb{K}^n$, where $\mathbb{K}$ is an algebraically closed field of characteristic zero. $\mathcal{Q}(\bar{t})$

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defines a rational parametrization between the r-dimensional affine space and $\mathcal{H}$:

$$\mathcal{Q} : \mathbb{K}^r \longrightarrow \mathcal{H} \subset \mathbb{K}^n; (a_1, \ldots, a_r) \longmapsto \mathcal{Q}(a_1, \ldots, a_r).$$

A natural question is to ask whether $\mathcal{Q}$ is injective over its domain of definition and whether it is surjective. Both questions are important in many applications, and usually appear quoted in the literature as the properness problem and the normality problem. More precisely, a normal parametrization is defined as follows:

Definition 1.1. *The rational affine parametrization $\mathcal{Q}(\bar{t})$ is called* normal *if the induced rational mapping $\mathcal{Q}$ is surjective, or equivalently if for all $P \in \mathcal{H}$ there exists $(a_1, \ldots, a_r) \in \mathbb{K}^r$ such that $\mathcal{Q}(a_1, \ldots, a_r) = P$.*

For the case of curves (either plane or spatial), both problems have been successfully addressed (see e.g. [1], [12] and [13]). Nevertheless, the case of surfaces have not been treated so extensively. Some exceptions are [6], where the properness problem is analyzed for some special types of surfaces, and [2], [3] where the normality question is studied. In [3] the case of algebraic varieties of arbitrary dimension is approached using Ritt-Wu's decomposition algorithm, providing normal parametrizations for some quadrics. In [2] normal parametrizations for the remaining quadrics are presented.

In this paper, we focus on the case of surfaces (i.e. $n = 3$, $r = 2$, $\mathcal{H}$ is a surface $\mathcal{S}$, and $\mathcal{Q}(\bar{t})$ is a parametrization $\mathcal{P}(s,t)$), and we consider the problem of deciding whether $\mathcal{P}(s,t)$ is normal, leaving for further research the problem of finding normal parametrizations. The problem for surfaces is already much complicated than for curves, for instance polynomial curve parametrizations are always normal, while polynomial surface parametrizations are not necessarily normal. For instance, the polynomial parametrization $(s^3, s(s+t), s(s-t))$, that corresponds to the surface defined by $8\,{x_1}^2 - {x_3}^3 - 3\,{x_3}^2 x_2 - 3\,x_3 {x_2}^2 - {x_2}^3$, does not reach the points $(0, \lambda, -\lambda) \in \mathcal{S}$ where $\lambda \neq 0$.

General elimination techniques can be applied to solve the problem as it has been remarked in [3]. Nevertheless, this type of methods is not feasible in practice because of the complexity of the proper close subset of $\mathcal{S}$ that one needs to analyze. Motivated by this fact, we propose an alternative approach. In this approach, we first find necessary conditions for a parametrization to be normal. This yields to the notion of pseudo-normal parametrization (see Def. 3.2). Furthermore, we provide an algorithmic method to check the pseudo-normality. The proper close subset of $\mathcal{S}$ to be analyzed is computed intersecting $\mathcal{S}$ with three polynomials directly derived from three univariate resultants; namely their leading coefficients. Moreover, under some hypothesis, the reduces the analysis to a close subset of $\mathcal{S}$ that is either empty or zero dimensional or consists of plane rational curves (see Section 3). As a remarkable corollary we get conditions on a polynomial parametrization to be pseudo-normal.

In a second step, we analyze conditions on a pseudo-normal parametrization to be normal. In particular, we show that in order to decide whether a pseudo-normal parametrization is normal one only needs to analyze the parameter values corresponding to the intersection points of the numerator and denominator of each component of the parametrization (see Theorem 4.2). As a consequence, we prove that every pseudo-normal polynomial parametrization is normal (see Corollary 4.4). In addition (see Section 4), we present an algorithm to decide the normality of certain class of parametrizations; namely, those where the algebraic sets (i.e. the empty set or a plane curve) defined by each numerator and denominator of the parametrization do not have a common affine intersection. In this algorithm, once the pseudo-normality have been checked, one only needs to analyze the behavior of the points in a close subset of $\mathcal{S}$ that is either empty or zero-dimensional or consists of plane curves.

To finish this introduction, we need to mention a couple of further details. In our analysis, we exclude rational cylinders over any coordinate plane. This is not a lose of generality as Section 2 shows. On the other hand, in the performance of the algorithmic methods, derived in Sections 3 and 4, we use the implicit equation of $\mathcal{S}$. The reason is somehow inherent to the problem, since we need to compute the intersection of certain varieties (in general planes) with $\mathcal{S}$ and, although we have a parametrization of $\mathcal{S}$, we cannot ensure that it provides all intersection points because we do not know whether the parametrization is normal. In order to compute the implicit equation we use the univariate resultant based method in [10].

2 Notation and General Assumptions

Throughout this paper, we will use the following **notation**. $\mathbb{K}$ is an algebraically closed field of zero characteristic. $\mathcal{P}(s,t)$ is a rational affine parametrization, non necessarily proper, in reduced form, of an algebraic surface $\mathcal{S}$ over $\mathbb{K}$. We write the components of $\mathcal{P}(s,t)$ as

$$\mathcal{P}(s,t) = \left(\frac{p_1(s,t)}{q_1(s,t)}, \frac{p_2(s,t)}{q_2(s,t)}, \frac{p_3(s,t)}{q_3(s,t)}\right),$$

where $p_i, q_i \in \mathbb{K}[s,t]$ and $\gcd(p_i, q_i) = 1$, $i = 1, 2, 3$. $F(x_1, x_2, x_3)$ is the defining polynomial of $\mathcal{S}$. Also, we consider the rational map $\mathcal{P} : \mathbb{K}^2 \longrightarrow \mathcal{S} : (s,t) \longmapsto \mathcal{P}(s,t)$. Note that $\mathcal{P}(s,t)$ is normal iff $\mathcal{P}(\mathbb{K}^2) = \mathcal{S}$.

For a polynomial ideal $\mathcal{J}$, we denote by $\mathbb{V}(\mathcal{J})$ the algebraic variety defined by $\mathcal{J}$ in the corresponding affine space over $\mathbb{K}$. Moreover, we denote by A^* the Zariski closure of a set A, and by $\overline{\mathcal{A}}$ the projective closure of an affine variety $\mathcal{A}$.

Moreover for a polynomial $f(x)$, with coefficient in a unique factorization domain, we denote by $\mathrm{lc}_x(f)$ its leading coefficient and by $\mathrm{coeff}(f, x^i)$ the coefficient of x^i in $f(x)$.

In addition, we associate to $\mathcal{P}(s,t)$ the polynomials

$$G_i(x_i, s, t) = p_i(s,t) - x_i q_i(s,t), \ \ i = 1, 2, 3,$$

as well as the algebraic varieties $\mathcal{C}_i = \mathbb{V}(p_i) \subset \mathbb{K}^2$, and $\mathcal{D}_i = \mathbb{V}(q_i) \subset \mathbb{K}^2$, $i = 1, 2, 3$. Note that each of these varieties is either the empty set or a plane curve.

We will approach the problem using elimination theory. For this purpose, we consider the ideal

$$\mathcal{I} =< G_1(x_1, s, t), G_2(x_2, s, t), G_3(x_3, s, t), wq(s,t) - 1 >\subset \mathbb{K}[w, s, t, x_1, x_2, x_3],$$

where $q = \mathrm{lcm}(q_1, q_2, q_3)$ as well as the algebraic variety $\mathcal{V} = \mathbb{V}(\mathcal{I}) \subset \mathbb{K}^6$, as well as the ideal

$$\tilde{\mathcal{I}} =< G_1(x_1, s, t), G_2(x_2, s, t), G_3(x_3, s, t) >\subset \mathbb{K}[s, t, x_1, x_2, x_3],$$

and the variety $\tilde{\mathcal{V}} = \mathbb{V}(\tilde{\mathcal{I}}) \subset \mathbb{K}^5$. Furthermore, we introduce the elimination ideals (say $\bar{x} = (x_1, x_2, x_3)$)

$$\mathcal{I}_1 = \mathcal{I} \cap \mathbb{K}[s, t, \bar{x}], \quad \mathcal{I}_2 = \mathcal{I} \cap \mathbb{K}[t, \bar{x}] \quad \mathcal{I}_3 = \mathcal{I} \cap \mathbb{K}[\bar{x}].$$

Finally, we consider the projection

$$\begin{array}{ccccccccc} & & \Pi_1 & & \Pi_2 & & \Pi_3 & \\ \Pi: & \mathbb{K}^6 & \longrightarrow & \mathbb{K}^5 & \longrightarrow & \mathbb{K}^4 & \longrightarrow & \mathbb{K}^3 \\ & (w, s, t, \bar{x}) & \longmapsto & (s, t, \bar{x}) & \longmapsto & (t, \bar{x}) & \longmapsto & \bar{x} \end{array}$$

In this situation, because of the Closure Theorem (see [4]), on has that

$$\mathbb{V}(\mathcal{I}_1) = \Pi_1(\mathcal{V})^*, \mathbb{V}(\mathcal{I}_2) = \Pi_2(\Pi_1(\mathcal{V}))^*, S = \mathbb{V}(\mathcal{I}_3) = \Pi(\mathcal{V})^* = \Pi_3(\Pi_2(\Pi_1(\mathcal{V})))^*.$$

In the paper we **assume** w.l.o.g. that:

1. in case $\mathcal{C}_i$ (resp. $\mathcal{D}_i$) is a curve, $(1:0:0) \notin \overline{\mathcal{C}_i}$ (resp. $(1:0:0) \notin \overline{\mathcal{D}_i}$); which implies that the leading coefficients w.r.t. s of numerators and denominators are constant. Note that this condition can be achieved by composing $\mathcal{P}(s,t)$ with a suitable affine linear transformation, and hence the normality character of $\mathcal{P}(s,t)$ is preserved. Therefore, it is not a loss of generality. We observe that a similar reasoning can be done imposing the condition w.r.t. $(0:1:0)$.

2. We will assume that $\mathcal{P}(s,t)$ is not a cylinder over a coordinate plane. This is not a loss of generality because we know how to decide whether $\mathcal{P}(s,t)$ defines or not such a cylinder (see Theorem 5 in [10]), and Theorem 2.1 (below) solves the normality question for this type of surfaces.

The Case of Cylinders

Let $\mathcal{S}$ be a cylinder over the x_1x_2-coordinate plane; we exclude the trivial case of planes of the type $x_i = \lambda$. Theorem 8, in [10], implies that if $a \in \mathbb{K}$ is such that $p_1(a,t)/q_1(a,t)$, and $p_2(a,t)/q_2(a,t)$ (similarly when substituting the parameter t) are not constant, then

$$\mathcal{Q}(t) = \left(\frac{p_1(a,t)}{q_1(a,t)}, \frac{p_2(a,t)}{q_2(a,t)}\right)$$

parametrizes the plane curve $\mathcal{S} \cap \mathbb{V}(x_3)$. Let $\mathcal{Q}(t)$ be proper, otherwise we apply a reparametrization algorithm. Then, from the results in [12], or in Section 6.3. in [13], one may check whether $\mathcal{Q}(t)$ is normal and, if not, computing a normal reparametrization of it. So, say that $\mathcal{Q}(t)$ is normal. Then, $(\mathcal{Q}(t), s)$ is a normal parametrization of $\mathcal{S}$. Therefore, we have proved the following theorem.

Theorem 2.1. *Every rational cylinder over a coordinate plane can always be normally parametrized.*

3 Pseudo-Normal Parametrizations

In this section, we study necessary conditions on a parametrization to be normal. This will yield to the notion of pseudo-normality (see Definition 3.2).

We start observing that $\Pi(\mathcal{V}) = \mathcal{P}(\mathbb{K}^2)$, and therefore $\mathcal{P}(s,t)$ is normal iff $\Pi(\mathcal{V}) = \mathcal{S}$. Or equivalently iff $\mathbb{V}(\mathcal{I}_3) = \Pi(\mathcal{V})$. In order to analyze necessary conditions, we study the set $\Pi_3(\Pi_2(\mathbb{V}(\mathcal{I}_1)))$; i.e. we analyze $\Pi_3(\Pi_2(\mathbb{V}(\mathcal{I}_1))) = \Pi_3(\Pi_2(\Pi_1(\mathcal{V})^*))$ instead of $\Pi(\mathcal{V}) = \Pi_3(\Pi_2(\Pi_1(\mathcal{V})))^*$.

Lemma 3.1. *If $\mathcal{S} \nsubseteq \Pi_3(\Pi_2(\mathbb{V}(\mathcal{I}_1)))$, then $\mathcal{P}(s,t)$ is not normal.*

Proof. Let $P \in \mathcal{S} \setminus \Pi_3(\Pi_2(\mathbb{V}(\mathcal{I}_1)))$. Then, $\forall s_0, t_0 \in \mathbb{K}$, $(s_0, t_0, P) \notin \mathbb{V}(\mathcal{I}_1)$. In particular, $\forall s_0, t_0 \in \mathbb{K}$, $(s_0, t_0, P) \notin \Pi_1(\mathcal{V})$. Thus, $P \notin \Pi(\mathcal{V})$, and hence $\mathcal{S} \neq \Pi(\mathcal{V})$. So, $\mathcal{P}(s,t)$ is not normal. □

The previous lemma, although providing a necessary condition for the normality, requires the first elimination ideal of $\mathcal{I}$. To avoid this, we consider a bigger set, namely $\Pi_3(\Pi_2(\tilde{\mathcal{V}}))$. Note that $\tilde{\mathcal{I}} \subseteq \mathcal{I}_1$, so $\mathbb{V}(\mathcal{I}_1) \subseteq \tilde{\mathcal{V}}$, and hence $\Pi_3(\Pi_2(\mathbb{V}(\mathcal{I}_1))) \subseteq \Pi_3(\Pi_2(\tilde{\mathcal{V}}))$. This motivates the following definition.

Definition 3.2. *$\mathcal{P}(s,t)$ is called* pseudo-normal *if $\mathcal{S} \subseteq \Pi_3(\Pi_2(\tilde{\mathcal{V}}))$.*

Next example shows that not all pseudo-normal parametrizations are normal

Example 3.3. *Let $\mathcal{P}(s,t) = (s, (t+s)/s, 0)$. Then $\mathcal{I}_1 =< x_3, t - x_2x_1 + x_1, s - x_1 >$. It is clear that $\mathcal{S} \subset \Pi_3(\Pi_2(\mathbb{V}(\mathcal{I}_1))) \subset \Pi_3(\Pi_2(\tilde{\mathcal{V}}))$ and hence is pseudo-normal. However, $\mathcal{P}(s,t)$ is not normal because $(0, \lambda, 0) \in \mathcal{S} \setminus \mathcal{P}(\mathbb{K}^2)$ where $\lambda \in \mathbb{K}$. In fact, in this case $\mathcal{I}_1 = \tilde{\mathcal{I}}$, and therefore $\Pi_3(\Pi_2(\mathbb{V}(\mathcal{I}_1))) = \Pi_3(\Pi_2(\tilde{\mathcal{V}}))$.*

Next lemma follows from Lemma 3.1.

Lemma 3.4. *If $\mathcal{P}(s,t)$ is normal then it is pseudo-normal.*

If $\mathcal{P}(s,t)$ is pseudo-normal, and $P \in \mathcal{S}$, then there exist $s_0, t_0 \in \mathbb{K}$ such that $(s_0, t_0, P) \in \tilde{\mathcal{V}}$. Now, there are two possibilities: either $q(s_0,t_0) \neq 0$, in which case $P \in \mathcal{P}(\mathbb{K}^2)$, or $q(s_0,t_0) = 0$ and we cannot ensure that $P \in \mathcal{P}(\mathbb{K}^2)$; compare to Example 3.3. In the sequel we will find sufficient conditions on $\mathcal{P}(s,t)$ to ensure that it is pseudo-normal and, if so, to exclude the second case above. For this purpose, we consider the following two varieties:

1. $\mathcal{Z}_1 = \mathbb{V}(\mathrm{lc}_t(R_{1,2}), \mathrm{lc}_t(R_{1,3}), \mathrm{lc}_t(R_{2,3}), F)$, where $R_{i,j} = \mathrm{Res}_s(G_i, G_j)$ for $i \neq j$. Observe that since $\forall k, \gcd(p_k, q_k) = 1$, then $R_{i,j} \neq 0$ for $i \neq j$.
2. For each $P \in \mathbb{K}^3$, $\mathcal{Z}_2(P) = \mathbb{V}(G_1(s,t,P), G_2(s,t,P), G_3(s,t,P))$.

Note that $\mathrm{lc}_t(R_{i,j}) \in \mathbb{K}[x_i, x_j]$, and hence $\mathcal{Z}_1$ is either empty or it is the intersection of three cylinders (each over each coordinate plane) and $\mathcal{S}$. Moreover, for each P, $\mathcal{Z}_2(P)$ is the intersection of three plane curves. We start showing that $\mathcal{Z}_1$ is contained in a 1-dimension variety.

Lemma 3.5. *Every non-empty component of $\mathbb{V}(\mathrm{lc}_t(R_{i,j})) \cap \mathcal{S}$, for $i \neq j$, has dimension 1.*

Proof. Let $\mathcal{Z}$ be a non-empty component of $\mathbb{V}(\mathrm{lc}_t(R_{i,j})) \cap \mathcal{S}$. By Theorem 6, pp. 76, in [11], $\dim(\mathcal{Z}) \geq 1$. But $\dim(\mathcal{Z}) \neq 2$ because $\mathcal{S}$ is irreducible, it is not a cylinder over the $x_i x_j$-coordinate plane, while $\mathrm{lc}_t(R_{i,j})$ defines either the empty variety or a cylinder of that type. □

Therefore, the following theorem holds.

Theorem 3.6. *If $\mathcal{Z}_1 \neq \emptyset$ then it is a proper close subset of $\mathcal{S}$.*

Then, we have the next theorem.

Theorem 3.7. *It holds that:*

1. *If $\deg_s(p_i) > \deg_s(q_i)$, for some $i = 1,2,3$, then $\mathcal{S} \setminus \mathcal{Z}_1 \subset \Pi_3(\Pi_2(\mathbb{V}(\mathcal{I}_1)))$. Moreover, $\mathcal{P}(s,t)$ is pseudo-normal iff $\forall P \in \mathcal{Z}_1, \mathcal{Z}_2(P) \neq \emptyset$.*
2. *If $\deg_s(p_i) \leq \deg_s(q_i)$, for $i = 1,2,3$, and $A = (a_1, a_2, a_3)$, where*

$$a_i = \frac{\mathrm{coeff}(p_i, s^{\deg_s(q_i)})}{\mathrm{lc}_s(q_i)},$$

   then $\mathcal{S} \setminus (\mathcal{Z}_1 \cup \{A\}) \subset \Pi_3(\Pi_2(\mathbb{V}(\mathcal{I}_1)))$. Moreover, $\mathcal{P}(s,t)$ is pseudo-normal iff $\forall P \in \mathcal{Z}_1 \cup \{A\}, \mathcal{Z}_2(P) \neq \emptyset$.

Proof. (1) We assume w.l.o.g. that $\deg_s(p_1) > \deg_s(q_1)$. Since $(1:0:0) \notin \overline{\mathcal{C}_1}$, then $\mathrm{lc}_s(G_1) \in \mathbb{K} \setminus \{0\}$. Thus, by the Extension Theorem (see [4]), for every $(t_0, P) \in \mathbb{V}(\mathcal{I}_2)$, there exists $s_0 \in \mathbb{K}$ such that $(s_0, t_0, P) \in \mathbb{V}(\mathcal{I}_1)$. Therefore, if $(t_0, P) \in \mathbb{V}(\mathcal{I}_2)$, then $P \in \Pi_3(\Pi_2(\mathbb{V}(\mathcal{I}_1))) \subset \Pi_3(\Pi_2(\tilde{\mathcal{V}}))$. Since $R_{i,j} \in \mathcal{I}_2$, there exists $f_1, \ldots, f_r \in \mathbb{K}[t, \bar{x}]$ such that $\mathcal{I}_2 = (R_{1,2}, R_{1,3}, R_{2,3}, f_1, \ldots, f_r)$. Let $P \in \mathcal{S} \setminus \mathcal{Z}_1$. Then $P \in \mathbb{V}(\mathcal{I}_3)$ and, by the Extension Theorem, there exists $t_0 \in \mathbb{K}$ such that $(t_0, P) \in \mathbb{V}(\mathcal{I}_2)$. So, $P \in \Pi_3(\Pi_2(\mathbb{V}(\mathcal{I}_1)))$.

Now, let $\mathcal{P}(s,t)$ be pseudo-normal and $P \in \mathcal{Z}_1$. Then, $P \in \mathcal{Z}_1 \subset \mathcal{S} \subset \Pi_3(\Pi_2(\tilde{\mathcal{V}}))$. So, there exist $s_0, t_0 \in \mathbb{K}$ such that $(s_0, t_0, P) \in \tilde{\mathcal{V}}$, and hence $(s_0, t_0) \in \mathcal{Z}_2(P)$. Conversely, let $P \in \mathcal{Z}_1$, and $(s_0, t_0) \in \mathcal{Z}_2(P)$. Then, $(s_0, t_0, P) \in \tilde{\mathcal{V}}$. So, $\mathcal{Z}_1 \subset \Pi_3(\Pi_2(\tilde{\mathcal{V}}))$, and hence $\mathcal{P}(s,t)$ is pseudo-normal.
(2) We first observe that, because of the general condition imposed to $\mathcal{C}_i$ and $\mathcal{D}_i$ in Section 2, A is a point in $\mathbb{K}^3$. Moreover, depending on whether $\deg_s(p_i) = \deg_s(q_i)$ or $\deg_s(p_i) < \deg_s(q_i)$, $\mathrm{lc}_s(G_i) = \mathrm{lc}_s(p_i) - x_i \mathrm{lc}_s(q_i)$ or $\mathrm{lc}_s(G_i) = x_i \mathrm{lc}_s(q_i)$. So, $\mathrm{lc}_s(G_i)$ only vanishes at A. Now, the proof follows as in (1). □

Theorem 3.7 shows how to computationally check whether a given surface parametrization is pseudo-normal. More precisely, the algorithm works as follows. We assume that the coefficients of $\mathcal{P}(s,t)$ belong to a computable field $\mathbb{L}$ and that $\mathbb{K}$ is its algebraic closure.

1. We decompose $\mathcal{Z}_1$ into irreducible components over $\mathbb{K}$; for instance, applying Gröbner basis.

2. Then, for each irreducible component $\mathcal{Z}_{1j}$ of $\mathcal{Z}_1$ we consider the field $\mathbb{F} := \mathbb{K}(\mathcal{Z}_{1j})$ of rational functions over $\mathcal{Z}_{1j}$. Observe that the arithmetic and zero-test in $\mathbb{F}$ can be carried out using Gröbner basis to decide the membership to the ideal of $\mathcal{Z}_{1j}$.

3. Now, we see a generic point in $\mathcal{Z}_{1j}$ as a point $P = (x_1, x_2, x_3) \in \mathbb{F}^3$. We need to check whether the $\mathcal{Z}_2(P)$ is empty or not. That is, whether the curves $G_i(s,t,P)$, defined over the algebraic closure of $\mathbb{F}$, have a common intersection or not. This can be done by performing generalized resultants in $\mathbb{F}[t][s]$ and gcds in $\mathbb{F}[t]$. Of course, when performing all these steps the zero-test application will provide zero-dimensional sets of points that will need to be treated separately, but this is not a problem computationally.

4. Additionally, the point A in Theorem 3.7 (2) might need to be checked, but this is again no restriction.

Remark 3.8. *We have emphasized (see Theorem 3.6) the fact that $\mathcal{Z}_1$ is either empty or proper in $\mathcal{S}$. An alternative possibility would be to work with a bigger close set containing $\mathcal{Z}_1$; for instance $\mathcal{S}$. This would avoid to decompose $\mathcal{Z}_1$ into* *irreducible components since $\mathcal{S}$ is already irreducible. Moreover, the field of rational functions over $\mathcal{S}$ is simpler to manage since the ideal is principal. So, where is the disadvantage? The difficulties appear in Step 3, when the zero-test generates particular cases. These particular cases would be, in general, of dimension 1 and this would imply to decompose into irreducible components, etc, this new set. In our examples, this 1-dimensional set was much more complicated than $\mathcal{Z}_1$, and indeed we were unable to compute its irreducible decomposition.*

Let us illustrate this process with two examples.

Example 3.9. *Let $\mathcal{P}(s,t) = (s^2, (t^4+s^4)/(s^3+2t), 1/(t+s^2))$. Then, $\mathrm{lc}_t(R_{1,2}) = 1$, one has that $\mathcal{Z}_1 = \emptyset$. Thus, by Theorem 3.7 (1), $\mathcal{P}(s,t)$ is pseudo-normal.*

Example 3.10. *Let*

$$\mathcal{P}(s,t) = \left(\frac{s^2-t+1}{s^2+t}, \frac{s^2+2t-s-1}{s^2-t}, \frac{s^2+t+s}{s^2+2t}\right).$$

Then,

$$\mathrm{lc}_t(R_{1,2}) = (2\,x_1x_2+x_1-3)^2\,,\ \mathrm{lc}_t(R_{1,3}) = (x_3x_1-3\,x_3+2)^2\,,\ \mathrm{lc}_t(R_{2,3}) = (3\,x_3x_2-2\,x_2-1)^2\,.$$

Moreover, the defining polynomial of the surface is

$F = 11-7\,x_1+x_2-22\,x_3+x_2x_3x_1-2\,{x_2}^2x_1+6\,x_2{x_3}^2x_1-18\,x_2{x_3}^2-x_3{x_1}^2+{x_1}^2-5\,x_1x_2+4\,{x_2}^2+2\,x_2{x_1}^2+15\,{x_3}^2-8\,{x_3}^2x_1+9\,x_3x_1+{x_3}^2{x_1}^2+9\,{x_3}^2{x_2}^2+15\,x_3x_2-12\,x_3{x_2}^2.$

The set $\mathcal{Z}_1$ only has one component one-dimensional

$$\{g_1 := 2\,x_1x_2+x_1-3,\ g_2 := x_3x_1-3\,x_3+2,\ g_3 := 3\,x_3x_2-2\,x_2-1\}.$$

Note that $\{g_1, g_2, g_3\}$ is the Gröbner basis w.r.t. the pure lexicographic order with $x_1 > x_2 > x_3$ of the ideal of $\mathcal{Z}_1$. Since we are in the conditions of Theorem 3.7 (2), we analyze whether for $P \in \mathcal{Z}_1 \cup \{A := (1,1,1)\}$ it holds that $\mathcal{Z}_2(P) \neq \emptyset$.

(1) $G_1(s,t,A) = 2t-1, G_2(s,t,A) = -3t+s+1, G_3(s,t,A) = t-s$, *so* $(1/2,1/2) \in \mathcal{Z}_2(A) \neq \emptyset$.

(2) Now, we analyze the problem for a generic element of $P \in \mathcal{Z}_1$. For this purpose, let $\mathbb{F} = \mathbb{K}(\mathcal{Z}_1)$, and $P = (x_1,x_2,x_3) \in \mathbb{F}^3$. We analyze $\mathcal{Z}_2(P)$. We distinguish two cases

(2.1) Let $x_1 = 1$ or $x_2 = 1$ or $x_3 = 1$. Using that $P \in \mathcal{Z}_1$, one gets that $P = A$ and this is done in (1).

(2.2) Let $x_1 \neq 1$, $x_2 \neq 1$, and $x_3 \neq 1$. Then,

$$G_1 = (x_1 - 1)s^2 + (x_1 t + t - 1),$$
$$G_2 = (x_2 - 1)s^2 + s + (1 - 2t - x_2 t),$$
$$G_3 = (x_3 - 1)s^2 - s + (2x_3 t - t),$$

and $\mathrm{NormF}(x_i - 1) = x_i - 1$, where NormF denotes the normal form w.r.t. the Gröbner basis $\{g_1, g_2, g_3\}$. Therefore $\deg_s(G_i) = 2$ as polynomials in $\mathbb{F}[t][s]$. In order to analyze the intersection, over the algebraic closure of $\mathbb{F}$, of the curves defined by G_1, G_2, G_3 we determine the generalized resultant w.r.t. s of G_1, G_2, G_3. That is, we compute the content w.r.t. t of $\mathrm{Res}_s(G_1, G_2 + wG_3)$ where w is a new variable. We get, after computing the normal form of the coefficients,

$$\mathrm{Res}_s(G_1, G_2 + wG_3) = a_0(t) + a_1(t)w + a_2 w^2$$

where

$$a_0 = (x_1^2 - 1)t + (8 - 4x_2 + x_2^2 - 6x_1 + x_1^2)$$
$$a_1 = (2 - 2x_1^2)t + (-\tfrac{4}{3} - \tfrac{2}{3}x_2 + 2x_3)$$
$$a_2 = (x_1^2 - 1)t + (-2x_3 + x_3^2 - x_1 + 2).$$

Moreover, $\gcd_{\mathbb{F}[t]}(a_1, a_2, a_3) = 1$, since $\gcd_{\mathbb{F}[t]}(a_1, a_3) = 1$ because

$$\mathrm{NormF}((8 - 4x_2 + x_2^2 - 6x_1 + x_1^2) - (-2x_3 + x_3^2 - x_1 + 2)) =$$

$$-\frac{79}{4} - 8\,{x_3}^2 - 13/2\,x_1 + 4\,x_2 + 28\,x_3 + {x_1}^4 + \frac{29}{4}\,{x_1}^2 - {x_2}^2 - 5\,{x_1}^3 \neq 0.$$

Therefore G_1, G_2, G_3 do not intersect. Thus, $\mathcal{Z}_2(P) = \emptyset$. So $\mathcal{P}(s,t)$ is not pseudo-normal, and hence neither normal.

The efficiency of the previous algorithmic process depends on how complicated $\mathcal{Z}_1$ is. In the sequel we show that, under certain conditions, the variety $\mathcal{Z}_1$ is either empty or decomposes as union of zero-dimensional components and plane rational curves; note that in this last case $\mathbb{F} = \mathbb{K}(h)$ where h is transcendental over $\mathbb{K}$.

Lemma 3.11. *Let $\overline{G_k}(x_k, s, t, u)$ be the homogenization of G_k w.r.t. $\{s, t\}$. If*

$$\gcd(\overline{G_i}(x_i, s, t, 0), \overline{G_j}(x_j, s, t, 0)) = 1, \quad \text{for } i \neq j,$$

then $\mathrm{lc}_t(R_{i,j})(x_i, x_j) = \mathrm{Res}_s(\overline{G_i}(x_i, s, 1, 0), \overline{G_j}(x_j, s, 1, 0))$.

Proof. Let $n_k = \deg_{\{s,t,u\}}(\overline{G_k})$, then $\deg_{t,u}(R_{i,j}) = n_i n_j$. The gcd condition implies that the projective curves defined, over the algebraic closure of $\mathbb{K}(\bar{x})$, by $\overline{G_i}$ and $\overline{G_j}$ do not have common points on the line at infinity $u = 0$. Therefore, $\mathrm{coeff}(R_{i,j}, t^{n_i n_j}) \neq 0$. So, if $R(x_i, x_j, t, u) = \mathrm{Res}_s(\overline{G_i}, \overline{G_j})$, then $\mathrm{lc}_t(R_{i,j}) = R(x_i, x_j, 1, 0)$. By hypothesis $(1:0:0) \notin \overline{\mathcal{C}_k}$ and $(1:0:0) \notin \overline{\mathcal{D}_k}$, so $\deg_s(\overline{G_k}) = \deg_s(\overline{G_k}(x_k, s, 1, 0))$. Now, the proof follows from the behavior of the resultant under specialization; see e.g. Lemma 4.3.1 in [14]. □

The next theorem extend Lemma 3.5, when the gcd condition in Lemma 3.11 is satisfied.

Theorem 3.12. *Let $\overline{G_k}$ be as in Lemma 3.11 and let $\overline{G_i}(x_i, s, t, 0), \overline{G_j}(x_j, s, t, 0)$ be coprime, where $i \neq j$.*

1. *If $\deg_s(p_i) = \deg_s(q_i)$, $\deg_s(p_j) = \deg_s(q_j)$, $\mathbb{V}(\mathrm{lc}_t(R_{i,j})) \cap \mathcal{S}$ is the union of curves.*
2. *If $\deg_s(p_i) > \deg_s(q_i)$, $\deg_s(p_j) > \deg_s(q_j)$, $\mathbb{V}(\mathrm{lc}_t(R_{i,j})) \cap \mathcal{S} = \emptyset$.*
3. *If $\deg_s(p_i) < \deg_s(q_i)$, $\deg_s(p_j) < \deg_s(q_j)$, $\mathbb{V}(\mathrm{lc}_t(R_{i,j})) \cap \mathcal{S}$ is the union of plane curves. Moreover, $\mathbb{V}(\mathrm{lc}_t(R_{i,j})) \cap \mathcal{S} = (\mathbb{V}(x_i) \cap \mathcal{S}) \cup (\mathbb{V}(x_j) \cap \mathcal{S})$.*
4. *If $\deg_s(p_i) > \deg_s(q_i)$, $\deg_s(p_j) = \deg_s(q_j)$, $\mathbb{V}(\mathrm{lc}_t(R_{i,j})) \cap \mathcal{S}$ is the union of plane curves, of the form $\mathbb{V}(\alpha x_j - \beta) \cap \mathcal{S}$, where α, $\beta \in \mathbb{K}$.*
5. *If $\deg_s(p_i) > \deg_s(q_i)$, $\deg_s(p_j) < \deg_s(q_j)$, $\mathbb{V}(\mathrm{lc}_t(R_{i,j})) \cap \mathcal{S}$ is the union of plane curves. Moreover, $\mathbb{V}(\mathrm{lc}_t(R_{i,j})) \cap \mathcal{S} = \mathbb{V}(x_j) \cap \mathcal{S}$.*
6. *If $\deg_s(p_i) < \deg_s(q_i)$, $\deg_s(p_j) = \deg_s(q_j)$, $\mathbb{V}(\mathrm{lc}_t(R_{i,j})) \cap \mathcal{S}$ is the union of plane curves, of the form $\mathbb{V}(\alpha x_j - \beta) \cap \mathcal{S}$, where α, $\beta \in \mathbb{K}$.*
7. *If $\deg_s(p_k) = \deg_s(q_k)$, for $k = 1, 2, 3$, and $\overline{G_1}(x_1, s, t, 0)$, $\overline{G_2}(x_2, s, t, 0)$, $\overline{G_3}(x_3, s, t, 0)$ are pairwise coprime, $\mathcal{Z}_1$ is either empty or zero-dimensional or a line or the rational curve parametrized by*

$$\left(\frac{p_1^h(s,1,0)}{q_1^h(s,1,0)}, \frac{p_2^h(s,1,0)}{q_2^h(s,1,0)}, \frac{p_3^h(s,1,0)}{q_3^h(s,1,0)}\right),$$

   where $p_k^h(s, t, u)$ and $q_k^h(s, t, u)$ are the homogenization of $p_k(s, t)$ and $q_k(s, t)$ respectively.

Proof. Let $\overline{g_k}(x_k, s) = \overline{G_k}(x_k, s, 1, 0)$, and let $p_k^h(s, t, u)$, $q_k^h(s, t, u)$ as in statement (7).

1. It follows from Theorem 3.5.
2. $\overline{g_i}(x_i, s) = p_i^h(s, 1, 0), \overline{g_j}(x_j, s) = p_j^h(s, 1, 0)$. So, by Lemma 3.11, $\mathrm{lc}_t(R_{i,j}) \in \mathbb{K} \setminus \{0\}$.
3. $\overline{g_i}(x_i, s) = q_i^h(s, 1, 0)x_i, \overline{g_j}(x_j, s) = q_j^h(s, 1, 0)x_j$. So, by Lemma 3.11, up to multiplication by constant,

$$\mathrm{lc}_t(R_{i,j}) = \mathrm{Res}_s(x_i q_i^h(s, 1, 0), x_j q_j^h(s, 1, 0)) = x_i^a x_j^b, \;\; a, b \in \mathbb{N}.$$

   Note that $a = \deg_s(q_j^h(s, 1, 0)) = \deg_s(q_j) > \deg_s(p_j) \geq 0$, and $b = \deg_s(q_i^h(s, 1, 0)) = \deg_s(q_i) > \deg_s(p_i) \geq 0$.

4. $\overline{g_i}(x_i, s) = p_i^h(s, 1, 0), \overline{g_j}(x_j, s) = q_j^h(s, 1, 0)x_j - p_j^h(s, 1, 0)$. Now the result follows from Lemma 3.11, and using that $\deg_s(p_i^h(s, 1, 0)) = \deg_s(p_i) > \deg_s(q_i) \geq 0$ and that $\deg_s(q_j^h(s, 1, 0)) = \deg_s(q_j) > 0$; the last inequality is due to the fact that $\deg_s(q_j) = \deg_s(p_j)$ and that $\mathcal{S}$ is not a cylinder over a coordinate plane.

5. $\overline{g_i}(x_i, s) = p_i^h(s, 1, 0), \overline{g_j}(x_j, s) = q_j^h(s, 1, 0)x_j$. So, by Lemma 3.11, up to multiplication by constant,

$$\mathrm{lc}_t(R_{i,j}) = \mathrm{Res}_s(p_i^h(s, 1, 0), x_j q_j^h(s, 1, 0)) = x_j^a, \quad a \in \mathbb{N}.$$

Note that, reasoning as in the previous case, $\deg_s(q_j^h) > 0$. Moreover, it holds that $a = \deg_s(p_i^h(s, 1, 0)) = \deg_s(p_i) > \deg_s(q_i) \geq 0$.

6. $\overline{g_i}(x_i, s) = q_i^h(s, 1, 0)x_i, \overline{g_j}(x_j, s) = q_j^h(s, 1, 0)x_j - p_j^h(s, 1, 0)$. So, by Lemma 3.11, up to multiplication by constant,

$$\mathrm{lc}_t(R_{i,j}) = \mathrm{Res}_s(x_i q_i^h(s, 1, 0), q_j^h(s, 1, 0)x_j - p_j^h(s, 1, 0)).$$

Note that $\deg_s(q_i^h(s, 1, 0)) = \deg_s(q_i) > \deg_s(p_i) \geq 0$, and $\deg_s(q_j^h(s, 1, 0)) = \deg_s(q_j) > 0$ because $\mathcal{S}$ is not a cylinder.

7. Let $C_k(s) = \mathrm{Content}_{x_k}(\overline{g_k})$ and $B_k(x_k, s) = \mathrm{Primpart}_{x_k}(\overline{g_k})$. Then, by Lemma 3.11, and up to multiplication by constants, it holds:

$$\mathrm{lc}_t(R_{i,j}) = \mathrm{Res}_s(B_i(x_i, s), B_j(x_j, s))$$

$$\mathrm{Res}_s(B_i(x_i, s), C_j(s))\mathrm{Res}_s(C_i(s), B_j(x_j, s)).$$

All the factors above factorize into linear polynomials (i.e. when cutting $\mathcal{S}$, they define plane curves) with the possible exception of the polynomial $S_{i,j}(x_i, x_j) = \mathrm{Res}_s(B_i(x_i, s), B_j(x_j, s))$. If both B_i, B_j do not depend on s, $S_{i,j}$ is constant. If either B_i or B_j does not depend on s, $S_{i,j}$ only depends on either x_j or x_i, and hence factors into linear factors too. Let B_i, B_j depend on s, and let $\alpha_i(s), \alpha_j(s)$ be the root of B_i, B_j, seen as polynomials in x_i and x_j, respectively. Then the square-free part of $S_{i,j}$, say $S_{i,j}^*$, is the implicit equation of the curve defined by $(\alpha_i(s), \alpha_j(s))$ (see Lemma 4.6. in [13]). We observe that $\alpha_k(s) = p_k^h(s, 1, 0)/q_k^h(s, 1, 0)$ for $k \in \{i, j\}$. Now, applying this reasoning to each of the leading coefficients $\mathrm{lc}_t(R_{1,2}), \mathrm{lc}_t(R_{1,3}), \mathrm{lc}_t(R_{2,3})$, and considering the intersection of the three of them with $\mathcal{S}$ one gets the result.

□

Corollary 3.13. *Let $\overline{G_k}$ be as in Lemma 3.11 and let $\overline{G_1}(x_1, s, t, 0)$, $\overline{G_2}(x_2, s, t, 0)$, and $\overline{G_3}(x_3, s, t, 0)$ be pairwise coprime. Then, $\mathcal{Z}_1$ is either empty or zero-dimensional or is union of a zero-dimensional set and several rational curves.*

Proof. Since the three polynomials are pairwise coprime, Theorem 3.12 is applicable to the leading coefficients of $R_{1,2}, R_{1,3}, R_{2,3}$. Now, the result follows from the structure of $\mathbb{V}(\mathrm{lc}_t(R_{i,j})) \cap \mathcal{S}$ in Theorem 3.12. □

Corollary 3.14. *If there exist $i, j \in \{1, 2, 3\}$, $i \neq j$, such that*

1. $\deg_s(p_i) > \deg_s(q_i)$, *and* $\deg_s(p_j) > \deg_s(q_j)$,
2. $\mathcal{C}_i$ *and* $\mathcal{C}_j$ *do not have common points at infinity (recall that $\mathcal{C}_i = \mathbb{V}(p_i)$),*

then $\mathcal{P}(s, t)$ is pseudo-normal.

Proof. By hypothesis (1) $\overline{g_i}(x_k, s) = p_i^h(s, 1, 0), \overline{g_j}(x_k, s) = p_j^h(s, 1, 0)$ where $\overline{g_k}, p_k^h, q_k^h$ are as in the proof of Theorem 3.12. Thus, by hypothesis (2), $\gcd(\overline{g_i}, \overline{g_j}) = 1$. Now, the result follows from Theorem 3.12 (2) and Theorem 3.7. □

Corollary 3.15. *Let $\mathcal{P}(s, t)$ be a polynomial parametrization. If there exist $i, j \in \{1, 2, 3\}$, $i \neq j$, such that $\mathcal{C}_i$ and $\mathcal{C}_j$ do not have common points at infinity, then $\mathcal{P}(s, t)$ is pseudo-normal.*

Proof. It follows from Corollary 3.14. □

Example 3.16. *We consider the surface $\mathcal{S}$ parametrized by*

$$\mathcal{P}(s, t) = \left(\frac{s^2 - 1}{1 + st + s^2}, \frac{s + t + 3}{4 + t - s}, \frac{t + s}{t^2 + 2\, s^2} \right).$$

We observe that $\mathcal{P}(s, t)$ satisfies the conditions in Section 2. Since $\overline{G_1}, \overline{G_2}, \overline{G_3}$ satisfy the conditions in Corollary 3.13, we expect a simple set $\mathcal{Z}_1$. In order to compute $\mathcal{Z}_1$, we first compute $\mathbb{V}(\mathrm{lc}_t(R_{1,2}), \mathrm{lc}_t(R_{1,3}), \mathrm{lc}_t(R_{2,3}))$ to afterwards intersect with $\mathcal{S}$.

$$\mathrm{lc}_t(R_{1,2}) = (x_2 - 1)\,(2\, x_1 x_2 - x_2 + 1)\,, \mathrm{lc}_t(R_{1,3}) = {x_3}^2 \left(1 + 3\, {x_1}^2 - 2\, x_1\right),$$

$$\mathrm{lc}_t(R_{2,3}) = x_3 \left(3 - 2\, x_2 + 3\, {x_2}^2\right).$$

So, $\mathbb{V}(\mathrm{lc}_t(R_{1,2}), \mathrm{lc}_t(R_{1,3}), \mathrm{lc}_t(R_{2,3}))$ is

$$\{(h, 1, 0)\} \cup \{((h - 1)/(2h), h, 0)\} \cup \{(1/3 + 1/3\, i\sqrt{2}, 1, h)\} \cup \{(1/3 - 1/3\, i\sqrt{2}, 1, h)\}.$$

Now, for computing the intersection with $\mathcal{S}$, we use its implicit equation (previously computed with the resultant-based algorithm in [10]) to get:

$$\mathcal{Z}_1 = \{(h, 1, 0)\} \cup \{((h - 1)/(2h), h, 0)\} \cup$$

$$\{(1/3 + 1/3\, i\sqrt{2}, 1, 0), (1/3 + 1/3\, i\sqrt{2}, 1, -\frac{1}{72}\,\frac{5 + 17\, i\sqrt{2}}{1 + i\sqrt{2}}),$$

$$(1/3 - 1/3\, i\sqrt{2}, 1, 0), (1/3 - 1/3\, i\sqrt{2}, 1, -\frac{1}{72}\, \frac{-5 + 17\, i\sqrt{2}}{-1 + i\sqrt{2}})\}.$$

Thus, the first line and the second rational curve are contained in the surface, while the third an fourth cut $\mathcal{S}$ into 4 points; 2 points each one. Now, we study $\mathcal{Z}_2(P)$ for $P \in \mathcal{Z}_1$. For $P = (h, 1, 0)$ we get

$$\mathcal{Z}_2(P) = \mathbb{V}(h + hts + hs^2 - s^2 + 1, -1 + 2\, s, -t - s)$$

which is empty unless $h = -3/4$. So, by Theorem 3.7 (1), $\mathcal{P}(s, t)$ is not pseudo-normal.

4 Normal Parametrizations

In this last section, we state necessary and sufficient conditions on a pseudo-normal parametrization to be normal. As a consequence, certain types of parametrizations are shown to be always normal. Finally, we derive an algorithm for certain type of surfaces. We start with the next theorem.

Theorem 4.1. *$\mathcal{P}(s, t)$ is normal iff $\forall P \in \mathcal{S}$, $\mathcal{Z}_2(P) \not\subseteq \mathbb{V}(\mathrm{lcm}(q_1, q_2, q_3))$.*

Proof. If $\mathcal{P}(s, t)$ is normal, then $\mathcal{S} = \mathcal{P}(\mathbb{K}^2)$. So, for every $P \in \mathcal{S}$, there exist $s_0, t_0 \in \mathbb{K}$ such that $(s_0, t_0) \in \mathcal{Z}_2(P)$ and $q(s_0, t_0) \neq 0$. Conversely, let $P \in \mathcal{S}$. Since $\mathcal{Z}_2(P) \not\subseteq \mathbb{V}(q)$, then there exists $(s_0, t_0) \in \mathcal{Z}_2(P) \setminus \mathbb{V}(q)$. Thus, $P = \mathcal{P}(s_0, t_0)$ and hence $\mathcal{P}(s, t)$ is normal. □

The next theorem introduces a new characterization for the normality. Differently to Theorem 4.1, it requires pseudo-normality but instead the subset of $\mathcal{S}$, that may need a further analysis, is simplified. We recall that $\mathcal{C}_i = \mathbb{V}(p_i)$ and that $\mathcal{D}_i = \mathbb{V}(q_i)$. Moreover, the curve intersections below are afin.

Theorem 4.2. *Let $\mathcal{P}(s, t)$ be pseudo-normal and $\mathcal{G} = \bigcup_{k=1}^{3} (\mathcal{C}_k \cap \mathcal{D}_k)$. The parametrization $\mathcal{P}(s, t)$ is normal iff*

$$\forall P \in \bigcup_{(s_0, t_0) \in \mathcal{G}} (\mathbb{V}(G_1(x_1, s_0, t_0), G_2(x_2, s_0, t_0), G_3(x_3, s_0, t_0)) \cap \mathcal{S}),$$

$\mathcal{Z}_2(P) \not\subseteq \mathbb{V}(\mathrm{lcm}(q_1, q_2, q_3))$.

Proof. If $\mathcal{P}(s, t)$ is normal, by Theorem 4.1, $\forall P \in \mathcal{S}$, $\mathcal{Z}_2(P) \not\subseteq \mathbb{V}(q)$. Therefore, the statement holds. Conversely, let $P \in \mathcal{S}$. Since $\mathcal{P}(s, t)$ is pseudo-normal, $\mathcal{Z}_2(P) \neq \emptyset$. Let $(s_0, t_0) \in \mathcal{Z}_2(P)$. If $q(s_0, t_0) \neq 0$, then $P \in \mathcal{P}(\mathbb{K}^2)$. If $q(s_0, t_0) = 0$, there exists $i \in \{1, 2, 3\}$ such that $q_i(s_0, t_0) = 0$ and $p_i(s_0, t_0) = G_i(P, s_0, t_0) = 0$. So, $(s_0, t_0) \in \mathcal{C}_i \cap \mathcal{D}_i \subset \mathcal{G}$ and, by hypothesis, there exists $(s_1, t_1) \in \mathcal{Z}_2(P)$ such that $q(s_1, t_1) \neq 0$. Thus, $P \in \mathcal{P}(\mathbb{K}^2)$. □

Corollary 4.3. *If $\mathcal{P}(s,t)$ is pseudo-normal, and $\bigcup_{k=1}^{3}(\mathcal{C}_k \cap \mathcal{D}_k) = \emptyset$, then it is normal.*

Proof. It follows from Theorem 4.2. □

Corollary 4.4. *Every pseudo-normal polynomial parametrization is normal.*

Proof. It follows from Corollary 4.3. □

Let $\mathcal{G}^* = \bigcap_{k=1}^{3}(\mathcal{C}_k \cap \mathcal{D}_k)$ and $\mathcal{G} = \bigcup_{k=1}^{3}(\mathcal{C}_k \cap \mathcal{D}_k)$. In the sequel, we show that if $\mathcal{G}^* = \emptyset$, Theorem 4.2 provides an algorithm to decide the normality of $\mathcal{P}(s,t)$; note that parametrizations with base points do not satisfy the above requirement (see [5] for the notion of base points). For this purpose, we assume w.l.o.g. that $\mathcal{S}$ is not a plane. Then, we proceed as follows:

1. Check whether $\mathcal{P}(s,t)$ is pseudo-normal (see Section 3). If it is not pseudo-normal, return that it is not normal.

2. Observe that, because $\gcd(p_i, q_i) = 1$, $\mathcal{G}$ is either empty or zero-dimensional. If $\mathcal{G} = \emptyset$ return that $\mathcal{P}(s,t)$ is normal.

3. For each $(s_0, t_0) \in \mathcal{G}$, since $\mathcal{S}$ is irreducible of degree at least 2 and $\mathcal{G}^* = \emptyset$,
$$\mathcal{Z}(s_0,t_0) := (\mathbb{V}(G_1(x_1,s_0,t_0), G_2(x_2,s_0,t_0), G_3(x_3,s_0,t_0)) \cap \mathcal{S}$$
is either empty or a proper close subset of $\mathcal{S}$; indeed it is either empty or zero-dimensional or a plane curve. Decompose $\mathcal{Z}(s_0,t_0)$ into irreducible components.

4. For each irreducible $\mathcal{Z}^*$ component of $\mathcal{Z}(s_0,t_0)$, consider a generic element (i.e. take $P = (x_1,x_2,x_3) \in \mathbb{F}^3$ where $\mathbb{F}$ is the field of rational functions of $\mathcal{Z}^*$) and check whether $\mathcal{Z}_2(P) \not\subseteq \mathbb{V}(\mathrm{lcm}(q_1,q_2,q_3))$. In order to do this one simply has to proceed as in the pseudo-normality test algorithm presented in Section 3.

Remark 4.5. *If $\mathcal{G}^* \neq \emptyset$ we still can apply Theorem 4.2. Nevertheless this implies that, for some (s_0,t_0), the $\mathcal{Z}(s_0,t_0) = \mathcal{S}$ and the difficulties commented in Remark 3.8 are also applicable here.*

Let see some examples.

Example 4.6. *We consider the surface parametrization*
$$\mathcal{P}(s,t) = \left(s^2 - 1, s + t + 3, \frac{t^2 + s^2}{t + 2s}\right).$$

We observe that $\mathcal{P}(s,t)$ does satisfy the conditions in Section 2. Moreover, by Corollary 3.14, $\mathcal{P}(s,t)$ is pseudo-normal. The set $\mathcal{G}$ is $\mathcal{G} = \{(0,0)\}$, and

$$\mathbb{V}(G_1(x_1,0,0), G_2(x_2,0,0), G_3(x_3,0,0)) \cap \mathcal{S} = \{(-1,3,h) \,|\, h \in \mathbb{C}\}.$$

Moreover, $\mathcal{Z}_2(-1,3,h) = \mathbb{V}(-s^2, -s-t, ht+2hs-t^2-s^2) = \{(0,0)\} \subseteq \mathbb{V}(t+2s)$. Therefore, $\mathcal{P}(s,t)$ is not normal. In fact, $\mathcal{P}(\mathbb{C}^2)$ is all $\mathcal{S}$ minus the line of equation $\{x_1 = -1, x_2 = 3\}$.

Example 4.7. *We consider the surface parametrization*

$$\mathcal{P}(s,t) = \left(\frac{s^2-1}{t+2\,s}, \frac{s+t+3}{t+2\,s}, \frac{t^2+s^2}{t+2\,s} \right).$$

We observe that $\mathcal{P}(s,t)$ does satisfy the conditions in Section 2. Moreover, by Corollary 3.14, $\mathcal{P}(s,t)$ is pseudo-normal. $\mathcal{G} = \{(0,0),(1,-2),(-1,2),(3,-6)\}$, and for $(s_0,t_0) \in \mathcal{G}$ one has $\mathbb{V}(G_1(x_1,s_0,t_0), G_2(x_2,s_0,t_0), G_3(x_3,s_0,t_0)) = \emptyset$. So $\mathcal{P}(s,t)$ is normal.

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