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Distance Bounds of ϵ–Points on Hypersurfaces *

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Abstract

ϵ–points were introduced by the authors (see [26], [27], [28]) as a generalization of the notion of approximate root of a univariate polynomial. The notion of ϵ–point of an algebraic hypersurface is quite intuitive. It essentially consists in a point such that when substituted in the implicit equation of the hypersurface gives values of small module. Intuition says that an ϵ-point of a hypersurface is a point close to it. In this paper, we formally analyze this assertion giving bounds of the distance of the ϵ-point to the hypersurface. For this purpose, we introduce the notions of height, depth and weight of an ϵ-point. The height and the depth control when the distance bounds are valid, while the weight is involved in the bounds.

1 Introduction

From the early beginnings of computer algebra, the achievements in symbolic computation have been related to many mathematical disciplines like linear algebra (e.g. homomorphic methods, fraction free techniques, etc), non-linear algebra (e.g. resultants, gcd, polynomial factorizations, Gröbner bases, etc), analysis (e.g. integration,

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computing with transcendental functions, solving differential equations, etc), algebraic geometry (e.g. singularities computation, implicitization and parametrization techniques, etc), etc.

In consequence of this development, symbolic algorithms have been used in some applications like, for instance in computer aided geometric design (see [19], [20]), providing exact answers when dealing with algorithmic questions on mathematical entities exactly given. This type of contributions have been, and are, important since they offer effective algorithmic solutions to applied problem, and indeed investigations in this direction constitute an active research branch of symbolic computation.

Nevertheless, in many practical applications, these symbolic approaches tend to be insufficient, since in practice most of data objects are given or become approximate. This fact implies that intrinsic mathematical properties of the original object may fail. This phenomenon has motivated an increasing interest of the research community, working on computational algebra and computational algebraic geometry, for the development of approximate algorithms; that is, algorithms that deal symbolically with mathematical inputs, that have suffered a modification. For instance, let us assume that we are dealing with an applied problem where one needs to factorize a polynomial, and in fact, because of the theory behind the experiment or the application, one knows that the output polynomial must be reducible. Now, say that because of errors in the measures, the data is perturbed and instead of getting the polynomial $f := x^2 - y^2$, which factors as $(x-y)(x+y)$, one gets $\overline{f} := 1.00001x^2+0.00002xy-1.00001y^2+0.00001$ that is irreducible. Every symbolic factorization algorithm will answer that $\overline{f}$ is irreducible, however $\overline{f}$ can be expressed as

$$\overline{f} = (1.00001x - y)(x + 1.00001y) + 0.00001,$$

which is "almost" reducible. An approximate factorization algorithm (see e.g. [7]) may recognize the above decomposition, and outputs that $\overline{f}$ factors approximately as $(1.00001x - y)(x + 1.00001y)$.

In algebra, approximate algorithms have been developed for computing polynomial greatest common divisors (see e.g. [6], [11], [25]), for finding zeros of multivariate systems (see e.g. [6], [12], [14]), for factoring polynomials (see e.g. [7], [16], [24], [29]), for the computation of Gröbner basis (see e.g. [23], [31]), etc. In algebraic geometry, approximate algorithms for computing singularities can be found in [2], [3], [9]; for implicitizating rational parametrizations in [8], [10]; for implicitization methods in [4], [15], [18], [26], [27], etc.

In this field an important, and usually hard, step is the error analysis of the algorithms. This analysis mostly consists in estimating how "close" the input and the output of the algorithm are. If one is working from an algebraic point of view, for instance with polynomial factorizations, this question may be approached by measuring relative errors of polynomials. However, when the objects are studied from the geometric point of view, the Euclidean metric has to be taken into account, for instance, by requiring that each geometric entity lies in the offset region of the other at some small distance (see Section 5 for further details).

A technique to guarantee that an algebraic hypersurface (in practice, an algebraic curve or surface) is within the offset region of another, is the use of ϵ–points (see Definition 1), and more precisely, metric properties of this type of points. ϵ–points were introduced by the authors (see [26], [27]) as a generalization of the notion of approximate root of a univariate polynomial. The notion of ϵ–point of an algebraic hypersurface is quite intuitive. It essentially consists in a point such that when substituted in the implicit equation of the hypersurface gives values of small module. This type of points play an important role in some algorithmic processes in algebraic geometry as the approximate parametrization (see [26], [27]).

Theoretical properties and algorithmic questions of ϵ–points have been studied by several authors for the univariate case. For instance, bound analysis of roots of univariate polynomials can be found in [5], [22], [24], formulae for separating small roots of univariate polynomials are given in [30], the problem of constructing univariate polynomials with exact roots at some specific ϵ–roots (see Section 2 for the notion of ϵ–root) is analyzed in [21], condition numbers of ϵ–roots are studied in [32], etc.

Intuition says that an ϵ-point of a hypersurface is a point close to it. To state formally this assertion, one need to estimate the distance of an ϵ-point to the hypersurface, for instance by giving bounds. In [28] bounds for the case of plane curves are provided. In this paper, beside the obvious advances from curves to hypersurfaces, we improve the bounds given in [28]. The particularization to curves of the bounds given here are sharper than those in [28], and describes better the phenomenon showing how the multiplicity is involved in the number of points being close to the ϵ-singularity. The main ideas allowing us to improve and to extend the bounds in [28] to hypersurfaces are the notions of height, depth and (local and global) weight of an ϵ-point.

The paper is structured as follows. In Section 2, we introduce the basic notions of the paper. Section 3 is devoted to the study of distance properties between ϵ–roots and exact roots of univariate polynomials over $\mathbb{C}$. Section 4 focuses on the general case of hypersurfaces. In this study we distinguish the cases of ϵ-singularities and simple ϵ-points. In addition in Sections 4 a joint experimental analysis, of the bounds given in Sections 3 and 4, is included. In Section 5 we show the connection of the problem with the use of offsets to error analysis of approximate algorithms in algebraic geometry. We finish with a section on conclusions and open questions. We also include an appendix with the input polynomials used in the experimental analysis presented in Section 4.

Notation: Throughout this paper, we use the notation $\underline{x} = (x_1, \ldots, x_n)$. We fix a tolerance $0 < \epsilon < 1$, and for polynomials in $\mathbb{C}[\underline{x}]$ we use the ∞–norm;

$$\| \sum_{i_1,\ldots,i_n \in I} c_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} \| = \max\{|c_{i_1,\ldots,i_n}|;\ i_1,\ldots,i_n \in I\}.$$

We also use the Euclidean norm $\| \cdot \|_2$ for points in the usual unitary space $\mathbb{C}^n$. In addition, we denote the partial derivatives of $p(\underline{x}) \in \mathbb{C}[\underline{x}]$ as:

$$p^{\overrightarrow{v}}(\underline{x}) := \frac{\partial^{i_1+\cdots+i_n} p}{\partial^{i_1} x_1 \cdots \partial^{i_n} x_n}(\underline{x})$$

with $\overrightarrow{v} = (i_1, \ldots, i_n) \in \mathbb{N}^n$. Note that, if $\overrightarrow{0}$ denotes the zero vector, then $p^{\overrightarrow{0}}(\underline{x}) = p(\underline{x})$. Moreover, note that, if $\overrightarrow{e_i}$ is the i-th canonical vector in $\mathbb{C}^n$ and $p(\underline{x}) \in \mathbb{C}[\underline{x}]$, then $p^{r\cdot \overrightarrow{e_i}}(\underline{x}) := \frac{\partial^r p}{\partial^r x_i}(\underline{x})$. Finally, for $\overrightarrow{v} = (i_1, \ldots, i_n) \in \mathbb{N}^n$ we write $|\overrightarrow{v}| = i_1 + \cdots + i_n$.

2 ϵ–Points on Hypersurfaces

In this section, we introduce the basic notions of the paper; namely, the concepts of ϵ-point, multiplicity, height and depth of an ϵ-point, and proper degree of a hypersurface. In addition, we also introduce the notion of weight.

We have already seen the intuitive meaning of ϵ–point of an algebraic hypersurface. The following definition states formally the concept.

Definition 1 *We say that $P^\star \in \mathbb{C}^n$ is an ϵ–*(affine) point *of an algebraic hypersurface $\mathcal{V}$, defined over $\mathbb{C}$ by a polynomial $f \in \mathbb{C}[\underline{x}]$, if it holds that*

$$\frac{|f(P^\star)|}{\|f\|} < \epsilon.$$

Note that in Definition 1, in order to control that the implicit equation is unique up to multiplication by non-zero constants, relative errors are taken. The next step, in this theoretical development, is the introduction of the notion of multiplicity of an ϵ-point. In (exact) algebraic geometry, the notion of multiplicity is usually introduced by considering the first order of derivation where the derivative does not vanish at the point. Therefore, it seems reasonable to define the multiplicity of an ϵ-point as the first order of derivation where the module of the evaluation of the derivative at the point, divided by the norm of the implicit equation, is greater or equal to ϵ. Nevertheless, if the notion of multiplicity is defined as above, it may happen that the order of derivation does not exist, and hence the multiplicity might not be well defined. For instance, let $0 < \epsilon < 1$, and let $\mathcal{L}$ be the line of equation $f = \frac{\epsilon}{2}\, x + \frac{\epsilon}{2}\, y - 1$. Now, $P^\star = (0, 2/\epsilon)$ is an exact simple point of $\mathcal{L}$, and therefore it is an ϵ-point of $\mathcal{L}$. However, for every $\overrightarrow{v} \in \mathbb{N}^2$ one has that

$$\frac{|f^{\overrightarrow{v}}(P^\star)|}{\|f\|} < \epsilon.$$

This phenomenon does not occur in (exact) algebraic geometry, because the total degree of the defining polynomial bounds the multiplicity of every point. However, when working with ϵ-points, it may happen that all the coefficients of the homogeneous form, of maximum degree, of the defining polynomial are smaller than the tolerance (see example above), and hence they are essentially considered as zero. In order to avoid this situation, we introduce the notion of proper degree.

Definition 2 *We say that a polynomial $f \in \mathbb{C}[\underline{x}]$ has* proper degree *d if the total degree of f is d, and there exists $\overrightarrow{v} = (i_1, \ldots, i_n) \in \mathbb{N}^n$, with $i_1 + \cdots + i_n = d$, such that*

$$\frac{\left|f^{\overrightarrow{v}}\right|}{\|f\|} > \epsilon.$$

We say that an algebraic hypersurface has proper degree *d if its defining polynomial has proper degree d.* □

We observe that, if $f(\underline{x})$ has proper degree d and $f_d(\underline{x})$ is its homogeneous form of degree d, then $d!\,\|f_d\| > \epsilon \cdot \|f\|$. Therefore, the proper degree does not depend on the Taylor representation of $f(\underline{x})$.

In the sequel, we always **assume** that the polynomials have proper degree. Moreover, we **assume** that $\mathcal{V}$ is a hypersurface over $\mathbb{C}$ of proper degree $d > 0$, defined by $f \in \mathbb{C}[\underline{x}]$. In this situation, we are ready to introduce the notion of multiplicity of an ϵ-point.

Definition 3 *Let $P^\star \in \mathbb{C}^n$ be an ϵ–point of $\mathcal{V}$. Then, we define the* multiplicity *of $P^\star$ as the smallest natural number $r \in \mathbb{N}$ satisfying that*

1. *for every $\overrightarrow{v} \in \mathbb{N}^n$, such that $0 \leq |\overrightarrow{v}| \leq r-1$, it holds that $\frac{|f^{\overrightarrow{v}}(P^\star)|}{\|f\|} < \epsilon$.*

2. *there exists $\overrightarrow{v} \in \mathbb{N}^n$, with $|\overrightarrow{v}| = r$, such that $\frac{|f^{\overrightarrow{v}}(P^\star)|}{\|f\|} \geq \epsilon$.*

If $r = 1$ we say that $P^\star$ is an ϵ–(affine) simple point *of $\mathcal{V}$. Otherwise, we say that $P^\star$ is an ϵ–*(affine) singularity of multiplicity *r of $\mathcal{V}$.* □

Remark 1 *Note that the multiplicity of an ϵ-point is well defined, and bounded by the proper degree.* □

When dealing with multivariate polynomials, we will use a particular case of ϵ–singularities, that are defined as follows:

Definition 4 *Let $P^\star \in \mathbb{C}^n$ be an ϵ–singularity of multiplicity r of $\mathcal{V}$. We say that $P^\star$ is a k-*pure ϵ–singularity of multiplicity r if

$$\frac{\left|f^{r\cdot\overrightarrow{e_k}}(P^\star)\right|}{\|f\|} \geq \epsilon.$$

Remark 2 *Note that every ϵ-simple point is pure.* □

In Definition 3, the k-order partial derivatives, with $k < r$, are required to be smaller than $\epsilon \|f\|$. The closeness of these values to zero plays an important role in the metric analysis. This fact motivates the next two definitions.

Definition 5 *Let $P^\star \in \mathbb{C}^n$ be an ϵ-point of $\mathcal{V}$ of multiplicity r, and let $\mathcal{D}_{P^\star}$ be the set of all partial derivatives of f, of order strictly smaller than r, non-vanishing at $P^\star$. Then, we define the* depth of $P^\star$ *as*

$$\operatorname{depth}(P^\star) = \begin{cases} \infty & \textit{if} \ \ \mathcal{D}_{P^\star} = \emptyset \\ \min\left\{ \log_\epsilon\left(\dfrac{|g(P^\star)|}{\|f\|}\right) \;\middle|\; g \in \mathcal{D}_{P^\star} \right\} & \textit{if} \ \ \mathcal{D}_{P^\star} \neq \emptyset \end{cases}$$

Remark 3 *We observe that:*

(i) *If $P^\star \in \mathbb{C}^n$ is an ϵ-point of multiplicity r of $\mathcal{V}$, from Definition 5, one has that*

$$\operatorname{depth}(P^\star) \leq \log_\epsilon\left(\frac{|g(P^\star)|}{\|f\|}\right)$$

for every g in $\mathcal{D}_{P^\star}$. Thus, since $0 < \epsilon < 1$, one gets that $\log_\epsilon(x)$ is a decreasing function and then,

$$\frac{|g(P^\star)|}{\|f\|} \leq \epsilon^{\operatorname{depth}(P^\star)}.$$

In addition, there exists $h \in \mathcal{D}_{P^\star}$ such that

$$\frac{|h(P^\star)|}{\|f\|} = \epsilon^{\operatorname{depth}(P^\star)}.$$

Therefore, for every $g \in \mathcal{D}_{P^\star}$, one has that

$$\operatorname{depth}(P^\star) \geq \min\left\{ \log_\epsilon\left(\frac{|g(P^\star)|}{\|f\|}\right)\right\} > \min\{\log_\epsilon(\epsilon)\} > 1.$$

Thus, $\epsilon^{\operatorname{depth}(P^\star)} < \epsilon$.

(ii) *$P^\star$ is an exact singularity of multiplicity r of $\mathcal{V}$ iff $\operatorname{depth}(P^\star) = \infty$. Moreover, the depth measures how close the ϵ–point of multiplicity r is to be an exact singularity of multiplicity r.* □

Definition 6 *Let $P^\star \in \mathbb{C}^n$ be an ϵ-point of $\mathcal{V}$ of multiplicity r, and let $\mathcal{D}_{P^\star,r}$ be the set of all r-order partial derivatives of f which value at $P^\star$ is greater or equal to $\epsilon \cdot \|f\|$. Then, we define the* height of $P^\star$ *as*

$$\operatorname{height}(P^\star) = \max\left\{ \log_\epsilon\left(\frac{|g(P^\star)|}{\|f\|}\right) \;\middle|\; g \in \mathcal{D}_{P^\star,r} \right\}.$$

□

Remark 4 *We observe that*

(i) If $P^\star \in \mathbb{C}^n$ is an ϵ-point of multiplicity r of $\mathcal{V}$, there exists $\overrightarrow{v} \in \mathbb{N}^n$, with $|\overrightarrow{v}| = r$, such that $f^{\overrightarrow{v}} \in \mathcal{D}_{P^\star,r}$. Hence

$$\text{height}(P^\star) \geq \log_\epsilon \left(\frac{\left| f^{\overrightarrow{v}}(P^\star) \right|}{\|f\|} \right),$$

which implies that

$$\frac{\left| f^{\overrightarrow{v}}(P^\star) \right|}{\|f\|} \geq \epsilon^{\text{height}(P^\star)}.$$

In addition, for every $g \in \mathcal{D}_{P^\star}$, one has that

$$\text{height}(P^\star) = \max \left\{ \log_\epsilon \left(\frac{|g(P^\star)|}{\|f\|} \right) \;\middle|\; g \in \mathcal{D}_{P^\star,r} \right\} \leq \max\{\log_\epsilon(\epsilon)\} = 1.$$

(ii) ϵ–points on algebraic hypersurfaces, as well as their multiplicity, depth and height, can be computed applying similar techniques to those used on [26] and [27], for the case of algebraic curves and surfaces, respectively. □

Finally, we introduce the notions of local and global weight, which apply to pure ϵ-singularities. The notion is not so intuitive as the concepts of height and depth, but it can be seen as a mean of the ratio of the pure partial derivatives at the ϵ-point till the order equals the multiplicity. The underline motivation of this concept follows from the algebraic manipulations required in the proofs of Theorem 1 (see Section 3) and Theorem 2 (see Section 4), where the distance bounds are derived.

Definition 7 *Let $P^\star \in \mathbb{C}^n$ be a k–pure ϵ–singularity of $\mathcal{V}$ of multiplicity r. Then, we define the* local weight of *$P^\star$, and we represent it as $\text{weight}_L(P^\star)$, as*

$$\text{weight}_L(P^\star) = \min_{j=1,\ldots,r} \{M_j(P^\star)\},$$

where for $j = 1, \ldots, r$,

$$\begin{cases} M_j(P^\star) = \max_{i=0,\ldots,j-1} \left\{ \left| \dfrac{j! \cdot f^{i \cdot \overrightarrow{e_k}}(P^\star)}{i! \cdot f^{j \cdot \overrightarrow{e_k}}(P^\star)} \right|^{\frac{1}{j-i}} \right\} & \text{if} \quad f^{j \cdot \overrightarrow{e_k}}(P^\star) \neq 0 \\[2ex] M_j(P^\star) = \infty & \text{if} \quad f^{j \cdot \overrightarrow{e_k}}(P^\star) = 0. \end{cases}$$

We define the global weight of *$P^\star$, and we represent it as $\text{weight}_G(P^\star)$ as*

$$\text{weight}_G(P^\star) = M_r(P^\star) = \max_{i=0,\ldots,r-1} \left\{ \left| \frac{r! \cdot f^{i \cdot \overrightarrow{e_k}}(P^\star)}{i! \cdot f^{r \cdot \overrightarrow{e_k}}(P^\star)} \right|^{\frac{1}{r-i}} \right\}.$$

Remark 5 *If* $P^\star \in \mathbb{C}^n$ *is an* ϵ*–simple point of* $\mathcal{V}$*, taking into account Remark 2, it holds* $\text{weight}_G(P^\star) = \text{weight}_L(P^\star) = M_1(P^\star)$. $\square$

The Univariate Case

We finish this section, showing how the preceding notions can be straightforwardly adapted to the univariate case, in terms of their roots. More precisely:

(i) We say that $a^\star \in \mathbb{C}$ is an ϵ–root of a polynomial $h(x) \in \mathbb{C}[x]$ if $\frac{|h(a^\star)|}{\|h\|} < \epsilon$.

(ii) We say that $h(x) = a_d x^d + \cdots + a_0 \in \mathbb{C}[x]$, where $a_d \neq 0$, has proper degree d if $|a_d| d! > \epsilon \cdot \|h\|$.

Now, let $h(x) \in \mathbb{C}[x]$ have proper degree, and let $a^\star \in \mathbb{C}$ be an ϵ-root of $h(x)$. Then:

(iii) We define the multiplicity $a^\star$ as the smallest $r \in \mathbb{N}$ such that

$$\frac{\left|h^{(i)}(a^\star)\right|}{\|h\|} < \epsilon \text{ for } 0 \leq i \leq r-1, \text{ and } \frac{\left|h^{(r)}(a^\star)\right|}{\|h\|} \geq \epsilon.$$

If $r = 1$ we say that $a^\star$ is an ϵ–simple root of $h(x)$; otherwise, we say that $a^\star$ is an ϵ–multiple root of $h(x)$.

(iv) Note that in the univariate case, every ϵ–root is pure in the sense of Definition 4.

(v) Let $a^\star$ have multiplicity r, and let $\mathcal{D}_{a^\star}$ be the set of all derivatives of h, of order strictly smaller than r, non-vanishing at $a^\star$. Then, we define the depth of $a^\star$ as

$$\text{depth}(a^\star) = \begin{cases} \infty & \text{if } \mathcal{D}_{a^\star} = \emptyset \\ \min\left\{ \log_\epsilon \left(\frac{|g(a^\star)|}{\|h\|} \right) \;\middle|\; g \in \mathcal{D}_{a^\star} \right\} & \text{if } \mathcal{D}_{a^\star} \neq \emptyset \end{cases}.$$

(vi) Let $a^\star$ have multiplicity r. We define the height of $a^\star$ as

$$\text{height}(a^\star) = \log_\epsilon \left(\frac{\left|h^{(r)}(a^\star)\right|}{\|h\|} \right).$$

(vii) Let $a^\star$ have multiplicity r. We define the local weight of $a^\star$ as $\text{weight}_L(a^\star) = \min_{j=1,\ldots,r} \{M_j(a^\star)\}$, where for $j = 1, \ldots, r$,

$$\begin{cases} M_j(a^\star) = \max_{i=0,\ldots,j-1} \left\{ \left| \frac{j! \cdot h^{(i)}(a^\star)}{i! \cdot h^{(j)}(a^\star)} \right|^{\frac{1}{j-i}} \right\} & \text{if } h^{(j)}(a^\star) \neq 0 \\ M_j(a^\star) = \infty & \text{if } h^{(j)}(a^\star) = 0. \end{cases}$$

We define the **global weight of $a^\star$** as

$$\mathrm{weight}_G(a^\star) = M_r(a^\star) = \max_{i=0,\ldots,r-1}\left\{\left|\frac{r!\cdot h^{(i)}(a^\star)}{i!\cdot h^{(r)}(a^\star)}\right|^{\frac{1}{r-i}}\right\}.$$

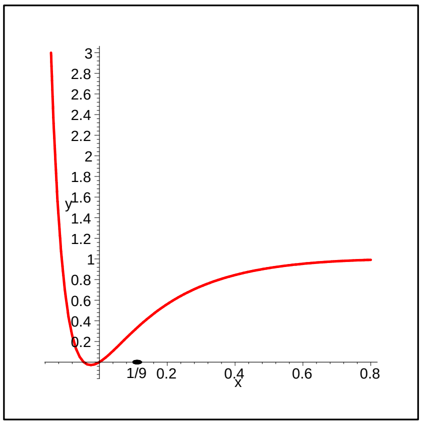
(viii) Note that if $a^\star$ is simple, then $\mathrm{weight}_G(a^\star) = \mathrm{weight}_L(a^\star) = M_1(a^\star)$.

3 Metric Properties of ϵ–Points of Univariate Polynomials

Intuition says that an ϵ–root might be close to a root of a polynomial. In this section we analyze this question, and we see that this assertion holds. Afterwards we extend the results to the general case. We start recalling two lemmas that can be found in [30]. For this purpose, we first introduce the following two rational functions that play an important role in this development:

$$\mathcal{R}_{\mathrm{in}}(x) = 2x\left(\frac{1}{1+3x}+\frac{16x}{(1+3x)^3}\right), \qquad \mathcal{R}_{\mathrm{out}}(x) = \frac{1}{2}-\frac{x(1-9x)}{2(1+3x)}-\frac{32x^2}{(1+3x)^3}.$$

We observe that for $x \in [0, 1/9]$, $\mathcal{R}_{\mathrm{in}}(x)$ and $\mathcal{R}_{\mathrm{out}}(x)$ are increasing and decreasing functions, respectively (see Fig. 1). We also note that in this interval, it holds that



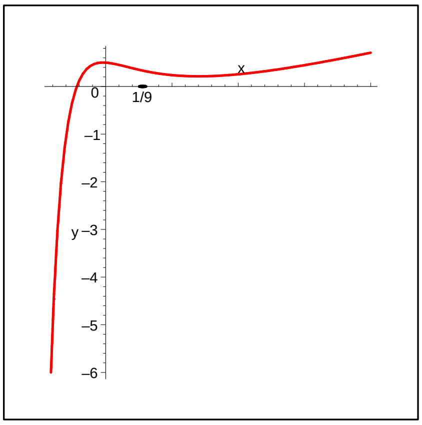


Figure 1: Left: The rational function $\mathcal{R}_{\mathrm{in}}(x)$. Right: The rational function $\mathcal{R}_{\mathrm{out}}(x)$.

$\mathcal{R}_{\mathrm{in}}(x) \le 6x$, and $\mathcal{R}_{\mathrm{out}}(x) \ge \frac{1}{2}-\frac{x}{2}-32x^2$.

The next two lemmas appear in [30].

Lemma 1 *Let*

$$P(x) = c_n x^n + \cdots + c_{m+1}x^{m+1} + x^m + \ell_{m-1}x^{m-1} + \cdots + \ell_0 \in \mathbb{C}[x],$$

where $n \geq m$. *Let*

$$\max\{|c_n|, \ldots, |c_{m+1}|\} \leq 1, \text{ and } \mu = \max\{|\ell_{m-1}|, |\ell_{m-2}|^{1/2}, \ldots, |\ell_0|^{1/m}\} < 1/9.$$

Then, $P(x)$ *has* m *roots inside a disc* D_{in} *of radius* R_{in}*, and* $n-m$ *roots outside a disc* D_{out} *of radius* R_{out}*, both located at the origin, where* $R_{\mathrm{in}} < \mathcal{R}_{\mathrm{in}}(\mu), R_{\mathrm{out}} > \mathcal{R}_{\mathrm{out}}(\mu)$.

Lemma 2 *Let*

$$P(x) = c_n x^n + \cdots + c_{m+1}x^{m+1} + c_m x^m + \ell_{m-1}x^{m-1} + \cdots + \ell_0 \in \mathbb{C}[x],$$

where $n \geq m$. *Let* $\mu = \dfrac{\beta}{\gamma} < 1/9$, *where*

$$\begin{aligned} \beta &= \max\{|\ell_{m-1}/c_m|, |\ell_{m-2}/c_m|^{1/2}, \ldots, |\ell_0/c_m|^{1/m}\}, \\ \gamma &= \max\{|c_{m+1}/c_m|, |c_{m+2}/c_m|^{1/2}, \ldots, |c_n/c_m|^{1/(n-m)}\}. \end{aligned}$$

Then, $P(x)$ *has* m *small roots inside a disc* D_{in} *of radius* R_{in}*, and* $n-m$ *roots outside a disc* D_{out} *of radius* R_{out}*, both located at the origin, where* $R_{\mathrm{in}} < \mathcal{R}_{\mathrm{in}}(\mu), R_{\mathrm{out}} > \mathcal{R}_{\mathrm{out}}(\mu)$.

In the sequel, in order to apply Lemmas 1 and 2 to our analysis, whenever we consider an ϵ-root $a^\star$ of multiplicity r of a univariate polynomial $h(x) \in \mathbb{C}[x]$ of degree d, we **assume** that ϵ is taken such that

$$\epsilon^{\mathrm{depth}(a^\star) - \mathrm{height}(a^\star)} < \frac{1}{9^d \cdot d!}.$$

Observe that this means that

$$\frac{\max_{i=0,\ldots,r-1}\left\{\left|h^{(i)}(a^\star)\right|\right\}}{\left|h^{(r)}(a^\star)\right|} < \frac{1}{9^d \cdot d!}.$$

Remark 6 *We observe that*

1. *Taking into account Remarks 3 and 4, one has that* $\mathrm{depth}(a^\star) - \mathrm{height}(a^\star) > 0$.
2. $\epsilon^{\mathrm{depth}(a^\star) - \mathrm{height}(a^\star)}$ *decreases when the exponent increases. Moreover, if* $\mathrm{depth}(a^\star)$ *increases, then the derivatives till order* $r-1$*, evaluated at* $a^\star$*, tend to zero, and if* $\mathrm{height}(a^\star)$ *decreases, the* r*–order derivative, evaluated at* $a^\star$*, increases its distance to zero.* □

In this situation, we are ready to start the analysis of distance properties of ϵ–roots. In the following results we **assume** that $h(x) \in \mathbb{C}[x]$ has proper degree $d > 0$.

Lemma 3 *Let $a^\star \in \mathbb{C}$ be an ϵ-root of multiplicity r of $h(x)$. It holds that*

$$\text{weight}_L(a^\star) \leq \text{weight}_G(a^\star) \leq \left(r! \cdot \epsilon^{\text{depth}(a^\star)-\text{height}(a^\star)}\right)^{1/r} < \frac{1}{9}.$$

Proof. First of all, note that $\text{weight}_L(a^\star) \leq \text{weight}_G(a^\star)$, and that (see Section 2):

$$|h^{(i)}(a^\star)| \leq \epsilon^{\text{depth}(a^\star)} \cdot \|h\|, \text{ for } i = 0, \ldots, r-1, \text{ and } |h^{(r)}(a^\star)| = \epsilon^{\text{height}(a^\star)} \cdot \|h\|.$$

Therefore,

$$\frac{|r! \cdot h^{(i)}(a^\star)|}{|i! \cdot h^{(r)}(a^\star)|} \leq \frac{r! \cdot \epsilon^{\text{depth}(a^\star)} \cdot \|h\|}{i! \cdot \epsilon^{\text{height}(a^\star)} \cdot \|h\|} \leq r! \cdot \epsilon^{\text{depth}(a^\star)-\text{height}(a^\star)}, \qquad i = 0, \ldots, r-1.$$

Moreover, since we have assumed that $\epsilon^{\text{depth}(a^\star)-\text{height}(a^\star)} < \dfrac{1}{9^d \cdot d!}$, we deduce that

$$r! \cdot \epsilon^{\text{depth}(a^\star)-\text{height}(a^\star)} < \frac{1}{9^d} < 1,$$

from where the results follows. □

Remark 7 *Taking into account Lemma 3 one has that*

(i) $\mathcal{R}_{\text{in}}(\text{weight}_L(a^\star)) \leq \mathcal{R}_{\text{in}}(\text{weight}_G(a^\star)) \leq$

$$\mathcal{R}_{\text{in}}\left(\left(r! \cdot \epsilon^{\text{depth}(a^\star)-\text{height}(a^\star)}\right)^{1/r}\right) \leq 6\left(r! \cdot \epsilon^{\text{depth}(a^\star)-\text{height}(a^\star)}\right)^{1/r}.$$

(ii) $\mathcal{R}_{\text{out}}(\text{weight}_L(a^\star)) \geq \mathcal{R}_{\text{out}}(\text{weight}_G(a^\star)) \geq \mathcal{R}_{\text{out}}\left(\left(r! \cdot \epsilon^{\text{depth}(a^\star)-\text{height}(a^\star)}\right)^{\frac{1}{r}}\right) \geq$

$$\frac{1 - \left(r!\epsilon^{\text{depth}(a^\star)-\text{height}(a^\star)}\right)^{\frac{1}{r}}}{2} - 32\left(r! \cdot \epsilon^{\text{depth}(a^\star)-\text{height}(a^\star)}\right)^{\frac{2}{r}}.$$

□

In these conditions, and using the terminology introduced in the subsection on univariate polynomials of Section 2, we present the following theorem, where distance bounds for ϵ–roots are given.

Theorem 1 *Let $a^\star \in \mathbb{C}$ be an ϵ-root of multiplicity r of $h(x)$, and let $s = \min\{j \in \{1,\dots,r\} \mid \mathrm{weight}_L(a^\star) = M_j(a^\star)\}$ (see item (vii) in subsection on univariate polynomials of Section 2 for the definition on $M_j(a^\star)$). Then, it holds that:*

1. *There exist r roots $a_1,\dots,a_r \in \mathbb{C}$ of $h(x)$ satisfying that:*
$$|a_j - a^\star| < \mathcal{R}_{\mathrm{in}}(\mathrm{weight}_G(a^\star)).$$
2. *There exist $d-r$ roots $b_1,\dots,b_{d-r} \in \mathbb{C}$ of $h(x)$ satisfying that:*
$$|b_j - a^\star| > \mathcal{R}_{\mathrm{out}}(\mathrm{weight}_G(a^\star)).$$
3. *There exist s roots $a_1,\dots,a_s \in \mathbb{C}$ of $h(x)$ satisfying that:*
$$|a_j - a^\star| < \mathcal{R}_{\mathrm{in}}(\mathrm{weight}_L(a^\star)) \leq \mathcal{R}_{\mathrm{in}}(\mathrm{weight}_G(a^\star)).$$
4. *There exist $d-s$ roots $b_1,\dots,b_{d-s} \in \mathbb{C}$ of $h(x)$ satisfying that:*
$$|b_j - a^\star| > \mathcal{R}_{\mathrm{out}}(\mathrm{weight}_L(a^\star)) \geq \mathcal{R}_{\mathrm{out}}(\mathrm{weight}_G(a^\star)).$$

Proof. Let us prove Statements 1 and 2. For this purpose, let $g(x)$ be the polynomial
$$g(x) = h(x+a^\star) = \sum_{i=0}^{d} \frac{h^{(i)}(a^\star)}{i!} x^i = \sum_{i=r}^{d} \frac{h^{(i)}(a^\star)}{i!} x^i + \sum_{i=0}^{r-1} \frac{h^{i)}(a^\star)}{i!} x^i,$$
and let
$$q(x) = \sum_{i=r}^{d} \frac{h^{(i)}(a^\star)}{i!} x^i, \quad \text{and} \quad \delta = \frac{h^{(r)}(a^\star)}{r!}.$$
Note that, since $a^\star$ has multiplicity r then $|\delta| > \frac{\epsilon\|h\|}{r!} > 0$. Now, we distinguish two different cases depending on either $\|q\| = |\delta|$ or $\|q\| \neq |\delta|$.

(a.) Let us assume that $\|q\| = |\delta|$. Then, we consider the polynomial $P(x) := \dfrac{g(x)}{\delta}$, and let us write it as
$$P(x) = c_d x^d + \cdots + c_{r+1}x^{r+1} + x^r + \ell_{r-1}x^{r-1} + \cdots + \ell_0,$$
where
$$c_i = \frac{h^{(i)}(a^\star)}{i!\delta}, \;\; i = r+1,\dots,d, \;\text{ and }\; \ell_i = \frac{h^{(i)}(a^\star)}{i!\delta}, \;\; i = 0,\dots,r-1.$$
Observe that $r \leq d$, because the polynomial $h(x)$ has proper degree d. In these conditions, the quantity μ introduced in Lemma 1 is equal to $\mathrm{weight}_G(a^\star)$, and by Lemma 3 it holds that $\mathrm{weight}_G(a^\star) < \frac{1}{9}$. On the other hand, using that $\|q\| = |\delta|$, one also has that $\max\{|c_d|,\dots,|c_{r+1}|\} \leq 1$. Therefore, hypotheses in Lemma 1 are satisfied, and hence one gets that

(a.1.) there exist r roots $x_0^1, \dots, x_0^r \in \mathbb{C}$ of $P(x)$ (and therefore of $g(x)$) such that for $j \in \{1, \dots, r\}$, it holds that

$$|x_0^j| < \mathcal{R}_{\text{in}}(\text{weight}_G(a^\star)).$$

(a.2.) there exist $d - r$ roots $y_0^1, \dots, y_0^{d-r} \in \mathbb{C}$ of $P(x)$ (and therefore of $g(x)$) such that for $j \in \{1, \dots, d - r\}$, it holds that

$$|y_0^j| > \mathcal{R}_{\text{out}}(\text{weight}_G(a^\star)).$$

(b.) Now, we assume that $||q|| \neq |\delta|$. In this case, we express the polynomial $g(x)$ as

$$g(x) = c_d x^d + \cdots + c_{r+1} x^{r+1} + c_r x^r + \ell_{r-1} x^{r-1} + \cdots + \ell_0,$$

where $c_i = \frac{h^{(i)}(a^\star)}{i!}$, $i = r, \dots, d$, and $\ell_i = \frac{h^{(i)}(a^\star)}{i!}$, $i = 0, \dots, r-1$. Now, we compute the quantity $\mu = \beta/\gamma$ of Lemma 2; i.e:

$$\beta = \max\{|\ell_{r-1}/c_r|, |\ell_{r-2}/c_r|^{1/2}, \dots, |\ell_0/c_r|^{1/r}\},$$

and

$$\gamma = \max\{|c_{r+1}/c_r|, |c_{r+2}/c_r|^{1/2}, \dots, |c_d/c_r|^{1/(d-r)}\}.$$

Observe that $\beta = \text{weight}_G(a^\star)$. Moreover, we note that since $||q|| \neq |\delta|$ then there exists $j \in \{r + 1, \dots, d\}$ such that $|c_j| > |c_r|$; note that $\delta = |c_r|$. Therefore,

$$\gamma = \max\{|c_{r+1}/c_r|, |c_{r+2}/c_r|^{1/2}, \dots, |c_d/c_r|^{1/(d-r)}\} > 1.$$

Hence, we deduce that $\mu = \beta/\gamma \leq \beta = \text{weight}_G(a^\star)$. Thus, by Lemma 3, it holds that $\mu < 1/9$, and therefore Lemma 2 can be applied. In this situation, we get that

(b.1.) there exist r roots $x_0^1, \dots, x_0^r \in \mathbb{C}$ of $g(x)$ such that for $j \in \{1, \dots, r\}$, it holds that

$$|x_0^j| < \mathcal{R}_{\text{in}}(\mu) \leq \mathcal{R}_{\text{in}}(\text{weight}_G(a^\star)).$$

(b.2.) there exist $d-r$ roots $y_0^1, \dots, y_0^{d-r} \in \mathbb{C}$ of $g(x)$ such that for $j \in \{1, \dots, d-r\}$, it holds that

$$|y_0^j| > \mathcal{R}_{\text{out}}(\mu) \geq \mathcal{R}_{\text{out}}(\text{weight}_G(a^\star)).$$

Finally since, in cases (a.1) and (b.1) $x_0^j \in \mathbb{C}$ are roots of $g(x)$ satisfying that

$$|x_0^j| < \mathcal{R}_{\text{in}}(\text{weight}_G(a^\star)), \quad j = 1, \dots, r,$$

one has that $a_j = x_0^j + a^\star \in \mathbb{C}$ for $j = 1, \dots, r$, are r roots of $h(x)$, and

$$|a_j - a^\star| = |x_0^j| < \mathcal{R}_{\text{in}}(\text{weight}_G(a^\star)), \quad j = 1, \dots, r.$$

Similarly since, in cases (a.2) and (b.2) $y_0^j \in \mathbb{C}$ are roots of $g(x)$ satisfying that

$$|y_0^j| > \mathcal{R}_{\mathrm{out}}(\mathrm{weight}_G(a^\star)), \quad j = 1, \ldots, d-r,$$

then $b_j = y_0^j + a^\star \in \mathbb{C}$ for $j = 1, \ldots, d-r$, are $d-r$ roots of $h(x)$, and

$$|b_j - a^\star| = |y_0^j| > \mathcal{R}_{\mathrm{out}}(\mathrm{weight}_G(a^\star)), \quad j = 1, \ldots, d-r.$$

Similarly, taking into account Lemma 3 and Remark 7, one gets statements 3 and 4. □

Example 1 *We take ϵ as $\epsilon = 10^{-7}$, and we consider the polynomial*

$$h(x) = x^5 + 0.5x^4 + x^2 + 10^{-7}x + 10^{-21}.$$

Observe that $a^\star = 0$ is an ϵ-root of $h(x)$ of multiplicity $r = 2$. Moreover,

$$\mathrm{depth}(a^\star) = \log_\epsilon(\epsilon) = 1, \qquad \mathrm{height}(a^\star) = \log_\epsilon(2) = -0.043,$$

and it holds that

$$\epsilon^{\mathrm{depth}(a^\star) - \mathrm{height}(a^\star)} = \epsilon < \frac{1}{9^5 \cdot 5!}.$$

In addition, since $\mathrm{weight}_L(a^\star) = \min_{j=1,2}\{M_j(a^\star)\}$, *where*

$$M_1(a^\star) = \left|\frac{h(a^\star)}{h^{(1)}(a^\star)}\right|, \quad M_2(a^\star) = \max_{i=0,1}\left\{\left|\frac{2! \cdot h^{(i)}(a^\star)}{i! \cdot h^{(2)}(a^\star)}\right|^{\frac{1}{2-i}}\right\},$$

we deduce that $\mathrm{weight}_L(a^\star) = M_1(a^\star) = \epsilon^2 = 10^{-14}$, *and therefore $s = 1$ (see Theorem 1). Thus, by Theorem 1 (3), one has that there exists a root of $h(x)$, say $a \in \mathbb{C}$, such that*

$$|a| = |a - a^\star| < \mathcal{R}_{in}(\mathrm{weight}_L(a^\star)) = 2\epsilon^2 = 2 \cdot 10^{-14}.$$

In fact, using numerical methods, one sees that $a = -0.10000001 \cdot 10^{-13}$. On the other hand, we also have that

$$\mathrm{weight}_G(a^\star) = M_2(a^\star) = \epsilon = 10^{-7}.$$

Thus, applying Theorem 1 (1), there exist two roots $a_1, a_2 \in \mathbb{C}$ of $h(x)$, satisfying

$$|a_i| = |a_i - a^\star| < \mathcal{R}_{in}(\mathrm{weight}_G(a^\star)) = 2\epsilon = 2 \cdot 10^{-7}.$$

In fact, using numerical methods (see e.g. [14], [24]), one sees that $a_1 = -0.10000001 \cdot 10^{-13}, a_2 = -0.9999999 \cdot 10^{-7}$. □

From Theorem 1, and taking into account that for ϵ–simple roots it holds that $\mathrm{weight}_L(a^\star) = \mathrm{weight}_G(a^\star)$, we deduce the following corollary.

Corollary 1 *Let $a^\star \in \mathbb{C}$ be an ϵ-root of multiplicity r of $h(x)$. Then, it holds that*

1. *There exists a root $a \in \mathbb{C}$ of $h(x)$ such that:*
$$|a - a^\star| < \mathcal{R}_{\text{in}}(\text{weight}_L(a^\star)) = \mathcal{R}_{\text{in}}(\text{weight}_G(a^\star)).$$

2. *There exist $d-1$ roots $b_1, \ldots, b_{d-1} \in \mathbb{C}$ of $h(x)$ such that for $j \in \{1, \ldots, d-1\}$ it holds that:*
$$|b_j - a^\star| > \mathcal{R}_{\text{in}}(\text{weight}_L(a^\star)) = \mathcal{R}_{\text{in}}(\text{weight}_G(a^\star)).$$

Example 2 *Let $\epsilon = 10^{-7}$. We consider the polynomial*

$$h(x) = x^5 + 0.5x^4 + 0.25x^2 + x + 10^{-7}.$$

Observe that $a^\star = 0$ is an ϵ-simple root. Moreover,

$$\text{depth}(a^\star) = \log_\epsilon(\epsilon) = 1, \qquad \text{height}(a^\star) = \log_\epsilon(1),$$

and $\epsilon^{\text{depth}(a^\star)-\text{height}(a^\star)} = \epsilon \leq \frac{1}{9^5 \cdot 5!}$. In these conditions, we have that

$$\text{weight}_L(a^\star) = \text{weight}_G(a^\star) = M_1(a^\star) = \left|\frac{h(a^\star)}{h^{(1)}(a^\star)}\right| = \epsilon = 10^{-7}.$$

Thus, applying Corollary 1, one has that there exists a root of $h(x)$ such that

$$|a| = |a - a^\star| < \mathcal{R}_{in}(\epsilon) = 2.0000026 \cdot 10^{-7}.$$

In fact, using numerical methods one obtains that $a = -0.1000000025 \cdot 10^{-6}$. □

4 Metric Properties of ϵ–Points on Hypersurfaces

In Section 3, we have seen that ϵ–roots of univariate polynomials are complex numbers close to the roots of the given polynomial. In this section, we focus on the general case of arbitrary hypersurfaces. In order to approach the general situation, we will reduce it to the univariate case. More precisely, when deriving distance bounds for ϵ–singularities, we will intersect the hypersurface with a line passing through the ϵ–singularity. Moreover, these lines will be taken parallel to one of the axes in $\mathbb{C}^n$. This is the reason why pure ϵ–singularities play an important role.

In our analysis Lemmas 1 and 2 will be applied. For this purpose, similarly as we did for the univariate case, in the sequel, whenever we consider an ϵ-singularity $P^\star$ of a polynomial $f(\underline{x})$ of degree d, we **assume** that ϵ is taken such that

$$\epsilon^{\text{depth}(P^\star)-\text{height}(P^\star)} < \frac{1}{9^d \cdot d!}.$$

Note that, by Remarks 3 and 4, one deduces that $\mathrm{depth}(P^\star) - \mathrm{height}(P^\star) > 0$. Also, throughout this section, we **assume** that $\mathcal{V}$ is an algebraic hypersurface of proper degree $d > 0$ over $\mathbb{C}$ defined by $f(\underline{x}) \in \mathbb{C}[\underline{x}]$.

In this situation, we analyze separately the case of pure ϵ-singularities, non-pure ϵ-singularities, and the case of ϵ-simple points.

Case of Pure ϵ–Singularities

We start with the following lemma that generalizes Lemma 3.

Lemma 4 *Let $P^\star \in \mathbb{C}^n$ be a k–pure ϵ–singularity of $\mathcal{V}$ of multiplicity r. Then it holds that*

$$\mathrm{weight}_L(P^\star) \leq \mathrm{weight}_G(P^\star) \leq \left(r! \cdot \epsilon^{\mathrm{depth}(P^\star)-\mathrm{height}(P^\star)}\right)^{1/r} < \frac{1}{9}.$$

Proof. Let $\alpha := \mathrm{depth}(P^\star)$, and $\beta := \mathrm{height}(P^\star)$. Since $\mathrm{weight}_L(P^\star) \leq \mathrm{weight}_G(P^\star)$, we proceed to prove the two last inequalities. For this purpose, we assume w.l.o.g that $|f^{r\cdot\overrightarrow{e_1}}(P^\star)| \geq \epsilon \cdot \|f(\underline{x})\|$. Now, by the results presented in Section 2, for $i \in \{0, \ldots, r-1\}$ it holds that $|f^{i\cdot\overrightarrow{e_1}}(P^\star)| \leq \epsilon^\alpha \cdot \|f\|$, and that $|f^{r\cdot\overrightarrow{e_1}}(P^\star)| \geq \epsilon^\beta \cdot \|f\|$. Therefore,

$$\frac{|r! \cdot f^{i\cdot\overrightarrow{e_1}}(P^\star)|}{|i! \cdot f^{r\cdot\overrightarrow{e_1}}(P^\star)|} \leq \frac{r! \cdot \epsilon^\alpha \cdot \|f\|}{i! \cdot \epsilon^\beta \cdot \|f\|} \leq r! \cdot \epsilon^{\alpha-\beta}, \quad i = 0, \ldots, r-1.$$

Moreover, since $\epsilon^{\alpha-\beta} < 1/(9^d \cdot d!)$, we deduce that $r! \cdot \epsilon^{\alpha-\beta} < 1/9^d < 1$, which implies that $\mathrm{weight}_G(P^\star) \leq (r! \cdot \epsilon^{\alpha-\beta})^{1/r}$. In addition note that, since $\epsilon^{\alpha-\beta} < 1/(9^d \cdot d!)$, one gets that

$$\mathrm{weight}_G(P^\star) \leq \left(r! \cdot \epsilon^{\alpha-\beta}\right)^{1/r} < \frac{1}{9^{d/r}} \leq \frac{1}{9}.$$

□

Remark 8 *Note that by Lemma 4, one has that*

(i) $\mathcal{R}_{\mathrm{in}}(\mathrm{weight}_L(P^\star)) \leq \mathcal{R}_{\mathrm{in}}(\mathrm{weight}_G(P^\star)) \leq \mathcal{R}_{\mathrm{in}}((r! \cdot \epsilon^{\mathrm{depth}(P^\star)-\mathrm{height}(P^\star)})^{1/r}) \leq 6(r! \cdot \epsilon^{\mathrm{depth}(P^\star)-\mathrm{height}(P^\star)})^{1/r}$.

(ii) $\mathcal{R}_{\mathrm{out}}(\mathrm{weight}_L(P^\star)) \geq \mathcal{R}_{\mathrm{out}}(\mathrm{weight}_G(P^\star)) \geq \mathcal{R}_{\mathrm{out}}\left(\left(r! \cdot \epsilon^{\mathrm{depth}(P^\star)-\mathrm{height}(P^\star)}\right)^{\frac{1}{r}}\right) \geq \frac{1-\left(r!\epsilon^{\mathrm{depth}(P^\star)-\mathrm{height}(P^\star)}\right)^{\frac{1}{r}}}{2} - 32\left(r!\epsilon^{\mathrm{depth}(P^\star)-\mathrm{height}(P^\star)}\right)^{\frac{2}{r}}$. □

The following theorem generalizes Theorem 1 for the case of pure ϵ–singularities.

Theorem 2 *Let $P^\star \in \mathbb{C}^n$ be a pure ϵ–singularity of $\mathcal{V}$ of multiplicity r, and let (see Definition 7 for the notion of $M_j(P^\star)$)*

$$s = \min\{j \in \{1, \ldots, r\} \mid \mathrm{weight}_L(P^\star) = M_j(P^\star)\}.$$

Then, it holds that

1. *There exist at least r points $P_1, \ldots, P_r \in \mathcal{V}$ such that, for $j = 1, \ldots, r$,*

$$\|P^\star - P_j\|_2 < \mathcal{R}_{\text{in}}(\text{weight}_G(P^\star)).$$

2. *There exist at least $d-r$ points $Q_1, \ldots, Q_{d-r} \in \mathcal{V}$ such that, for $j = 1, \ldots, d-r$,*

$$\|P^\star - Q_j\|_2 > \mathcal{R}_{\text{out}}(\text{weight}_G(P^\star)).$$

3. *There exists at least s points $P_1, \ldots, P_s \in \mathcal{V}$ such that, for $j = 1, \ldots, s$,*

$$\|P^\star - P_j\|_2 < \mathcal{R}_{\text{in}}(\text{weight}_L(P^\star)) \leq \mathcal{R}_{\text{in}}(\text{weight}_G(P^\star)).$$

4. *There exists at least $d-s$ points $Q_1, \ldots, Q_{d-s} \in \mathcal{V}$ such that, for $j = 1, \ldots, d-s$,*

$$\|P^\star - Q_j\|_2 > \mathcal{R}_{\text{out}}(\text{weight}_L(P^\star)) \geq \mathcal{R}_{\text{out}}(\text{weight}_G(P^\star)).$$

Proof. First, let us prove Statements (1) and (2). For this purpose, we assume w.l.o.g that $|f^{r\cdot\overrightarrow{e_1}}(P^\star)| \geq \epsilon \cdot \|f(\underline{x})\|$. In these conditions, we express the polynomial $f(\underline{x})$ as

$$f(\underline{x}) = \sum_{i_1+\cdots+i_n=0}^{d} \frac{f^{\overrightarrow{v}}(P^\star)}{i_1! \cdots i_n!}(x_1 - a_1^\star)^{i_1} \cdots (x_n - a_n^\star)^{i_n},$$

where $\overrightarrow{v} = (i_1, \ldots, i_n) \in \mathbb{N}^n$, and $P^\star = (a_1^\star, \ldots, a_n^\star)$. Furthermore, let $g(t)$ be the univariate polynomial

$$g(t) = f(t + a_1^\star, a_2^\star, \ldots, a_n^\star) = \sum_{i=0}^{d} \frac{f^{i\cdot\overrightarrow{e_1}}(P^\star)}{i!} t^i \in \mathbb{C}[t],$$

and let

$$q(t) = \sum_{i=r}^{d} \frac{f^{i\cdot\overrightarrow{e_1}}(P^\star)}{i!} t^i \quad \text{and} \quad \delta = \frac{f^{r\cdot\overrightarrow{e_1}}(P^\star)}{r!}.$$

Note that, $|\delta| > \frac{\epsilon\|f\|}{r!} > 0$. Then, we distinguish two different cases depending on either $\|q\| = |\delta|$ or $\|q\| \neq |\delta|$.

(a.) Let us assume that $\|q\| = |\delta|$. Then, we consider the polynomial $P(t) := \frac{1}{\delta} g(t)$, and let us write it as (note that $r \leq d$; see Remark 1)

$$P(t) = c_d t^d + \cdots + c_{r+1} t^{r+1} + t^r + \ell_{r-1} t^{r-1} + \cdots + \ell_0,$$

where

$$c_i = \frac{f^{i\cdot\overrightarrow{e_1}}(P^\star)}{i!\delta}, \;\; i = r+1, \ldots, d, \quad \text{and} \quad \ell_i = \frac{f^{i\cdot\overrightarrow{e_1}}(P^\star)}{i!\delta}, \;\; i = 0, \ldots, r-1.$$

In these conditions, the quantity μ introduced in Lemma 1 is equal to $\text{weight}_G(P^\star)$, and by Lemma 4 it holds that $\text{weight}_G(P^\star) < \frac{1}{9}$. On the other hand, using that $\|q\| = |\delta|$, one also has that $\max\{|c_d|, \ldots, |c_{r+1}|\} \leq 1$. Therefore, by Lemma 1, one gets that:

(a.1.) there exist r roots $a_1^1, \dots, a_1^r \in \mathbb{C}$ of $P(t)$ (and therefore of $g(t)$) such that for $j \in \{1, \dots, r\}$, it holds that $|a_1^j| < \mathcal{R}_{\text{in}}(\text{weight}_G(P^\star))$.

(a.2.) there exist $d - r$ roots $b_1^1, \dots, b_1^{d-r} \in \mathbb{C}$ of $P(t)$ (and therefore of $g(t)$) such that for $j \in \{1, \dots, d - r\}$, it holds that $|b_1^j| > \mathcal{R}_{\text{out}}(\text{weight}_G(P^\star))$.

(b.) Now, we assume that $||q|| \neq |\delta|$. In this case, we express the polynomial $g(t)$ as

$$g(t) = c_d t^d + \cdots + c_{r+1} t^{r+1} + c_r t^r + \ell_{r-1} t^{r-1} + \cdots + \ell_0,$$

where

$$c_i = \frac{f^{i \cdot \overrightarrow{e_1}}(P^\star)}{i!}, \quad i = r, \dots, d, \quad \text{and} \quad \ell_i = \frac{f^{i \cdot \overrightarrow{e_1}}(P^\star)}{i!}, \quad i = 0, \dots, r - 1.$$

We compute the quantity $\mu = \dfrac{\beta}{\gamma}$ of Lemma 2; i.e:

$$\begin{aligned} \beta &= \max\{|\ell_{r-1}/c_r|, |\ell_{r-2}/c_r|^{1/2}, \dots, |\ell_0/c_r|^{1/r}\}, \\ \gamma &= \max\{|c_{r+1}/c_r|, |c_{r+2}/c_r|^{1/2}, \dots, |c_d/c_r|^{1/(d-r)}\}. \end{aligned}$$

Observe that $\beta = \text{weight}_G(P^\star)$. Moreover, since $\|q\| \neq |\delta|$, there exists $j \in \{r + 1, \dots, d\}$ such that $|c_j| > |c_r|$; note that $\delta = |c_r|$. Therefore, $\gamma > 1$. Hence, we deduce that $\mu = \frac{\beta}{\gamma} \leq \beta = \text{weight}_G(P^\star)$. Thus, by Lemma 3, it holds that $\mu < 1/9$, and therefore Lemma 2 can be applied. Furthermore, one gets that

(b.1.) there exist r roots $a_1^1, \dots, a_1^r \in \mathbb{C}$ of $g(t)$ such that, for $j \in \{1, \dots, r\}$, it holds that $|a_1^j| < \mathcal{R}_{\text{in}}(\mu) \leq \mathcal{R}_{\text{in}}(\text{weight}_G(P^\star))$.

(b.2.) there exist $d-r$ roots $b_1^1, \dots, b_1^{d-r} \in \mathbb{C}$ of $g(t)$ such that, for $j \in \{1, \dots, d-r\}$, it holds that $|b_1^j| > \mathcal{R}_{\text{out}}(\mu) \geq \mathcal{R}_{\text{out}}(\text{weight}_G(P^\star))$.

Finally, since in cases (a.1) and (b.1), $a_1^j \in \mathbb{C}$ are roots of $g(t)$ such that

$$|a_1^j| < \mathcal{R}_{\text{in}}(\text{weight}_G(P^\star)), \quad j = 1, \dots, r,$$

one has that $P_j = (a_1^j + a_1^\star, a_2^\star, \dots, a_n^\star) \in \mathcal{V}$, and

$$\|P^\star - P_j\|_2 = |a_1^j| < \mathcal{R}_{\text{in}}(\text{weight}_G(P^\star)), \quad j = 1, \dots, r.$$

Similarly, since in cases (a.2) and (b.2), $b_1^j \in \mathbb{C}$ are roots of $g(t)$ such that

$$|b_1^j| > \mathcal{R}_{\text{out}}(\text{weight}_G(P^\star)), \quad j = 1, \dots, d - r,$$

then $Q_j = (b_1^j + a_1^\star, a_2^\star, \dots, a_n^\star) \in \mathcal{V}$, and

$$\|P^\star - Q_j\|_2 = |b_1^j| > \mathcal{R}_{\text{out}}(\text{weight}_G(P^\star)), \quad j = 1, \dots, d - r.$$

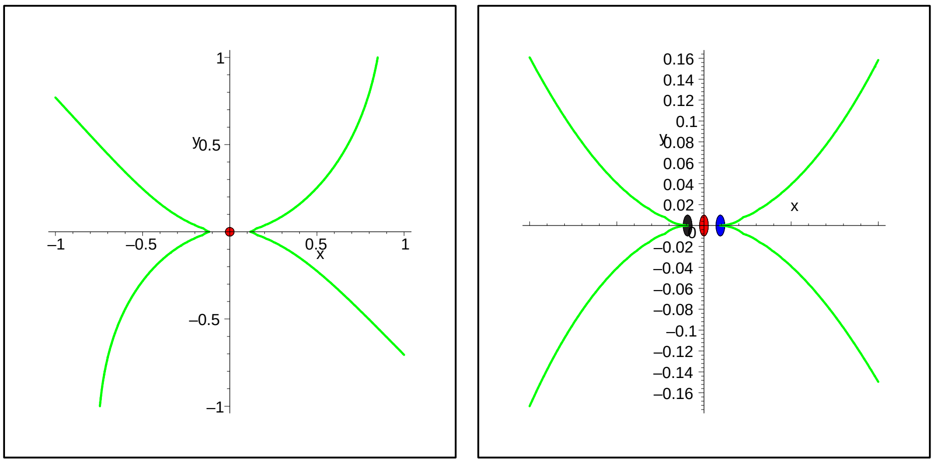


Figure 2: **Left**: The curve $\mathcal{V}$ and the ϵ–point $P^\star$ (red color); **Right**: The ϵ–point $P^\star$ (red color), and the exact points P_1 (blue color) and P_2 (black color) on the curve $\mathcal{V}$

In addition, taking into account that $\mathrm{weight}_G(P^\star) < 1/9$ and Lemma 4, one gets the statements (1) and (2). Similarly, taking into account Lemma 4 and Remark 8, one gets Statements (3) and (4). □

In the following, we illustrate Theorem 2 by means of two examples. The first one deals with a plane curve, and the second with a surface.

Example 3 *We consider the curve $\mathcal{V}$ defined by the polynomial*

$$f(x_1,x_2) = 5x_1^4 - x_2^4 + 5x_2^3x_1 - x_2^2x_1 - 5x_2^2 - 0.00001 + 0.000045x_2 \in \mathbb{C}[x_1,x_2],$$

and let $\epsilon = 0.0001$. Note that $P^\star = (0,0)$ is an ϵ–singularity of multiplicity 2 of $\mathcal{V}$ (see Fig. 2, Left). The actual computation of this type of points can be approach, for instance applying the techniques presented in Section 2 in [26]. Moreover, it holds that $P^\star$ is a 2–pure singularity since $|f^{2\cdot\overrightarrow{e_2}}(P^\star)| \geq \epsilon\cdot\|f(\underline{x})\|$. In addition $\mathrm{depth}(P^\star) = 1.261439373, \mathrm{height}(P^\star) = 0.07525749892$. *Thus*

$$\epsilon^{\mathrm{depth}(P^\star)-\mathrm{height}(P^\star)} < \frac{1}{9^4\cdot 4!}.$$

Now, in order to apply Theorem 2, we compute

$$\mathrm{weight}_L(P^\star) = \mathrm{weight}_G(P^\star) = \max_{i=0,1}\left\{\left|\frac{2!\cdot f^{i\cdot\overrightarrow{e_2}}(P^\star)}{i!\cdot f^{2\cdot\overrightarrow{e_2}}(P^\star)}\right|^{\frac{1}{2-i}}\right\} = 0.001414213562.$$

Thus, by statement (1), one has that there exist at least two exact points $P_1, P_2 \in \mathbb{C}^2$ of $\mathcal{V}$ such that for $j=1,2$, it holds that

$$\|P^\star - P_j\|_2 < \mathcal{R}_{\mathrm{in}}(\mathrm{weight}_G(P^\star)) = 0.002879670098.$$

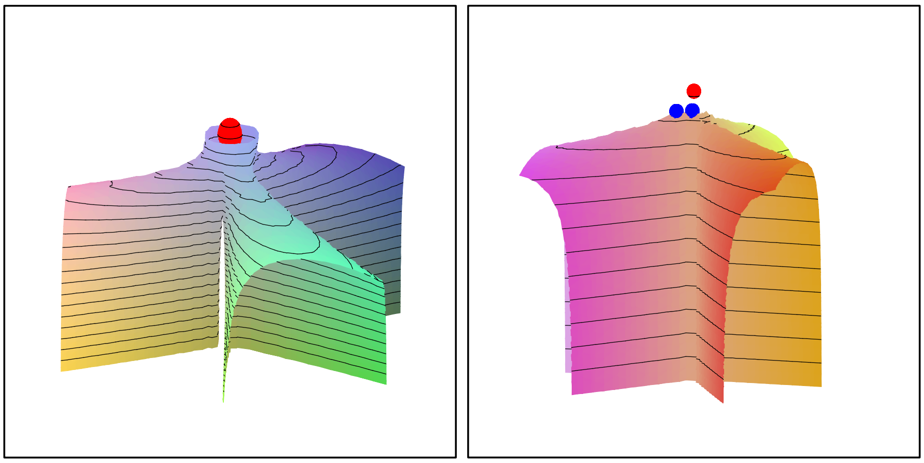


Figure 3: Left: The surface $\mathcal{V}$ and the ϵ–point $P^\star$ (red color); Right: The ϵ–point $P^\star$ (red color), and the exact points P_1 and P_2 (blue color) on the surface $\mathcal{V}$

In fact, applying numerical techniques (see e.g. [6], [13], [14], [17], [19], [29]), one may approximate points in the intersection of $\mathcal{V}$, and the axes $x_2 = 0$ (see Fig. 2, right.) □

Example 4 *We consider the surface $\mathcal{V}$ defined by the polynomial*

$$f(x_1,x_2,x_3) = x_1^4 + 2x_1^2x_2^2 + x_2^4 + 9x_1x_3^2x_2^2 - 3x_1^3x_3^2 +$$

$$0.0001x_1 + 0.0001x_2 + 0.0001x_1^2 + 0.0001x_3^2 - 10^{-7} \in \mathbb{C}[x_1,x_2,x_3],$$

and let $\epsilon = 0.0001$. Note that $P^\star = (0,0,0)$ is an ϵ–singularity of multiplicity 4 of $\mathcal{V}$ (see Fig. 3, left). The actual computation of this type of points can be approach, for instance applying the techniques presented in Section 3 in [27]. Moreover, it holds that $P^\star$ is a 2–pure singularity since $|f^{4\cdot\overrightarrow{e_2}}(P^\star)| \geq \epsilon \cdot \|f(\underline{x})\|$. In addition, $\mathrm{depth}(P^\star) = 1.744954692, \mathrm{height}(P^\star) = 0.01278813061$. *Thus*

$$\epsilon^{\mathrm{depth}(P^\star)-\mathrm{height}(P^\star)} < \frac{1}{9^5 \cdot 5!}.$$

Now, in order to apply Theorem 2,

$$\mathrm{weight}_L(P^\star) = M_1(P^\star) = \left|\frac{f(P^\star)}{f^{\overrightarrow{e_2}}(P^\star)}\right| = 0.001,$$

and

$$\mathrm{weight}_G(P^\star) = M_4(P^\star) = \max_{i=0,1,2,3}\left\{\left|\frac{4!\cdot f^{i\cdot\overrightarrow{e_2}}(P^\star)}{i!\cdot f^{4i\cdot\overrightarrow{e_2}}(P^\star)}\right|^{\frac{1}{2-i}}\right\} = 0.0177827941.$$

Hence, by Statement (1), one has that there exist at least 4 *exact points* $P_1, P_2, P_3, P_4 \in \mathbb{C}^3$ *of* $\mathcal{V}$ *such that for* $j = 1, 2, 3, 4$*, it holds that*

$$\|P^\star - P_j\|_2 < \mathcal{R}_{\text{in}}(\text{weight}_G(P^\star)) = 0.04242264150.$$

Now, applying Statement (3), one has that there exist at least one exact point $P \in \mathbb{C}^3$ *of* $\mathcal{V}$ *such that*

$$\|P^\star - P\|_2 < \mathcal{R}_{\text{in}}(\text{weight}_L(P^\star)) = 0.002025731666.$$

In fact, one may approximate points in the intersection of $\mathcal{V}$*, and the axe* $x_1 = x_3 = 0$ *(see Fig. 3, right.)* □

Case of Non-Pure ϵ-Singularities

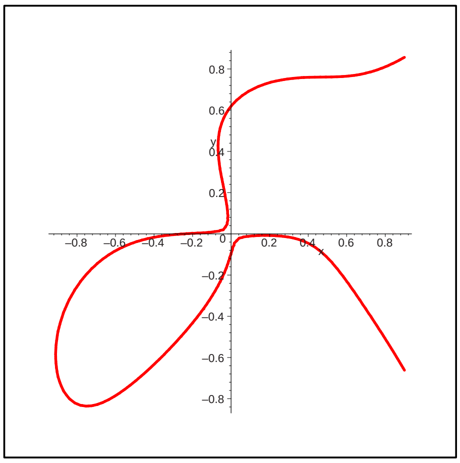
Let $P^\star \in \mathbb{C}^n$ be a non-pure ϵ–singularity of $\mathcal{V}$ of multiplicity r. Then, the strategy consists in performing a linear change of coordinates such that $P^\star$ is transformed into a pure ϵ-singularity of the new hypersurface. For this purpose, we first apply a translation mapping $P^\star$ into the origin, and afterwards we apply an isometry in the Euclidean space $\mathbb{C}^n$ such that the origin is a pure ϵ–singularity. In this way, Euclidean distances are preserved, and therefore bounds in the previous subsection are valid. However, we recall that we work with two different norms, namely the Euclidean and the ∞–norm. The Euclidean norm is used to measure distances between points, while the ∞–norm is involved in the concept of ϵ-point. With the isometry performance, one controls the Euclidean norm, however the ∞–norm may behave improperly under these affine movements, and therefore it might happen that the new point is not anymore an ϵ-singularity of the new hypersurface (see Example 5). In order to avoid this difficulty we proceed as follows. Let $\mathcal{T}$ be the translation mapping $P^\star$ into the origin, and let $\mathcal{O}$ be an isometry guaranteeing such that $\|g\| \geq \|f\|$, where $g(\underline{x}) \in \mathbb{C}[\underline{x}]$ is the defining polynomial of the transformed hypersurface $\mathcal{W}$ of $\mathcal{V}$ by $\mathcal{O} \circ \mathcal{T}$. Then, we get that

$$|g(0)| = |f(P^\star)| \leq \epsilon\|f\| \leq \epsilon\|g\|,$$

and therefore 0 is a pure ϵ–singularity of $\mathcal{W}$. In order to choose the isometry satisfying the above requirement, we take $\mathcal{O}$ generic. Note that the set of isometries satisfying the above condition forms a semi-algebraic set in the variety of isometries. More precisely, we identify set of isometries $\{A := (a_{i,j})_{1\leq i,j\leq n} | A \cdot A^T = I\}$ as the algebraic set Σ of $\mathbb{C}^{2n}$ defined by the equations $\sum_{i=1}^{n} a_{j,i}a_{k,i} = 0$ for $j \neq k$, and $\sum_{j=1}^{n} a_{1,j}^2 = 1$. Now, let $g(\underline{x}) = f(\mathcal{O}(\mathcal{T}(\underline{x})))$, where $\mathcal{O}$ is taken generic. Then, if $b(a_{1,1}, \ldots, a_{n,n})$ is a non-zero coefficient of $g(\underline{x})$, all isometries in $\Sigma \cap \{A \in \Sigma / b(a_{1,1}, \ldots, a_{n,n}) \geq \|f\|\}$ are valid in our process.

Example 5 *We consider the curve* $\mathcal{V}$ *defined by the polynomial*

$$f(x_1, x_2) = -x_1^5 + x_2^5 - x_2^3 + x_1^2x_2 + 0.9x_2^4 + 0.5x_1^2x_2^2 - x_1x_2 + 0.0001x_1 + 0.00001x_2 - 0.00099,$$



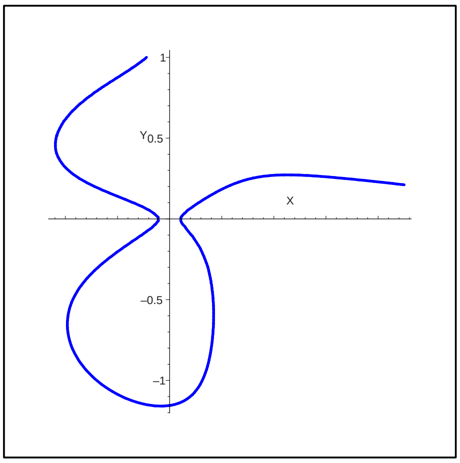


Figure 4: Left: Curve $\mathcal{V}$. Right: Curve $\mathcal{W}$.

and let $\epsilon = 0.001$. *The point* $P^\star = (0,0)$ *is a non–pure* ϵ*–singularity of multiplicity* 2 *of* $\mathcal{V}$ *(see Fig. 4, left). We apply to* $\mathcal{V}$ *a rotation of center* $P^\star$*, and angle* $\pi/4$*. We get the curve* $\mathcal{W}$ *defined by the polynomial (see Fig. 4, right)*

$g(x_1,x_2) = -1.414213562x_1^2x_2 - 3.535533905x_1^3x_2^2 - 1.767766953x_1x_2^4 - 0.00099 + 0.375x_2^4 + 0.375x_1^4 + 0.5x_1^2 - 0.5x_2^2 - x_1^3x_2 + 1.25x_1^2x_2^2 - x_1x_2^3 - 0.3535533905x_1^5 + 0.00006363961029x_1 + 0.00007778174591x_2 + 1.414213562x_1x_2^2.$

It holds that $P^\star$ *is* 1*–pure* ϵ*–singularity of multiplicity* 2 *of* $\mathcal{W}$*. Note that* $\|f\| = 1$*, and* $\|g\| = 3.535533905$*. Therefore, we may apply Theorem 2 (Statement 3) to deduce that there exist at least two points* $P_1, P_2 \in \mathcal{W}$ *such that* $\|P^\star - P_i\|_2 < \mathcal{R}_{\text{in}}(\text{weight}_L(P^\star)) = 0.1220206068$*. Hence, there exist at least two points* $Q_1, Q_2 \in \mathcal{V}$ *such that*

$$\|P^\star - Q_i\|_2 < 0.1220206068.$$

However, if we apply a rotation of center $P^\star$*, and angle* $5\pi/9$*, we get the surface* $\mathcal{W}$ *defined by the polynomial*

$g(x_1,x_2) = -0.9264698986x_2^5 - 0.9261541212x_1^5 + 0.1710100717x_2^2 + 0.5027170459x_2x_1^3 + 0.9254165784x_1^3 + 0.8368240892x_2x_1^2 - 0.8066342291x_1x_2^2 - 0.1631759112x_2^3 + 0.95522409x_1^4 - 0.171010072x_1^2 + 0.587733334x_1^2x_2^2 + 0.01553146731x_2^4 + 0.181323241x_2^3x_1 + 0.9396926208x_2x_1 - 0.8211461571x_2x_1^4 - 0.2372191128x_2^2x_1^3 - 0.3387840032x_2^3x_1^2 + 0.00009674429352x_2 - 0.00002721289530x_1 - 0.00099 + 0.8121918419x_2^4x_1.$

Now, $P^\star$ *is not an* ϵ*–point of* $\mathcal{W}$*. Note that* $\|f\| = 1$*,* $\|g\| = 0.955224088$*, and*

$$|g(P^\star)| = |f(P^\star)| = 0.00099 > \epsilon \cdot \|g\| = 0.000955224088.$$

Case of ϵ–Simple Points

We start observing that ϵ–simple points are always pure (see Remark 2). In addition, by Remark 5, if $P^\star$ is an ϵ-simple point then $\text{weight}_L(P^\star) = \text{weight}_G(P^\star)$. Thus, from Theorem 2, we get the following corollary.

Corollary 2 *Let $P^\star \in \mathbb{C}^n$ be an ϵ–simple point of $\mathcal{V}$. Then, it holds that*

1. *There exist at least one point $P \in \mathcal{V}$ such that:*

$$\|P^\star - P\|_2 < \mathcal{R}_{\text{in}}(\text{weight}_L(P^\star)) = \mathcal{R}_{\text{in}}(\text{weight}_G(P^\star)).$$

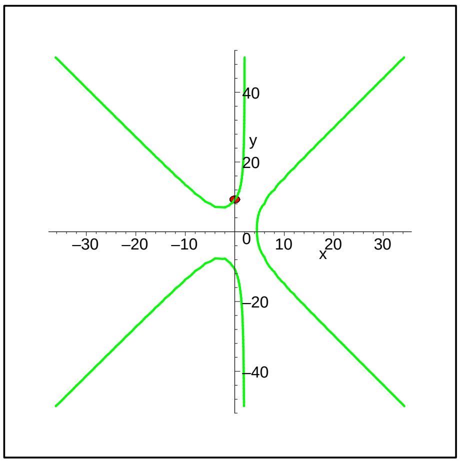
2. *There exist at least $d-1$ points $Q_1, \ldots, Q_{d-1} \in \mathcal{V}$ such that:*

$$\|P^\star - Q_j\|_2 > \mathcal{R}_{\text{out}}(\text{weight}_L(P^\star)) = \mathcal{R}_{\text{out}}(\text{weight}_G(P^\star)).$$

Example 6 *We consider the curve $\mathcal{V}$ defined by the polynomial*

$$f(x_1, x_2) = 2.00009x_1^3 - x_2^2x_1 + 2.000005x_2^2 + 2.9999x_2 - 200 - 0.00001x_1^2,$$

and let the tolerance $\epsilon = 0.0001$. Note that $P^\star = (0, 9.27814782)$ is an ϵ–simple point



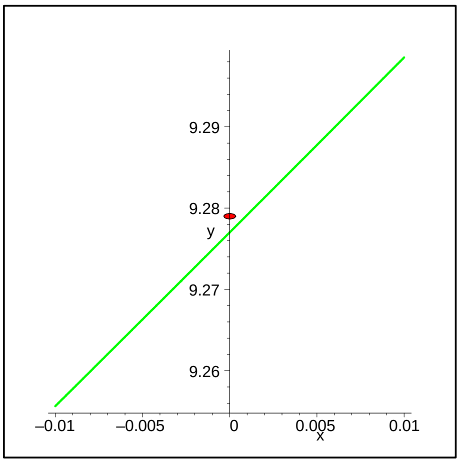


Figure 5: Left: The curve $\mathcal{V}$ and the ϵ–point $P^\star$ (red color). Right: The curve $\mathcal{V}$ plotted in a neighborhood of the ϵ–point $P^\star$ (red color).

of $\mathcal{V}$ (see Fig. 5). Moreover $\text{depth}(P^\star) = 1.874995929, \text{height}(P^\star) = 0.1744373389.$ *Thus*

$$\epsilon^{\text{depth}(P^\star) - \text{height}(P^\star)} < \frac{1}{9^3 \cdot 3!}.$$

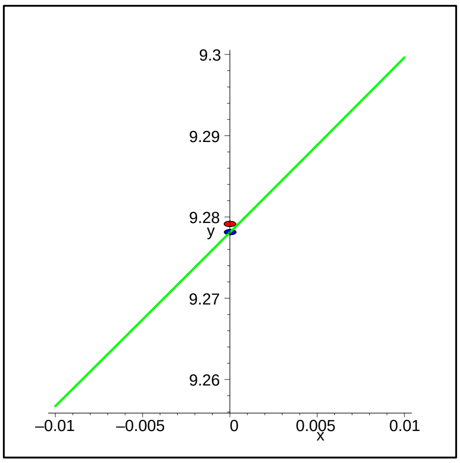


Figure 6: The ϵ–point $P^\star$ (red color), and the exact point P of $\mathcal{V}$ (blue color)

In this case, we get that

$$\mathrm{weight}_L(P^\star) = \mathrm{weight}_G(P^\star) = M_1(P^\star) = \left| \frac{f(P^\star)}{f^{\overrightarrow{e_2}}(P^\star)} \right| = 0.00004985974793.$$

Therefore, by Corollary 2, we deduce that there exists at least one point $P \in \mathcal{V}$ such that

$$\|P^\star - P_j\|_2 < \mathcal{R}_{\mathrm{in}}(\mathrm{weight}_G(P^\star)) = 0.00009978409824.$$

In fact, applying numerical techniques one may approximate points in $\mathcal{V} \cap \{x_1 = 0\}$ (see Fig. 6). □

Experimental Analysis

In the following table we illustrate the results obtained in Theorems 1 (Statement 3) and 2 (Statement 3). The polynomials have been taken randomly but ensuring that the degree is proper and that $P^\star = (0, \ldots, 0) \in \mathbb{R}^\kappa$ is an ϵ-point of the hypersurface defined by the polynomial. The polynomials used in the table appear in the Appendix. Important properties of the polynomials, that might affect to the experiment, vary in the inputs. For instance, different value for the ∞-norm of the polynomials have been considered (see column 6 in Table 1), the degree varies from 3 to 30 (see column 2), the order of multiplicity of the ϵ-point varies from 1 to 20 (see column 3), and the dimension of the hypersurface (i.e. the number of variables of the polynomial) varies from 1 to 7 (see column 4). Also, in column 5, we have written the number of points of the hypersurface lying within the disk centered at P^* and radius the bound (see Statement 3 in Theorems 1 and 2). In addition, we have considered a fixed tolerance $\epsilon = 10^{-3}$. We have repeated the experiment taking the same polynomials and different values of ϵ, and no significant difference has been detected.

Input	d	r	κ	s	$\|\|f\|\|$	B	ℓ
I	3	2	5	2	1	0.02288854288	0.01022113068
II	5	4	5	1	1	0.03317076276	0.01424281458
III	7	6	2	6	1	0.5746730482	0.2040285094
IV	9	6	7	6	8	0.7639038282	0.3216218617
V	28	10	7	10	32	0.9149384356	0.4729876659
VI	9	5	3	5	1	0.6085878356	0.2149485609
VII	9	5	6	5	7	0.6028871304	0.2182010583
VIII	8	6	5	6	5	0.5419193264	0.1779080163
IX	9	3	7	3	8	0.1167031511	0.04272819426
X	20	12	6	12	32	0.9726209592	0.5785706591
XI	3	1	1	1	1	0.00008703593936	0.00004349343199
XII	5	3	1	3	6	0.05935119610	0.02376382352
XIII	10	7	1	1	1	0.3244798282	0.1103038339
XIV	10	7	1	7	20	0.5878797718	0.2083571007
XV	15	9	1	2	1	0.6892482506	0.2517971845
XVI	20	1	1	1	18	0.00005517544520	0.00002757786468
XVII	20	10	1	10	1	0.7708109636	0.3128446102
XVIII	20	10	1	2	40	0.5390290592	0.1896647505
XIX	30	5	1	5	1	0.5423311092	0.1905043473
XX	30	20	1	1	1	0.9143898198	0.5323385708

Table 1: d =degree of the polynomial; r =multiplicity of P^*; κ =dimension of the hypersurface; $s = \min\{j \in \{1,\ldots,r\} \mid \mathrm{weight}_L(P^\star) = M_j(P^\star)\}$ (see Theorem 1 or 2); $B = \mathcal{R}_{\mathrm{in}}(\mathrm{weight}_L(P^\star))$ (see Theorem 1 (3) and 2 (3); ℓ =approximation of the module of the closest exact point.

In the above table, one notes that the degree, dimension, multiplicity, and ∞-norm do not seem to affect significantly to the bound. Moreover, observing the column seven and eight one sees that the ratio ℓ/B is essentially 0.5, which is consistent with the numerical test of the sharpness of the bound formula provided in [30] by means of the Smith's disk (see Section 3 in [30]). This is not surprising, since the ultimate corner stone in our reasoning is Sasaki-Terui's bound. Another interesting phenomenon, remarkable from the experiment, is that when the number s of close points to P^* was not 1, all of them are very close together, and therefore it corresponds to the intuitive idea that P^* explodes in different points when the multiplicity is not 1.

5 Application to Error Analysis of Geometric Approximate Algorithms

In the error analysis of approximate algorithms one estimates the "closeness" of the input and the output. The precise notion of "closeness" depends on the problem that

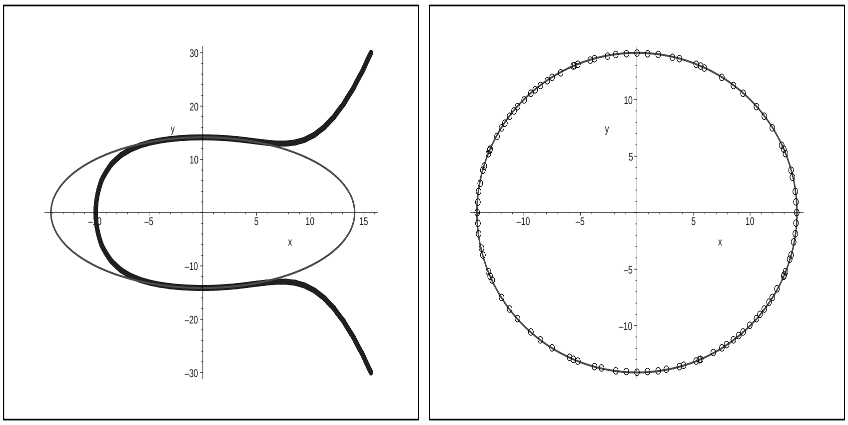


Figure 7: Left: $\mathcal{C}_1$ and $\mathcal{C}_2$; Right: $\mathcal{C}_1$ (in continuous trace) and $\mathcal{C}_3$ (in dot trace)

one is dealing with. For instance, when computing gcds, resultants or factorizations the "closeness" is measured in terms of relative errors of the polynomials. Nevertheless, it may happen that even though the relative errors are small the algebraic varieties defined by the polynomials are not close in terms of the Euclidean distance. For example, let us consider three real plane algebraic curves $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ given, respectively, be the polynomials

$$\begin{aligned} f_1(x,y) &= x^2 + y^2 - 200, \\ f_2(x,y) &= 0.9995x^2 + y^2 - 200 - 0.001x^5, \\ f_3(x,y) &= 0.9995x^2 + 0.999y^2 - 200.0005. \end{aligned}$$

The relative errors of the polynomials are small:

$$\frac{\|f_1 - f_2\|_\infty}{\|f_2\|_\infty} = 0.0000049999875, \quad \frac{\|f_1 - f_3\|_\infty}{\|f_3\|_\infty} = 0.0000049999875.$$

However, when plotting the curves (see Fig. 7) one realizes that $\mathcal{C}_1$, $\mathcal{C}_2$ are not close, but $\mathcal{C}_1$, $\mathcal{C}_3$ are close. In order to deal with this problem, the error is measured in terms of offsets.

Offsets to hypersurfaces play an important role in many practical applications in computer aided geometric design, and have been extensively studied both from the theoretical and algorithmic point of view. Let $\mathcal{V}$ be a hypersurface in $\mathbb{C}^n$, then the offset to $\mathcal{V}$ at distance $d \in \mathbb{C}$ is essentially the envelope of all the spheres centered at the points of $\mathcal{V}$ with fixed radius d; i.e. the envelope of the spheres $\sum_{i=1}^n (x_i - y_i)^2 = d^2$ where $\underline{x} \in \mathcal{V}$ (for a formal definition of offset we refer to [1]). The offset to $\mathcal{V}$ is again a hypersurface with at most two algebraic components that correspond to the intuitive idea of internal an external offset. The region between these two parts of the offset (i.e. the external and internal analytic components) is called the offset region. More formally, the offset region to $\mathcal{V}$ is defined as the union of the sets $\{\bar{y} \in \mathbb{C}^n \mid \sum_{i=1}^n (x_i - y_i)^2 \leq d^2\}$, where $\underline{x} \in \mathcal{V}$. Similarly, one introduces the offset region to a subset of $\mathcal{V}$.

In this situation, and coming back to the discussion on error analysis, the notion of "closeness" between two hypersurfaces is introduced by requiring that each hypersurface lies in the offset region to the other at an small distance that depends on the tolerance. The metric properties on ϵ-point developed in the previous sections can be applied to this problem. More precisely, the next corollary shows how two hypersurfaces are locally related, in terms of their offsets when ϵ–points appear.

Corollary 3 *Let $\mathcal{V}$ and $\mathcal{V}^\star$ be two algebraic hypersurfaces in $\mathbb{C}^n$ of proper degree, and let $Q \in \mathcal{V}$. If Q is an ϵ–simple point of $\mathcal{V}^\star$, then in an Euclidean neighborhood of Q, $\mathcal{V}$ is contained in the offset region of $\mathcal{V}^\star$ at distance d where*

$$d \leq n \cdot \mathcal{R}_{\text{in}}(\text{weight}_G(Q)) \leq n \cdot \mathcal{R}_{\text{in}}(\epsilon^{\text{depth}(Q)-\text{height}(Q)}).$$

Proof. Let $Q = (a_1^\star, \ldots, a_n^\star)$. Since Q is an ϵ–simple point of $\mathcal{V}^\star$, by Corollary 2, one deduces that there exists $P = (a_1, \ldots, a_n) \in \mathcal{V}^\star$ such that

$$\|Q - P\|_2 \leq \mathcal{R}_{\text{in}}(\text{weight}_G(Q)) \leq \mathcal{R}_{\text{in}}(\epsilon^{\text{depth}(Q)-\text{height}(Q)}).$$

In this situation, we consider the tangent hyperplane to $\mathcal{V}^\star$ at P; i.e $T^\star(x_1, \ldots, x_n) = n_{x_1}(x_1 - a_1) + \cdots + n_{x_n}(x_n - a_n)$, where $(n_{x_1}, \ldots, n_{x_n})$ is the unitary normal vector to $\mathcal{V}^\star$ at P. Then, we bound the value $\|T^\star(Q)\|_2$ by

$$\|T^\star(Q)\|_2 \leq |n_{x_1}| \cdot |a_1 - a_1^\star| + \cdots + |n_{x_n}| \cdot |a_n - a_n^\star| \leq$$

$$\|Q - P\|_2(|n_{x_1}| + \cdots + |n_{x_n}|) \leq n \cdot \mathcal{R}_{\text{in}}(\text{weight}_G(Q)) \leq n \cdot \mathcal{R}_{\text{in}}(\epsilon^{\text{depth}(Q)-\text{height}(Q)}).$$

Therefore, reasoning as in Subsection 2.2 of [13] one deduces that, $\mathcal{V}$ is contained in the offset region of $\mathcal{V}^\star$ at distance at most

$$n \cdot \mathcal{R}_{\text{in}}(\text{weight}_G(Q)) \leq n \cdot \mathcal{R}_{\text{in}}(\epsilon^{\text{depth}(Q)-\text{height}(Q)}).$$

□

6 Conclusions

Given an ϵ-point P of a hypersurface $\mathcal{V}$, we have proved the existence of points on $\mathcal{V}$ at small distance of P, and bounds for this distance have been presented. These bounds, beside the obvious extension to hypersurfaces, generalize and improve those bounds given in [28] by means of the notions of height, depth and weight of an ϵ-point. Also we have seen how to apply these results to estimate the offset-region-measure when analyzing the error in approximate geometric algorithms. In addition, examples and experimental analysis show that the bounds are quite satisfactory.

Nevertheless, there are open problems to address. For instance, the distance analysis for non-pure ϵ-singularities is based on the application of a generic isometry. This isometry has to be taken such that the ∞-norm is controlled. The problem on how linear changes of coordinates transform ϵ-point needs a deeper study. Also, although the implementation of our method is quite efficient, an asymptotic complexity analysis is still required. Finally, one can also mention as an open problem the corresponding analysis of these concepts and bounds for the case of algebraic varieties of arbitrary dimension, as for instance space curves.

7 Appendix

INPUT I: $f(x_1,\ldots,x_5) = x_1^2 + 0.00002742221911x_1 + 0.00001619118552x_2 + 0.00001619118552x_3 + 0.00001619118552x_4 + 0.00001619118552x_5 + 0.0001044795612 + 0.5x_1x_2 + x_1^3 + x_2x_3^2$,

INPUT II: $f(x_1,\ldots,x_5) = x_1^4 + 0.0008547008547x_1 + 0.004273504274x_2 + 0.0008547008547x_3x_2x_1 + 0.0008547008547x_4^3 + 0.0008547008547x_5 + 0.00001179968849x_1^2x_5 - 0.0000221508318 + 0.00001176373710x_2^2x_4 + 0.5x_1x_2 + x_1^5 + x_2x_1x_3x_4x_5 + x_2x_3^4x_1^4 + 0.0008547008547x_1 - 0.00001221508318 + x_1^5$,

INPUT III: $f(x_1,x_2) = x_1^6 + 0.0001431050212x_1^5 + 0.00008944063825x_2 + 0.00001788812765x_2^3x_1 + 0.0000337397564x_2^3 + 0.00001788812765x_1 + 0.00001733462765x_1^2x_2 - 0.00007583513442 + 0.5x_1x_2 + x_2^2x_1^2 + x_2^3x_1^4$,

INPUT IV: $f(x_1,\ldots,x_7) = x_1^6 + 0.00009134401297x_1^5 + 0.00005709000811x_2 + 0.00001141800162x_5^2x_2x_7 + 0.00001141800162x_6^3 + 0.00001141800162x_1 + 0.00001141800162x_7^4 + 0.00005709000811x_6x_5^3 + 0.00001312611572x_1^2x_3 + 0.00001083963839x_1x_2x_3 + 5x_1x_2 - 6x_5x_1^2x_6 + x_2^3x_1^4x_6^2 + x_7^9 + 8x_6x_3^2x_7 + x_1x_3^5x_5 + 6x_3^9 + 0.001109717799 - x_2^3 + 5x_6x_1x_7^3x_1^6 + 0.00009134401297x_1^5 + 0.001109717799 + 0.00001141800162x_1$,

INPUT V: $f(x_1,\ldots,x_7) = 0.00003371885221x_1 + 0.0001685942611x_2 + 5x_1x_2 - x_2^3 + 0.00003371885221x_2^6x_3x_7 + 0.00009991008093x_1^2x_3 + 5.x_5x_6x_7 - 6x_7^2x_3x_1^6 + x_1^{10} + 0.00003371885221x_1^9 + x_6^{15} + 9x_1^{13} + 0.0002697508177x_1^5 + x_2^3x_1^{10}x_6^{15} + x_4x_3x_2^9 + x_4^{12} + 0.001093290476 + x_1x_3^5x_5 + 0.00004120822516x_1x_2x_3 + 0.00003371885221x_6^3 + 0.00003371885221x_7^4 + x_7^9 + 6x_3^9 + 0.0001685942611x_6x_5^3 + 5x_6x_1x_7^3 + 32x_6x_3^2x_7 - 6x_5x_1^2x_6 + 0.00003371885221x_5^2x_2x_7 + 0.00003371885221x_1 + 0.001093290476 + x_1^{10} + 0.00003371885221x_1^9 + 9x_1^{13} + 0.0002697508177x_1^5$,

INPUT VI: $f(x_1,x_2,x_3) = x_1^5 + 0.0003985662386x_1 + 0.00001111951252x_2 + 0.00001111951252x_1^2x_2^3x_3 + 0.00001111951252x_2^9 + 0.00001111951252x_3x_2^6 + 0.00001111951252x_1^3 + 0.00002223902504x_1^2x_3 + 0.0000555975626x_1^4x_2^3 - 0.0005456132693 + x_1^2x_2^3x_3^3 + 0.5x_1x_2^8 + x_1^9 + x_2^8x_3$, INPUT VII: $f(x_1,\ldots,x_6) = x_1^9 + 0.00006920255588x_1 + 0.0001153375931x_2 + 0.00002306751863x_3x_2x_1 - 0.00002306751863x_2x_4^3 - 0.00009227007451x_5 + 0.00001561938678x_1^2x_5 + 0.0005108556833 + 0.00001529145513x_2^2x_4 + 0.5x_1x_2 + x_1^5 + x_2x_1^2x_3x_4x_5 + x_2x_3^4 - 4x_1^2x_2^3x_3^4 - 7x_1^2x_2^3x_3x_6 + x_6^2x_2^3x_3 + 2x_1^2x_2^3x_3 - 0.00000001478902079x_6^3x_1x_2x_4x_5x_3 + 3x_6x_1x_2^4 - 0.8x_4^3x_5x_2x_6x_1^9 + 0.00006920255588x_1 + 0.0005108556833 + x_1^5$,

INPUT VIII: $f(x_1,\ldots,x_5) = x_1^6 + 0.0008967604529x_1^5 + 0.0005604752830x_2 + 0.1120950566x_5^2x_2x_1 + 0.0001120950566x_2^3 + 0.0001120950566x_4 + 0.0001120950566x_1 - x_5^7 + 0.00002110684284x_1^2x_2 + 0.0000479631312 + 0.0000212553404x_2^2x_3 + 0.5x_1x_2 + x_2^2x_1^2 - x_5x_2^3x_1^4 + x_3^7 + x_3^6x_2 - 5x_5x_2x_4^5$,

INPUT IX: $f(x_1,\ldots,x_7) = x_1^3 + 0.0000883372717x_2^5 + 0.00005521079481x_2 + 0.00001104215896x_5^2x_2x_7 + 0.00001104215896x_6^3 + 0.00001104215896x_1 + 0.00001104215896x_7^4 + .5521079481x_6x_5^3 + 0.0001894657067x_1^2x_3 + 0.00001040528589x_1x_2x_3 + 5x_1x_2 - 6x_5x_1^2x_6 + x_2^3x_1^4x_6^2 + x_7^9 + 8x_6x_3^2x_7 + x_1x_3^5x_5 + 6x_3^9 - 0.00007848061529 - x_2^3 + 5x_6x_1x_7^3x_1^3 - 0.00007848061529 + 0.00001104215896x_1$,

INPUT X: $f(x_1,\ldots,x_6) = 0.00004398697985x_1 + 0.0002199348993x_2 + 9x_1^{13} + x_2^3x_1^{10}x_6^7 + 5.x_1x_2 + 0.0003518958388x_1^5 + 0.00004398697985x_1^9 + x_6^{15} + 0.00004398697985x_6^3 - 6.x_5x_1^2x_6 + 0.00002230251126x_1x_2x_3 + 0.00001022097753x_1^2x_3 + x_1x_3^5x_5 + 6.x_3^9 + 0.00004398697985x_5^2x_2x_4 + 32x_6x_3^2x_4 + 5x_6x_1x_4^3 - 6x_4^2x_3x_1^6 + 0.00004398697985x_2^6x_3x_4 + 5x_5x_6x_4 + x_1^{12} + 0.4398697985x_4^4 + 0.008665511265 + x_4^9 + 0.0002199348993x_6x_5^3 - x_2^3 + 0.00004398697985x_1 + 9x_1^{13} + 0.0003518958388x_1^5 + 0.00004398697985x_1^9 + 0.008665511265 + x_1^{12}$,

INPUT XI: $f(x_1) = 0.1136363637x_1^2 - 0.1190476191x_1^3 + 0.000217466945 + x_1$,

INPUT XII: $f(x_1) = -0.0909090909x_1^4 + 0.2x_1^5 - 0.00008154943932 + 0.0000651989655x_1 - 0.000861821316x_1^2 - 6x_1^3$,

INPUT XIII: $f(x_1) = 0.01123595506x_1^8 - 0.01369863014x_1^9 + .1428571429x_1^{10} + 0.00001432172319 - 0.0001323977228x_1 + 0.00001199918406x_1^2 - 0.00004831151263x_1^3 + 0.00001569316720x_1^4 - 0.00001043373016x_1^5 + 0.00001221687395x_1^6 + x_1^7$,

INPUT XIV: $f(x_1) = 2.222222222x_1^8 - 0.350877193x_1^9 + 0.2352941176x_1^{10} + 0.0003865481252 - 0.0002516862982x_1 + 0.0003549119818x_1^2 - 0.0005534340584x_1^3 + 0.0003536755734x_1^4 - 0.0003309121594x_1^5 + 0.00020908036x_1^6 + 20x_1^7$,

INPUT XV: $f(x_1) = 0.02941176471x_1^{10} - 0.01315789474x_1^{11} + 0.01075268817x_1^{12} - 0.03703703704x_1^{13} + 0.05882352941x_1^{14} - 0.01351351351x_1^{15} + 0.00001191213608 -$

$0.00001204993493x_1 + 0.0001653439153x_1^2 - 0.00001899876508x_1^3 + 0.00002998590662x_1^4 - 0.00001444460494x_1^5 + 0.00002989089822x_1^6 - 0.00001248891609x_1^7 + 0.0001653986107x_1^8 + x_1^9,$

INPUT XVI: $f(x_1) = -0.6206896552x_1^2 + 1.1250x_1^3 - 0.3214285715x_1^4 + 2.25x_1^5 - 0.3461538461x_1^6 + 0.2857142857x_1^7 - 0.1836734693x_1^8 + 0.2022471911x_1^9 - 0.4186046511x_1^{10} + 0.1836734693x_1^{11} - 0.1818181818x_1^{12} + 0.6428571428x_1^{13} - 0.6428571428x_1^{14} + 0.2399999999x_1^{15} - 0.6923076923x_1^{16} + 2.250x_1^{17} - 0.2168674699x_1^{18} + 0.5999999999x_1^{19} - 0.3396226415x_1^{20} - 0.0004964010921 - 18x_1,$

INPUT XVII: $f(x_1) = -0.01075268817x_1^{11} + 0.02380952381x_1^{12} - 0.01136363636x_1^{13} + 0.01754385965x_1^{14} - 0.09090909091x_1^{15} + 0.01204819277x_1^{16} - 0.0625x_1^{17} + 0.01818181818x_1^{18} - 0.01886792453x_1^{19} + 0.01086956522x_1^{20} + 0.00001417012654 - 0.00001992666985x_1 + 0.00001174094773x_1^2 - 0.00001880936706x_1^3 + 0.00001120586291x_1^4 - 0.00003301201637x_1^5 + 0.00002951506744x_1^6 - 0.00002135383301x_1^7 + 0.00005931198102x_1^8 - 0.00001876243011x_1^9 + x_1^{10},$

INPUT XVIII: $f(x_1) = -2.666666667x_1^{11} + 0.4878048780x_1^{12} - 0.5970149252x_1^{13} + 0.9523809524x_1^{14} - 0.8695652172x_1^{15} + 2.222222222x_1^{16} - 0.8888888888x_1^{17} + 0.9756097560x_1^{18} - 0.4651162792x_1^{19} + 0.8333333332x_1^{20} + 0.001004722194 - 0.001236705417x_1 + 0.02803083392x_1^2 - 0.002983738624x_1^3 + 0.0006585012512x_1^4 - 0.000443739392x_1^5 + 0.002176752286x_1^6 - 0.0009497352612x_1^7 + 0.003305785124x_1^8 - 0.0006363853312x_1^9 + 40x_1^{10},$

INPUT XIX: $f(x_1) = -x_1^{29} + 0.1x_1^{26} - 0.02941176471x_1^{27} + 0.07142857143x_1^{28} + 0.01724137931x_1^{24} - 0.02173913043x_1^{21} + 0.01724137931x_1^{22} - .625x_1^{23} - 0.00001151861408x_1 + 0.01639344262x_1^{30} - 0.015625x_1^{25} + x_1^5 + 0.00002830215379x_1^2 + 0.00003690309248x_1^4 + 0.0625x_1^{14} - 0.05555555556x_1^{15} - 0.08333333333x_1^{11} + 0.01694915254x_1^{12} - .1250000000x_1^{13} + 0.0002528445006 - 0.00001455159267x_1^3 + 0.02439024390x_1^{16} - 0.05x_1^{17} + 0.01851851852x_1^{18} - 0.03125x_1^{19} + 0.01724137931x_1^{20} + 0.01098901099x_1^8 - 0.02040816327x_1^9 + 0.04x_1^{10} - 0.05555555556x_1^7 + 0.01612903226x_1^6,$

INPUT XX: $f(x_1) = -0.09090909091x_1^{29} + 0.01351351351x_1^{26} - 0.01162790698x_1^{27} + 0.01136363636x_1^{28} + 0.01204819277x_1^{24} - 0.03225806452x_1^{21} + 0.01470588235x_1^{22} - 0.08333333333x_1^{23} - 0.00002411265432x_1 + 0.02777777778x_1^{30} - 0.05x_1^{25} - 0.00001336362421x_1^5 + 0.00003751782096x_1^2 + 0.0000457582136x_1^4 + 0.00005596597269x_1^{14} - 0.00001267121986x_1^{15} - 0.0000127995085x_1^{11} + 0.00003225390272x_1^{12} - 0.00004152306606x_1^{13} - 0.000029563058x_1^3 + 0.0001011736139x_1^{16} - 0.00001643574446x_1^{17} + 0.00003368364322x_1^{18} - 0.00001679514956x_1^{19} + x_1^{20} + 0.0000301295571x_1^8 - 0.00002055751994x_1^9 + 0.00001020929045x_1^{10} - 0.00002059859518x_1^7 + 0.00003032692424x_1^6 + 0.0000121673744.$

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