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S.Pérez-Díaz · J.R.Sendra · C.Villarino

Finite Piecewise Polynomial Parametrization of Plane Rational Algebraic Curves

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Abstract We present an algorithm with the following characteristics: given a real non-polynomial rational parametrization $\mathcal{P}(t)$ of a plane curve and a tolerance $\epsilon > 0$, $\mathbb{R}$ is decomposed as union of finitely many intervals, and for each interval I of the partition, with the exception of some isolating intervals, the algorithm generates a polynomial parametrization $\mathcal{P}_I(t)$. Moreover, as an option, one may also input a natural number N and then the algorithm returns polynomial parametrizations with degrees smaller or equal to N. In addition, we present an error analysis where we prove that the curve piece $\mathcal{C}_I = \{\mathcal{P}(t) \,|\, t \in I\}$ is in the offset region of $\mathcal{C}_I^* = \{\mathcal{P}_I(t) \,|\, t \in I\}$ at distance at most $\sqrt{2}\epsilon$, and conversely.

Keywords Piecewise Polynomial Parametrization · Rational Algebraic Curves · Error Analysis

1 Introduction

Rational plane algebraic curves are curves accepting a parametric representation by means of a pair of univariate rational functions. This type of curves are precisely those having genus zero. There exist algorithmic methods for deciding whether a given plane curve is rational, and if so, for computing a rational parametrization (see e.g. [11], [12]). These curves play an important

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S. Pérez-Díaz, J. R. Sendra and C. Villarino
Dpto. de Matemáticas. Universidad de Alcalá. E-28871 Madrid, Spain
Tel.: +34-91-8854902
Fax: +34-91- 8854951
E-mail: sonia.perez, rafael.sendra, carlos.villarino@uah.es

role in many practical applications, in particular in computer aided geometric design, and many authors have addressed multiple problems related to them (see e.g. [4]).

Within the set of rational curves, one may consider curves parametrizable by polynomials (i.e. polynomial curves). Polynomial curves have an additional interest in practical applications, since the non-existence of denominators avoids the possible unstable behavior of the parametrization, when the parameter takes values close to the roots of the denominators. Polynomial curves are characterized as those rational plane curves having only one place at infinity (see [1]). Therefore, one cannot approach globally the problem of computing polynomial parametrizations of every genus zero curve. Nevertheless, one may try to compute local piecewise polynomial parametrizations. In this paper, we deal with this problem. More precisely, let $\epsilon > 0$ be a fixed tolerance, and let $\mathcal{P}(t)$ be a real non-polynomial rational parametrization of a plane curve $\mathcal{C}$, then the problem consists in:

(i) decomposing the parameter space $\mathbb{R}$ as a union of finitely many intervals,
(ii) and for each interval I of the partition, computing a polynomial parametrization $\mathcal{P}_I(t)$ such that the "curve pieces" $\mathcal{C}_I = \{\mathcal{P}(t)\,|\,t \in I\}$ and $\mathcal{C}_I^* = \{\mathcal{P}_I(t)\,|\,t \in I\}$ are close (we refer to [9] and [10] for the notion of closeness).

In order to solve the problem, one may need to approximate real roots of polynomials. In that case, isolating intervals appear in the partition. To these intervals none polynomial parametrization is assigned. Alternatively, one may return a sequence of polynomial parametrizations that converge to the curve in those regions.

There exist methods to approach the problem. For instance, one may apply to both rational components of $\mathcal{P}(t)$ the well-known Approximation Theorem of Weierstrass in combination with Bernstein-polynomials (see e.g. [5]). Also, in [7], the authors present a Bézier-like approach, based on the so called hybrid polynomials.

In this paper, we present an alternative approach based on polynomial sequences uniformly converging to the rational functions. An important property of our method is that, for a given natural number N satisfying certain minimal requirements, the algorithm generates polynomial parametrizations which degrees are bounded by N. We also present an error analysis where we provide an explicit a priori bound of the closeness of the input and the output. In fact, if $\mathcal{P}_I(t)$ is the output polynomial parametrization, we prove that $\mathcal{C}_I \subset \mathcal{O}_{\sqrt{2}\epsilon}(\mathcal{C}_I^\star)$, and $\mathcal{C}_I^\star \subset \mathcal{O}_{\sqrt{2}\epsilon}(\mathcal{C}_I)$, where $\mathcal{O}_d$ denotes the offset region at distance d. We refer to [2] for the notion of offset.

The structure of this paper is as follows. In Section 2 we describe the method for polynomially approximating a rational function in a compact interval. In Section 3 we deal with the same problem, but showing how to control the degree of the output. In Section 4 we apply these results to derive the piecewise polynomial parametrization algorithm. In Section 5 we present the error analysis, and in Section 6 we compare our method with Sederberg-Kakimoto's method and Bernstein based method.

2 Polynomial Approximation of a Rational Function in a Compact Interval

In this section, we analyze how to polynomially approximate $\chi(t) \in \mathbb{R}(t)\backslash\mathbb{R}[t]$ in a compact interval $K \subset \mathbb{R}$, where $\chi(t)$ is continuous. The strategy is as follows. First we define an infinite family of polynomial sequences (see Definition 1), and we prove that each of these sequences approximates $\chi(t)$. Afterwards, in order to choose the best sequence in the family, we introduce the notion of order of convergence (see Definition 2). Then, we find a sequence in the family minimizing the order of convergence. This leads to the notion of "approximating polynomial sequence" of $\chi(t)$ (see Definition 3). Finally, under the criterion of minimizing the degree, we compute a polynomial in that sequence approximating $\chi(t)$ under the given tolerance.

Definition 1 Let $\chi(t) \in \mathbb{R}(t) \setminus \mathbb{R}[t]$ be continuous in a compact interval $K \subset \mathbb{R}$. Let $\chi(t)$ be expressed as $\chi(t) = q(t)+r(t)/\chi_2(t)$, where $q, r, \chi_2 \in \mathbb{R}[t]$, $\gcd(r, \chi_2) = 1, \deg(r) < \deg(\chi_2)$, and $\chi_2(t) > 0$ for every $t \in K$. We define the parametric polynomial sequence $\{\{H^{\chi(t),K}(t, x, n)\}_{n\in\mathbb{N}}\}_{x\in\mathbb{R}}$ associated to $\chi(t)$ in K as

$$H^{\chi(t),K}(t, x, n) = q(t) + x + \frac{r(t) - x\chi_2(t)}{\max_K\{\chi_2(t)\}} \sum_{k=0}^{n} u(t)^k,$$

where $u(t) = 1 - \frac{\chi_2(t)}{\max_K\{\chi_2(t)\}}$. Furthermore, we define the parametric error sequence $\{\{R^{\chi(t),K}(t, x, n)\}_{n\in\mathbb{N}}\}_{x\in\mathbb{R}}$ as

$$R^{\chi(t),K}(t, x, n) = \chi(t) - H^{\chi(t),K}(t, x, n).$$

Remark 1 Note that using Euclidean division, and taking into account that χ_2 does not vanish at K, one can always express $\chi(t)$ as in Definition 1. □

Now, we prove the basic properties of the parametric polynomial sequence.

Theorem 1 *Let $\chi(t)$ and K be as in Definition 1. It holds that:*

1. $\forall\, t \in K$, $0 \leq u(t) < 1$.
2. $\forall\, \lambda \in \mathbb{R}$, $H^{\chi(t)+\lambda,K} = H^{\chi(t),K} + \lambda$, *and* $|R^{\chi(t)+\lambda,K}| = |R^{\chi(t),K}|$.
3. $\forall\, \lambda \in \mathbb{R}$, $H^{\lambda\chi(t),K} = \lambda H^{\chi(t),K}$, *and* $|R^{\lambda\chi(t),K}| = |\lambda|\,|R^{\chi(t),K}|$.
4. $\forall\, t, x \in \mathbb{R}$, $|R^{\chi(t),K}| = \left|\frac{r(t)}{\chi_2(t)} - x\right| u(t)^{n+1}$.
5. $\forall\, x_0 \in \mathbb{R}$, $\{H^{\chi(t),K}(t, x_0, n)\}_{n\in\mathbb{N}}$ *converges uniformly to* $\chi(t)$ *in* K.

Proof. (1), (2), and (3) are immediate. In order to prove (4) and (5), let $U := \sum_{k=0}^{n} u(t)^k$, $M := \max_K\{\chi_2\}$, and $m := \min_K\{\chi_2\}$. Then:

$$|R^{\chi(t),K}| = \left|\chi - q - x - \frac{(r - x\chi_2)U}{M}\right| = \left|\frac{r}{\chi_2} - x - \frac{(r - x\chi_2)U}{M}\right| =$$

$$\frac{|r - x\chi_2|}{M}\left|\frac{M}{\chi_2} - U\right| = \frac{|r - x\chi_2|}{M}\left|\frac{1}{1-u} - U\right| = \frac{|r - x\chi_2|}{M}\left|\frac{u^{n+1}}{1-u}\right| = \left|\frac{r}{\chi_2} - x\right| u^{n+1}.$$

To prove (5) we take $x_0 \in \mathbb{R}$ and we show that $\{R^{\chi(t),K}(t,x_0,n)\}_{n\in\mathbb{N}}$ converges uniformly to 0 in K. For $t \in K$ it holds that

$$|R^{\chi(t),K}(t,x_0,n)| = \left|\frac{r(t)}{\chi_2(t)} - x_0\right| u(t)^{n+1} \leq \max_K\{|\frac{r}{\chi_2} - x_0|\} \left(1 - \frac{m}{M}\right)^{n+1}.$$

Now, the result follows taking into account that $0 < 1 - m/M < 1$. □

By Theorem 1 (5), for every $x_0 \in \mathbb{R}$, $\{H^{\chi(t),K}(t,x_0,n)\}_{n\in\mathbb{N}}$ converges uniformly to $\chi(t)$ in K. Therefore, polynomials in this sequence can be used as polynomial approximation of $\chi(t)$. Indeed, for any $x_0 \in \mathbb{R}$, and for the given tolerance $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and $t \in K$ then $|R^{\chi(t),K}(t,x_0,n)| \leq \epsilon$. Furthermore, Theorem 1 (2) implies that the polynomial approximation provided by this sequence is invariant under translations, which is a very intuitive property to be required to any approximation process. Nevertheless, statement (3) shows that the approximation is affected by homotecies, which is not surprising.

Since we have infinitely many different polynomial sequences to be used in the approximation, namely $\{\{H^{\chi(t),K}(t,x_0,n)\}_{n\in\mathbb{N}}\}_{x_0\in\mathbb{R}}$, the natural question is how to choose $x_0 \in \mathbb{R}$ such that the convergence is accelerated. In order to be more precise in our claims, we introduce the following notion.

Definition 2 Let $J \subset \mathbb{R}$, let $\delta > 0$, and let $\{g(t,n)\}_{n\in\mathbb{N}}$ be a real functional sequence uniformly converging to $g(t)$ in J. Then, we define the δ-order of convergence of $\{g(t,n)\}_{n\in\mathbb{N}}$ in J at the smallest $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, with $n \geq n_0$, and for all $t \in J$, it holds that $|g(t,n) - g(t)| \leq \delta$.

Note that the convergence accelerates when the order decreases. In our situation, given the tolerance $\epsilon > 0$, the question is how to choose $x_0 \in \mathbb{R}$ such that the ϵ-order of convergence of $\{H^{\chi(t),K}(t,x_0,n)\}_{n\in\mathbb{N}}$ in K is as small as possible. For this purpose, we minimize the upper bound of the error given by Theorem 1 (4). Let $t \in K$, then it holds that

$$|R^{\chi(t),K}(t,x,n)| = \left|\frac{r(t)}{\chi_2(t)} - x\right| u(t)^{n+1} \leq \max_K\{|\frac{r(t)}{\chi_2(t)} - x|\} \cdot \max_K\{u(t)^{n+1}\}.$$

Note that both function, $\frac{r(t)}{\chi_2(t)} - x$ and $u(t)^{n+1}$, are continuous in K, and hence both maximums exist. In this situation, we observe that the second maximum does not depend on x while the first does. Thus, we minimize $\max_K\{|\frac{r(t)}{\chi_2(t)} - x|\}$. This is done in the following lemma.

Lemma 1 *Let $\chi(t)$ and K be as in Definition 1, and let $\mathfrak{M} : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as $\mathfrak{M}(x) = \max_K\{|r(t)/\chi_2(t) - x|\}$. $\mathfrak{M}$ reaches its minimum value at $x_0 = \frac{1}{2}(\max_K\{r/\chi_2\} + \min_K\{r/\chi_2\})$, and $\mathfrak{M}(x_0) = \frac{1}{2}(\max_K\{r/\chi_2\} - \min_K\{r/\chi_2\})$.*

Proof. Let $M := \max_K\{\frac{r}{\chi_2}\}$ and $m := \min_K\{\frac{r}{\chi_2}\}$. Then $\mathfrak{M}(x_0) = \frac{1}{2}(M-m)$ because $\mathfrak{M}(x) = \max\{|M - x|, |m - x|\}$. Now, assume that there exists x_0^*

such that $\mathfrak{M}(x_0^*) < \mathfrak{M}(x_0)$. Then, we deduce that: (1) $|M - x_0^*| < \mathfrak{M}(x_0)$; (2) $|m - x_0^*| < \mathfrak{M}(x_0)$. From (1) one has that $M - x_0^* < \mathfrak{M}(x_0)$, and hence $\frac{1}{2}(M+m) < x_0^*$. From (2) one has that $-\mathfrak{M}(x_0) < m - x_0^*$, and hence $x_0^* < \frac{1}{2}(M+m)$, which is a contradiction. Thus, $\mathfrak{M}$ reaches the minimum at x_0. □

Taking into account Lemma 1, we introduce the following concept.

Definition 3 Let $\chi(t)$ and K be as in Definition 1. We define the approximating polynomial sequence $\{P^{\chi(t),K}(t,n)\}_{n\in\mathbb{N}}$ associated to $\chi(t)$ in K as

$$P^{\chi(t),K}(t,n) = H^{\chi(t),K}(t,x_0,n),$$

where x_0 is as in Lemma 1. Furthermore, we define the approximating error sequence $\{\mathcal{R}^{\chi(t),K}(t,n)\}_{n\in\mathbb{N}}$ as $\mathcal{R}^{\chi(t),K}(t,n) = R^{\chi(t),K}(t,x_0,n)$.

The following theorem follows from Theorem 1 and Lemma 1.

Theorem 2 *Let $\chi(t)$ and K be as in Definition 1. It holds that:*

1. $\{P^{\chi(t),K}(t,n)\}_{n\in\mathbb{N}}$ *converges uniformly to* $\chi(t)$ *in* K.
2. *The error sequence* $\{\mathcal{R}^{\chi(t),K}(t,n)\}_{n\in\mathbb{N}}$ *is preserved under translations.*
3. $\forall\, t \in K,\ |\mathcal{R}^{\chi(t),K}(t,n)| \le \frac{1}{2}(\max_K\{\frac{r(t)}{\chi_2(t)}\} - \min_K\{\frac{r(t)}{\chi_2(t)}\})(1 - \frac{\min_K\{\chi_2(t)\}}{\max_K\{\chi_2(t)\}})^{n+1}$.

Observe that $\{P^{\chi(t),K}(t,n)\}_{n\in\mathbb{N}}$ has been chosen among the sequences $\{\{H^{\chi(t),K}(t,x,n)\}_{n\in\mathbb{N}}\}_{x\in\mathbb{R}}$ under the criterion of minimizing the ϵ-orders of convergence. Now, if n_0 is the ϵ-order of convergence of $\{P^{\chi(t),K}(t,n)\}_{n\in\mathbb{N}}$ in K, for $n \ge n_0$, the polynomial $P^{\chi(t),K}(t,n)$ approximates $\chi(t)$ under tolerance ϵ. Therefore, $P^{\chi(t),K}(t,n_0)$ is the polynomial in $\{P^{\chi(t),K}(t,n)\}_{n\in\mathbb{N}}$ approximating $\chi(t)$ in K with smallest degree. In the following algorithm we estimate n_0.

Sub-Algorithm 1. [Estimation of the ϵ-order of convergence]

INPUT: $(\epsilon, \chi(t), K)$ where $\epsilon > 0$; $\chi(t) = \frac{\chi_1(t)}{\chi_2(t)} \in \mathbb{R}(t)\setminus\mathbb{R}[t]$ is in reduced form; and $K \subset \mathbb{R}$ is a compact interval where $\chi(t)$ is continuous.

OUTPUT: an estimation of the ϵ-order of convergence of $\{P^{\chi(t),K}(t,n)\}_{n\in\mathbb{N}}$ in K.

Remark: We denote by $\mathsf{SAlg}_1(\epsilon,\chi,K)$ the output.

1. If K is a point, then RETURN 0.
2. Take $t_0 \in K$. If $\chi_2(t_0) < 0$, THEN for $i = 1,2$ replace χ_i by $-\chi_i$. If $\deg(\chi_1) < \deg(\chi_2)$ THEN $q := 0, r := \chi_1$ ELSE take q, r as the quotient and remainder of χ_1 divided by χ_2.
3. $M := \max_K\{\frac{r}{\chi_2}\}, m := \min_K\{\frac{r}{\chi_2}\}$, $M^* := \max_K\{\chi_2\}, m^* := \min_K\{\chi_2\}$, $\alpha := \frac{M-m}{2}$, $\beta := \frac{M^*-m^*}{M^*}$. RETURN $\max\{0, \lceil \log_\beta(\epsilon/\alpha) - 1\rceil\}$. □

Remark 2 Note that $C(x) = \alpha\beta^{x+1}$ is decreasing. Hence, if $n_0 := \mathsf{SAlg}_1(\epsilon,\chi,K)$, for every $n \in \mathbb{N}$ such that $n < n_0$ then $C(n) > \epsilon$. □

Based on the previous results we can derive an algorithm for approximating, by a polynomial, a rational function defined in a compact interval.

Algorithm 1: [Polynomial approximation in a compact].

INPUT: $(\epsilon, \chi(t), K)$ where $\epsilon > 0$; $\chi(t) = \frac{\chi_1(t)}{\chi_2(t)} \in \mathbb{R}(t) \setminus \mathbb{R}[t]$ is in reduced form; $K \subset \mathbb{R}$ is a compact interval where $\chi(t)$ is continuous.

OUTPUT: a polynomial approximating $\chi(t)$ in K under precision ϵ.

Remark: We denote by $\mathsf{Alg}_1(\epsilon, \chi, K)$ the output.

1. Take $t_0 \in K$. If $\chi_2(t_0) < 0$, THEN for $i = 1, 2$ replace χ_i by $-\chi_i$. If $\deg(\chi_1) < \deg(\chi_2)$ THEN $q := 0, r := \chi_1$ ELSE take q, r as the quotient and remainder of χ_1 divided by χ_2.
2. $M := \max_K\{\frac{r}{\chi_2}\}$, $m := \min_K\{\frac{r}{\chi_2}\}$, $M^* := \max_K\{\chi_2\}$, $x_0 := \frac{M+m}{2}$, $n_0 := \mathsf{SAlg}_1(\epsilon, \chi, K)$.
3. RETURN $P^{\chi(t),K}(t, n_0) := q(t) + x_0 + \frac{r(t) - x_0\,\chi_2(t)}{M^*} \sum_{k=0}^{n_0} (1 - \frac{\chi_2(t)}{M^*})^k$. □

The next theorem follows from Theorem 2.

Theorem 3 *Let* $p(t) = \mathsf{Alg}_1(\epsilon, \chi(t), K)$*. Then* $|\chi(t) - p(t)| \leq \epsilon, \ \forall\ t \in K$*.*

3 Degree Control of the Approximation

In Section 2 we have seen how to polynomially approximate a rational function in a compact interval where it is continuous. In this section we show how to decompose the given compact interval such that the degree of the approximating polynomial is smaller or equal to a given natural number N.

Throughout this section we use the following notation: ϵ, K and $\chi(t)$ are as in the input of Algorithm 1. $N \in \mathbb{N}$ is such that $N \geq \max\{\deg(\chi_1) - \deg(\chi_2), \deg(\chi_2)\}$. Associated with N we consider the auxiliary number $N_1 := \lfloor -1 + \frac{N}{\deg(\chi_2)} \rfloor$. Note that by assumption $N \geq \deg(\chi_2)$, and hence $N_1 \in \mathbb{N}$. Also, we express K as $K = [\lambda_1, \mu]$.

The next lemma gives a criterion to ensure that the degree of the approximation is smaller or equal to N.

Lemma 2 *Let* $p = \mathsf{Alg}_1(\epsilon, \chi, K)$*. If* $\mathsf{SAlg}_1(\epsilon, \chi, K) \leq N_1$ *then* $\deg(p) \leq N$*.*

Proof. It follows from $\deg(p) \leq \max\{\deg(\chi_1) - \deg(\chi_2), (\mathsf{SAlg}_1(\epsilon, \chi, K) + 1)\deg(\chi_2)\}$. □

Therefore, we assume w.l.o.g. that $\mathsf{SAlg}_1(\epsilon, \chi, K) > N_1$. In this situation, the strategy consists in decomposing K as $[\lambda_1, \mu] = [\lambda_1, \lambda_2] \cup \cdots \cup [\lambda_\ell, \lambda_{\ell+1}]$, where $\lambda_{\ell+1} = \mu$, and such that $\mathsf{SAlg}_1(\epsilon, \chi, [\lambda_i, \lambda_{i+1}]) \leq N_1$, which implies by Lemma 2 that $\deg(\mathsf{Alg}_1(\epsilon, \chi, [\lambda_i, \lambda_{i+1]}) \leq N$. The next lemma shows how to proceed.

Lemma 3 *Let $K' = [\gamma_1, \gamma_2] \subset K$. If $\mathsf{SAlg}_1(\epsilon, \chi, K') > N_1$, there exists a unique $\gamma \in (\gamma_1, \gamma_2)$ such that $\mathsf{SAlg}_1(\epsilon, \chi, [\gamma_1, \gamma]) = N_1$. Furthermore, if $x \in (\gamma_1, \gamma)$ then $\mathsf{SAlg}_1(\epsilon, \chi, [\gamma_1, x]) \leq N_1$.*

Proof. For $x \in K'$ let $K'(x) := [\gamma_1, x]$, $M(x) := \max_{K'(x)}\{\frac{r}{\chi_2}\}, m(x) := \min_{K'(x)}\{\frac{r}{\chi_2}\}, M^*(x) := \max_{K'(x)}\{\chi_2\}$, and $m^*(x) := \min_{K'(x)}\{\chi_2\}$. Also, let $\alpha(x) = \frac{1}{2}(M(x) - m(x))$ and $\beta(x) = 1 - m^*(x)/M^*(x)$. Then we introduce the real function

$$\mathfrak{T}_{K'} : K' \longrightarrow \mathbb{R} : x \mapsto \mathfrak{T}_{K'}(x) = \alpha(x)\beta(x)^{N_1+1} - \epsilon.$$

In this situation, we first prove that $\mathfrak{T}_{K'}$ is nondecreasing and continuous in K'. The continuity of $\mathfrak{T}$ follows from the continuity of $\chi(t)$. To show that $\mathfrak{T}_{K'}(x)$ is nondecreasing in K', we prove that $S(x) := \mathfrak{T}_{K'}(x) + \epsilon$ is an increasing function. For this purpose, let $x_1, x_2 \in K'$ such that $x_1 < x_2$. Then $K'(x_1) \subset K'(x_2)$. Therefore, $m(x_2) \leq m(x_1) < M(x_1) \leq M(x_2)$, and $0 < m^*(x_2) \leq m^*(x_1) < M^*(x_1) \leq M^*(x_2)$. Thus $0 < M(x_1) - m(x_1) \leq M(x_2) - m(x_2)$, and $0 < \frac{m^*(x_2)}{M^*(x_2)} \leq \frac{m^*(x_1)}{M^*(x_1)}$, $0 < 1 - \frac{m^*(x_1)}{M^*(x_1)} \leq 1 - \frac{m^*(x_2)}{M^*(x_2)} < 1$. Therefore $S(x_1) \leq S(x_2)$.
Now, we observe that $\mathfrak{T}_{K'}(\gamma_1) = -\epsilon < 0$. Moreover, since $\mathsf{SAlg}_1(\epsilon, \chi, K') > N_1$, by Remark 2 to Sub-Algorithm 1, one has that $\mathfrak{T}_{K'}(\gamma_2) > 0$. Therefore, there exists a unique $\gamma \in (\gamma_1, \gamma_2)$ such that $\mathfrak{T}_{K'}(\gamma) = 0$. Hence, since $N_1 \in \mathbb{N}$, one has that $\mathsf{SAlg}_1(\epsilon, \chi, [\gamma_1, \gamma]) = N_1$. Finally, if $x \in (\gamma_1, \gamma)$ then $[\gamma_1, x] \subset [\gamma_1, \gamma]$. Thus $\mathsf{SAlg}_1(\epsilon, \chi, [\gamma_1, x]) \leq \mathsf{SAlg}_1(\epsilon, \chi, [\gamma_1, \gamma]) = N_1$ □

<u>Sub-Algorithm 2.</u> [Computation of γ in Lemma 3]

INPUT: $(\epsilon, \chi(t), N_1, K')$ where $\epsilon > 0$; $\chi(t) \in \mathbb{R}(t) \setminus \mathbb{R}[t]$ is expressed as $\chi(t) = q(t) + r(t)/\chi_2(t)$, with $q, r, \chi_2 \in \mathbb{R}[t]$, $\gcd(r, \chi_2) = 1, \deg(r) < \deg(\chi_2)$; $N_1 \in \mathbb{N}$ is defined as above; and $K' = [\gamma_1, \gamma_2] \subset \mathbb{R}$ is such that $\chi_2(t) > 0$ for every $t \in K'$, $r/\chi_2, \chi_2$ are monotone in K', and $\mathsf{SAlg}_1(\epsilon, \chi, K') > N_1$.

OUTPUT: $\gamma \in (\gamma_1, \gamma_2)$ such that $\mathsf{SAlg}_1(\epsilon, \chi, [\gamma_1, \gamma]) \leq N_1$.

<u>Remark:</u> We denote by $\mathsf{SAlg}_2(\epsilon, \chi, N_1, K')$ the output.

1. $\mathcal{T}(x_1, x_2, x_3, x_4) := \frac{1}{2}(\frac{r}{\chi_2}(x_1) - \frac{r}{\chi_2}(x_2))\left(1 - \frac{\chi_2(x_3)}{\chi_2(x_4)}\right)^{N_1+1} - \epsilon$.
2. If $\frac{r}{\chi_2}$, χ_2 are increasing in K', approximate the root γ in K' of the numerator of the reduced form of $\mathcal{T}(x, \gamma_1, \gamma_1, x)$. RETURN γ.
3. If $\frac{r}{\chi_2}$, χ_2 are decreasing in K', approximate the root γ in K' of the numerator of the reduced form of $\mathcal{T}(\gamma_1, x, x, \gamma_1)$. RETURN γ.
4. If $\frac{r}{\chi_2}$ is increasing, and χ_2 is decreasing in K', approximate the root γ in K' of the numerator of the reduced form of $\mathcal{T}(x, \gamma_1, x, \gamma_1)$. RETURN γ.
5. If $\frac{r}{\chi_2}$ is decreasing, and χ_2 is increasing in K', approximate the root γ in K' of the numerator of the reduced form of $\mathcal{T}(\gamma_1, x, \gamma_1, x)$. RETURN γ. □

In this situation, we approach the problem as follows. We are given $\epsilon, \chi(t), N$ and $K = [\lambda_1, \mu]$ such that $\mathsf{SAlg}_1(\epsilon, \chi, K) > N_1$. We assume that $r/\chi_2, \chi_2$ are monotone in K, otherwise we first decompose K. Then, let

$\lambda_2 := \mathsf{SAlg}_2(\epsilon, \chi, N_1, K)$. Now, we introduce the new compact $K_1 := [\lambda_2, \mu]$. If $\mathsf{SAlg}_1(\epsilon, \chi, K_1) \leq N_1$ the process ends, and K decomposes as $K = [\lambda_1, \lambda_2] \cup K_1$. Otherwise, let $\lambda_3 := \mathsf{SAlg}_2(\epsilon, \chi, N_1, K_1)$. Then, we introduce $K_2 := [\lambda_3, \mu]$. If $\mathsf{SAlg}_1(\epsilon, \chi, K_2) \leq N_1$ the process ends, and $K = [\lambda_1, \lambda_2] \cup [\lambda_2, \lambda_3] \cup K_2$. Otherwise, Sub-Algorithm 2 is applied to $(\epsilon, \chi, N_1, K_2)$, etc. Finally note that, by Lemma 2, Algorithm 1 applied to ϵ, χ, and each of the generated compacts, provide an approximating polynomial of degree smaller or equal than N; which is our goal.

Obviously, in order to derive an algorithm from the previous reasoning we have to prove that the process ends. This is done in the next theorem.

Theorem 4 *The set $\{\lambda_i\}$, introduced in the above process, is finite.*

Proof. Let us assume that the process introduces infinitely many λ_i. Then, we get an increasing sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ in K, and an infinite sequence of intervals $\{K_i := [\lambda_{i+1}, \mu]\}_{i\geq 0}$. Associated to each K_i, we consider the function $\mathfrak{T}_{K_i}(x)$ introduced in the proof of Lemma 3. Moreover, $\{\lambda_i\}_{i\in\mathbb{N}}$ is convergent because it is increasing and bounded. Note that this implies that the length of $[\lambda_i, \lambda_{i+1}]$ converges to zero. Moreover, by construction, one has that $\mathfrak{T}_{K_i}(\lambda_{i+2}) = 0$. Now, taking limits in this last equality one gets that $\epsilon = 0$ which contradicts the hypothesis that $\epsilon > 0$. □

Applying the previous ideas, we derive an alternative algorithm to Algorithm 1, where the degrees of the output polynomials are bounded by N. For simplicity, we present the algorithm assuming that the compact interval satisfies the monotony conditions required in Sub-Algorithm 2.

Algorithm 2: [Polynomial approximation in a compact with degree control].

INPUT: $(\epsilon, \chi(t), N, K)$ where $\epsilon > 0$; $\chi(t) \in \mathbb{R}(t) \setminus \mathbb{R}[t]$ is expressed as $\chi(t) = q(t) + r(t)/\chi_2(t)$, with $q, r, \chi_2 \in \mathbb{R}[t]$, $\gcd(r, \chi_2) = 1, \deg(r) < \deg(\chi_2)$; $N \in \mathbb{N}$ is such that $N \geq \max\{\deg(\chi_1) - \deg(\chi_2), \deg(\chi_2)\}$; and $K := [\lambda_1, \mu] \subset \mathbb{R}$ is a compact interval such that χ_2 is positive in K, and $r/\chi_2, \chi_2$ are monotone.

OUTPUT: a finite decomposition of K in compact intervals, and for each interval a polynomial of degree at most N that approximates $\chi(t)$ with precision ϵ.

Remark: We denote by $\mathsf{Alg}_2(\epsilon, \chi, N, K)$ the output.

1. $\mathcal{F} := \emptyset,\ N_1 := \lfloor \frac{N}{\deg(\chi_2)} - 1 \rfloor$.
2. While $\mathsf{SAlg}_1(\epsilon, \chi, [\lambda_1, \mu]) > N_1$ do $\ll \lambda_2 := \mathsf{SAlg}_2(\epsilon, \chi, N_1, [\lambda_1, \mu])$, $p(t) := \mathsf{Alg}_1(\epsilon, \chi, [\lambda_1, \lambda_2])$, append $[[\lambda_1, \lambda_2], p(t)]$ to $\mathcal{F}$, replace λ_1 by λ_2. $\gg$
3. $p(t) := \mathsf{Alg}_1(\epsilon, \chi, [\lambda_1, \mu])$. Append $[[\lambda_1, \mu], p(t)]$ to $\mathcal{F}$. RETURN $\mathcal{F}$. □

The next theorem follows from the previous results and from Theorem 2.

Theorem 5 *Let $\mathcal{F} = \mathsf{Alg}_2(\epsilon, \chi(t), N, K)$, and $[K', p(t)] \in \mathcal{F}$. Then $|\chi(t) - p(t)| \leq \epsilon$ for every $t \in K'$ and $\deg(p(t)) \leq N$.*

4 Finite Piecewise Polynomial Parametrization

In this section we apply the preceding results to derive a piecewise polynomial parametrization algorithm for real rational affine plane curves. Throughout this section, $\mathcal{C}$ is a non-polynomial rational real affine curve, and $\epsilon > 0$ is the tolerance used in the process. Moreover, $\mathcal{P}(t)$ is a real rational parametrization of $\mathcal{C}$. We assume w.l.o.g. that none of the components of $\mathcal{P}(t)$ is polynomial. In addition, we assume that $\mathcal{P}(t)$ is expressed as

$$\mathcal{P}(t) = (\chi(t), \xi(t)) = \left(\frac{\chi_1(t)}{\chi_2(t)}, \frac{\xi_1(t)}{\xi_2(t)}\right) = \left(q_1(t) + \frac{r_1(t)}{\chi_2(t)}, q_2(t) + \frac{r_2(t)}{\xi_2(t)}\right),$$

where $q_i, r_i, \chi_i, \xi_i \in \mathbb{R}[t]$, $\gcd(r_1, \chi_2) = \gcd(r_2, \xi_2) = 1$, $\deg(r_1) < \deg(\chi_2)$, and $\deg(r_2) < \deg(\xi_2)$.

In this situation, we proceed as follows. First we show how to decompose the parameter space. Secondly we assign polynomials to each subset of the decomposition. Finally, we derive the algorithm, and we illustrate it by an example.

In order to decompose $\mathbb{R}$ we start determining the unbounded part of the partition. Since $\lim_{t\to\infty} r_1(t)/\chi_2(t) = 0$, there exists $B_{\chi,\epsilon} \in \mathbb{R}^+$ such that for every $t \in \mathbb{R}$ with $|t| > B_{\chi,\epsilon}$ it holds that $|\frac{r_1(t)}{\chi_2(t)}| \leq \epsilon$. Similarly for $r_2(t)/\xi_2(t)$. Then, if $B = \max\{B_{\chi,\epsilon}, B_{\xi,\epsilon}\}$, the unbounded part of the decomposition is taken as $I_0 = (-\infty, -B) \cup (B, \infty)$. For the actual computation of I_0, we observe that B can be taken as an upper bounds of the roots, in module, of $r_1(t) \pm \epsilon\chi_2(t)$ and $r_2(t) \pm \epsilon\, \xi_2(t)$.

The decomposition of the compact interval $\mathbb{R} \setminus I_0 = [-B, B]$ is done depending on whether $h(t) := \mathrm{lcm}(\chi_2, \xi_2)$ has real roots or not. If $h(t)$ has no real roots, we decompose $\mathbb{R}$ as $\mathbb{R} = I_0 \cup K_1$, where $K_1 := [-B, B]$. Now, let $h(t)$ have real roots. Before discussing the details we first ensure that these roots belong to $(-B, B)$.

Lemma 4 *The real roots of $h(t)$ belong to the interval $(-B, B)$.*

Proof. Let $\theta \in \mathbb{R}$ be a root of $h(t)$. Then $\lim_{t\to\theta^+} r_1(t)/\chi_2(t) = \infty$ and $\lim_{t\to\theta^-} r_1(t)/\chi_2(t) = \infty$; similarly for r_2/ξ_2. By the construction of B, for $t \in \mathbb{R}$ with $|t| > B$, $|r_1(t)/\chi_2(t)| \leq \epsilon$, $|r_2(t)/\xi_2(t)| \leq \epsilon$. Thus, $\theta \in (-B, B)$. □

In this situation, since in general the roots cannot be computed exactly, we isolate them (see e.g. [6]). Therefore, $\mathbb{R}$ is decomposed as $\mathbb{R} = I_0 \cup_{i=0}^{m+1} K_i \cup_{i=1}^{i=m} J_i$, where I_0 is as above, K_i are compact intervals, and J_i are open intervals isolating the real roots of $h(t)$. The length of J_i can be taken as small as the particular problem requires. Here this length is taken smaller or equal to ϵ.

In addition to the decomposition process described above, in both cases, one might have to consider a further decomposition of the compacts K_i. This is due to the degree control of the polynomial approximations (see Section 3). More precisely, each K_i is decomposed such that $r_1/\chi_2, r_2/\xi_2, \chi_2, \xi_2$ are monotone. In this process, similarly as it has happened with the real roots of $h(t)$, one might also need to isolate real roots. Therefore, new isolating open intervals might appear. Moreover, during the applications of Algorithm 2, the compacts intervals might be also decomposed as union of new compact intervals. All these details are clarified in Algorithm 3.

Now, let us see how to assign the polynomial approximations. Let us consider that $\mathbb{R}$ has been decomposed as $\mathbb{R} = I_0 \cup_{i=0}^{\ell_1} K_i \cup_{i=0}^{\ell_2} J_i$, where $I_0 = (-B, B)$, K_i are compact intervals and J_i are open intervals, of length smaller or equal to ϵ. J_i are introduced either when isolating real roots of $h(t)$ or when decomposing the compacts for guaranteing the monotony property. Then, the polynomial assignment is as follows: we assign (q_1, q_2) to I_0. For each K_i, we apply Algorithm 1 to (ϵ, χ, K_i) and to (ϵ, ξ, K_i). Let $p_1(t)$ and $p_2(t)$ the outputs. Then, we assign (p_1, p_2) to K_i.

All these ideas are summarized in the following algorithm.

Algorithm 3: [Piecewise polynomial parametrization].

INPUT: $(\epsilon, \mathcal{P}(t), N)$ where $\epsilon > 0$; $\mathcal{P}(t) = (\chi(t), \xi(t)) = (\frac{\chi_1(t)}{\chi_2(t)}, \frac{\xi_1(t)}{\xi_2(t)})$ is a rational parametrization in reduced form where $\chi, \xi \in \mathbb{R}(t) \backslash \mathbb{R}[t]$, and $N \in \mathbb{N} \cup \{\infty\}$ is such that $N \geq \max\{\deg(\chi_1) - \deg(\chi_2), \deg(\xi_1) - \deg(\xi_2), \deg(\chi_2), \deg(\xi_2)\}$.

OUTPUT: a list $\mathcal{F}$. Each element of $\mathcal{F}$ is of the form $[A, \mathcal{Q}(t)]$ where $\mathcal{Q}(t)$ is a polynomial parametrization (that eventually may be a point), and A is a subset of $\mathbb{R}$ where t takes values. Moreover, if $N \in \mathbb{N}$ then $\deg(\mathcal{Q}(t)) \leq N$ else there is no degree control.

Remark: We denote by $\mathsf{Alg}_3(\epsilon, \mathcal{P}(t), N)$ the output.

1. If $\deg(\chi_1) < \deg(\chi_2)$ THEN $q_1 := 0$ and $r_1 := \chi_1$, ELSE take q_1, r_1 as the quotient and remainder of χ_1 divided by χ_2. Similarly with ξ generating q_2, r_2.
2. [Non-Bounded Part] Compute an upper bound B of the roots in module of $r_1(t) \pm \epsilon\chi_2(t)$ and $r_2(t) \pm \epsilon\xi_2(t)$. $I_0 := (-\infty, -B) \cup (B, \infty)$. Append $[I_0, (q_1, q_2)]$ to $\mathcal{F}$.
3. Decide whether $h(t) := \operatorname{lcm}(\chi_2(t), \xi_2(t))$ has real roots or not. $\mathcal{K} := \emptyset$.
4. [Absence of Real Roots: Partition Step] If $h(t)$ does not have real roots do If $N = \infty$ THEN append $[-B, B]$ to $\mathcal{K}$ ELSE decompose $[-B, B]$ in compact intervals such that $r_1/\chi_2, r_2/\xi_2, \chi_2, \xi_2$ are monotone, and append them to $\mathcal{K}$.
5. [Existence of Real Roots: Partition Step] If $h(t)$ has real roots, isolate them. Let $[-B, B] = \cup_{i=0}^{m+1} K_i \cup_{i=1}^{m} J_i$, with K_i compacts and J_i the isolating open intervals.
   If $N = \infty$ THEN append $K_1, \ldots, K_{m+1}$ to $\mathcal{K}$ ELSE decompose each K_i in compact intervals such that $r_1/\chi_2, r_2/\xi_2, \chi_2, \xi_2$ are monotone, and append them to $\mathcal{K}$.
6. [Degree Control] If $N = \infty$ THEN $\mathcal{K}' := \mathcal{K}$ ELSE
   6.1. $\mathcal{K}' := \emptyset$, $N_1 := \lfloor -1 + N/\deg(\chi_2) \rfloor$, $N_1^* := \lfloor -1 + N/\deg(\xi_2) \rfloor$.
   6.2. For every $K := [\lambda_1, \mu] \in \mathcal{K}$ do
      6.2.1. $n := \mathsf{SAlg}_1(\epsilon, \chi, [\lambda_1, \mu])$, $n^* := \mathsf{SAlg}_1(\epsilon, \xi, [\lambda_1, \mu])$.

6.2.2. If $n \leq N_1, n^* > N_1^*$ do $\ll \lambda_2 := \mathsf{SAlg}_2(\epsilon, \xi, N_1^*, K)$, append $[\lambda_1, \lambda_2]$ to $\mathcal{K}'$, replace K by $[\lambda_2, \mu]$ and go to step 6.2.1. $\gg$
6.2.3. If $n > N_1, n^* \leq N_1^*$ do $\ll \lambda_2 := \mathsf{SAlg}_2(\epsilon, \chi, N_1, K)$, append $[\lambda_1, \lambda_2]$ to $\mathcal{K}'$, replace K by $[\lambda_2, \mu]$ and go to step 6.2.1. $\gg$
6.2.4. If $n > N_1, n^* > N_1^*$ do $\ll \lambda_2 := \min\{\mathsf{SAlg}_2(\epsilon, \chi, N_1, K), \mathsf{SAlg}_2(\epsilon, \xi, N_1^*, K)\}$, append $[\lambda_1, \lambda_2]$ to $\mathcal{K}'$, replace K by $[\lambda_2, \mu]$ and go to step 6.2.1. $\gg$

7. [Polynomial Assignment] For every $K \in \mathcal{K}^{'}$ do $\ll p_1(t) := \mathsf{Alg}_1(\epsilon, \chi, K)$, $p_2(t) := \mathsf{Alg}_1(\epsilon, \xi, K)$, append $[K, (p_1, p_2)]$ to $\mathcal{F}$. $\gg$
8. RETURN $\mathcal{F}$.

Remark 3 In general the output polynomial parametrizations of the algorithm may not join smoothly. This phenomenon can be solved by using polynomial parametric blending. For this purpose, we refer to [8]. □

We illustrate Algorithm 3 by an example. The example has been performed with Maple 8, using floating point arithmetic of 5 digits.

Example 1 We consider the curve $\mathcal{C}$ given by the parametrization

$$\mathcal{P}(t) = \left(\frac{t^3}{t^2 + 0.3}, \frac{\left(t^2 + 0.1\right) t}{t - 0.1} \right).$$

Let $\epsilon = 0.2$. We apply Algorithm 3 first to $(\epsilon, \mathcal{P}(t), \infty)$, and afterwards to $(\epsilon, \mathcal{P}(t), 10)$. We start with $(\epsilon, \mathcal{P}(t), \infty)$. Applying Steps 1 and 2 one gets $B := 1.2623$, and therefore $I_0 := (-\infty, -1.2623) \cup (1.2623, \infty)$. In addition, we assign to I_0 the parametrization $\mathcal{P}_{I_0}(t) = (q_1(t), q_2(t)) := (t, t^2 + 0.1t + 0.11)$. In Step 3 one gets $h(t) := (t^2 + 0.3)(t - 0.1)$. Since $h(t)$ has real roots, we apply Step 5, and $[-B, B]$ is decomposed as

$$[-B, B] := K_1 \cup K_2 \cup J, \quad \text{where}$$

$K_1 := [-1.2623, 0.086667], K_2 := [0.11333, 1.2623]$, $J := (0.086667, 0.11333)$. Note that J isolates the real root of $h(t)$. Since $N = \infty$ we go to Step 7, where Algorithm 1 provides the following polynomial parametrizations (see Figure 1). For K_1 and K_2, we get, respectively,

$$\mathcal{P}_{K_1}(t) = (0.84156t + 0.079658 - 0.049993t^2,$$
$$t^2 + 0.1t - 0.30655 + 0.73405(0.03065 - 0.41655t)(\sum_{k=0}^{72}(0.9266 + 0.73405t)^k),$$

$$\mathcal{P}_{K_2}(t) = (0.84156t - 0.16096 + 0.10102t^2,$$
$$t^2 + 0.1t + 0.52733 + 0.86036(0.05273 - 0.41733t)(\sum_{k=0}^{61}(1.086 - 0.86036t)^k).$$

Observe that the degree of the polynomials approximations are $2, 73, 62$. Now, we apply Algorithm 3 to $(\epsilon, \mathcal{P}(t), 10)$. Steps 1,2,3 are as above, and hence B, I_0 and $\mathcal{P}_{I_0}(t)$ states the same. In Step 5, beside the decomposition generated

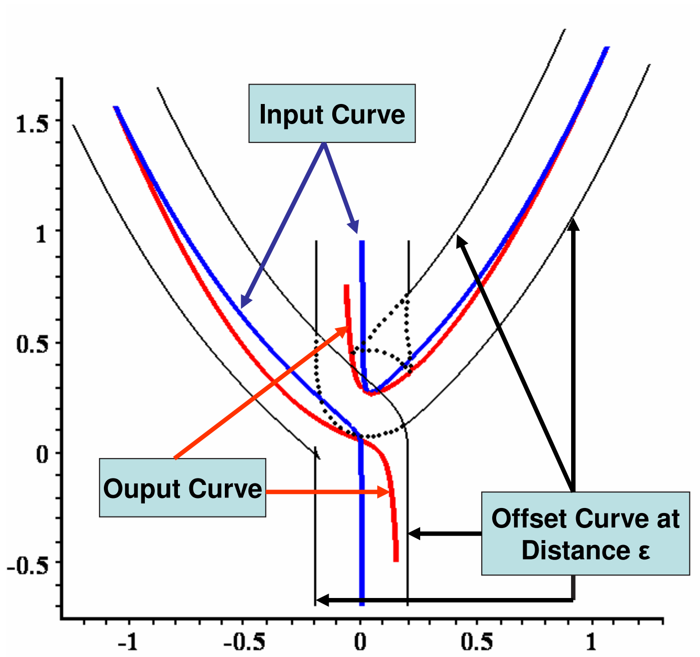

Fig. 1 $\mathcal{P}(t)$, $\mathcal{P}_{K_1}$, $\mathcal{P}_{K_2}$ and offset region of tolerance

by the real roots of $h(t)$, we decompose $[-B, B]$ taking care of the monotony. We get $[-B, B] = K_{1,1} \cup K_{1,2} \cup K_{1,3} \cup K_{2,1} \cup K_{2,2} \cup J_1 \cup J_2 \cup J_3 \cup J_4$, where

$$K_{1,1} = [-1.2623, -0.56105], K_{1,2} = [-0.53439, -0.013333],$$
$$K_{1,3} = [0.01333, 0.086667], K_{2,1} = [0.11333, 0.53439], K_{2,2} = [0.56105, 1.2623],$$
$$J_1 = (-0.56105, -0.53439), J_2 = (-0.01333, 0.01333),$$
$$J_3 = (0.086667, 0.11333), J_4 = (0.53439, 0.56105).$$

Note that the compacts K_1, K_2, introduced in the first part of the example, are now decomposed as $K_1 := K_{1,1} \cup J_1 \cup K_{1,2} \cup J_2 \cup K_{1,3}$ and $K_2 := K_{2,1} \cup J_4 \cup K_{2,2}$. In Step 6 the compacts are decomposed again such that the degree inequality holds. In this case, $K_{1,1}, K_{1,2}, K_{1,3}$ and $K_{2,2}$ do not decompose while $K_{2,1}$ is expressed as $K_{2,1} = K_{2,1,1} \cup K_{2,1,2}$ where $K_{2,1,1} := [0.11333, 0.30920]$ and $K_{2,1,2} := [0.30920, 0.53439]$. Therefore, the parameter space is decomposed as

$$\mathbb{R} = I_0 \cup K_{1,1} \cup K_{1,2} \cup K_{1,3} \cup K_{2,1,1} \cup K_{2,1,2} \cup K_{2,2} \cup_{i=1}^{4} J_i.$$

Applying Step 7 one gets the following polynomial parametrizations.

$$\begin{aligned}
\mathcal{P}_{K_{1,1}}(t) &= (0.19936 + 0.84156t - 0.12511t^2, 0.090476 + 0.090929t + t^2)\\
\mathcal{P}_{K_{1,2}}(t) &= (0.070008 + 0.48769t - 0.24514t^2, 0.044477 + 0.009836t + t^2)\\
\mathcal{P}_{K_{1,3}}(t) &= (-0.001195 + 0.02443t + 0.15913t^2,\\
&\quad -0.0002 - 0.93770t - 17.092t^2 + 254.84t^3 - 8435.9t^4)\\
\mathcal{P}_{K_{2,1,1}}(t) &= (-0.04146 + 0.24166t + 0.43372t^2,\\
&\quad 27.306 - 829.34t + 11620t^2 - 97418t^3 + 539430t^4 - 2056600t^5 +\\
&\quad 5462200t^6 - 9970300t^7 + 11965000t^8 - 8520100t^9 + 2733400t^{10})\\
\mathcal{P}_{K_{2,1,2}}(t) &= (-0.12394 + 0.48769t + 0.43398t^2, 0.18324 + 0.010329t + t^2)\\
\mathcal{P}_{K_{2,2}}(t) &= (-0.19936 + 0.84156t + 0.12511t^2, 0.13756 + 0.085665t + t^2).
\end{aligned}$$

Note that all parametrizations have degree 2 with the exception of $\mathcal{P}_{K_{1,3}}(t)$ that has degree 4 and $\mathcal{P}_{K_{2,1,1}}(t)$ that has degree 10.

5 Error Analysis

In the previous example we have seen that the output polynomial pieces are close to the input curve. In this section, we state properly this behavior analyzing the error. Indeed, we prove that the generated portions of polynomial curves lie within the offset region of the input curve at distance at most $\sqrt{2}\,\epsilon$, and conversely

For this purpose, let $\epsilon, \mathcal{C}$ and $\mathcal{P}(t)$ be as in Section 4. Furthermore, let $\mathcal{F} := \mathsf{Alg}_3(\epsilon, \mathcal{P}(t), N)$. Therefore, $\mathcal{F}$ is a list, and each element of $\mathcal{F}$ is a pair of the form $[\,I,\, \mathcal{Q}(t)\,]$ where $\mathcal{Q}(t)$ is a polynomial parametrization (that eventually may be a constant point), and I is a subset of $\mathbb{R}$ where t takes values. In addition, for $[\,I,\, \mathcal{Q}(t)\,] \in \mathcal{F}$, we use the notation $\mathcal{C}_I^* := \{\mathcal{Q}(t) \mid t \in I\}$ and $\mathcal{C}_I := \{\mathcal{P}(t) \mid t \in I\}$. In this situation, we prove the following theorem.

Theorem 6 *Let $[\,I,\, \mathcal{Q}(t)\,] \in \mathcal{F}$. Then, it holds that:*
(1) For every $P \in \mathcal{C}_I$ there exists $P^\star \in \mathcal{C}_I^\star$ such that $\|P - P^\star\|_2 \le \sqrt{2}\,\epsilon$.
(2) For every $P^\star \in \mathcal{C}_I^\star$, there exists $P \in \mathcal{C}_I$ such that $\|P - P^\star\|_2 \le \sqrt{2}\,\epsilon$.

Proof. We prove statement (1). Statement (2) follows similarly. $[\,I,\, \mathcal{Q}(t)\,]$ has been generated in either Step 2 or Step 7 of Algorithm 3. We treat each situation separately.

[Case of Step 2.] In this case, $I := (-\infty, -B) \cup (B, \infty)$ and $\mathcal{Q}(t) := (q_1(t), q_2(t))$. Let $\Delta_1(t) = |\chi(t) - q_1(t)|$ and $\Delta_2(t) = |\xi(t) - q_2(t)|$. Taking into account how B is computed, for $t \in I$ it holds that $\Delta_1(t) \le \epsilon$, and $\Delta_2(t) \le \epsilon$. Therefore, for every $P := \mathcal{P}(t_0) \in \mathcal{C}_I$, we take $P^\star := \mathcal{Q}(t_0) \in \mathcal{C}_I^\star$ and

$$\|P - P^\star\|_2 = \sqrt{\Delta_1(t_0)^2 + \Delta_2(t_0)^2} \le \sqrt{2}\,\epsilon.$$

[Case of Step 7.] In this case I is a compact interval, and $\mathcal{Q}(t) = (p_1(t), p_2(t))$ where $p_1(t) := \mathsf{Alg}_1(\epsilon, \chi(t), I)$ and $p_2 := \mathsf{Alg}_1(\epsilon, \xi(t), I)$. Let $\Delta_1(t) = |\chi(t) - p_1(t)|$ and $\Delta_2(t) = |\xi(t) - p_2(t)|$. Then, by Theorem 3 one gets $\Delta_1(t) \le \epsilon$, and $\Delta_2(t) \le \epsilon$ for $t \in I$. Now, for every $P := \mathcal{P}(t_0) \in \mathcal{C}_I$, we take $P^\star := \mathcal{Q}(t_0) \in \mathcal{C}^\star$ and the result follows as in the previous case. □

Now we prove, in terms of offsets, that the piecewise parametrization generated by Algorithm 3 is close to the original curve. For this purpose, if $\mathcal{D}$ is a piece of a plane curve, we denote by $\mathcal{O}_d(\mathcal{D})$ the classical offset region of $\mathcal{D}$ at distance d; i.e. it is the union of all close disks of radius d centered at points of $\mathcal{D}$.

Corollary 1 *Let* $[\,I,\ \mathcal{Q}(t)\,] \in \mathcal{F}$. *Then,* $\mathcal{C}_I \subset \mathcal{O}_{\sqrt{2}\epsilon}(\mathcal{C}_I^\star)$ *and* $\mathcal{C}_I^\star \subset \mathcal{O}_{\sqrt{2}\epsilon}(\mathcal{C}_I)$.

Proof. We prove the first inclusion; the second follows similarly. Let $P = \mathcal{P}(t_0) \in \mathcal{C}_I$ where $t_0 \in I$. By Theorem 6, there exists $P^\star = \mathcal{Q}(t_0) \in \mathcal{C}_I^\star$ such that $\|P - P^\star\|_2 \leq \sqrt{2}\epsilon$. Thus $P \in \mathcal{O}_{\sqrt{2}\epsilon}(\mathcal{C}_I^\star)$. □

6 Comparison of Methods

In the introduction we have mentioned Sederberg–Kakimoto's method (see [3], [7]), and Bernstein based method (see [5], pp.127). In this section, we briefly compare our method with them analyzing their main advantages and disadvantages.

The first method is based on the combination of Bernstein-polynomials with the Approximation Theorem of Weierstrass. In this case, an error bound of the approximation is given in terms of the modulus of continuity of the rational functions in the compact interval. This standard method has the advantage of representing the solution in terms of Bézier polynomial. However, it has two main computational difficulties when the denominators of $\mathcal{P}(t)$ have real roots. The first one is that the degree of the polynomials can be very high if the real roots are near the extremes of the interval. The second difficulty is that, in order to guarantee a good distance bound in the error analysis, the compact intervals have to be taken sufficiently far from the real roots.

In [7] the authors present an alternative Bézier-like approach, based on the so called hybrid polynomials, that solves partially the difficulties remarked in the previous paragraph. The approach in [7] is based on the observation that any rational function can be written as a Bézier rational function, and any Bézier rational function can be expressed as a Bézier polynomial with one rational function moving control point, which is itself a Bézier rational function. The polynomials with a rational function moving control point are called hybrid polynomial. The approach presented in [7] provides very low degree in the polynomials. Nevertheless, the main disadvantage of Sederberg-Kakimoto's approach is the control of the error and the convergence of the method. In [3] the convergence is analyzed and necessary conditions are given. However, the error analysis of the method does not provide a priori bound easily computable from the input. This implies that each time the resulting bound is not satisfactory, the process needs to be repeated either changing the interval or increasing the degree of the polynomials.

The direct application of our approach (i.e. when N is taken as infinity in Algorithm 3) may also generate high degree polynomials. Nevertheless, the method allows to control the degree of the output. Indeed, if N in Algorithm 3 is taken as a natural number, then the degree of the polynomials are bounded by N. Another important advantage of our method is its good behavior in terms of error analysis. We provide an explicit bound for the error that depends only on the tolerance. Therefore, in our method, bounds for the degrees and for the error are known in advance. This implies that none iteration of the process is required.

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