Asymptotes and Perfect Curves

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Abstract

We develop a method for computing all the *generalized asymptotes* of a real plane algebraic curve $\mathcal{C}$ implicitly defined by an irreducible polynomial $f(x,y) \in \mathbb{R}[x,y]$. The approach is based on the notion of perfect curve introduced from the concepts and results presented in Blasco and Pérez-Díaz (2013). In addition, we study some properties concerning perfect curves and in particular, we provide a necessary and sufficient condition for a plane curve to be perfect. Finally, we show that the equivalent class of generalized asymptotes for a branch of a plane curve can be described as an affine space $\mathbb{R}^m$ for a certain m.

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1. Introduction

In this paper, we consider $\mathcal{C}$ an irreducible real plane algebraic curve over $\mathbb{C}$ implicitly defined by an irreducible polynomial $f(x,y) \in \mathbb{R}[x,y]$. That is, we work over the field of complex numbers $\mathbb{C}$, but $\mathcal{C}$ has infinitely many points in the affine plane over $\mathbb{R}$. Since every irreducible real curve has a real defining polynomial, we assume that $\mathcal{C}$ is defined by the irreducible polynomial $f(x,y) \in \mathbb{R}[x,y]$ (see Chapter 7 in Sendra et al. (2007)). The assumption of reality is included because of the nature of the problem, but the theory can be similarly developed for the case of complex non-real curves.

Under these conditions, we deal with the problem of computing the asymptotes of the infinity branches of $\mathcal{C}$. This question is very important

in the study of real plane algebraic curves because asymptotes contain much of the information about the behavior of the curves in the large. For instance, determining the asymptotes of a curve is an important step in sketching its graph.

Intuitively speaking, the asymptotes of some branch of a real plane algebraic curve reflect the status of this branch at the points with sufficiently large coordinates. In analytic geometry, an asymptote of a curve is a line such that the distance between the curve and the line approaches zero as they tend to infinity. In some contexts, such as algebraic geometry, an asymptote is defined as a line which is tangent to a curve at infinity.

More precisely, let $\mathcal{C}$ be a real plane algebraic curve, and B an infinity branch of $\mathcal{C}$. A line ℓ is called the asymptote of $\mathcal{C}$ at B, if for every $\epsilon \in \mathbb{R}^+$, there exists $M \in \mathbb{R}^+$ such that $d(P, \ell) < \epsilon$, for every $P \in B$ with $\|P\| > M$.

If B can be defined by some explicit equation of the form $y = f(x)$ (or $x = g(y)$), where f (or g) is a continuous function on an infinite interval, it is easy to decide whether $\mathcal{C}$ has an asymptote at B by analyzing the existence of the limits of certain functions when $x \to \infty$ (or $y \to \infty$). Moreover, if these limits can be computed, we may obtain the equation of the asymptote of $\mathcal{C}$ at B. However, if this branch B is implicitly defined and its equation cannot be converted into an explicit form, both the decision and the computation of the asymptote of $\mathcal{C}$ at B require some others tools.

Determining the asymptotes of an implicit algebraic plane curve is a topic considered in many text-books on analysis (see for instance Maxwell (1962)). In Kečkić (2000), is presented a fast and a simple method for obtaining the asymptotes of a curve defined by an irreducible polynomial, with emphasis on second order polynomials. In Zeng (2007), an algorithm for computing all the linear asymptotes of a real plane algebraic curve $\mathcal{C}$ implicitly defined, is obtained. More precisely, one may decide whether a branch of $\mathcal{C}$ has an asymptote, compute all the asymptotes of $\mathcal{C}$, and determine those branches whose asymptotes are the same. By this algorithm, all the asymptotes of $\mathcal{C}$ may be represented via polynomial real root isolation.

An algebraic plane curve may have more general curves than lines describing the status of a branch at the points with sufficiently large coordinates. This motivates that in this paper, we are interested in analyzing and computing these *generalized asymptotes*. Intuitively speaking, we say that a curve $\widetilde{\mathcal{C}}$

is a *generalized asymptote* (or *g-asymptote*) of another curve $\mathcal{C}$ if the distance between $\widetilde{\mathcal{C}}$ and $\mathcal{C}$ tends to zero as they tend to infinity, and $\mathcal{C}$ can not be approached by a new curve of lower degree.

In this paper, we present an algorithm for computing all the g-asymptotes of a real algebraic plane curve $\mathcal{C}$ defined by an irreducible polynomial $f(x, y) \in \mathbb{R}[x, y]$. For this purpose, we use the results in Blasco and Pérez-Díaz (2013), where the notions of convergent branches (that is, branches that get closer as they tend to infinity) and approaching curves are introduced. In addition, in Blasco and Pérez-Díaz (2013), we also provide some results that characterize whether two implicit algebraic plane curves approach each other at the infinity, and we present a method to compare the asymptotic behavior of two curves (i.e., the behavior at the infinity). In particular, we prove that if two plane curves have the same *asymptotic behavior*, the Hausdorff distance between them is finite. Some of these results are summarized in this paper (see Section 2).

The study of approaching curves and convergent branches leads to the notions of *perfect curve* (a curve of degree d that cannot be approached by any curve of degree less than d) and *g-asymptote* (a perfect curve that approaches another curve at an infinity branch). These concepts are introduced in Section 3. In this section, we also develop an algorithm that computes a g-asymptote for each infinity branch of a given curve. In Section 4, we provide some necessary and sufficient conditions for a curve to be perfect. In particular, we show that a perfect curve admits a polynomial parametrization. In Section 5, we observe that "proximity" is an equivalence relation for the set of perfect curves. Hence, an infinity branch of a given curve does not have, in general, a unique g-asymptote, but a whole "equivalence class" defined by infinitely many curves. We show that all the curves in a same class have the same degree d, and that any class is isomorphic to $\mathbb{R}^{d(d-1)/2}$. Finally, we present some results that allows us to obtain, under certain assumptions, all the curves within a class.

2. Notation and Previous Results

In this section, we introduce the notion of *infinity branch*, *convergent branches* and *approaching curves*, and we obtain some properties which allow us to compare the behavior of two implicit algebraic plane curves at the

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infinity. For more details on these concepts and results, we refer to Blasco and Pérez-Díaz (2013).

Throughout the paper, we consider an irreducible algebraic affine plane curve $\mathcal{C}$ over $\mathbb{C}$ defined by the irreducible polynomial $f(x, y) \in \mathbb{R}[x, y]$. Let $\mathcal{C}^*$ be its corresponding projective curve defined by the homogeneous polynomial

$$F(x, y, z) = f_d(x, y) + z f_{d-1}(x, y) + z^2 f_{d-2}(x, y) + \cdots + z^d f_0 \in \mathbb{R}[x, y, z],$$

where $d := \deg(\mathcal{C})$. We assume that $(0 : 1 : 0)$ is not an infinity point of $\mathcal{C}^*$; otherwise, we may consider a linear change of coordinates.

We note that we work over $\mathbb{C}$, but we assume that the curve has infinitely many points in the affine plane over $\mathbb{R}$ and then, $\mathcal{C}$ has a real defining polynomial (see Chapter 7 in Sendra et al. (2007)). We recall that the assumption of reality is included because of the nature of the problem, but the theory developed in this paper can be applied for the case of complex non-real curves.

Let $P = (1 : m : 0)$, $m \in \mathbb{C}$ be an infinity point of $\mathcal{C}^*$, and we consider the curve defined by the polynomial $g(y, z) = F(1 : y : z)$. We compute the series expansion for the solutions of $g(y, z) = 0$. There exist exactly $\deg_Y(g)$ solutions given by different Puiseux series that can be grouped into conjugacy classes. More precisely, if

$$\varphi(z) = m + a_1 z^{N_1/N} + a_2 z^{N_2/N} + a_3 z^{N_3/N} + \cdots \in \mathbb{C} \ll z \gg, \quad a_i \neq 0, \forall i \in \mathbb{N},$$

where $N \in \mathbb{N}$, $N_i \in \mathbb{N}$, $i = 1, \ldots,$ and $0 < N_1 < N_2 < \cdots$, is a Puiseux series such that $g(\varphi(z), z) = 0$, and $\nu(\varphi) = N$ (i.e., N is the *ramification index* of φ), the series

$$\varphi_j(z) = m + a_1 c_j^{N_1} z^{N_1/N} + a_2 c_j^{N_2} z^{N_2/N} + a_3 c_j^{N_3} z^{N_3/N} + \cdots$$

where $c_j^N = 1$, $j = 1, \ldots, N$, are called the *conjugates* of φ. The set of all (distinct) conjugates of φ is called the *conjugacy class* of φ, and the number of different conjugates of φ is $\nu(\varphi)$ (see Duval (1989)).

Since $g(\varphi(z), z) = 0$ in some neighborhood of $z = 0$ where $\varphi(z)$ converges, there exists $M \in \mathbb{R}^+$ such that

$$F(1 : \varphi(t) : t) = g(\varphi(t), t) = 0, \quad \text{for } t \in \mathbb{C} \text{ and } |t| < M,$$

which implies that $F(t^{-1} : t^{-1}\varphi(t) : 1) = f(t^{-1}, t^{-1}\varphi(t)) = 0$, for $t \in \mathbb{C}$ and $0 < |t| < M$. We set $t^{-1} = z$, and we obtain that

$$f(z, r(z)) = 0, \quad z \in \mathbb{C} \text{ and } |z| > M^{-1}, \qquad \text{where}$$

$$r(z) = z\varphi(z^{-1}) = mz + a_1 z^{1-N_1/N} + a_2 z^{1-N_2/N} + a_3 z^{1-N_3/N} + \cdots, \quad a_i \neq 0, \forall i \in \mathbb{N}$$

$N, N_i \in \mathbb{N}, \ i = 1, \ldots,$ and $0 < N_1 < N_2 < \cdots.$

Reasoning similarly with the N different series in the conjugacy class, $\varphi_1, \ldots, \varphi_N$, we get

$$r_i(z) = z\varphi_i(z^{-1}) = mz + a_1 c_i^{N_1} z^{1-N_1/N} + a_2 c_i^{N_2} z^{1-N_2/N} + a_3 c_i^{N_3} z^{1-N_3/N} + \cdots$$

where $c_1, \ldots, c_N$ are the N complex roots of $x^N = 1$.

Under these conditions, we introduce the following definition of branch.

Definition 1. *An infinity branch of an affine plane curve $\mathcal{C}$ associated to the infinity point $P = (1 : m : 0)$, $m \in \mathbb{C}$, is a set $B = \bigcup\limits_{j=1}^{N} L_j$, where $L_j = \{(z, r_j(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$, $M \in \mathbb{R}^+$, and*

$$r_j(z) = z\varphi_j(z^{-1}) = mz + a_1 c_j^{N_1} z^{1-N_1/N} + a_2 c_j^{N_2} z^{1-N_2/N} + a_3 c_j^{N_3} z^{1-N_3/N} + \cdots$$

where $N, N_i \in \mathbb{N}, \ i = 1, \ldots, \ 0 < N_1 < N_2 < \cdots,$ and $c_j^N = 1, \ j = 1, \ldots, N.$ The subsets $L_1, \ldots, L_N$ are called the leaves of the infinity branch B.

Remark 1. *We observe that:*

1. *An infinity branch is uniquely determined from one leaf, up to conjugation. That is, if $B = \bigcup\limits_{i=1}^{N} L_i$, where $L_i = \{(z, r_i(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_i\}$, and*

   $$r_i(z) = z\varphi_i(z^{-1}) = mz + a_1 z^{1-N_1/N} + a_2 z^{1-N_2/N} + a_3 z^{1-N_3/N} + \cdots$$

   then $r_j = r_i, \ j = 1, \ldots, N$, up to conjugation; i.e.

   $$r_j(z) = z\varphi_j(z^{-1}) = mz + a_1 c_j^{N_1} z^{1-N_1/N} + a_2 c_j^{N_2} z^{1-N_2/N} + a_3 c_j^{N_3} z^{1-N_3/N} + \cdots$$

   where $N, N_i \in \mathbb{N}$, and $c_j^N = 1, \ j = 1, \ldots, N.$

2. *We may represent $L_i = \{(z, r_i(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}, i = 1, \ldots, N$, where $M := \max\{M_1, \ldots, M_N\}$.*
3. *By abuse of notation, we say that N is the ramification index of the branch B, and we write $\nu(B) = N$. Note that B has $\nu(B)$ leaves.*

Let $\psi(t) := \varphi(t^N)$, where $\varphi(z)$ is a series expansion for a solution of $g(y, z) = 0$. Observe that $(1 : \psi(t) : t^N)$ is a local projective parametrization, with center at P, of the projective curve $\mathcal{C}^*$. Thus, from $\psi_i(t) := \varphi_i(t^N)$, $i = 1, \ldots, N$ (φ_i are the N different series in the conjugacy class of φ), we obtain N equivalent local projective parametrizations, $(1 : \psi_i(t) : t^N)$ (note that they are equivalent since $\varphi_1, \ldots \varphi_N$ belong to the same conjugacy class). Therefore, the leaves of B are all associated to a unique infinity place.

Conversely, from a given infinity place defined by a local projective parametrization $(1 : \psi(t) : t^N)$ (see Theorem 2.5.3 in Sendra et al. (2007)), we obtain N Puiseux series, $\varphi_j(t) = \psi(c_j t^{1/N})$, $c_j^N = 1$, that provide different expressions $r_j(z) = z\varphi_j(z^{-1})$, $j = 1, \ldots, N$. Hence, the infinity branch B is defined by the leaves $L_i = \{(z, r_i(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$, $i = 1, \ldots, N$.

From the above discussion, we deduce that there exists a one-to-one relation between infinity places and infinity branches. In addition, we can say that each infinity branch is associated to a unique infinity point given by the center of the corresponding infinity place. Reciprocally, taking into account the above construction, we get that every infinity point has associated, at least, one infinity branch. Hence, every algebraic plane curve has, at least, one infinity branch. Furthermore, every algebraic plane curve has a finite number of branches.

In the following, we introduce the notions of convergent branches and approaching curves. Intuitively speaking, two infinity branches converge if they get closer as they tend to infinity. This concept will allow us to analyze whether two curves approach each other. For further details see Blasco and Pérez-Díaz (2013).

Definition 2. *Two infinity branches, B and $\overline{B}$, are convergent if there exist two leaves $L = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\} \subset B$ and $\overline{L} = \{(z, \overline{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > \overline{M}\} \subset \overline{B}$ such that $\lim_{z \to \infty}(\overline{r}(z) - r(z)) = 0$. In this case, we say that the leaves L and $\overline{L}$ converge.*

Observe that two convergent infinity branches are associated to the same

infinity point (see Remark 4.5 in Blasco and Pérez-Díaz (2013)).

In Lemma 1, we characterize the convergence of two given infinity branches (see Lemma 4.2, and Proposition 4.6 in Blasco and Pérez-Díaz (2013)).

Lemma 1. *The following statements hold:*

- *Two leaves $L = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$ and $\overline{L} = \{(z, \overline{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > \overline{M}\}$ are convergent if and only if the terms with non negative exponent in the series $r(z)$ and $\overline{r}(z)$ are the same.*

- *Two infinity branches B and $\overline{B}$ are convergent if and only if for each leaf $L \subset B$ there exists a leaf $\overline{L} \subset \overline{B}$ convergent with L, and reciprocally.*

Note that two convergent branches may be contained in the same curve or they may belong to different curves. In this second case, we will say that *these curves approach each other*. More precisely, we have the following definition.

Definition 3. *Let $\mathcal{C}$ be an algebraic plane curve with an infinity branch B. We say that a curve $\overline{\mathcal{C}}$ approaches $\mathcal{C}$ at its infinity branch B if there exists one leaf $L = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\} \subset B$ such that $\lim_{z \to \infty} d((z, r(z)), \overline{\mathcal{C}}) = 0$.*

In the following, we state some important results concerning two curves that approach each other. These results are proved in Blasco and Pérez-Díaz (2013) (see Theorem 4.11 and Corollary 4.13).

Theorem 1. *Let $\mathcal{C}$ be a plane algebraic curve with an infinity branch B. A plane algebraic curve $\overline{\mathcal{C}}$ approaches $\mathcal{C}$ at B if and only if $\overline{\mathcal{C}}$ has an infinity branch, $\overline{B}$, such that B and $\overline{B}$ are convergent.*

Remark 2. *It holds that:*

1. *Relation of "proximity" is a symmetric relation. That is, $\overline{\mathcal{C}}$ approaches $\mathcal{C}$ at some infinity branch B if and only if $\mathcal{C}$ approaches $\overline{\mathcal{C}}$ at some infinity branch $\overline{B}$. In the following, we say that $\mathcal{C}$ and $\overline{\mathcal{C}}$ approach each other or that they are approaching curves.*
2. *Two approaching curves have a common infinity point.*
3. *$\overline{\mathcal{C}}$ approaches $\mathcal{C}$ at an infinity branch B iff for every leaf $L = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\} \subset B$, it holds that $\lim_{z \to \infty} d((z, r(z)), \overline{\mathcal{C}}) = 0$.*

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Corollary 1. *Let $\mathcal{C}$ be an algebraic plane curve with an infinity branch B. Let $\overline{\mathcal{C}}_1$ and $\overline{\mathcal{C}}_2$ be two different curves that approach $\mathcal{C}$ at B. Then:*

1. *$\overline{\mathcal{C}}_i$ has an infinity branch $\overline{B}_i$ that converges with B, for $i = 1, 2$.*
2. *$\overline{B}_1$ and $\overline{B}_2$ are convergent. Then, $\overline{\mathcal{C}}_1$ and $\overline{\mathcal{C}}_2$ approach each other.*

Taking into account that an infinity branch B is uniquely determined from one leaf, up to conjugation (see statement 1 in Remark 1), and that the results stated above hold for any leaf of B, for the sake of simplicity, in the following,

$$B = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$$

stands for the infinity branch whose leaves are obtained by conjugation on

$$r(z) = mz + a_1 z^{1-N_1/N} + a_2 z^{1-N_2/N} + a_3 z^{1-N_3/N} + \cdots, \quad a_i \neq 0, \forall i \in \mathbb{N}$$

$N, N_i \in \mathbb{N}$, $i = 1, \ldots$, and $0 < N_1 < N_2 < \cdots$. We also will prove that the results obtained throughout the paper hold for any leaf.

3. Asymptotes and perfect curves

Given an algebraic plane curve $\mathcal{C}$ and a infinity branch B of $\mathcal{C}$, in Blasco and Pérez-Díaz (2013) we analyze whether $\mathcal{C}$ can be approached at B by a new curve $\overline{\mathcal{C}}$. Intuitively speaking, if $\mathcal{C}$ is approached at B by $\overline{\mathcal{C}}$, and $\deg(\overline{\mathcal{C}}) < \deg(\mathcal{C})$, one may say that $\mathcal{C}$ *degenerates*, since $\mathcal{C}$ behaves at the infinity as a curve of less degree.

For instance, a hyperbola is a curve of degree 2 that has two real asymptotes, which implies that the hyperbola degenerates, at the infinity, in two lines. The behavior of an ellipse is similar; in this case, the infinity branches are complex but they can also be approached by (complex) lines. However, the asymptotic behavior of a parabola is different, since at the infinity, the parabola cannot be approached by any line. This motivates the following definition.

Definition 4. *A curve of degree d is a perfect curve if it cannot be approached by any curve of degree less than d.*

A curve that is not perfect can be approached by other curves of less degree. If these curves are perfect, we call them *g-asymptotes*. More precisely, we have the following definition.

Definition 5. *Let $\mathcal{C}$ be a curve with an infinity branch B. A g-asymptote (generalized asymptote) of $\mathcal{C}$ at B is a perfect curve that approaches $\mathcal{C}$ at B.*

Note that the notion of *g-asymptote* is similar to the classical concept of asymptote. The difference is that a g-asymptote is not necessarily a line, but a perfect curve. Actually, it is a generalization, since every line is a perfect curve (this fact follows from Definition 4). Thus, the algorithm presented in Section 3.1 also computes the asymptotes being lines (see Example 1). Throughout the paper we refer to *g-asymptote* simply as *asymptote*.

In order to clarify this notion, let us consider a plane curve $\mathcal{C}$ defined by the irreducible polynomial

$$f(x,y) = -yx - y^2 - x^3 + 2x^2y + x^2 - 2y \in \mathbb{R}[x,y].$$

$\mathcal{C}$ has degree 3, and two infinity branches. In Figure 1, one can check that these infinity branches are approached by the parabola $y - 2x^2 + 3/2x + 15/8 = 0$, and the line $y - x/2 + 1/8 = 0$.
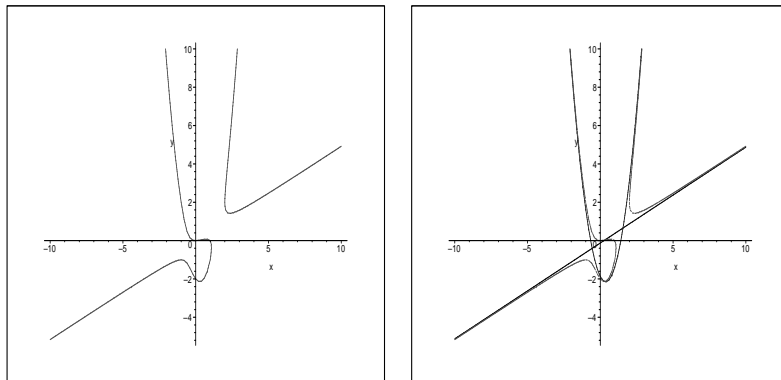

Figure 1: Curve $\mathcal{C}$ (left) approached by a parabola and a line (right).

Later we see that other plane curves of degree 3, like $y - x^3 = 0$ or $y^2 - x^3 = 0$, cannot be approached by any curve of degree less than 3. That is, they are perfect curves.

Remark 3. *The degree of an asymptote is less or equal than the degree of the curve it approaches. In fact, an asymptote of a curve $\mathcal{C}$ at a branch B has minimal degree among all the curves that approach $\mathcal{C}$ at B. Indeed, let $\mathcal{D}$ be an asymptote of $\mathcal{C}$ at B and let $\mathcal{D}'$ be another curve that approaches $\mathcal{C}$ at*

B. From Corollary 1, $\mathcal{D}'$ approaches $\mathcal{D}$, and since $\mathcal{D}$ is perfect, we conclude that $\deg(\mathcal{D}') \geq \deg(\mathcal{D})$.

In the following, we show that every infinity branch of a given algebraic plane curve has, at least, one asymptote (see Theorem 2). In order to prove this property, we first need to show some previous results. For this purpose, let $\mathcal{C}$ be a plane curve and $B = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$ an infinity branch of $\mathcal{C}$ associated to $P = (1 : m : 0)$. From Definition 1, we have that

$$r(z) = mz + a_1 z^{-N_1/N+1} + \cdots + a_k z^{-N_k/N+1} + a_{k+1} z^{-N_{k+1}/N+1} + \cdots \quad (1)$$

where $a_1, a_2, \ldots \in \mathbb{C} \setminus \{0\}, m \in \mathbb{C}, N, N_1, N_2 \ldots \in \mathbb{N}$, and $0 < N_1 < N_2 < \cdots$. In addition, let $N_k \leq N < N_{k+1}$, i.e. the terms $a_j z^{-N_j/N+1}$ with $j \geq k+1$ have negative exponent. Note that $\nu(B) = N$.

Lemma 2. *Let $\mathcal{C}$ be a plane curve containing an infinity branch of the form in (1), and let $f(x, y) \in \mathbb{R}[x, y]$ be the irreducible polynomial defining implicitly $\mathcal{C}$. It holds that $(y - mx)^N$ divides $f_d(x, y)$.*

Proof: Let $r_1, \ldots, r_N$ be the conjugates of r. That is,

$$r_i(z) = z\varphi_i(z^{-1}) = mz + a_1 c_i^{N_1} z^{1-N_1/N} + a_2 c_i^{N_2} z^{1-N_2/N} + a_3 c_i^{N_3} z^{1-N_3/N} + \cdots,$$

where $c_i^N = 1$, and φ_i, $i = 1, \ldots, N$, are the series expansions for solutions of $g(y, z) = F(1 : y : z) = 0$. Now, we see $g(y, z)$ as a polynomial in the variable y, and we denote it by $g_z(y) \in \mathbb{C} \ll z \gg [y]$. From Theorem 4.2 in Verger-Gaugry (2011), we deduce that $(y - \varphi_i(z))$ divides $g_z(y)$, for $i = 1, \ldots, N$. Hence, $x(y/x - \varphi_i(z/x))$ divides $x^d F(1 : y/x : z/x) = F(x : y : z)$ w.r.t the variable y. In addition, since the factors $(y - x\varphi_i(z/x))$ are all different (they are obtained by conjugation), we get that $\prod_{i=1}^N (y - x\varphi_i(z/x))$ divides $F(x : y : z)$. Observe that since

$$\varphi_i(x^{-1}) = m + a_1 c_i^{N_1} x^{-N_1/N} + a_2 c_i^{N_2} x^{-N_2/N} + a_3 c_i^{N_3} x^{-N_3/N} + \cdots,$$

we get that

$$x(y/x - \varphi_i(z/x)) = y - (mx + a_1 c_i^{N_1} x^{1-N_1/N} z^{N_1/N} + a_2 c_i^{N_2} x^{1-N_2/N} z^{N_2/N} + \cdots) =$$

$$y - mx - p_i(x, z), \quad \text{where } p_i(x, z) := a_1 c_i^{N_1} x^{1-N_1/N} z^{N_1/N} + a_2 c_i^{N_2} x^{1-N_2/N} z^{N_2/N} + \cdots,$$

and $p_i(x, 0) = 0$. Thus,

$$\prod_{i=1}^{N}(y - x\varphi_i(z/x)) = \prod_{i=1}^{N}(y - mx - p_i(x, z))$$

divides $F(x : y : z)$, and if we set $z = 0$, since $p_i(x, 0) = 0$, we deduce that $\prod_{i=1}^{N}(y - mx) = (y - mx)^N$ divides $F(x : y : 0) = f_d(x, y)$. $\qquad \square$

Remark 4. *From the proof of Lemma 2, we get that $h(x, y) := \prod_{i=1}^{N}(y - r_i(x))$ divides $f(x, y)$ as polynomials in the variable y. Indeed: since $\prod_{i=1}^{N}(y - x\varphi_i(z/x)) = \prod_{i=1}^{N}(y - mx - p_i(x, z))$ divides $F(x : y : z)$, we set $z = 1$, and we deduce that $h(x, y)$ divides $f(x, y)$ w.r.t. the variable y.*

In the following, we write equation (1) defining a branch B as

$$r(z) = mz + a_1 z^{-n_1/n+1} + \cdots + a_k z^{-n_k/n+1} + a_{k+1} z^{-N_{k+1}/N+1} + \cdots \quad (2)$$

where $\gcd(N, N_1, \ldots, N_k) = b, N_j = n_j b, N = nb, j = 1, \ldots, k$. That is, we have simplified the non negative exponents such that $\gcd(n, n_1, \ldots, n_k) = 1$. Note that $0 < n_1 < n_2 < \cdots, n_k \leq n$, and $N < n_{k+1}$, i.e. the terms $a_j z^{-N_j/N+1}$ with $j \geq k + 1$ have negative exponent. We denote these terms as

$$A(z) := \sum_{\ell=k+1}^{\infty} a_\ell z^{-q_\ell}, \quad q_\ell = -N_\ell/N + 1 \in \mathbb{Q}^+, \ \ell \geq k + 1.$$

Under these conditions, we introduce the definition of degree of a branch B as follows:

Definition 6. *Let $B = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$ defined by (2) an infinity branch associated to $P = (1 : m : 0), m \in \mathbb{C}$. We say that n is de degree of B, and we denote it by $\deg(B)$.*

Proposition 1. *Let $\overline{\mathcal{C}}$ be a curve that approaches $\mathcal{C}$ at its infinity branch B. Let $\overline{f} \in \mathbb{R}[x, y]$ be the implicit polynomial of $\overline{\mathcal{C}}$. Then, $(y - mx)^n$ divides the homogeneous form of maximum degree of $\overline{f}(x, y)$.*

Proof: Using Theorem 1, we get that $\overline{\mathcal{C}}$ has an infinity branch $\overline{B} = \{(z, \overline{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > \overline{M}\}$ convergent with $B = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| >$

M}. From Lemma 1, we deduce that the terms with non negative exponent in the series $r(z)$ and $\bar{r}(z)$ are the same, and hence $\overline{B}$ is a branch of degree n of the form given in (2). Thus, Lemma 2 states that the homogeneous form of maximum degree of $\overline{f}(x,y)$ is divided by $(y - mx)^{\nu(\overline{B})}$. Now, the result follows taking into account that, always, $\deg(\overline{B}) = n \leq \nu(\overline{B})$. $\qquad\square$

Remark 5. *From Lemma 2, we deduce that a curve $\mathcal{C}$ containing an infinity branch B of degree n has degree at least $\nu(\overline{B}) \geq n$. Furthermore, from Proposition 1, we also have that $\deg(\overline{\mathcal{C}}) \geq n$ for any curve $\overline{\mathcal{C}}$ approaching $\mathcal{C}$ at B.*

3.1. Construction of an asymptote

Taking into account the results presented above, we have that any curve $\overline{\mathcal{C}}$ approaching $\mathcal{C}$ at B should have an infinity branch $\overline{B} = \{(z, \bar{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > \overline{M}\}$ such that the terms with non negative exponent in $r(z)$ and $\bar{r}(z)$ are the same. In the simplest case, if $A = 0$ (i.e. there are not terms with negative exponent; see equation (2)), we obtain

$$\tilde{r}(z) = mz + a_1 z^{-n_1/n+1} + a_2 z^{-n_2/n+1} + \cdots + a_k z^{-n_k/n+1}, \qquad (3)$$

where $a_1, a_2, \ldots \in \mathbb{C} \setminus \{0\}, m \in \mathbb{C}, n, n_1, n_2 \ldots \in \mathbb{N}, \gcd(n, n_1, \ldots, n_k) = 1$, and $0 < n_1 < n_2 < \cdots$. Note that $\tilde{r}$ has the same terms with non negative exponent that r, and $\tilde{r}$ does not have terms with negative exponent.

Let $\widetilde{\mathcal{C}}$ be the plane curve containing the branch $\widetilde{B} = \{(z, \tilde{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > \widetilde{M}\}$ (note that $\widetilde{\mathcal{C}}$ is unique since two different algebraic curves have finitely many common points). Observe that

$$\widetilde{\mathcal{Q}}(t) = (t^n, mt^n + a_1 t^{n-n_1} + \cdots + a_k t^{n-n_k}) \in \mathbb{C}[t]^2, \qquad (4)$$

where $n, n_1, \ldots, n_k \in \mathbb{N}, \gcd(n, n_1, \ldots, n_k) = 1$, and $0 < n_1 < \cdots < n_k$, is a polynomial parametrization of $\widetilde{\mathcal{C}}$, and it is proper (see Lemma 3). In Theorem 2, we prove that $\widetilde{\mathcal{C}}$ is an asymptote of $\mathcal{C}$ at B.

Lemma 3. *The parametrization given in (4) is proper (i.e. invertible).*

Proof: Let us assume that $\widetilde{\mathcal{Q}}$ is not proper. Then, there exists $R(t) \in \mathbb{C}[t]$, with $\deg(R) = r > 1$, and $\mathcal{Q}(t) = (q_1(t), q_2(t)) \in \mathbb{C}[t]^2$, such that $\mathcal{Q}(R) = \widetilde{\mathcal{Q}}$ (see Binder (1996), Hong and Schicho (1998) or Manocha and Canny (1991)). In particular, we have that $q_1(R(t)) = t^n$, which implies that

$$q_1(t) = (t - R(0))^k, \quad \text{and} \quad R(t) = t^r + R(0), \quad rk = n.$$

Let us consider $R^\star(t) = R(t) - R(0) = t^r \in \mathbb{C}[t]$, and

$$\mathcal{Q}^\star(t) = \mathcal{Q}(t + R(0)) = (t^k, q_2^\star(t)) = (t^k, c_0 + c_1 t + c_2 t^2 + \ldots + c_u t^u) \in \mathbb{C}[t]^2.$$

Then, $\mathcal{Q}^\star(R^\star) = \mathcal{Q}(R) = \widetilde{\mathcal{Q}}$, and in particular $q_2^\star(R^\star) = q_2^\star(t^r) = mt^n + a_1 t^{n-n_1} + a_2 t^{n-n_2} + \cdots + a_k t^{n-n_k}$. That is,

$$c_0 + c_1 t^r + c_2 t^{2r} + \ldots + c_u t^{ur} = mt^n + a_1 t^{n-n_1} + a_2 t^{n-n_2} + \cdots + a_k t^{n-n_k}.$$

From this equality, and taking into account that r divides n (recall that $rk = n$), we deduce that r divides n_j, $j = 1, \ldots, k$. This is impossible, because $r > 1$, and $\gcd(n, n_1, \ldots, n_k) = 1$. Therefore, we conclude that $\widetilde{\mathcal{Q}}$ is proper. $\qquad\square$

The next result states a property concerning the implicit polynomial of $\widetilde{\mathcal{C}}$ (compare with Lemma 2).

Lemma 4. *Let $\widetilde{\mathcal{C}}$ be the plane curve containing the infinity branch given in (3). Let $\tilde{f}(x, y) \in \mathbb{R}[x, y]$ be the implicit polynomial defining $\widetilde{\mathcal{C}}$. It holds that the homogeneous form of maximum degree of $\tilde{f}(x, y)$ is $(y - mx)^n$.*

Proof: First, we consider the polynomial proper parametrization (see Lemma 3) defining $\widetilde{\mathcal{C}}$, and introduced in (4):

$$\widetilde{\mathcal{Q}}(t) = (t^n, mt^n + a_1 t^{n-n_1} + \cdots + a_k t^{n-n_k}) \in \mathbb{C}[t]^2.$$

Now, we distinguish two different cases:

1. If $m = 0$, i.e. B is associated to the infinity point $P = (1 : 0 : 0)$, we apply the results in Sendra et al. (2007) (see Chapter 4), and one has that, up to constants in $\mathbb{R} \setminus \{0\}$,

$$\tilde{f}(x, y) = \text{resultant}_t(x - t^n, y - p(t)) = \prod_{i=1}^{n} (y - p(\alpha_i))$$

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where $p(t) = mt^n + a_1 t^{n-n_1} + \cdots + a_k t^{n-n_k}$, and $\alpha_1, \ldots, \alpha_n$ are the n roots of the equation $x - t^n = 0$. Hence, since $\deg(p) = n - n_1$, we get that the maximum exponent of $p(\alpha_i)$ is $(n - n_1)/n < 1$ and then, the form of maximum degree of $\tilde{f}(x, y)$ is y^n.

2. Let $m \neq 0$, and then B is associated to the infinity point $P = (1 : m : 0)$. In this case, we apply the linear change of variables, $x = X - mY$, $y = mX + Y$, and the infinity point moves to $(1 : 0 : 0)$. By applying case 1, we get that the homogeneous form of maximum degree of $\tilde{f}(X - mY, mX + Y)$ is Y^n. Finally, undoing the change, we get that the homogeneous form of maximum degree of $\tilde{f}(x, y)$ is $(y - mx)^n$. $\square$

Remark 6. *From Lemma 4, we deduce that* $\deg(\widetilde{\mathcal{C}}) = n$.

Theorem 2. *The curve* $\widetilde{\mathcal{C}}$ *is an asymptote of* $\mathcal{C}$ *at* B.

Proof: Taking into account the construction of $\widetilde{\mathcal{C}}$, we have that $\widetilde{\mathcal{C}}$ approaches $\mathcal{C}$ at B. Therefore, we only need to show that $\widetilde{\mathcal{C}}$ is perfect, i.e. that $\widetilde{\mathcal{C}}$ cannot be approached by any curve with degree less than $\deg(\widetilde{\mathcal{C}})$.

For this purpose, we first note that $\widetilde{\mathcal{C}}$ admits the polynomial parametrization given by the form in (4). Then, using the results in Manocha and Canny (1991), we deduce that the unique infinity branch of $\widetilde{\mathcal{C}}$ is $\widetilde{B}$. In addition, we observe that by construction, $\widetilde{B}$ and B are convergent.

Under these conditions, let us consider a plane curve, $\overline{\mathcal{C}}$, that approaches $\widetilde{\mathcal{C}}$ at $\widetilde{B}$. From Theorem 1, we get that $\overline{\mathcal{C}}$ has an infinity branch $\overline{B}$ convergent with $\widetilde{B}$. Since $\widetilde{B}$ and B are convergent, from Corollary 1, we deduce that $\overline{B}$ and B are convergent which implies that $\overline{\mathcal{C}}$ approaches $\mathcal{C}$ at B. Now, from Remarks 5 and 6, we deduce that $\deg(\overline{\mathcal{C}}) \geq n = \deg(\widetilde{\mathcal{C}})$. Therefore, we conclude that $\widetilde{\mathcal{C}}$ is perfect. $\square$

We have shown that, for any infinity branch B of a plane curve $\mathcal{C}$, there always exists an asymptote that approaches $\mathcal{C}$ at B. Furthermore, we have provided a method to obtain it. From these results, we obtain the following algorithm that computes an asymptote for each infinity branch of a given plane curve.

We assume that we have prepared the input curve $\mathcal{C}$, such that by means of a suitable linear change of coordinates, $(0 : 1 : 0)$ is not an infinity point of $\mathcal{C}$.

Algorithm Asymptotes Construction.

Given an irreducible real plane algebraic curve $\mathcal{C}$ implicitly defined by an irreducible polynomial $f(x, y) \in \mathbb{R}[x, y]$, the algorithm computes one asymptote for each of its infinity branches.

1. Compute the infinity points of $\mathcal{C}$. Let $P_1, ..., P_n$ be these points.
2. For each $P_i := (1 : m_i : 0)$ do:

2.1. Compute the infinity branches of $\mathcal{C}$ associated to P_i. Let $B_j = \{(z, r_j(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_j\}$, $j = 1, \ldots, s_i$, be these branches, where r_j is written as in equation (2). That is,

$$r_j(z) = m_i z + a_{1,j} z^{-n_{1,j}/n_j+1} + \cdots + a_{k_j,j} z^{-n_{k_j,j}/n_j+1} + A_j(z),$$

$$A_j(z) = \sum_{\ell=k_j+1}^{\infty} a_{\ell,j} z^{-q_{\ell,j}}, \quad q_{\ell,j} = -N_{\ell,j}/N_j + 1 \in \mathbb{Q}^+, \;\; \ell \geq k_j + 1,$$

$a_{1,j}, a_{2,j}, \ldots \in \mathbb{C} \setminus \{0\}$, $n_j, n_{1,j}, \ldots \in \mathbb{N}$, $0 < n_{1,j} < n_{2,j} < \cdots$, $n_{k_j} \leq n_j$, $N_j < n_{k_j+1}$, and $\gcd(n_j, n_{1,j}, \ldots, n_{k_j,j}) = 1$.

2.2. For each branch B_j, $j = 1, \ldots, s_i$ do:

2.2.1. Consider $\tilde{r}_j$ as in equation (3). That is,

$$\tilde{r}_j(z) = m_i z + a_{1,j} z^{-n_{1,j}/n_j+1} + \cdots + a_{k_j,j} z^{-n_{k_j,j}/n_j+1}$$

Note that $\tilde{r}$ has the same terms with non negative exponent that r, and $\tilde{r}$ does not have terms with negative exponent.

2.2.2. Return the asymptote $\widetilde{\mathcal{C}}_j$ defined by the proper parametrization (see Lemma 3), $\widetilde{Q}_j(t) = (t^{n_j}, \tilde{r}_j(t^{n_j})) \in \mathbb{C}[t]^2$, and the implicit polynomial (see Chapter 4 in Sendra et al. (2007)):

$$\tilde{f}_j(x, y) = \mathrm{Res}_t(x - t^{n_j}, y - \tilde{r}_j(t^{n_j})) \in \mathbb{C}[x, y].$$

Correctness. The algorithm Asymptotes Construction outputs an asymptote $\widetilde{\mathcal{C}}$ that is independent of leaf chosen to define the branch $B = \{(z, r(z)) \in \mathbb{C}^2 :$

$z \in \mathbb{C}, |z| > M\}$ (see Section 2). Indeed: let $\widetilde{\mathcal{C}}$ be an asymptote obtained by the algorithm, and defined by the proper parametrization $\widetilde{Q}(t) = (t^n, \tilde{r}(t^n))$, where

$$\tilde{r}(z) = mz + a_1 z^{-n_1/n+1} + \cdots + a_k z^{-n_k/n+1}, \qquad \text{and}$$

$$r(z) = mz + a_1 z^{-n_1/n+1} + \cdots + a_k z^{-n_k/n+1} + A(z), \quad A = \sum_{\ell=k+1}^{\infty} a_\ell z^{-q_\ell}, \ q_\ell \in \mathbb{Q}^+,$$

$a_1, a_2, \ldots \in \mathbb{C}\backslash\{0\}, \ n, n_1, \ldots \in \mathbb{N}, 0 < n_1 < n_2 < \cdots,$ and $\gcd(n, n_1, \ldots, n_k) = 1, N_j = n_j b, \ N = nb, \ j = 1, \ldots, k$ (see equations (2) and (3)). Now, let

$$r_s(z) = mz + a_1 c_s^{n_1 b} z^{-n_1/n+1} + \cdots + a_k c_s^{n_k b} z^{-n_k/n+1} + A_s(z), \ A_s = \sum_{\ell=k+1}^{\infty} a_\ell c_s^{n_\ell} z^{-q_\ell},$$

where $c_s^N = 1, \ s = 1, \ldots, N$. That is, $r_s = r$, up to conjugation. Then, the parametrization obtained by algorithm using r_s is $\widetilde{Q}_s(t) = (t^n, \tilde{r}_s(t^n))$, where

$$\tilde{r}_s(z) = mz + a_1 c_s^{n_1 b} z^{-n_1/n+1} + \cdots + a_k c_s^{n_k b} z^{-n_k/n+1}.$$

Since

$$\widetilde{Q}(t) = (t^n, \tilde{r}(t^n)) = (t^n, mt^n + a_1 t^{-n_1+n} + \cdots + a_k z^{-n_k+n}), \quad \text{and}$$

$$\widetilde{Q}_s(t) = (t^n, \tilde{r}_s(t^n)) = (t^n, mt^n + a_1 c_s^{n_1 b} t^{-n_1+n} + \cdots + a_k c_s^{n_k b} t^{-n_k+n}),$$

and taking into account that $c_s^N = c_s^{nb} = 1$, we deduce that $\widetilde{Q}_s(c_s^b t) = \widetilde{Q}(t)$. Therefore, both parametrizations, $\widetilde{Q}_s$ and $\widetilde{Q}$, define the same asymptote $\widetilde{\mathcal{C}}$.

In the following, we illustrate algorithm Asymptotes Construction with an example.

Example 1. *Let $\mathcal{C}$ be the curve of degree $d = 4$ defined by the irreducible polynomial*

$$f(x, y) = 2y^3 x - y^4 + 2y^2 x - y^3 - 2x^3 + x^2 y + 3 \in \mathbb{R}[x, y].$$

We apply algorithm Asymptotes Construction to compute the asymptotes of $\mathcal{C}$.

    Step 1: *We have that $f_4(x, y) = 2y^3 x - y^4$. Hence, the infinity points are $P_1 = (1 : 2 : 0)$ and $P_2 = (1 : 0 : 0)$.*

    We start by analyzing the point P_1:

Step 2.1: *The only infinity branch associated to P_1 is $B_1 = \{(z, r_1(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_1\}$, where*

$$r_1(z) = 2z + \frac{3z^{-3}}{8} - \frac{9z^{-4}}{64} + \frac{27z^{-5}}{512} - \frac{81z^{-6}}{4096} + \cdots$$

(we compute r_1 using the algcurves *package included in the computer algebra system* Maple*).*

Step 2.2.1: *We compute $\tilde{r}_1(z)$, and we have that $\tilde{r}_1(z) = 2z$.*

Step 2.2.2: *The parametrization of the asymptote $\widetilde{\mathcal{C}}_1$ is given by $\widetilde{Q}_1(t) = (t, 2t) \in \mathbb{R}[t]^2$, and the polynomial defining implicitly $\widetilde{\mathcal{C}}_1$ is*

$$\tilde{f}_1(x, y) = y - 2x \in \mathbb{R}[x, y].$$

Now, we focus on the point P_2:

Step 2.1: *The only infinity branch associated to P_2 is $B_2 = \{(z, r_2(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_2\}$, where*

$$r_2(z) = z^{2/3} - \frac{1}{3} + \frac{z^{-2/3}}{9} - \frac{2z^{-4/3}}{81} + \cdots.$$

Step 2.2.1: *We obtain that $\tilde{r}_2(z) = z^{2/3} - \frac{1}{3}$.*

Step 2.2.2: *The parametrization of the asymptote $\widetilde{\mathcal{C}}_2$ is given by $\widetilde{Q}_2(t) = (t^3, t^2 - 1/3) \in \mathbb{R}[t]^2$, and the polynomial defining implicitly $\widetilde{\mathcal{C}}_2$ is*

$$\tilde{f}_2(x, y) = -x^2 + y^3 + y^2 + 1/3y + 1/27 \in \mathbb{R}[x, y].$$

In Figure 2, we plot the curve $\mathcal{C}$, and the asymptotes $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$.

4. Some results on perfect curves

An indispensably tool in the development of the results presented in this paper is the notion of perfect curve. In this section we try to understand better this concept. For this purpose, in the following we show some properties concerning perfect curves. We start with a necessary condition for a curve to be perfect.
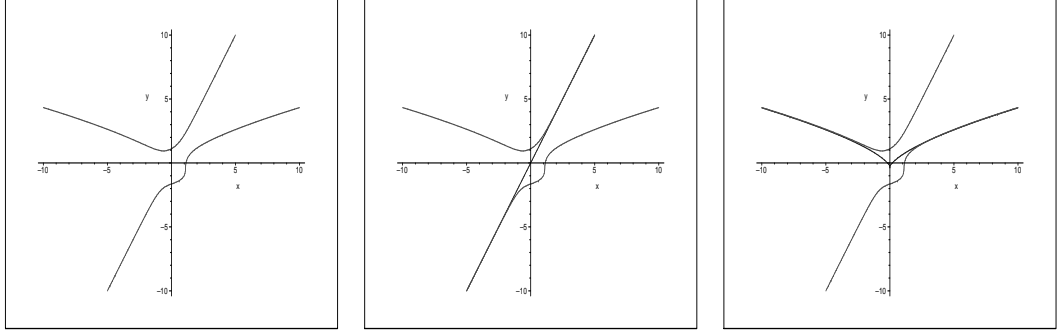
Figure 2: Curve $\mathcal{C}$ (left), asymptote $\widetilde{\mathcal{C}}_1$ (center) and asymptote $\widetilde{\mathcal{C}}_2$ (right).

Proposition 2. *A perfect curve is polynomial (i.e. it admits a polynomial parametrization).*

Proof: Taking into account the results in Manocha and Canny (1991), one has that a curve is polynomial if and only if it has only one infinity branch. Thus, in order to prove the proposition, we only need to show that a perfect curve cannot have more than one infinity branch.

For this purpose, let $\mathcal{C}$ be a perfect curve defined by the polynomial $f(x,y)$, and let us assume that $\mathcal{C}$ has two different infinity branches, $B_j = \{(z, r_j(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_j\}$, $j = 1, 2$, where r_j are as in equation (1), i.e.

$$r_1 = m_1 z + a_1 z^{-\frac{r_1}{N_1}+1} + a_2 z^{-\frac{r_2}{N_1}+1} + \cdots, \quad r_2 = m_2 z + b_1 z^{-\frac{s_1}{N_2}+1} + b_2 z^{-\frac{s_2}{N_2}+1} + \cdots,$$

and $\nu(B_j) = N_j$, $j = 1, 2$ (see Remark 1).

Now, we consider

$$h_1(x,y) = \prod_{j=1}^{N_1}(y - r_{1,j}(x)) \qquad h_2(x,y) = \prod_{j=1}^{N_2}(y - r_{2,j}(x))$$

where $r_{i,1}(z), \ldots r_{i,N_i}(z)$ are the conjugates of $r_i(z), i = 1, 2$. It holds that h_1 and h_2 divide f w.r.t. the variable y (see Remark 4). In addition, h_1 and h_2 do not have common factors since $r_{1,i}$ and $r_{2,j}$ belong to different conjugacy

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classes for every $i \in \{1, \ldots, N_1\}$, and $j \in \{1, \ldots, N_2\}$. Thus, we deduce that

$$\deg(\mathcal{C}) \geq \deg_y(f) \geq N_1 + N_2 > N_1 \geq n_1, \quad \text{where } \deg(B_1) = n_1.$$

In addition, from Theorem 2, we have that $\mathcal{C}$ can be approached at B_1 by an asymptote of degree n_1. Therefore, we get that $\mathcal{C}$ is not perfect which contradicts the assumption. Hence, we conclude that a perfect curve cannot have more than one infinity branch. $\qquad\square$

In the following example, we show that the reciprocal of Proposition 2 is not true. More precisely, we consider a polynomial curve $\mathcal{C}$, and we show that $\mathcal{C}$ is not a perfect curve.

Example 2. *Let $\mathcal{C}$ be the curve defined by the polynomial parametrization*

$$\mathcal{P}(t) = (t^4 + t, t^2) \in \mathbb{R}[t]^2.$$

We compute the implicit polynomial of $\mathcal{C}$ by applying for instance the results in Sendra et al. (2007) (see Chapter 4). We have that

$$f(x, y) = \text{resultant}_t(x - t^4 - t, y - t^2) = -y + x^2 - 2xy^2 + y^4 \in \mathbb{R}[x, y].$$

We apply Algorithm Asymptotes Construction *to determine the asymptotes of $\mathcal{C}$. We first observe that $\mathcal{C}$ only has the infinity point $P = (1 : 0 : 0)$. We compute the infinity branch associated to P (we use the* algcurves *package included in* Maple*), and we get that $B = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$, where*

$$r(z) = z^{1/2} + \frac{1}{2} z^{-1/4} - \frac{1}{64} z^{-7/4} + \frac{1}{128} z^{-10/4} + \cdots.$$

Now, we consider $\tilde{r}(z) = z^{1/2}$, and we obtain the asymptote $\widetilde{\mathcal{C}}$ defined by the parametrization $\widetilde{Q}(t) = (t^2, t)$. The polynomial defining implicitly $\widetilde{\mathcal{C}}$ is $\tilde{f}(x, y) = y^2 - x$.

Thus, the curve $\mathcal{C}$ that has degree 4, is approached by $\widetilde{\mathcal{C}}$ that has degree 2. Therefore, $\mathcal{C}$ is not perfect. In Figure 3, we plot $\mathcal{C}$ and the asymptote $\widetilde{\mathcal{C}}$.

We remark that $\widetilde{\mathcal{C}}$ approaches $\mathcal{C}$ at every leaf of the branch B (see statement 3 in Remark 2). More precisely, B has two real and two complex leaves (see Remark 4.7 in Blasco and Pérez-Díaz (2013)). The real leaves are convergent with the leaf $\tilde{r}_1(z) = z^{1/2}$ of the parabola $\widetilde{\mathcal{C}}$, and the complex leaves are convergent with the leaf $\tilde{r}_2(z) = -z^{1/2}$ of $\widetilde{\mathcal{C}}$.
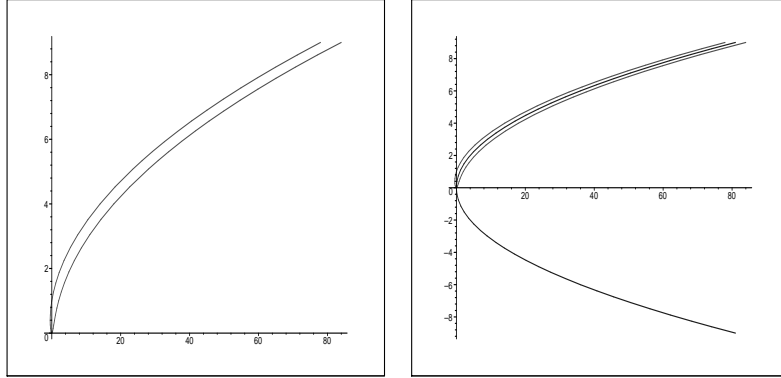
Figure 3: Curve $\mathcal{C}$ (left), and asymptote $\widetilde{\mathcal{C}}$ (right).

Proposition 2 states that a perfect curve has only one infinity branch but this condition does not ensure that the curve is perfect. In the following, we provide a characterization of perfect curves (see Proposition 3). For this purpose, we first remark some properties obtained from the definitions and results introduced before.

Remark 7.
1. *Two approaching perfect curves have the same degree (see Definition 4).*
2. *Two asymptotes that approach the same branch have the same degree (see Statement 1 above and Corollary 1).*
3. *Any asymptote that approaches a curve at a branch of degree n has degree n (see Remark 6 and Theorem 2).*
4. *From statement above and Lemma 2, we deduce that if B is a branch of a perfect curve then $\nu(B) = \deg(B)$.*

Proposition 3. *Let $\mathcal{C}$ be an algebraic plane curve of degree n. $\mathcal{C}$ is perfect if and only if it has a unique infinity branch B, and $\deg(B) = n$.*

Proof: First, we assume that $\mathcal{C}$ is perfect. From Proposition 2, we get that $\mathcal{C}$ only has one infinity branch, B. It holds that $\deg(B) \geq n$; otherwise $\mathcal{C}$ would be approached by asymptotes of degree less than n (see Remark 7, statement 3). Moreover, from Remark 5, we get that $\deg(B) \leq n$. Thus, we deduce that $\deg(B) = n$.
Now, let $\mathcal{C}$ be such that it has a unique infinity branch B, and $\deg(B) = n$. Let us assume that $\mathcal{C}$ is not perfect. Then, there exists a curve $\overline{\mathcal{C}}$, with

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$\deg(\overline{\mathcal{C}}) < n$, that approaches $\mathcal{C}$ at B. From Remark 3, we get that the asymptotes of $\mathcal{C}$ at B have degree less than n, which contradicts Remark 7 (statement 3). Therefore, we conclude that $\mathcal{C}$ is perfect. $\qquad \square$

Example 3. *Let $\mathcal{C}$ be the plane curve defined by the irreducible polynomial $f(x,y) = y^n - x^m \in \mathbb{R}[x,y]$, $n, m \in \mathbb{N}$, $n > m$. Let us prove that $\mathcal{C}$ is perfect. For this purpose, we first observe that $\gcd(n,m) = 1$; otherwise, $n = n_1 k$, $m = m_1 k$, $k \geq 2$, and $(y^{n_1} - x^{m_1})$ divides f which is impossible because f is irreducible.*

Note that $\mathcal{C}$ has a unique infinity branch B, since it admits the polynomial parametrization (t^n, t^m). In addition, B is given by $r(z) = z^{m/n}$, and thus $\deg(B) = n$. Then, by applying Proposition 3, we conclude that $\mathcal{C}$ is perfect.

The following proposition states a sufficient condition for a curve to be perfect, which can be checked without computing any infinity branch. This result will play an important role in Section 5.

Proposition 4. *Let $\mathcal{C}$ be an algebraic plane curve with a unique infinity point P, and let P be regular. Then $\mathcal{C}$ is perfect.*

Proof: Let $\deg(\mathcal{C}) = d$, and we assume that $d \geq 2$ (the case of lines is trivial). Let us prove that $\mathcal{C}$ satisfies the conditions of Proposition 3; i.e, $\mathcal{C}$ has a unique infinity branch, say B, and $\deg(B) = d$.

We assume w.l.o.g that $P = (1 : 0 : 0)$ (otherwise, we apply a linear change of coordinates). Since P is regular, there exists a unique place centered at P and hence, there exists a unique infinity branch associated to P (see Section 2). Therefore the first condition holds.

Now, we focus on the second condition. Let $B = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$, $M \in \mathbb{R}^+$, be the unique infinity branch of $\mathcal{C}$, where

$$r(z) = a_1 z^{-N_1/N+1} + \cdots + a_k z^{-N_k/N+1} + a_{k+1} z^{-N_{k+1}/N+1} + \cdots,$$

$a_1, a_2, \ldots \in \mathbb{C} \setminus \{0\}$, $N, N_1, N_2 \ldots \in \mathbb{N}$, $0 < N_1 < N_2 < \cdots$, and $N_k \leq N < N_{k+1}$ (see equation (1)). Let $\gcd(N, N_1, \ldots, N_k) = b$, $N_j = n_j b$, $N = nb$, $j = 1, \ldots, k$ (see equation (2)); that is, $\deg(B) = n$ (see Definition 6). In the following, we prove that $n = d$. For this purpose, we analyze how $r(z)$ is obtained. Let

$$f(x,y) = y^d + f_{d-1}(x,y) + f_{d-2}(x,y) + \cdots + f_1(x,y) + f_0,$$

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where $f_i(x, y)$ is the homogeneous form of degree i. In addition, let

$$f_{d-1}(x, y) = b_0 x^{d-1} + b_1 x^{d-2} y + \cdots + b_{d-2} xy^{d-2} + b_{d-1} y^{d-1}, \quad \text{and}$$

$$f_{d-2}(x, y) = c_0 x^{d-2} + c_1 x^{d-3} y + \cdots + c_{d-3} xy^{d-3} + c_{d-2} y^{d-2}.$$

The projective curve associated to $\mathcal{C}$ is given by:

$$F(x : y : z) = y^d + z f_{d-1}(x, y) + z^2 f_{d-2}(x, y) + \cdots + z^{d-1} f_1(x, y) + z^d f_0.$$

We observe that $b_0 \neq 0$. Indeed: since $d \geq 2$, we note that

$$\frac{\partial F}{\partial y}(P) = \frac{\partial f_d}{\partial y}(1, 0) = 0, \quad \text{and} \quad \frac{\partial F}{\partial z}(P) = f_{d-1}(1, 0) = b_0.$$

Furthermore, from Euler's formula, we have that $\frac{\partial F}{\partial x}(P) = 0$. Then, since P is regular, we deduce that $b_0 \neq 0$.

Under these conditions, in order to compute $r(z)$, we consider the polynomial (see Section 2):

$$g(y, z) = F(1 : y : z) = y^d + z f_{d-1}(1, y) + \cdots + z^d f_0$$

which can be written as

$$g(y, z) = y^d + z(b_0 + b_1 y + \cdots + b_{d-2} y^{d-2} + b_{d-1} y^{d-1}) +$$

$$z^2 (c_0 + c_1 y + \cdots + c_{d-3} y^{d-3} + c_{d-2} y^{d-2}) + \cdots.$$

In addition, we have that $r(z) = z\varphi(z^{-1})$, where $\varphi(z)$ is a series expansion for a solution of $g(y, z) = F(1 : y : z) = 0$; that is, $g(\varphi(z), z) = 0$, where $\varphi(z) = a_1 z^{N_1/N} + a_2 z^{N_2/N} + a_3 z^{N_3/N} + \cdots$ (see Section 2). Hence, the expression

$$g(\varphi(z), z) = (a_1 z^{N_1/N} + a_2 z^{N_2/N} + \cdots)^d +$$

$$z \left(b_0 + b_1 (a_1 z^{N_1/N} + a_2 z^{N_2/N} + \cdots) + b_2 (a_1 z^{N_1/N} + a_2 z^{N_2/N} + \cdots)^2 + \cdots \right) +$$

$$z^2 \left(c_0 + c_1 (a_1 z^{N_1/N} + a_2 z^{N_2/N} + \cdots) + c_2 (a_1 z^{N_1/N} + a_2 z^{N_2/N} + \cdots)^2 + \cdots \right) + \cdots$$

vanishes for $|z| < M^{-1}$ which implies that terms with a common exponent must cancel.

Since $b_0 \neq 0$, the terms with lowest order in $g(\varphi(z), z)$ are $b_0 z$ and $(a_1 z^{N_1/N})^d = a_1^d z^{N_1 d/N}$. Thus, they must cancel and then, $N = N_1 d$ and

$n = n_1 d$ (note that $N = nb$, and $N_1 = n_1 b$). Hence, since $n_1 \geq 1$, we get that $n \geq d$. On the other hand, Remark 5 states that $d \geq N \geq n$, so $n = d$.

Therefore, $\mathcal{C}$ has a unique infinity branch B, and $\deg(B) = d$. From Proposition 3, we conclude that $\mathcal{C}$ is a perfect curve. $\qquad\square$

Remark 8. *The reciprocal of Proposition 4 is not true. For instance, the curve $y^3 - x = 0$ is perfect (see Example 3) but its unique infinity point, $(1:0:0)$, is singular.*

5. Families of asymptotes

In Section 3, given an algebraic plane curve $\mathcal{C}$, and an infinity branch B of $\mathcal{C}$, we prove that $\mathcal{C}$ can be approached by an asymptote at B. In the following, we see that this asymptote may not be unique. In fact, in most of cases, there are infinitely many asymptotes associated to a given infinity branch.

For instance, let $\mathcal{C}$ be the curve defined by the polynomial $f(x,y) = y^2 - x$ and, for each $k \in \mathbb{R}$, let $\mathcal{D}_k$ be the curve defined by $g_k(x,y) = y^2 - x + k$. It holds that $\mathcal{D}_k$ approaches $\mathcal{C}$, for every $k \in \mathbb{R}$. Indeed: note that both curves admit a polynomial parametrization, so each of them has only one infinity branch. Furthermore, these branches are defined by $r(z) = z^{1/2}$, and $r_k(z) = z^{1/2} - k/(2z^{1/2}) - k^2/(8z^{3/2}) + \cdots = (z-k)^{1/2}$, respectively. Clearly, these branches are convergent, since $\lim_{z \to \infty}(r(z) - r_k(z)) = 0$. Thus, $\{\mathcal{D}_k\}_{k \in \mathbb{R}}$ defines an infinite family of curves that approach $\mathcal{C}$ at its unique infinity branch (see Figure 4).
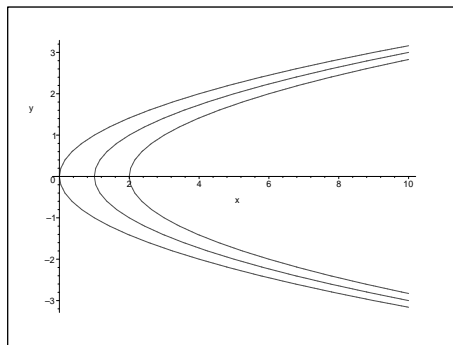

Figure 4: Curves $\mathcal{C}$, $\mathcal{D}_1$ and $\mathcal{D}_2$.

Note that the relation of *proximity* is actually an equivalence relation for the set of curves having only one infinity branch. Clearly, it is reflexive, since every curve approaches itself. Moreover, statement 1 in Remark 2 ensures the symmetry.

Transitivity does not hold for a general set of curves. More precisely, let $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ be three plane curves such that $\mathcal{C}_1$ approaches $\mathcal{C}_2$ at an infinity branch B, and $\mathcal{C}_2$ approaches $\mathcal{C}_3$ at a different infinity branch B^*. We can not get that $\mathcal{C}_1$ approaches $\mathcal{C}_3$. However, if we consider curves with only one infinity branch, B and B^* are the same and then, $\mathcal{C}_1$ approaches $\mathcal{C}_3$ at B. In this case, transitivity holds and then, we have an equivalence relation for the set of curves having only one infinity branch. In the following, we refer to the equivalence classes associated to this equivalence relation as *proximity classes*.

We observe that this property can be applied to perfect curves since, from Proposition 2, perfect curves only have one infinity branch. That is, we consider the above property restricted to the set of perfect curves and then, given a perfect curve $\mathcal{C}$, the set of perfect curves approaching $\mathcal{C}$ determines a proximity class. Therefore, the set of asymptotes that approach a curve at a given infinity branch is also a proximity class.

In this section, we provide a method that allows, in same cases, to compute the curves in a proximity class. For this purpose, we first introduce the following definition.

Definition 7. *A regular perfect curve is a curve having a unique infinity point, which is regular.*

From Proposition 4, regular perfect curves are a subset of the set of perfect curves. In the following, we prove a nice property of this subset which will allow us to decide whether two regular perfect curves approach each other by comparing their implicit polynomials.

More precisely, Theorem 3 states that a polynomial defining implicitly a regular perfect curve has some *irrelevant terms* (i.e. terms that do not affect the asymptotic behavior of the curve). If we modify the coefficients of these terms, we obtain a new perfect curve that approaches the original one. Hence, we can construct infinitely many different asymptotes approaching a given plane curve simply by manipulating its irrelevant terms.

Theorem 3. *Let $\mathcal{C}$ be a regular perfect curve defined by an irreducible polynomial $f \in \mathbb{R}[x,y]$ of degree d. The asymptotic behavior of $\mathcal{C}$ is completely determined by f_d and f_{d-1}.*

Proof: Taking into account Proposition 2, we have that $\mathcal{C}$ has only one infinity branch. Let $B = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$ be this branch. From Proposition 3, we get that $\deg(B) = d$ which implies that $r(z)$ can be written as

$$r(z) = mz + a_1 z^{1-1/d} + a_2 z^{1-2/d} + \cdots + a_{d-1} z^{1-(d-1)/d} + a_d + A(z)$$

where $m, a_1, a_2, \ldots \in \mathbb{C}$, and $A(z) = \sum_{j=d+1}^{\infty} a_j z^{-q_j}$, $q_j \in \mathbb{Q}^+$ for $j \geq d+1$. Note that this notation is slightly different from that of (2); here, some coefficients a_i might be zero.

In order to describe the asymptotic behavior of $\mathcal{C}$, we just need to compute the terms with non negative exponent in $r(z)$. That is, we need to determine the coefficients $m, a_1, \ldots a_d$. The first of them, m, can be obtained directly from the homogeneous form of maximum degree, $f_d(x,y) = (y - mx)^d$. In the following we assume w.l.o.g. that $m = 0$; that is, B is associated to the infinity point $P = (1 : 0 : 0)$ (otherwise, we apply a linear change of coordinates).

The coefficients $a_1, a_2, \ldots a_d$ can be obtained by applying the procedure described in the proof of Proposition 4. The condition to be held is that $g(\varphi(z), z) = 0$ for $|z| < M^{-1}$, where

$$\varphi(z) = z r(z^{-1}) = a_1 z^{1/d} + a_2 z^{2/d} + \cdots + a_{d-1} z^{(d-1)/d} + a_d z + z A(z^{-1}).$$

Reasoning as in the proof of Proposition 4, we get that the terms with a common exponent in the expression

$$g(\varphi(z), z) = (a_1 z^{1/d} + a_2 z^{2/d} + \cdots)^d +$$

$$+ z \left(b_0 + b_1(a_1 z^{1/d} + a_2 z^{2/d} + \cdots) + b_2(a_1 z^{1/d} + a_2 z^{2/d} + \cdots)^2 + \cdots \right) +$$

$$+ z^2 \left(c_0 + c_1(a_1 z^{1/d} + a_2 z^{2/d} + \cdots) + c_2(a_1 z^{1/d} + a_2 z^{2/d} + \cdots)^2 + \cdots \right) + \cdots$$

must cancel. In addition, we also have that $b_0 \neq 0$.

First, we consider the terms with minimum exponent. They are $b_0 z$ and $a_1^d z$. Then, $b_0 + a_1^d = 0$ which implies that $a_1 = (-b_0)^{1/d}$. We set $\mathsf{A}_1 := a_1$ (A_1 represents that a_1 has now a fixed value).

We substitute the value of a_1 in $g(\varphi(z), z)$, and we reason similarly to compute a_2. Note that the terms $b_0 z$ and $a_1^d z$ have been canceled, and a_1 has a fixed value $\mathsf{A}_1 \neq 0$ (note that $b_0 \neq 0$). Hence, the terms with minimum exponent are $b_1 \mathsf{A}_1 z^{1+\frac{1}{d}}$ and $d\mathsf{A}_1^{d-1} a_2 z^{1+\frac{1}{d}}$. Then, $b_1 \mathsf{A}_1 + d\mathsf{A}_1^{d-1} a_2 = 0$ which implies that the value of a_2 is determined from b_1. We have that $a_2 = \frac{-b_1}{d\mathsf{A}_1^{d-2}}$. We set $\mathsf{A}_2 := a_2$.

Once we have a_1 and a_2, we compute a_3. For this purpose, we substitute the values of a_1, a_2 in $g(\varphi(z), z)$, and the terms with minimum exponent are $b_2 \mathsf{A}_1^2$, $b_1 \mathsf{A}_2$, $\binom{d}{2} \mathsf{A}_1^{d-2} \mathsf{A}_2^2$ and $d\mathsf{A}_1^{d-1} a_3$ (all of them are multiplied by $z^{1+\frac{2}{d}}$). Thus, we have that

$$b_2 \mathsf{A}_1^2 + b_1 \mathsf{A}_2 + \binom{d}{2} \mathsf{A}_1^{d-2} \mathsf{A}_2^2 + d\mathsf{A}_1^{d-1} a_3 = 0.$$

Again, we obtain an equation where a_3, b_2 and a set of constants appear. Then, the value of a_3 is determined from b_2 (we recall that $\mathsf{A}_1 \neq 0$).

In the following, we prove that once the values of a_j, $j = 1, \ldots, i-1$ are obtained, reasoning as above, we get an equation where a_i, b_{i-1} and a set of constants appear. For this purpose, we first observe that the term with minimum exponent including a_i is $d\mathsf{A}_1^{d-1} a_i z^{1+\frac{i-1}{d}}$, and the term with minimum exponent including b_{i-1} is $b_{i-1} \mathsf{A}_1^{i-1} z^{1+\frac{i-1}{d}}$ (note that $\mathsf{A}_1 \neq 0$). Hence, both coefficients, a_i and b_{i-1}, appear in the equation multiplied by $z^{1+\frac{i-1}{d}}$. The remainder elements involved in this equation are constants computed in the previous steps. Thus, a_i is obtained from b_{i-1}.

Therefore, we conclude that the asymptotic behavior of the curve is determined by the coefficient m, obtained from $f_d(x, y)$, and the coefficients $b_0, b_1, \ldots, b_{d-1}$, obtained from $f_{d-1}(x, y)$. $\qquad\square$

Theorem 3 implies that any modification in the terms of f of degree less or equal to $d - 2$, does not affect the asymptotic behavior of the curve (we refer to them as the *irrelevant terms*). In addition, from the proof of Theorem 3, we also get that for regular perfect curves, there exists a one to one correspondence between the coefficients $m, a_1, a_2, \ldots a_d$ (that determine the asymptotic behavior of the curve), and the coefficients of f_d and f_{d-1}. Hence, we deduce the following corollary.

Corollary 2. *Two regular perfect curves approach each other if and only if their terms of degree d and $d-1$ are the same.*

In the following, we consider two approaching perfect curves $\mathcal{C}$ and $\overline{\mathcal{C}}$. From statement 2 in Remark 2, we have that both curves have a unique common infinity point P. In the next corollary, we prove that P is a regular point of $\mathcal{C}$ iff P is a regular point of $\overline{\mathcal{C}}$. Thus, we deduce that a perfect curve having a singular infinity point cannot be approached by a regular perfect curve.

Corollary 3. *Let $\mathcal{C}$ and $\overline{\mathcal{C}}$ be two approaching perfect curves. $\mathcal{C}$ is a regular perfect curve if and only if $\overline{\mathcal{C}}$ is a regular perfect curve.*

Proof: Let $\mathcal{C}$ be defined by the polynomial

$$f(x,y) = (y - mx)^d + f_{d-1}(x,y) + f_{d-2}(x,y) + \cdots + f_1(x,y) + f_0$$

where $f_i(x,y)$ is the homogeneous form of degree i, and $d \geq 2$ (the case of lines trivially holds). The projective curve associated to $\mathcal{C}$ is given by

$$F(x:y:z) = (y-mx)^d + z f_{d-1}(x,y) + z^2 f_{d-2}(x,y) + \cdots + z^{d-1} f_1(x,y) + z^d f_0.$$

Thus, $\frac{\partial F}{\partial y}(P) = 0$ ($d \geq 2$), $\frac{\partial F}{\partial z}(P) = f_{d-1}(1,m)$, and $\frac{\partial F}{\partial x}(P) = 0$ (apply Euler's formula). Then, P is regular if and only if $f_{d-1}(1,m) \neq 0$. Finally, the result follows by applying Corollary 2. $\qquad \square$

Given a regular perfect curve $\mathcal{C}$ of degree d defined by a polynomial $f(x,y)$, we may compute all the curves in its proximity class simply by modifying the irrelevant terms in f (see Corollaries 2 and 3).

For instance, let $\mathcal{C}$ be the curve defined by the polynomial

$$f(x,y) = x^3 + 3x^2 y + 3xy^2 + y^3 + 2x^2 + y - 3 \in \mathbb{R}[x,y].$$

$\mathcal{C}$ has only one infinity point, $P = (1 : -1 : 0)$, which is regular. Thus, $\mathcal{C}$ is a regular perfect curve. The curves within the proximity class of $\mathcal{C}$ are implicitly defined by the polynomials

$$x^3 + 3x^2 y + 3xy^2 + y^3 + 2x^2 + ax + by + c, \quad a,b,c \in \mathbb{R}.$$

Note that any curve that belongs to this proximity class can be associated uniquely to a vector $(a,b,c) \in \mathbb{R}^3$. This remark motivates the following proposition.

Proposition 5. *Let $\mathcal{C}$ be a regular perfect curve of degree d. The proximity class of $\mathcal{C}$ is isomorphic to $\mathbb{R}^{\frac{(d-1)d}{2}}$.*

Proof: From Remark 7, statement 1, we have that all the curves in the same proximity class have the same degree d. Thus, the result follows taking into account that the number of irrelevant terms in a generic polynomial of degree d is $\frac{(d-1)d}{2}$, and that any curve that belongs to this proximity class can be associated uniquely to a vector in $\mathbb{R}^{\frac{(d-1)d}{2}}$. $\qquad\square$

In the following, we show that Theorem 3 provides a method for computing all the asymptotes of a curve $\mathcal{C}$ at an infinity branch B associated to a regular infinity point.

Theorem 4. *Let $\mathcal{C}$ be a plane curve, and P a regular infinity point of $\mathcal{C}$. Let B be an infinity branch of $\mathcal{C}$ associated to P, and $\widetilde{\mathcal{C}}$ the asymptote of $\mathcal{C}$ at B obtained from algorithm Asymptotes Construction. It holds that:*

1. *P is a regular point of $\widetilde{\mathcal{C}}$.*

2. *The asymptotes of $\mathcal{C}$ at B are the curves within the proximity class of $\widetilde{\mathcal{C}}$.*

Proof: Statement 2 is deduced from Corollary 2. So, in the following, we prove statement 1. For this purpose, let $P = (1 : 0 : 0)$ (otherwise, we consider a linear change of variables), and $\deg(B) = n$. Let $B = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$, where

$$r(z) = a_1 z^{1-1/n} + a_2 z^{1-2/n} + \cdots + a_{n-1} z^{1-(n-1)/n} + a_n + A(z)$$

$a_1, a_2, \ldots \in \mathbb{C}$, and $A(z) = \sum_{j=n+1}^{\infty} a_j z^{-q_j}$, $q_j \in \mathbb{Q}^+$ for $j \geq n+1$ (note that this notation is slightly different from that of (2); here, some coefficients a_i might be null). From Algorithm Asymptotes Construction, we get that a proper polynomial parametrization of $\widetilde{\mathcal{C}}$ is given by

$$\widetilde{Q}(t) = (q_1(t), q_2(t)) = (t^n, a_1 t^{-1+n} + a_2 t^{-2+n} + \cdots + a_n) \in \mathbb{C}[t]^2.$$

Under these conditions, we apply the results in Pérez-Díaz (2007) and Lemma 4, and we have that the multiplicity of the point P in $\widetilde{\mathcal{C}}$ is

$$\deg(\widetilde{\mathcal{C}}) - \deg\left(\frac{1}{q_2(t)}\right) = n - \deg(q_2).$$

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Taking into account these previous results, we have to prove that $\deg(q_2) = n - 1$, which is equivalent to $a_1 \neq 0$.

For this purpose, since P is an infinity point of $\mathcal{C}$, the projective curve associated to $\mathcal{C}$ is given by

$$F(x:y:z) = f_d(x,y) + zf_{d-1}(x,y) + z^2 f_{d-2}(x,y) + \cdots + z^{d-1}f_1(x,y) + z^d f_0,$$

where

$$f_d(x,y) = y^s \prod_{j=1}^{d-s}(y - m_j x) = y^d + \ell_{d-1}y^{d-1}x + \cdots + \ell_s y^s x^{d-s}, \ \ s \geq 1,$$

$$f_{d-1}(x,y) = b_0 x^{d-1} + b_1 x^{d-2}y + \cdots + b_{d-2}xy^{d-2} + b_{d-1}y^{d-1}.$$

If $s = 1$, since $s \geq N \geq n$ (see Lemma 2), we get that $\deg(B) = n = 1$ which implies that $\widetilde{\mathcal{C}}$ is a line. Thus, P is a regular point in $\widetilde{\mathcal{C}}$. So, let $s \geq 2$. Then, $\frac{\partial F}{\partial x}(P) = 0$ (apply Euler's formula), $\frac{\partial F}{\partial z}(P) = f_{d-1}(1,0) = b_0$, and $\frac{\partial F}{\partial y}(P) = \frac{\partial f_d}{\partial y}(1,0) = 0$ ($s \geq 2$) which implies that P is a regular point in $\mathcal{C}$ if and only if $b_0 \neq 0$.

In the following, we prove that $b_0 \neq 0$ implies that $a_1 \neq 0$ and then, the result holds. For this purpose, we observe that the coefficients $a_1, a_2, \ldots a_n$ can be obtained by applying the procedure described in the proof of Proposition 4. Since $g(\varphi(z), z) = F(1, \varphi(z), z) = 0$, where

$$\varphi(z) = zr(z^{-1}) = a_1 z^{1/n} + a_2 z^{2/n} + \cdots + a_{n-1}z^{(n-1)/n} + a_n z + zA(z^{-1}),$$

reasoning as in the proof of Proposition 4, we get that the terms with a common exponent in the expression $g(\varphi(z), z) =$

$$(a_1 z^{1/n} + a_2 z^{2/n} + \cdots)^d + \ell_{d-1}(a_1 z^{1/n} + a_2 z^{2/n} + \cdots)^{d-1} + \cdots + \ell_s (a_1 z^{1/n} + a_2 z^{2/n} + \cdots)^s$$

$$+ z\left(b_0 + b_1(a_1 z^{1/n} + a_2 z^{2/n} + \cdots) + b_2(a_1 z^{1/n} + a_2 z^{2/n} + \cdots)^2 + \cdots\right) + \cdots$$

must cancel.

If $b_0 \neq 0$, the first non-zero term with minimum exponent appearing is $b_0 z$. Thus, there exists $j \in \mathbb{N}$ such that $js/n = 1$, and $b_0 + a_j^s = 0$, and $a_i = 0$, $i = 1, \ldots, j - 1$. Since $n \leq N \leq s$ (see Lemma 2), we deduce that $j = 1$, $n = s$, and $a_1 = (-b_0)^{1/s} \neq 0$, as we wanted to prove. $\qquad\square$

Remark 9. *If $\widetilde{\mathcal{C}}$ is a line, there are no irrelevant terms. In this case the asymptote is unique.*

In the following, we illustrate these results with an example.

Example 4. *We consider the curve $\mathcal{C}$ introduced in Example 1, and defined by the polynomial*

$$f(x,y) = 2y^3x - y^4 + 2y^2x - y^3 - 2x^3 + x^2y + 3 \in \mathbb{R}[x,y].$$

In Example 1, we obtain the asymptote $\tilde{\mathcal{D}}$ of $\mathcal{C}$ at the infinity branch B associated to $(1:0:0)$. We get that $\tilde{\mathcal{D}}$ is defined by the polynomial

$$\tilde{f}(x,y) = y^3 - x^2 + y^2 + 1/3y + 1/27 \in \mathbb{R}[x,y].$$

From Proposition 4, we deduce that $\tilde{\mathcal{D}}$ is a regular perfect curve, and all the asymptotes of $\mathcal{C}$ at B are the curves defined by polynomials

$$y^3 - x^2 + y^2 + ax + by + c, \quad a,b,c \in \mathbb{R}.$$

These curves determine a proximity class isomorphic to $\mathbb{R}^3$ (Proposition 5).

In Figure 4, we plot $\mathcal{C}$, and the asymptotes $\tilde{\mathcal{D}}_i$, $i = 1,2,3$ defined by the polynomials

$$\tilde{f}_1(x,y) = y^3 - x^2 + y^2 - y + 1/27, \qquad \tilde{f}_2(x,y) = y^3 - x^2 + y^2 + 1/3y - 2x + 1/27$$

$$\tilde{f}_3(x,y) = y^3 - x^2 + y^2.$$

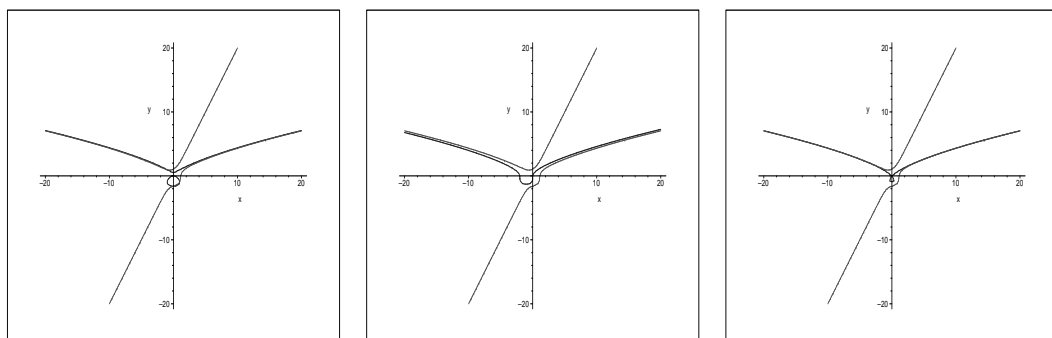Figure 5: Curve $\mathcal{C}$, and asymptotes $\tilde{\mathcal{D}}_1$ (left), $\tilde{\mathcal{D}}_2$ (center) and $\tilde{\mathcal{D}}_3$ (right).

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