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Numerical Proper Reparametrization of Parametric Plane Curves☆

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Abstract

We present an algorithm for reparametrizing algebraic plane curves from a numerical point of view. More precisely, given a tolerance $\epsilon > 0$ and a rational parametrization $\mathcal{P}$ of a plane curve $\mathcal{C}$ with perturbed float coefficients, we present an algorithm that computes a parametrization $\mathcal{Q}$ of a new plane curve $\mathcal{D}$ such that $\mathcal{Q}$ is an *ϵ–proper reparametrization* of $\mathcal{D}$. In addition, the error bound is carefully discussed and we present a formula that measures the "closeness" between the input curve $\mathcal{C}$ and the output curve $\mathcal{D}$.

Keywords: Rational Curve, Approximately Improper, Proper Reparametrization

1. Introduction

Let $\mathcal{P}(t)$ be a rational affine parametrization of an algebraic plane curve $\mathcal{C}$ over the complex field $\mathbb{C}$. Associated with $\mathcal{P}(t)$, we have the rational map $\phi_{\mathcal{P}} : \mathbb{C} \to \mathcal{C}; t \to \mathcal{P}(t)$, where $\phi_{\mathcal{P}}(\mathbb{C}) \subset \mathcal{C}$ is dense. $\phi_{\mathcal{P}}$ is a birational map if $\mathcal{P}$ is proper. That is, except for a finite number of points, for almost every point $p \in \mathcal{C}$, there is exactly one parameter value $t_0 \in \mathbb{C}$ such that $\mathcal{P}(t_0) = p$. Geometrically, $\mathcal{P}$ proper means that $\mathcal{P}$ traces the curve once. If $\mathcal{P}$ is not proper, there is more than one parameter value corresponding to a generic point on $\mathcal{C}$. Lüroth's Theorem shows constructively that it is always possible to reparametrize an improperly parametrized curve to a proper one.

Proper parameterizations are crucial to many practical problems in computer aided geometric design (CAGD), such as visualization (see [17, 18]). Particularly, proper parameterizations ensure the validity of the resultant technique in the implicitization problem (see [12, 15, 28, 32]). Therefore, the proper reparametrization problem has received extensive research (see [8, 22, 23, 31, 33] for examples).

The problem of proper reparametrization for curves has widely been discussed from the symbolic point of view. More precisely, *given the field of complex numbers $\mathbb{C}$, and a rational parametrization $\mathcal{P}(t) \in \mathbb{C}(t)^2$ of an algebraic plane curve $\mathcal{C}$ with exact coefficients,*

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one computes a rational proper parametrization $\mathcal{Q}(t) \in \mathbb{C}(t)^2$ *of* $\mathcal{C}$*, and a rational function* $R(t) \in \mathbb{C}(t) \setminus \mathbb{C}$ *such that* $\mathcal{P}(t) = (\mathcal{Q} \circ R)(t)$*.* Nevertheless, in many practical applications, symbolic (or exact) approaches tend to be insufficient, since in practice object data are usually given approximately. As a consequence, hybrid symbolic-numerical algorithms have stepped onto stage.

Briefly speaking, given a tolerance $\epsilon > 0$, and an irreducible affine algebraic plane curve $\mathcal{C}$ defined by a parametrization $\mathcal{P}$ with perturbed float coefficients that is "*nearly improper*" (i.e. improper within the tolerance ϵ), one looks for a rational curve $\mathcal{D}$ defined by a parametrization $\mathcal{Q}$, such that $\mathcal{Q}$ is proper and almost all points of the rational curve $\mathcal{D}$ are in the "*vicinity*" of $\mathcal{C}$. The notion of vicinity can be illustrated by the offset region restricted by the external and internal offset to $\mathcal{C}$ at distance ϵ (see Section 4 for details). Therefore, the problem reduces to find a properly parameterized curve $\mathcal{D}$ that lies within the offset region of $\mathcal{C}$. For instance, assume that we are given a tolerance $\epsilon = 0.2$, and a curve $\mathcal{C}$ defined by the parametrization

$$\mathcal{P}(t) = \left(\frac{1.999t^2 + 3.999t + 2.005 - 0.003t^4 + 0.001t^3}{2.005 + 0.998t^4 + 4.002t^3 + 6.004t^2 + 3.997t}, \frac{0.001 - 0.998t^4 - 4.003t^3 - 5.996t^2 - 4.005t}{2.005 + 0.998t^4 + 4.002t^3 + 6.004t^2 + 3.997t} \right).$$

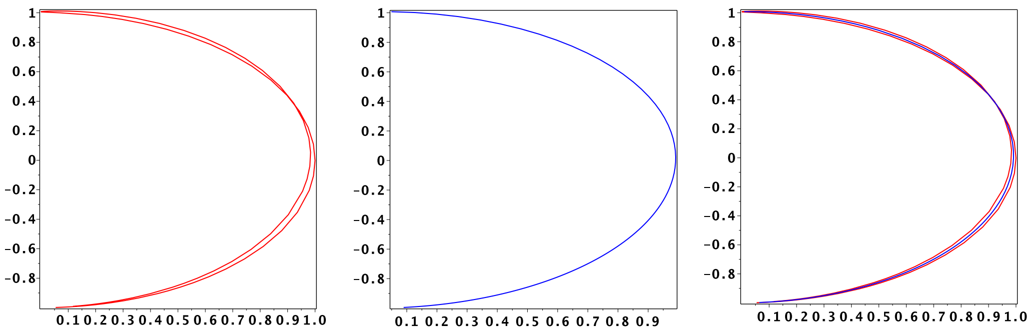


Figure 1: Input curve $\mathcal{C}$ (left), curve $\mathcal{D}$ (center), curves $\mathcal{C}$ and $\mathcal{D}$ (right)

One may check that $\mathcal{P}$ is proper from the symbolic point of view; but it is nearly improper (numerically speaking), since for almost all points $p := \mathcal{P}(s_0) \in \mathcal{C}$, $s_0 \in \mathbb{C}$, there exist two values of the parameter t, given by the approximate roots of the equation $.4901606943t^2 + .2393271335\,10^{-8}(2202769s_0 + 417838122)t - .4954325182s_0^2 - s_0 = 0$, such that $\mathcal{P}(t)$ is "*almost equal*" to $\mathcal{P}(s_0)$. Our method provides an ϵ*–proper reparametrization* of $\mathcal{D}$

$$\mathcal{Q}(t) = \left(\frac{-0.00139214373770521\,t^2 - 0.455587113115768\,t + 0.230804565878748}{0.472790306463932\,t^2 - 0.475516806696674\,t + 0.233345983511073}, \right.$$

$$\left. \frac{-0.472791477433681\,t^2 + 0.473001908925789\,t - 0.00421763512489261}{0.472790306463932\,t^2 - 0.475516806696674\,t + 0.233345983511073} \right).$$

In Figure 1, one may check that $\mathcal{C}$ and $\mathcal{D}$ are "*close*".

To relate the tolerance with the vicinity region, one can either approach from analyzing locally the condition number of the implicit equations (see [13]), estimating the Hausdorff distance (see e.g. [29, 30]) or studying whether for almost every point p on the original curve, there exists a point q on the output curve such that the distance of p and q is significantly smaller than the tolerance (see e.g. [24, 25]). The error analysis we present in this paper will be based on the third approach, and we shall derive upper bounds for the distance of the offset region.

Approximate algorithms have been developed for many applied numerical topics, such as computing approximate greatest common divisor (gcd) [3, 4, 10, 21, 40], finding zeros of multivariate systems [5, 10], factoring polynomials [9, 14], etc. In addition, computing approximate parametrizations for algebraic curves and surfaces has been investigated. For instance, in [2], the authors construct a C^1-continuous piecewise (m, n) rational ϵ-approximation of a real algebraic plane curve. Using the Weierstrass Preparation Theorem, Newton power series factorizations, and modified rational Padé approximations, the authors construct a locally approximate rational parametric representations for all real branches of the given algebraic plane curve. In [6], a novel approach for computing an approximate parameterization of a whole closed space algebraic curve from a small number of approximating arcs is presented. [7] proposes an algorithm that subdivides the given curve into arcs, and then approximates the arcs with curves parametrized by rational functions of low degree. In [19], a method for computing an approximate parameterization of a planar algebraic curve by a rational Bézier (spline) curve is described. The approach is based on the minimization of a suitable non-linear objective function, which takes into account both the distance from the curve and the positivity of the weight function (i.e., the numerator of the rational parametric representation). In [20], a method of approximating a segment of the intersection curve of two implicitly defined surfaces by a rational parametric curve is presented. The method includes predictor and corrector steps. The corrector step is formulated as an optimization problem, where the objective function is the approximation of integral of the squared Euclidean distance of the curve to the intersection curve. The predictor step is based on simple extrapolation and on a differential equation. In [24, 25, 29, 30], the authors deal with mathematical objects given approximately. They develop numerical theory parallel to that for exact algebraic varieties. More precisely, given an algebraic variety $\mathcal{V}$ (curve or surface) implicitly defined, they analyze the existence of a new variety $\overline{\mathcal{V}}$ that is close to $\mathcal{V}$. A rational parametrization of $\overline{\mathcal{V}}$ is computed. In [16], the authors propose an algorithm of computing a piecewise conic parametric curve as an approximation of the input implicit plane or space curve. For a given space curve, [34] provides its certified cubic B-spline approximation with topology and geometric feature preserved. In [38], new algorithms are presented to approximate digitized curves by piecewise circular arcs with geometric continuity G^0 or G^1. First, iterative methods are proposed to solve for the best single arc and biarc problems of a digitized curve with respect to the maximum norm. Then, an approximate arcwise curve of G^0 or G^1 continuity is constructed with the approximate error controlled within a given tolerance $\epsilon > 0$. [39] is devoted to implicit planar curves (not necessary algebraic), when the region of interest is a certain box. This paper proposes an all-at-once approach, which relies on an evolution process: starting from the bounding box of the domain of interest, a closed B-spline curve is moved gradually towards the given implicit curve.

Despite of the importance and the popularity of developing approximate algorithms for algebraic varieties, the work on reparametrizing a given parametric curve with perturbed float coefficients is rare. To our knowledge, only a heuristic algorithm was proposed in [31], which, however, lacks of the error analysis and the detection of the improperness of a numerical curve within a given tolerance.

In this paper, we aim at developing a proper parameterization technique for algebraic plane curves given numerically. Our work will be based on the existing symbolic algorithm given by [22], but it generalizes the corresponding theory to the numerical situation. However, far beyond a natural generalization, our work consists of numerical reparametrization and error analysis. We extend the concept of tracing index, which is the cardinality of a generic fibre of a parameterization (see Chapter 4 in [32]) and it is used to characterize the properness of a parametrization to the numerical situation. Numerically, the *approximate tracing index* is defined as the number of parameter values mapped to points *in a neighborhood of a generic point* of a given plane curve. We provide an algorithm that computes an approximate curve for the originally given one that is defined by a new proper parametrization, and we measure the closeness of the two curves through error analysis. We prove that our obtained new approximate curve always lies in the offset region of the original one (and reciprocally).

The paper is organized as follows. In Section 2, the symbolic algorithm of proper reparameterization presented in [22] is briefly reviewed. In Section 3, the definitions of approximate tracing index (ϵ–index) and ϵ–numerical reparametrization are proposed. The ϵ–numerical reparametrization is constructed and proved to be ϵ–proper. Afterwards in Section 4, the closeness of the reparametrized curve and the originally given one is measured through the error analysis. In Section 5, a numerical algorithm and concrete examples are given. We conclude in Section 6 with the topics for further study.

2. Symbolic Algorithm of Reparametrization for Curves

Before studying the approximate case, we briefly review the notions and algorithm of symbolically reparameterizing curves presented in [22]. The idea will be reformulate the results obtained in [22] to the case in which the input is given approximately (see Section 3).

Consider the field of the complex numbers $\mathbb{C}$, and a rational algebraic plane curve $\mathcal{C}$ over $\mathbb{C}$ defined by a rational parametrization $\mathcal{P}(t) \in \mathbb{C}(t)^2$. The parametrization $\mathcal{P}$ is proper if and only if the map $\mathcal{P} : \mathbb{C} \longrightarrow \mathcal{C} \subset \mathbb{C}^2; t \longmapsto \mathcal{P}(t)$ is birational; or equivalently, if for almost every point on $\mathcal{C}$ and for almost all values of the parameter in $\mathbb{C}$ the map $\mathcal{P}$ is rationally bijective. The notion of properness can also be stated algebraically in terms of fields of rational functions. In fact, a rational parametrization $\mathcal{P}$ is proper if and only if the induced monomorphism $\phi_{\mathcal{P}}$ on the fields of rational functions $\phi_{\mathcal{P}} : \mathbb{C}(\mathcal{C}) \longrightarrow \mathbb{C}(t); R(x, y) \longmapsto R(\mathcal{P}(t))$ is an isomorphism. Therefore, $\mathcal{P}$ is proper if and only if the map $\phi_{\mathcal{P}}$ is surjective; that is, $\phi_{\mathcal{P}}(\mathbb{C}(\mathcal{C})) = \mathbb{C}(\mathcal{P}(t)) = \mathbb{C}(t)$. Lüroth's Theorem implies that any rational curve over $\mathbb{C}$ can be properly parametrized (see [1, 32, 35]). [22, 31] further show how to reparameterize a given improper parameterization to a proper one.

Intuitively speaking, $\mathcal{P}$ is proper if and only if $\mathcal{P}(t)$ traces $\mathcal{C}$ only once. In this sense, we may generalize the above notion by introducing the notion of tracing index of $\mathcal{P}(t)$ as follows: we say that $k \in \mathbb{N}$ is the *tracing index* of $\mathcal{P}(t)$, and we denote it by $\text{index}(\mathcal{P})$, if all but finitely many points on $\mathcal{C}$ are generated, via $\mathcal{P}(t)$, by k parameter values; i.e. $\text{index}(\mathcal{P})$ represents the number of times that $\mathcal{P}(t)$ traces $\mathcal{C}$. Hence, the birationality of $\phi_{\mathcal{P}}$, i.e. the properness of $\mathcal{P}(t)$, is characterized by tracing index one (for further details see [32]).

We hereby present some preliminaries on resultants. The univariate resultant of two polynomials $A, B \in \mathbb{C}[x_1, \ldots, x_n, t]$ denoted by $\text{Res}_t(A, B)$, is defined as the determinant of the Sylvester matrix associated with A and B with respect to t. Clearly, $\text{Res}_t(A, B) \in \mathbb{C}[x_1, \ldots, x_n]$, and $\text{Res}_t(A, B) = 0$ if and only if A and B have a common factor on t. Note that $\text{Res}_t(A, B)$ is contained in the ideal generated by A and B. Hence, if $A(\alpha, b) = B(\alpha, b) = 0$ at $\alpha = (\alpha_1, \ldots, \alpha_n)$, we have $\text{Res}_t(A, B)(\alpha) = 0$. Reciprocally, if $\text{Res}_t(A, B)(\alpha) = 0$, we get that $\text{lc}(A, t)(\alpha) = \text{lc}(B, t)(\alpha) = 0$, or there exists $b \in \mathbb{C}$ such that $A(\alpha, b) = B(\alpha, b) = 0$, where $\text{lc}(A, t)$ denotes the leading coefficient of A with respect to t (for more details see for instance Chapter 3 in [11], or Sections 5.8 and 5.9 in [36]).

The following property of resultants will play an important role in our later analysis: if $\alpha \in \mathbb{C}^n$ is such that $\deg_t(\varphi_\alpha(A)) = \deg_t(A)$, and $\deg_t(\varphi_\alpha(B)) = \deg_t(B) - k$ then,

$$\varphi_\alpha(\text{Res}_t(A, B)) = \varphi_\alpha(\text{lc}(A, t))^k \text{Res}_t(\varphi_\alpha(A), \varphi_\alpha(B)),$$

where φ_α is the natural evaluation homomorphism

$$\varphi_\alpha : \mathbb{C}[x_1, \ldots, x_n, t] \longrightarrow \mathbb{C}[x_1, \ldots, x_n, t]; A(x_1, \ldots, x_n, t) \longmapsto A(\alpha_1, \ldots, \alpha_n, t)$$

(see Lemma 4.3.1, pp. 96, in [37]).

Finally, given $R(t) = r_1(t)/r_2(t) \in \mathbb{C}(t)$, where $\gcd(r_1, r_2) = 1$, we define $\deg(R)$ as the maximum of $\deg(r_1)$ and $\deg(r_2)$.

We next outline the algorithm developed in [22], which computes a rational proper reparametrization of an improperly parametrized algebraic plane curve. The algorithm is valid over any field (here we consider the field of complex numbers $\mathbb{C}$), and involves the computation of polynomial gcds and univariate resultants. We shall adopt the resultant approach, which is efficient and produces no extra factors (see step 5 in the algorithm). Of course other elimination approach, such as Gröbner basis, will also work.

Symbolic Algorithm Reparametrization for Curves.

INPUT: a rational affine parametrization $\mathcal{P}(t) = (p_{11}(t)/p_{12}(t),\, p_{21}(t)/p_{22}(t)) \in \mathbb{C}(t)^2$, with $\gcd(p_{i1}, p_{i2}) = 1$, $i = 1, 2$, of an algebraic plane curve $\mathcal{C}$.
OUTPUT: a rational proper parametrization $\mathcal{Q}(t) \in \mathbb{C}(t)^2$ of $\mathcal{C}$, and a rational function $R(t) \in \mathbb{C}(t) \setminus \mathbb{C}$ such that $\mathcal{P}(t) = (\mathcal{Q} \circ R)(t)$.

1. Compute $H_j(t, s) = p_{j1}(t)p_{j2}(s) - p_{j1}(s)p_{j2}(t), \quad j = 1, 2$.
2. Determine the polynomial $S(t, s) = \gcd(H_1(t, s), H_2(t, s)) = C_m(t)s^m + \cdots + C_0(t)$.
3. If $\deg_t(S) = 1$, RETURN $\mathcal{Q}(t) = \mathcal{P}(t)$, and $R(t) = t$. Otherwise go to step 4.
4. Consider a rational function $R(t) = \frac{C_i(t)}{C_j(t)} \in \mathbb{C}(t)$, such that $C_j(t)$, $C_i(t)$ are two of the polynomials obtained in step 2 such that $\gcd(C_j, C_i) = 1$, and $C_jC_i \notin \mathbb{C}$ (see Lemma 3, Theorem 1, and Section 3 in [22]).
5. For $k = 1, 2$, compute the polynomials
$$L_k(s, x_k) = \mathrm{Res}_t(G_k(t, x_k), sC_j(t) - C_i(t)) = (q_{k2}(s)x_k - q_{k1}(s))^{\deg(R)},$$
where $G_k(t, x_k) = x_kp_{k2}(t) - p_{k1}(t)$.
6. RETURN $\mathcal{Q}(t) = (q_{11}(t)/q_{12}(t),\, q_{21}(t)/q_{22}(t)) \in \mathbb{C}(t)^2$, and $R(t) = C_i(t)/C_j(t)$.

Remark 1. *It holds that* $\mathrm{index}(\mathcal{P}) = \deg_t(S)$. *Furthermore, for all but a finite number of values of the variable s,* $\deg_t(S) = \deg_t(\gcd(H_1(t, \alpha), H_2(t, \alpha)))$ *(see Subsection 4.3 in [32]).*

Example 1. *Let* $\mathcal{C}$ *be the rational curve defined by the parametrization*

$$\mathcal{P}(t) = \left(\frac{p_{11}(t)}{p_{12}(t)}, \frac{p_{21}(t)}{p_{22}(t)}\right) =$$

$$\left(\frac{10t^4 + 13t^3 + 17t^2 + 3t^5 + 24t + 11 + t^6}{(t^3 + 2)(t^2 + 3t + 7 + 3t^3)}, -\frac{2t^4 - t^3 - 9t^2 + 5 + t^6 + t^5}{(t^3 + 2)^2}\right).$$

In step 1 of the algorithm, we compute the polynomials

$H_1(t, s) = 270t - 270s + 216t^2 + 39t^3 + 107t^4 - 19t^6 + 31t^5 - 26t^6s^3 - 49t^6s^2 + 26t^5s^3 - 31t^4s^2 + 91t^4s^3 - 11t^5s^2 - 195t^3s^2 + 195t^2s^3 + 234ts^3 + 19s^6 - 31s^5 - 107s^4 - 39s^3 - 216s^2 - 54ts^2 + 12ts^4 + 66ts^6 + 6ts^5 + 31t^2s^4 + 54t^2s + 49t^2s^6 + 11t^2s^5 - 91t^3s^4 - 234t^3s + 26t^3s^6 - 26t^3s^5 - 12t^4s + 27t^4s^6 + t^4s^5 - 27t^6s^4 - 66t^6s - 8t^6s^5 - t^5s^4 - 6t^5s + 8t^5s^6,$

$H_2(t, s) = 36t^2 + 24t^3 - 8t^4 + t^6 - 4t^5 - 5t^6s^3 - 9t^6s^2 - 4t^5s^3 - 8t^4s^3 - 36t^3s^2 + 36t^2s^3 - s^6 + 4s^5 + 8s^4 - 24s^3 - 36s^2 + 9t^2s^6 + 8t^3s^4 + 5t^3s^6 + 4t^3s^5 - 2t^4s^6 + 2t^6s^4 + t^6s^5 - t^5s^6.$

Now, we determine $S(t, s)$. *We obtain*

$$S(t, s) = C_0(t) + C_1(t)s + C_2(t)s^2 + C_3(t)s^3,$$

where $C_0(t) = -6t - 2t^2 + t^3$, $C_1(t) = 3t^3 + 6$, $C_2(t) = t^3 + 2$, *and* $C_3(t) = -t^2 - 3t - 1$.

Since $\deg_t(S) > 1$, *we go to step 4, and we consider*

$$R(t) = \frac{C_3(t)}{C_2(t)} = \frac{-t^2 - 3t - 1}{t^3 + 2}.$$

Note that $\gcd(C_2, C_3) = 1$. *Now, we compute the polynomials*

$$L_1(s, x_1) = \mathrm{Res}_t(G_1(t, x_1), sC_2(t) - C_3(t)) = -961(-3x_1 + sx_1 + 1 - 3s + s^2)^3,$$
$$L_2(s, x_2) = \mathrm{Res}_t(G_2(t, x_2), sC_2(t) - C_3(t)) = -961(-x_2 - 1 + s + s^2)^3,$$

where $G_i(t, x_i) = x_i p_{i2}(t) - p_{i1}(t)$, $i = 1, 2$ *(see step 5). Finally, in step 6, the algorithm outputs the proper parametrization* $\mathcal{Q}(t)$, *and the rational function* $R(t)$

$$\mathcal{Q}(t) = \left(-\frac{1 - 3t + t^2}{t - 3}, -1 + t + t^2\right), \qquad R(t) = \frac{-t^2 - 3t - 1}{t^3 + 2}.$$

3. The Problem of Numerical Reparametrization for Curves

The problem of numerical reparametrization for curves can be stated as follows:

- Given the field $\mathbb{C}$ of complex numbers, a tolerance $\epsilon > 0$, and a rational parametrization $\mathcal{P}(t) = (p_{11}(t)/p_{12}(t), p_{21}(t)/p_{22}(t)) \in \mathbb{C}(t)^2$ of an algebraic plane curve $\mathcal{C}$ that is *approximately improper* (see Definition 1).

- Find a rational parametrization $\mathcal{Q}(t) \in \mathbb{C}(t)^2$ of an algebraic plane curve $\mathcal{D}$, and a rational function $R(t) \in \mathbb{C}(t) \setminus \mathbb{C}$ such that $\mathcal{Q}$ is an ϵ*–proper reparametrization* of $\mathcal{D}$ (see Definition 4).

- Measure the closeness between $\mathcal{C}$ and $\mathcal{D}$ (see Section 4).

The *idea* in this section is to adapt the symbolic algorithm in Section 2 to the case in which the input and output are not assumed to be exact. Instead, we deal with mathematical objects that are given approximately, probably because they proceed from an exact data that has been perturbed under some previous measuring process or manipulation. Note that, in many practical applications, for instance in the frame of computer aided geometric design, most of data objects are given approximately.

The structure of this section is as follows: first, we introduce some previous definitions as the notion of ϵ–index, and ϵ–numerical reparametrization (see Subsection 3.1). Afterwards, in Subsection 3.2, we obtain some properties that characterize whether a rational parametrization is an ϵ–proper reparametrization. Using these results, in Subsection 3.3, we show how to construct the rational function R that will be used to compute the ϵ–proper reparametrization $\mathcal{Q}$ in Subsection 3.4. Finally, in Subsection 3.5 we prove some properties concerning the parametrizaton $\mathcal{Q}$. These properties will play an important role in the development of Section 4.

3.1. Notation and Preliminary Definitions

In order to start with the problem proposed, we consider a tolerance $\epsilon > 0$, and a rational parametrization of a given algebraic plane curve $\mathcal{C}$

$$\mathcal{P}(t) = (p_1(t), p_2(t)) = \left(\frac{p_{11}(t)}{p_{12}(t)}, \frac{p_{21}(t)}{p_{22}(t)}\right) \in \mathbb{C}(t)^2, \quad \epsilon\text{–gcd}(p_{j1}, p_{j2}) = 1, \; j = 1, 2,$$

where ϵ–gcd(p_{j1}, p_{j2}) denotes the approximate gcd for the polynomials p_{j1} and p_{j2} (we remind that $\mathcal{P}$ is expected to be given with perturbed float coefficients). We assume that index$(\mathcal{P}) = 1$. Observe that we are working numerically and then, with probability almost one $\deg_t(S) = 1$, where S is the polynomial introduced in Section 2. Otherwise, if index$(\mathcal{P}) > 1$, we may apply Symbolic Algorithm Reparametrization for Curves in Section 2.

We also consider the polynomials

$$S_\epsilon^{\mathcal{PQ}}(t, s) = \epsilon\text{–gcd}(H_1^{\mathcal{PQ}}, H_2^{\mathcal{PQ}}), \quad H_j^{\mathcal{PQ}}(t, s) = p_{j1}(t)q_{j2}(s) - q_{j1}(s)p_{j2}(t), \; j = 1, 2,$$

where s is a new variable, and

$$\mathcal{Q}(t) = (q_1(t), q_2(t)) = \left(\frac{q_{11}(t)}{q_{12}(t)}, \frac{q_{21}(t)}{q_{22}(t)}\right) \in \mathbb{C}(t)^2, \quad \epsilon\text{–gcd}(q_{j1}, q_{j2}) = 1, \; j = 1, 2$$

is a rational parametrization of a new plane curve. Note that since ϵ–gcd$(p_{j1}, p_{j2}) = 1, \; j = 1, 2$, then $S_\epsilon^{\mathcal{PQ}}(t, s) \in \mathbb{C}[t, s] \setminus \mathbb{C}[s]$, and $S_\epsilon^{\mathcal{PQ}}(t, s) \in \mathbb{C}[t, s] \setminus \mathbb{C}[t]$.

Observe that the polynomials $H_j^{\mathcal{PQ}}$, $j = 1, 2$, generalize the polynomials H_j, $j = 1, 2$, introduced in step 1 of the symbolic algorithm presented in Section 2. More precisely, we have that $H_j(t, s) = H_j^{\mathcal{PP}}(t, s)$, $j = 1, 2$.

Once the polynomials $H_j^{\mathcal{PP}}$, $j = 1, 2$, are introduced, we observe that in step 2 of the symbolic algorithm one computes gcd(H_1, H_2). From this fact, one gets the idea that under our conditions and taking into account that we are working with mathematical objects that are assumed to be given approximately, we have to compute the approximate gcd of $H_1^{\mathcal{PP}}$ and $H_2^{\mathcal{PP}}$ (that is, ϵ–gcd$(H_1^{\mathcal{PP}}, H_2^{\mathcal{PP}})$) instead of gcd$(H_1, H_2)$ (note that the gcd of two not exact input polynomials is always 1).

Therefore, at this point, we need to generalize the concept of tracing index (see Section 2) to the numerical situation. For this purpose, in the following definition, we introduce the notion of *approximate tracing index of* $\mathcal{P}$.

Definition 1. *We define the* approximate tracing index *of* $\mathcal{P}$ *as* $\deg_t(S_\epsilon^{\mathcal{PP}})$, *where*

$$S_\epsilon^{\mathcal{PP}}(t, s) = \epsilon\text{–}gcd(H_1^{\mathcal{PP}}, H_2^{\mathcal{PP}}), \quad H_j^{\mathcal{PP}}(t, s) = p_{j1}(t)p_{j2}(s) - p_{j1}(s)p_{j2}(t), \; j = 1, 2.$$

We denote it as ϵ–index$(\mathcal{P})$. Furthermore, $\mathcal{P}$ is said to be approximately improper *or* ϵ–improper *if ϵ–index$(\mathcal{P}) > 1$. Otherwise, $\mathcal{P}$ is said to be* approximately proper *or* ϵ–proper.

Remark 2. *Obviously the notion of* approximate tracing index *generalizes the concept of* tracing index. *In particular, if ϵ–index$(\mathcal{P}) = 1$ then* index$(\mathcal{P}) = 1$.

Remark 3. *In order to compute the polynomial $S_\epsilon^{\mathcal{PP}}$, one should note that:*

1. *There are different ϵ–gcd algorithms proposed for inexact polynomials (see for instance, [3, 4, 10, 21, 40]). Some typical algorithms of univariate polynomials are included in the mathematical softwares, for example,* `Maple` *provides some ϵ–gcd algorithms in the package SNAP. We here introduce the ϵ–gcd algorithm for a pair of univariate numeric polynomials by using QR factoring. It is implemented in* `Maple` *as the function* QRGCD. *The* QRGCD(f, g, x, ϵ) *function returns univariate numeric polynomials u, v, d such that d is an ϵ–gcd for the input polynomials f and g, and u, v satisfy (with high probability)*

$$\|uf + vg - d\| < \epsilon\|(f, g, u, v, d)\|, \quad \|f - df_1\| < \epsilon\|f\|, \quad \textit{and} \quad \|g - dg_1\| < \epsilon\|g\|,$$

where the polynomials f_1 and f_2 are cofactors of f and g with respect to the divisor d, $\|\cdot\| \in \mathbb{R}$ denotes the infinity norm (that is, $\|h\|$ computes the maximum of the absolute values of the coefficients of a polynomial $h(x) \in \mathbb{C}[x]$, with respect to the variable x; for complex coefficients, $\|h\|$ finds the maximum of the absolute values of the real and imaginary parts), and $\|(f, g, u, v, d)\| := \max\{\|f\|, \|g\|, \|u\|, \|v\|, \|d\|\}$.
2. *In the symbolic situation, one can get the tracing index with probability one by counting the common solutions for a specialized s_0 (see Remark 1). For the numerical situation, we can fix $s = s_0 \in \mathbb{C}$ as a specialization and find the ϵ–gcd for two univariate polynomials $H_1^{\mathcal{PP}}(t, s_0)$ and $H_2^{\mathcal{PP}}(t, s_0)$, under tolerance ϵ. Hence, we first can compute the approximate tracing index by the specialization and then, we can recover an ϵ–gcd defined by the polynomial $S_\epsilon^{\mathcal{PP}}(t, s)$. More precisely, $S_\epsilon^{\mathcal{PP}}(t, s)$ can be found from several $S_\epsilon^{\mathcal{PP}}(t, s_k), k = 1, \ldots, n$, whose degrees equal to the approximate tracing index. The polynomial $S_\epsilon^{\mathcal{PP}}(t, s)$ can be computed using least squares method while n is greater than the number of the unterminated coefficients (see the method presented in [31]). Note that the approximate tracing index is related to the selected ϵ, and the used ϵ–gcd algorithm.*

In the following example, we show how to compute the polynomial $S_\epsilon^{\mathcal{PP}}$ for a given algebraic plane curve defined by a parametrization $\mathcal{P}$ with perturbed float coefficients.

Example 2. *Let $\epsilon = 0.01$, and the rational curve $\mathcal{C}$ defined by the parametrization*

$$\mathcal{P}(t) = \left(\frac{p_{11}(t)}{p_{12}(t)}, \frac{p_{21}(t)}{p_{22}(t)}\right) =$$

$$\left(\frac{t^4 - .2502500000 + .0005000000000\,t}{t^4 + .2500000000 + .0002500000000\,t^2}, \frac{t^2 - .0002500000000}{t^4 + .2500000000 + .0002500000000\,t^2}\right).$$

$\mathcal{C}$ is a quartic curve but approximately a multiple conic curve (see Figure 2). Using the SNAP package included in Maple, one has that ϵ–gcd$(p_{j1}, p_{j2}) = 1,\ j = 1, 2$. We compute

the polynomials

$$H_1^{\mathcal{P}\mathcal{P}}(t,s) = 8004000\,s^4 + 4000\,s^4\,t^2 - 8004000\,t^4 - 1001\,t^2 + 8000\,s\,t^4 + 2000\,s + 2\,s\,t^2 - 4000\,t^4\,s^2 + 1001\,s^2 - 8000\,t\,s^4 - 2000\,t - 2\,t\,s^2,$$

$$H_2^{\mathcal{P}\mathcal{P}}(t,s) = 16000000\,t^4\,s^2 + 4000001\,s^2 - 4000\,t^4 - 4000001\,t^2 - 16000000\,s^4\,t^2 + 4000\,s^4.$$

Now, we determine the polynomial $S_\epsilon^{\mathcal{P}\mathcal{P}}$*, and we get*

$$S_\epsilon^{\mathcal{P}\mathcal{P}}(t,s) \approx_\epsilon 52160t^2 + 83t + (-83t - 83)s - 52077s^2.$$

Then, ϵ*–index*$(\mathcal{P}) = \deg_t(S_\epsilon^{\mathcal{P}\mathcal{P}}) = 2$ *(see Definition 1).*

Once the polynomial $S_\epsilon^{\mathcal{P}\mathcal{P}}$ is computed, we consider the rational function $R(t)$ similarly as in step 4 of the symbolic algorithm (the details will be developed in Subsection 3.3), and in step 5 we compute the same resultant (the details will be developed in Subsection 3.4). Again, since we are working with approximate mathematical objects, the resultant does not factor as in the symbolic case (see Theorem 2). That is, if the input is an exact parametrization, the symbolic algorithm in Section 2 outputs the parametrization $\mathcal{Q}(t) = (q_{11}(t)/q_{12}(t),\, q_{21}(t)/q_{22}(t)) \in \mathbb{C}(t)^2$, where

$$L_k(s,x_k) = \mathrm{Res}_t(G_k(t,x_k), sC_j(t) - C_i(t)) = (q_{k2}(s)x_i - q_{k1}(s))^{\deg(R)},$$

and $G_k(t,x_k) = x_k p_{k2}(t) - p_{k1}(t),\ k = 1,2$. However, in our case, $q_{k1}(s)/q_{k2}(s)$ will not be exact roots of the polynomials $L_k(s,x_k)$ (see Theorem 2) but ϵ*–roots* or ϵ*–points* (see [24]). Thus, one may expect that a small perturbation of L_k, provides a new polynomial that factorizes as above and the root of this new polynomial provides the output parametrization.

In order to develop this idea, we need to go into detail about the notion of ϵ*–point*. This concept was introduced in [24] as follows: given a tolerance $\epsilon > 0$, and a non-zero polynomial $A \in \mathbb{C}[t,s]$, we say that $(t_0,s_0) \in \mathbb{C}^2$ is an ϵ–point of A, if $|A(t_0,s_0))| \leq \epsilon \|A\|$, where $\|\cdot\| \in \mathbb{R}$ denotes the infinity norm (that is, $\|A\|$ computes the maximum of the absolute values of the coefficients of a polynomial $A \in \mathbb{C}[t,s]$, with respect to the variables t,s; for complex coefficients, $\|A\|$ finds the maximum of the absolute values of the real and imaginary parts), and $|\cdot|$ is the absolute value. An ϵ–point $(t_0,s_0) \in \mathbb{C}^2$ of a polynomial $A \in \mathbb{C}[t,s]$ is represented as $A(t_0,s_0) \approx_\epsilon 0$ (for further details in this notion see [24, 25, 26, 29]).

In Definition 2, we generalize this concept and in particular the operator $\approx_\epsilon$. More precisely, we present the notion of ϵ–point for points of the form $(t, r(t)) \in \mathbb{C}(t)^2$. For this purpose, in the following, $\mathrm{num}(\tau)$ represents the numerator of a rational function $\tau(t) \in \mathbb{C}(t)$. Furthermore, throughout the paper, we use the infinity norm, and we denote it as $\|\cdot\|$ (see the paragraph above). Note that since in our situation all the norms are equivalent, one may reason similarly with a different norm.

Definition 2. *Given two non-zero polynomials $A_i \in \mathbb{C}[t,s]$ with $\|A_i\| = 1$, $i = 1,2$, we say that $A_1 \approx_\epsilon A_2$, if $\|A_1 - A_2\| \leq \epsilon$ and $\deg_t(A_1) = \deg_t(A_2)$, $\deg_s(A_1) = \deg_s(A_2)$. Furthermore, given $r(t) \in \mathbb{C}(t)$, and a non-zero polynomial $A \in \mathbb{C}[t,s]$, we say that $A(t, r(t)) \approx_\epsilon 0$ if $\|\mathrm{num}(A(t, r(t)))\| \leq \epsilon \|A\|$.*

Now, we observe that from the symbolic point of view, it holds that $\mathcal{P} = \mathcal{Q} \circ R$ if and only if $S(t, R(t)) = 0$ (see Section 2). In Definition 3, we translate this fact to the numerical field. For this purpose, we use Definition 2.

Definition 3. *We say that $\mathcal{P}(t) \sim_\epsilon (\mathcal{Q} \circ r)(t)$ if $S_\epsilon^{\mathcal{P}\mathcal{Q}}(t, r(t)) \approx_\epsilon 0$, where $r(t) \in \mathbb{C}(t)$.*

Remark 4. *Throughout the paper, we assume that $\mathcal{P}(t) \not\sim_\epsilon (a,b) \in \mathbb{C}^2$. Thus, we have that $\deg_t(S_\epsilon^{\mathcal{P}\mathcal{P}}) \geq 1$. Indeed: note that $H_j^{\mathcal{P}\mathcal{P}}(t,s) \approx_\epsilon (t-s)N_j(t,s)$, where $N_j \in \mathbb{C}[t,s]$, $j = 1,2$. It holds that $N_j \neq 0$, $j = 1,2$; otherwise, $S_\epsilon^{\mathcal{P}\mathcal{P}}(t,s) \approx_\epsilon 0$, and in particular $S_\epsilon^{\mathcal{P}\mathcal{P}}(t, s_0) \approx_\epsilon 0$ for $s_0 \in \mathbb{C}$ satisfying that $p_{12}(s_0)p_{22}(s_0) \neq 0$. Then, $\mathcal{P}(t) \sim_\epsilon \mathcal{P}(s_0) \in \mathbb{C}^2$ which is impossible, and thus $N_j \neq 0$, $j = 1,2$. Hence, $S_\epsilon^{\mathcal{P}\mathcal{P}}(t,s) \approx_\epsilon (t-s)N(t,s)$, where $N \in \mathbb{C}[t,s] \setminus \{0\}$.*

Remark 5. *From Remark 4, we get that ϵ–index$(\mathcal{P}) = 1$ if and only if $S_\epsilon^{\mathcal{P}\mathcal{P}}(t,s) \approx_\epsilon (t-s)$.*

Now, we are ready to introduce the notions of *ϵ–numerical reparametrization* and *ϵ–proper reparametrization*. For this purpose, we use Definition 3.

Definition 4. *Let $\mathcal{P}(t) = (p_1(t), p_2(t)) \in \mathbb{C}(t)^2$ be a rational parametrization of a given plane curve $\mathcal{C}$. We say that a parametrization $\mathcal{Q}(t) = (q_1(t), q_2(t)) \in \mathbb{C}(t)^2$ is an ϵ–numerical reparametrization of $\mathcal{P}(t)$ if there exists $R(t) = M(t)/N(t) \in \mathbb{C}(t) \setminus \mathbb{C}$, ϵ–gcd$(M,N) = 1$, such that $\mathcal{P} \sim_\epsilon \mathcal{Q} \circ R$. In addition, if ϵ–index$(\mathcal{Q}) = 1$, then we say that $\mathcal{Q}$ is an ϵ–proper reparametrization of $\mathcal{P}$.*

Example 3. *In Example 5, we will show that*

$$\mathcal{Q}(t) = \left(\frac{t^2 + .000005006649227t - .2494538109}{t^2 - .0002445955365t + .2492042101}, \frac{-.9984087427t - .0002529376363}{t^2 - .0002445955365t + .2492042101} \right)$$

is an ϵ–proper reparametrization of the parametrization $\mathcal{P}$ introduced in Example 2. More precisely, it holds that $\mathcal{P} \sim_\epsilon \mathcal{Q} \circ R$, where $R(t) = \frac{52160t^2 + 83t}{-52077}$ (see Example 4). In Figure 3, we plot the input curve $\mathcal{C}$ and the output curve $\mathcal{D}$.

3.2. Some Previous Results

Using the concepts introduced in Subsection 3.1, we obtain some important properties that will be used to construct the ϵ–numerical reparametrization in Subsections 3.3 and 3.4. In particular, we characterize whether a given ϵ–numerical reparametrization $\mathcal{Q}$ is ϵ–proper (see Definition 4). We start with a technical result that will play an important role to prove the main theorem (Theorem 1).

Proposition 1. *Let $\mathcal{Q}(t) = (q_1(t), q_2(t)) \in \mathbb{C}(t)^2$, $q_j = q_{j1}/q_{j2}$, ϵ–gcd$(q_{j1}, q_{j2}) = 1$, $j = 1, 2$ be such that $\mathcal{Q}(t) \not\approx_\epsilon (a, b) \in \mathbb{C}^2$. Let $R(t) = M(t)/N(t) \in \mathbb{C}(t) \setminus \mathbb{C}$, ϵ–gcd$(M, N) = 1$. Up to constants in $\mathbb{C} \setminus \{0\}$, it holds that*

$$S_\epsilon^{\mathcal{Q}(R)\mathcal{Q}(R)}(t, s) \approx_\epsilon \mathrm{num}(S_\epsilon^{\mathcal{Q}\mathcal{Q}}(R(t), R(s))).$$

Proof. From the definition of $S_\epsilon^{\mathcal{Q}\mathcal{Q}}(t, s)$, there are $M_1, M_2 \in \mathbb{C}[t, s]$ satisfying that

$$H_j^{\mathcal{Q}\mathcal{Q}}(t, s) \approx_\epsilon S_\epsilon^{\mathcal{Q}\mathcal{Q}}(t, s)M_j(t, s),\ j = 1, 2, \quad \text{and} \quad \epsilon\text{–gcd}(M_1, M_2) = 1 \tag{1}$$

(see Definition 2). Now, taking into account the definition of $S_\epsilon^{\mathcal{Q}(R)\mathcal{Q}(R)}$, one gets that

$$S_\epsilon^{\mathcal{Q}(R)\mathcal{Q}(R)}(t, s) = \epsilon\text{–gcd}(H_1^{\mathcal{Q}(R)\mathcal{Q}(R)}(t, s), H_2^{\mathcal{Q}(R)\mathcal{Q}(R)}(t, s)). \tag{2}$$

In addition, it holds that

$$\epsilon\text{–gcd}(H_1^{\mathcal{Q}(R)\mathcal{Q}(R)}(t, s), H_2^{\mathcal{Q}(R)\mathcal{Q}(R)}(t, s)) = \epsilon\text{–gcd}(\mathrm{num}(H_1^{\mathcal{Q}\mathcal{Q}}(R(t), R(s))), \mathrm{num}(H_2^{\mathcal{Q}\mathcal{Q}}(R(t), R(s)))). \tag{3}$$

In order to prove (3), we assume that $\deg(q_{i2}) \geq \deg(q_{i1})$ (otherwise, we reason similarly), and we consider $q_{ij}^*(x, y) \in \mathbb{C}[x, y]$ the homogenization of the polynomial $q_{ij}(x) \in \mathbb{C}[x]$ with respect to the variable x, and $\alpha := \deg(q_{i2}) - \deg(q_{i1})$. Under these conditions, equality (3) follows since

$$H_i^{\mathcal{Q}(R)\mathcal{Q}(R)}(t, s) = \mathrm{num}\left(\frac{q_{i1}(R(t))}{q_{i2}(R(t))} - \frac{q_{i1}(R(s))}{q_{i2}(R(s))}\right) =$$

$$N(t)^\alpha q_{i1}^*(M(t), N(t))q_{i2}^*(M(s), N(s)) - N(s)^\alpha q_{i1}^*(M(s), N(s))q_{i2}^*(M(t), N(t)),$$

and

$$\mathrm{num}(H_i^{\mathcal{Q}\mathcal{Q}}(R(t), R(s))) = \mathrm{num}\left(q_{i1}(R(t))q_{i2}(R(s)) - q_{i1}(R(s))q_{i2}(R(t))\right) =$$

$$\mathrm{numer}\left(\frac{q_{i1}^*(M(t), N(t))q_{i2}^*(M(s), N(s))}{N(t)^{\deg(q_{i1})}N(s)^{\deg(q_{i2})}} - \frac{q_{i1}^*(M(s), N(s))q_{i2}^*(M(t), N(t))}{N(t)^{\deg(q_{i2})}N(s)^{\deg(q_{i1})}}\right) =$$

$$N(t)^\alpha q_{i1}^*(M(t), N(t))q_{i2}^*(M(s), N(s)) - N(s)^\alpha q_{i1}^*(M(s), N(s))q_{i2}^*(M(t), N(t)).$$

Thus, from the above equalities, one deduces that

$$S_\epsilon^{\mathcal{Q}(R)\mathcal{Q}(R)}(t, s) \overbrace{=}^{(2)\text{ and }(3)} \epsilon\text{–gcd}(\mathrm{num}(H_1^{\mathcal{Q}\mathcal{Q}}(R(t), R(s))), \mathrm{num}(H_2^{\mathcal{Q}\mathcal{Q}}(R(t), R(s)))) \overbrace{\approx_\epsilon}^{(1)}$$

$$\epsilon\text{–gcd}(\text{num}(S_\epsilon^{\mathcal{Q}\mathcal{Q}}(R(t),R(s)))\text{num}(M_1(R(t),R(s))),\text{num}(S_\epsilon^{\mathcal{Q}\mathcal{Q}}(R(t),R(s)))\text{num}(M_2(R(t),R(s))))$$
$$= \text{num}(S_\epsilon^{\mathcal{Q}\mathcal{Q}}(R(t),R(s)))M(t,s),$$

where $M(t,s) := \epsilon\text{–gcd}(\text{num}(M_1(R(t),R(s))),\text{num}(M_2(R(t),R(s))))$. Since $\epsilon\text{–gcd}(M_1,M_2) = 1$, we have that $M(t,s) = 1$, and we conclude that

$$S_\epsilon^{\mathcal{Q}(R)\mathcal{Q}(R)}(t,s) \approx_\epsilon \text{num}(S_\epsilon^{\mathcal{Q}\mathcal{Q}}(R(t),R(s))). \qquad \square$$

In the following, we consider $\mathcal{Q}(t) \in \mathbb{C}(t)^2$ an ϵ–numerical reparametrization of $\mathcal{P}(t)$. Hence, $\mathcal{P} \sim_\epsilon \mathcal{Q} \circ R$ where $R(t) = M(t)/N(t) \in \mathbb{C}(t) \setminus \mathbb{C}$ (see Definition 4). Under these conditions, in the following results we characterize whether $\mathcal{Q}$ is ϵ–proper. This characterization is based in ϵ–index$(\mathcal{P})$ (see Theorem 1 and Corollary 1), and in the polynomial $S_\epsilon^{\mathcal{P}\mathcal{P}}$ (see Corollary 2).

Theorem 1. *$\mathcal{Q}$ is ϵ–proper if and only if ϵ–index$(\mathcal{P}) = \deg(R)$.*

Proof. If ϵ–index$(\mathcal{Q}) = 1$, from Proposition 1 and Remark 5, one gets that

$$S_\epsilon^{\mathcal{P}\mathcal{P}}(t,s) \approx_\epsilon \text{num}(S_\epsilon^{\mathcal{Q}\mathcal{Q}}(R(t),R(s))) \approx_\epsilon \text{num}(R(t)-R(s)) = M(t)N(s) - M(s)N(t).$$

Therefore, ϵ–index$(\mathcal{P}) = \deg_t(S_\epsilon^{\mathcal{P}\mathcal{P}}) = \deg(R)$ (see Definitions 1 and 2). Reciprocally, from Proposition 1, we have that

$$S_\epsilon^{\mathcal{P}\mathcal{P}}(t,s) \approx_\epsilon \text{num}(S_\epsilon^{\mathcal{Q}\mathcal{Q}}(R(t),R(s)))$$

which implies that $\deg_t(S_\epsilon^{\mathcal{P}\mathcal{P}}) = \deg_t(S_\epsilon^{\mathcal{Q}\mathcal{Q}})\deg(R)$ (see Definition 2). Therefore, if ϵ–index$(\mathcal{P}) = \deg(R)$, then $\deg_t(S_\epsilon^{\mathcal{Q}\mathcal{Q}}) = 1$ and thus $\mathcal{Q}$ is ϵ–proper (see Definition 1). $\square$

Corollary 1. *It holds that ϵ–index$(\mathcal{P}) = \epsilon$–index$(\mathcal{Q})\deg(R)$.*

Proof. Reasoning as in the proof of Theorem 1, one gets that $\deg_t(S_\epsilon^{\mathcal{P}\mathcal{P}}) = \deg_t(S_\epsilon^{\mathcal{Q}\mathcal{Q}})\deg(R)$. Thus, from Definition 1, we conclude that ϵ–index$(\mathcal{P}) = \epsilon$–index$(\mathcal{Q})\deg(R)$. $\square$

Corollary 2. *$\mathcal{Q}$ is ϵ–proper if and only if*

$$S_\epsilon^{\mathcal{P}\mathcal{P}}(t,s) \approx_\epsilon \text{num}(R(t)-R(s)) = M(t)N(s) - M(s)N(t).$$

Proof. If ϵ–index$(\mathcal{Q}) = 1$, reasoning as in the proof of Theorem 1, one deduces that $S_\epsilon^{\mathcal{P}\mathcal{P}}(t,s) \approx_\epsilon M(t)N(s) - M(s)N(t)$. Reciprocally, if $S_\epsilon^{\mathcal{P}\mathcal{P}}(t,s) \approx_\epsilon \text{num}(R(t)-R(s))$, we get that ϵ–index$(\mathcal{P}) = \deg_t(S_\epsilon^{\mathcal{P}\mathcal{P}}) = \deg(R)$ (see Definition 2). Thus, Corollary 1 implies that $\mathcal{Q}$ is ϵ–proper. $\square$

3.3. Construction of the Rational Function $R(t)$

In this section, we construct a rational function $R(t) \in \mathbb{C}(t) \setminus \mathbb{C}$ such that there exists an ϵ–proper reparametrization of $\mathcal{P}$. That is, there exists $\mathcal{Q}$ such that $\mathcal{P} \sim_\epsilon \mathcal{Q} \circ R$ and $\mathcal{Q}$ is ϵ–proper (see Theorem 2 and Corollary 3 in Subsection 3.4). Hence, we are addressing the existence of the ϵ–proper reparameterization.

For this purpose, we first note that in the symbolic case, once the polynomial $S(t,s)$ is computed (see step 2 of the algorithm presented in Section 2), we get $R(t)$ as follows: let $S(t,s) = \gcd(H_1(t,s), H_2(t,s)) = C_m(t)s^m + \cdots + C_0(t)$, where $H_j(t,s) = p_{j1}(t)p_{j2}(s) - p_{j1}(s)p_{j2}(t)$, $j = 1,2$. Then, we consider $R(t) = C_i(t)/C_j(t)$ such that $C_j(t)$, $C_i(t)$ are obtained from the polynomial $S(t,s)$ satisfying that $\gcd(C_j, C_i) = 1$, and $C_jC_i \notin \mathbb{C}$ (see step 4 in the symbolic algorithm presented in Section 2). We note that in the symbolic situation, Lemma 3 in [22] states that, up to constants in $\mathbb{C} \setminus \{0\}$,

$$S(t,s) = \mathrm{num}\left(\frac{C_i(t)}{C_j(t)} - \frac{C_i(s)}{C_j(s)}\right) = C_m(t)s^m + C_{m-1}(t)s^{m-1} + \cdots + C_0(t),$$

and $S(t,s) = \mathrm{num}(R(t) - R(s))$. This is the reason why one considers $R(t) = C_i(t)/C_j(t)$.

From the numerical point of view, the idea is similar as in the symbolic case. More precisely, we first write

$$S_\epsilon^{\mathcal{PP}}(t,s) = C_m(t)s^m + C_{m-1}(t)s^{m-1} + \cdots + C_0(t). \tag{4}$$

Now, from Corollary 2, we have that

$$S_\epsilon^{\mathcal{PP}}(t,s) \approx_\epsilon \mathrm{num}(R(t) - R(s)),$$

where $R(t) = M(t)/N(t) \in \mathbb{C}(t) \setminus \mathbb{C}$ is the unknown rational function we are looking for.

Taking into account that, up to constants in $\mathbb{C} \setminus \{0\}$,

$$\mathrm{num}\left(\frac{C_i(t)}{C_j(t)} - \frac{C_i(s)}{C_j(s)}\right) = C_m(t)s^m + C_{m-1}(t)s^{m-1} + \cdots + C_0(t), \tag{5}$$

where C_i, C_j are such that $C_iC_j \notin \mathbb{C}$, and $\gcd(C_i,\, C_j) = 1$ (see Lemma 3 in [22]), and reasoning as in the symbolic case, we consider

$$R(t) = \frac{C_i(t)}{C_j(t)} \in \mathbb{C}(t) \setminus \mathbb{C},$$

where C_i and C_j are from (4) satisfying that:

1) $C_iC_j \notin \mathbb{C}$, 2) ϵ–$\gcd(C_i,\, C_j) = 1$.

Under these conditions, we observe that from equality (5), it holds that $S_\epsilon^{\mathcal{PP}}(t,s) \approx_\epsilon \mathrm{num}(R(t) - R(s))$.

In the following proposition, we prove that the polynomials C_i, C_j satisfying conditions 1) and 2) above exist.

Proposition 2. *Let*

$$S_\epsilon^{\mathcal{PP}}(t,s) = C_m(t)s^m + C_{m-1}(t)s^{m-1} + \cdots + C_0(t).$$

It holds that there exist C_i and C_j, $i,j \in \{0,\ldots,m\}$, $i \neq j$, satisfying that

1) $C_iC_j \notin \mathbb{C}$,

2) ϵ*–gcd*$(C_i,\, C_j) = 1$.

Proof. First, taking into account Remark 4, we have that $S_\epsilon^{\mathcal{PP}}(t,s) \approx_\epsilon (t-s)N(t,s)$. Thus, there exists $C_i \notin \mathbb{C}$ for some $i = 0,\ldots,m$, and $m \geq 1$. Then, condition 1 holds. Now, let us suppose that $C_0 \notin \mathbb{C}$, and let us prove that there exists $i \in \{1,\ldots,m\}$ such that ϵ–gcd$(C_0,\, C_i) = 1$. Let us assume that ϵ–gcd$(C_0,\, C_i) \neq 1$ for every $i \in \{1,\ldots,m\}$. Then, we get that $C_i \notin \mathbb{C}$ for every $i \in \{0,\ldots,m\}$ and hence, each pair of polynomials C_i, C_j satisfies condition 1. If ϵ–gcd$(C_i,\, C_j) = 1$ for some $i,j \in \{0,\ldots,m\}$, $i \neq j$, we get that condition 2 holds. Otherwise, ϵ–gcd$(C_i,\, C_j) \neq 1$ for every $i,j \in \{0,\ldots,m\}$, $i \neq j$, and thus $M(t) :=$ ϵ–gcd$(C_1,\ldots,C_m) \in \mathbb{C}(t) \setminus \mathbb{C}$. Let $t_0 \in \mathbb{C}$ such that $M(t_0) = 0$, and $p_{12}(t_0)p_{22}(t_0) \neq 0$ (since we have approximate mathematical objects and in exact computation gcd$(C_0,\ldots,C_m) = 1$, we get that this t_0 exists). Then, $S_\epsilon^{\mathcal{PP}}(t_0,s) \approx_\epsilon 0$ which implies that $\mathcal{P}(t) \sim_\epsilon \mathcal{P}(t_0) \in \mathbb{C}^2$ (note that $p_{12}(t_0)p_{22}(t_0) \neq 0$). This is impossible, since by Remark 4, $\mathcal{P}(t) \not\sim_\epsilon (a,b) \in \mathbb{C}^2$. □

In Example 4, we consider the plane curve introduced in Example 2, and we show how to construct the rational function $R(t)$.

Example 4. *Continuing with Example 2, we get that*

$$S_\epsilon^{\mathcal{PP}}(t,s) \approx_\epsilon C_0(t) + C_1(t)s + C_2(t)s^2,$$

where $C_0(t) = 52160t^2 + 83t,\ C_1(t) = -83t - 83,$ *and* $C_2(t) = -52077$. *Observe that C_0 and C_2 satisfy the conditions in Proposition 2. Thus, we may consider the rational function*

$$R(t) = \frac{C_0(t)}{C_2(t)} = \frac{52160t^2 + 83t}{-52077}.$$

3.4. Construction of the ϵ-Numerical Reparametrization $\mathcal{Q}(t)$

In the following, we consider the rational function $R(t) = \frac{C_i(t)}{C_j(t)} \in \mathbb{C}(t) \setminus \mathbb{C}$ computed in Subsection 3.3. Then, we have that $S_\epsilon^{\mathcal{PP}}(t,s) \approx_\epsilon \mathrm{num}(R(t) - R(s))$. Hence, from Corollary 2, if $\mathcal{Q}$ is such that $\mathcal{P} \sim_\epsilon \mathcal{Q} \circ R$, then $\mathcal{Q}$ is ϵ–proper. Thus, in the following, we focus on the computation of $\mathcal{Q}$ satisfying that $\mathcal{P} \sim_\epsilon \mathcal{Q} \circ R$.

For this purpose, we observe that in the symbolic case, once the rational function R is computed (see step 4 of the algorithm in Section 2), we consider the polynomials

$$L_k(s,x_k) = \mathrm{Res}_t(G_k(t,x_k), sC_j(t) - C_i(t)) = (q_{k2}(s)x_k - q_{k1}(s))^{\deg(R)},$$

where $G_k(t,x_k) = x_k p_{k2}(t) - p_{k1}(t)$ for $k=1,2$ (see step 5 of the algorithm in Section 2). Thus, the output parametrization $\mathcal{Q}$ is given by the roots with respect to the variable x_k of the polynomial L_k. That is, $\mathcal{Q}(t) = (q_{11}(t)/q_{12}(t),\, q_{21}(t)/q_{22}(t)) \in \mathbb{C}(t)^2$ (see step 6 in the symbolic algorithm presented in Section 2).

From the numerical point of view, the idea is similar as in the symbolic case. More precisely, we compute the same resultants as in the symbolic case, and the "approximate" roots of these polynomials with respect to the variable x_k will provide the parametrization $\mathcal{Q}$ (see Theorem 2).

Theorem 2. *Let*

$$L_k(s,x_k) = \mathrm{Res}_t(G_k(t,x_k), sC_j(t) - C_i(t)), \ \textit{where } G_k(t,x_k) = x_k p_{k2}(t) - p_{k1}(t), \ k=1,2.$$

If

$$L_k(s,x_k) = (x_k q_{k2}(s) - q_{k1}(s))^\ell + \epsilon^\ell W_k(s,x_k), \quad \|\mathrm{num}(W_k(R,p_k))\| \le \|H_k^{\mathcal{PQ}}\|^\ell, \ k=1,2,$$

where $\ell := \deg(R)$, *and* ϵ*–gcd*$(q_{k1},q_{k2}) = 1$, *then* $\mathcal{Q}(s) = \left(\frac{q_{11}(s)}{q_{12}(s)}, \frac{q_{21}(s)}{q_{22}(s)}\right)$ *is an* ϵ*-numerical reparametrization of* $\mathcal{P}$.

Proof. First, we observe that $L_k \neq 0$ (otherwise, G_k and $sC_j(t) - C_i(t)$ have a common factor depending on t, which is impossible because $\gcd(C_i, C_j) = 1$). In addition, it holds that $\deg_{x_k}(L_k) = \deg(R)$. Indeed, since

$$L_k(s,x_k) = \mathrm{Res}_t(G_k(t,x_k), sC_j(t) - C_i(t)),$$

we get that, up to constants in $\mathbb{C}(s) \setminus \{0\}$,

$$L_k(s,x_k) = \prod_{\{\alpha_\ell \,|\, sC_j(\alpha_\ell) - C_i(\alpha_\ell) = 0\}} G_k(\alpha_\ell, x_k),$$

(see Sections 5.8 and 5.9 in [36]), and thus

$$\deg_{x_k}(L_k) = \deg_t(sC_j(t) - C_i(t)) \deg_{x_k}(G_k(t,x_k)) = \deg(R).$$

In addition, from $\deg_{x_k}(L_k) = \deg(R)$, we deduce that $\deg_{x_k}(W_k) \le \ell$. In fact, since we are working with approximate mathematical objects, we may assume without loss of generality that $\deg_{x_k}(W_k) = \ell$.
Now, taking into account the properties of the resultant (see Section 2), one has that

$$0 = L_k(R(t), p_k(t)) = (p_k(t) q_{k2}(R(t)) - q_{k1}(R(t)))^\ell + \epsilon^\ell W_k(R(t), p_k(t)).$$

Then,

$$\mathrm{num}(H_k^{\mathcal{PQ}}(t, R(t)))^\ell = \epsilon^\ell e_k(t), \ \text{ where } \ e_k := -W_k(R(t), p_k(t)) p_{k2}(t)^\ell C_j^{\ell \deg(q_k)}, \qquad k=1,2.$$

Since $\deg_{x_k}(W_k) = \ell$, and $\deg_s(W_k) = \ell \deg(q_k)$ (see Corollary 4), one has that

$$e_k = -\mathrm{num}(W_k(R(t), p_k(t))) \in \mathbb{C}[t]$$

(i.e. the denominator of $W_k(R(t), p_k(t))$ is canceled with $p_{k2}(t)^\ell C_j(t)^{\ell \deg(q_k)}$). Therefore, from $\mathrm{num}(H_k^{\mathcal{PQ}}(t, R(t)))^\ell = \epsilon^\ell e_k(t)$, and taking into account that $\|\mathrm{num}(W_k(R, p_k))\| \leq \|H_k^{\mathcal{PQ}}\|^\ell$, we get that

$$\|\mathrm{num}(H_k^{\mathcal{PQ}}(t, R(t)))\|^\ell = \epsilon^\ell \|e_k\| \leq \epsilon^\ell \|H_k^{\mathcal{PQ}}\|^\ell,$$

which implies that $H_k^{\mathcal{PQ}}(t, R(t)) \approx_\epsilon 0$ (see Definition 2). Thus, $S_\epsilon^{\mathcal{PQ}}(t, R(t)) \approx_\epsilon 0$, and then $\mathcal{P}(t) \sim_\epsilon (\mathcal{Q} \circ R)(t)$. □

Remark 6. *From the proof of Theorem 2, we get that* $\deg_{x_k}(L_k) = \deg_{x_k}(W_k) = \deg(R) = \ell$

Remark 7. *If the tolerance in Theorem 2 changes (that is, instead ϵ we have $\overline{\epsilon}$), Theorem 2 provides an $\overline{\epsilon}$-numerical reparametrization of $\mathcal{P}$. More precisely, if*

$$L_k(s, x_k) = (x_k q_{k2}(s) - q_{k1}(s))^\ell + \overline{\epsilon}^\ell \, W_k(s, x_k), \quad \|\mathrm{num}(W_k(R, p_k))\| \leq \|H_k^{\mathcal{PQ}}\|^\ell,$$

where $\ell = \deg(R)$ and ϵ–gcd$(q_{k1}, q_{k2}) = 1$, then $\mathcal{Q}(s) = \left(\frac{q_{11}(s)}{q_{12}(s)}, \frac{q_{21}(s)}{q_{22}(s)}\right)$ is an $\overline{\epsilon}$–numerical reparametrization of $\mathcal{P}$.

3.5. Properties of the ϵ–Numerical Reparametrization $\mathcal{Q}(t)$

Let $\mathcal{Q}$ be the ϵ–numerical reparametrization of $\mathcal{P}$ computed in Theorem 2. In the following, we present some corollaries obtained from Theorem 2, where some properties concerning $\mathcal{Q}$ are provided (for this purpose, results in Subsection 3.2 are used). In particular, we show that $\mathcal{Q}$ is ϵ–proper (see Corollary 3) and we prove that $\deg(\mathcal{P}) = \deg(\mathcal{Q})\deg(R)$ (see Corollary 4). This last equality also holds in the symbolic case, and it shows the expected property that the degree of the rational functions defining the ϵ–proper parametrization $\mathcal{Q}$ is lower than the non ϵ–proper input parametrization $\mathcal{P}$.

In addition, we show how the parametrization $\mathcal{Q}$ can be computed from the expression obtained in Theorem 2 (see Corollary 6).

Corollary 3. *$\mathcal{Q}$ is ϵ–proper.*

Proof. Since $R(t) = \frac{C_i(t)}{C_j(t)} \in \mathbb{C}(t) \setminus \mathbb{C}$ is such that $S_\epsilon^{\mathcal{PP}}(t, s) \approx_\epsilon \mathrm{num}(R(t) - R(s))$, and $\mathcal{Q}$ is an ϵ-numerical reparametrization of $\mathcal{P}$ (see Theorem 2), from Corollary 2, we conclude that $\mathcal{Q}$ is ϵ–proper. □

Remark 8. *Corollaries 1 and 3 imply that $\ell = \epsilon$–index$(\mathcal{P})$, where $\ell = \deg(R)$ is introduced in Theorem 2.*

Corollary 4. *It holds that* $\deg(\mathcal{P}) = \deg(\mathcal{Q})\deg(R)$.

Proof. First, we observe that $\deg_s(W_k) = \deg(p_k)$, for $k = 1, 2$. Indeed, since

$$L_k(s, x_k) = \mathrm{Res}_t(G_k(t, x_k), sC_j(t) - C_i(t)),$$

we get that, up to constants in $\mathbb{C}(x_k) \setminus \{0\}$,

$$L_k(s, x_k) = \prod_{\{\beta_\ell \,|\, G_k(\beta_\ell, x_k)=0\}} sC_j(\beta_\ell) - C_i(\beta_\ell),$$

(see Sections 5.8 and 5.9 in [36]), and thus

$$\deg_s(L_k) = \deg_s(sC_j(t) - C_i(t))\deg_t(G_k(t, x_k)) = \deg(p_k).$$

Since we are working with approximate mathematical objects, we may assume that $\deg_s(W_k) = \deg_s(L_k)$. On the other side, from Theorem 2, we have that

$$L_k(s, x_k) = (x_k q_{k2}(s) - q_{k1}(s))^\ell + \epsilon^\ell W_k(s, x_k), \quad k = 1, 2.$$

Since we are working numerically, we may assume without loss of generality that

$$\deg_s(W_k) = \deg_s((x_k q_{k2}(s) - q_{k1}(s))^\ell) = \ell \deg(q_k).$$

Therefore, $\ell \deg(q_k) = \deg(p_k)$, $k = 1, 2$, which implies that

$$\deg(\mathcal{P}) = \deg(\mathcal{Q})\ell = \deg(\mathcal{Q})\deg(R)$$

(from Theorem 2, we have that $\ell = \deg(R)$). □

Corollary 5. *Under the conditions of Theorem 2, it holds that*

$$\mathrm{Res}_t(p_{k2}(t), sC_j(t) - C_i(t)) = q_{k2}(s)^\ell + \epsilon^\ell b_k(s), \quad b_k \in \mathbb{C}[s]$$

and $q_{k2}(s)^\ell + \epsilon^\ell b_k(s) \neq 0$*, for* $k = 1, 2$.

Proof. From Theorem 2 and Corollary 4, we have that

$$L_k(s, x_k) = (x_k q_{k2}(s) - q_{k1}(s))^\ell + \epsilon^\ell W_k(s, x_k), \;\; k = 1, 2,$$

and $\deg_s(W_k) = \ell \deg(q_k) = \deg(p_k)$. Let $L_k^*(s, x_k, x_3)$ be the homogeneous form of the polynomial $L_k(s, x_k)$ with respect to the variable x_k. Using the specialization resultant property (see Section 2), we deduce that

$$L_k^*(s, x_k, x_3) = \mathrm{Res}_t(x_k p_{k2}(t) - x_3 p_{k1}(t), sC_j(t) - C_i(t)) =$$

$$= (x_k q_{k2}(s) - x_3 q_{k1}(s))^\ell + \epsilon^\ell b_k(s) x_k^\ell + \epsilon^\ell x_3 U_k^*(s, x_k, x_3),$$

where $U^*(s, x_k, x_3)$ denotes the homogeneous form of the polynomial $W_k(s, x_k) - b_k(s)x_k^\ell \in (\mathbb{C}[s])[x_k]$, and b_k is the leading coefficient of W_k with respect to x_k (that is, b_k is the coefficient of W_k with respect to x_k^ℓ; note that by Remark 6, we have that $\deg_{x_k}(W_k) = \deg(R) = \ell$). Hence, from the specialization resultant property (see Section 2), we get

$$L_k^*(s, 1, 0) = \mathrm{Res}_t(p_{k2}(t), sC_j(t) - C_i(t)) = q_{k2}(s)^\ell + \epsilon^\ell b_k(s), \quad k = 1, 2.$$

Finally, we note that $q_{k2}(s)^\ell + \epsilon^\ell b_k(s) \neq 0$, for $k = 1, 2$. Otherwise, $\mathrm{Res}_t(p_{k2}(t), sC_j(t) - C_i(t)) = 0$ which would imply that $\gcd(p_{k2}, C_i, C_j) \neq 1$ (see the properties of the resultant in Section 2). This is impossible since $\gcd(C_i, C_j) = 1$ (see Subsection 3.3). $\square$

In the following, we consider the parametrization

$$\widetilde{\mathcal{Q}}(s) = (\widetilde{q}_1(s), \widetilde{q}_2(s)) = \left(\frac{\widetilde{q}_{11}(s)}{\widetilde{q}_{12}(s)}, \frac{\widetilde{q}_{21}(s)}{\widetilde{q}_{22}(s)}\right) =$$

$$\left(\frac{q_{11}(s)q_{12}(s)^{\ell-1} + \epsilon^\ell a_1(s)/\ell}{q_{22}(s)^\ell + \epsilon^\ell b_1(s)}, \frac{q_{21}(s)q_{22}(s)^{\ell-1} + \epsilon^\ell a_2(s)/\ell}{q_{22}(s)^\ell + \epsilon^\ell b_2(s)}\right).$$

Observe that $\widetilde{\mathcal{Q}}$ can be further simplified by removing the approximate gcd from the numerator and denominator (for instance, one may use QRGCD algorithm to compute an approximate gcd of two univariate polynomials; see statement 1 in Remark 3). The simplification of $\widetilde{\mathcal{Q}}$ will provide the searched rational parametrization $\mathcal{Q}(s) = \left(\frac{q_{11}(s)}{q_{12}(s)}, \frac{q_{21}(s)}{q_{22}(s)}\right)$.

Corollary 6 shows how $\widetilde{\mathcal{Q}}$ can be easily computed from the polynomial L_k introduced in Theorem 2

Corollary 6. *Under the conditions of Theorem 2, it holds that*

1. $\deg_{x_k}\left(\frac{\partial^{\ell-1} L_k}{\partial^{\ell-1} x_k}(s, x_k)\right) = 1$ *and* $\frac{\partial^{\ell-1} L_k}{\partial^{\ell-1} x_k}(s, \widetilde{q}_k(s)) = 0, \; k = 1, 2.$
2. $\frac{\partial^{\ell-1} L_k}{\partial^{\ell-1} x_k} = \frac{-\mathrm{coeff}(L_k, x_k^{\ell-1})/\ell}{\mathrm{coeff}(L_k, x_k^\ell)}$, $k = 1, 2$, *where* coeff(pol, var) *denotes the coefficient of a polynomial pol with respect to var.*

Proof. In order to prove statement 1, we first note that from Theorem 2,

$$L_k(s, x_k) = (x_k q_{k2}(s) - q_{k1}(s))^\ell + \epsilon^\ell W_k(s, x_k), \quad k = 1, 2,$$

where $\deg_s(W_k) = \ell \deg(q_k) = \deg(p_k)$ (see Corollary 4). Thus,

$$\frac{\partial^{\ell-1} L_k}{\partial^{\ell-1} x_k}(s, x_k) = \ell! x_k q_{k2}(s)^\ell - \ell! q_{k1}(s) q_{k2}(s)^{\ell-1} + \ell! \epsilon^\ell x_k b_k(s) - (\ell - 1)! \epsilon^\ell a_k(s),$$

where $b_k(s)$ is the coefficient of W_k with respect to x_k^ℓ (by Remark 6, we have that $\deg_{x_k}(W_k) = \deg(R) = \ell$), and $a_k(s)$ is the coefficient of W_k with respect to $x_k^{\ell-1}$. Note that from Corollary 5, we have that $q_{k2}(s)^\ell + \epsilon^\ell b_k(s) \neq 0$ and thus, $\deg_{x_k}\left(\dfrac{\partial^{\ell-1} L_k}{\partial^{\ell-1} x_k}(s, x_k)\right) = 1$. In addition, clearly one has that $\dfrac{\partial^{\ell-1} L_k}{\partial^{\ell-1} x_k}(s, \widetilde{q}_k(s)) = 0, \;\; k = 1, 2$.
Statement 2 is obtained from statement 1 using that:

$$\mathrm{coeff}(L_k, x_k^\ell) = q_{k2}(s)^\ell + \epsilon^\ell b_k(s), \;\text{ and }\; \mathrm{coeff}(L_k, x_k^{\ell-1}) = -\ell q_{k2}(s)^{\ell-1} q_{k1}(s) - \epsilon^\ell a_k(s), \;\; k = 1, 2. \;\square$$

Remark 9. *In the following, we denote by $\widetilde{\mathcal{D}}$ the plane curve defined by the parametrization $\widetilde{\mathcal{Q}}(s) = \left(\frac{\widetilde{q}_{11}(s)}{\widetilde{q}_{12}(s)}, \frac{\widetilde{q}_{21}(s)}{\widetilde{q}_{22}(s)}\right)$ introduced above. In addition, let $\mathcal{D}$ be the plane curve defined by the parametrization $\mathcal{Q}(s) = \left(\frac{q_{11}(s)}{q_{12}(s)}, \frac{q_{21}(s)}{q_{22}(s)}\right)$ obtained by removing the approximate gcd from the numerator and denominator of each component of $\widetilde{\mathcal{Q}}$ (see statement 1 in Remark 3). In Section 4, we will analyze the relationship between the input curve $\mathcal{C}$ and $\widetilde{\mathcal{D}}$ (see Subsection 4.1). Afterwards, since the simplification modifies the geometry (at infinity) of the reparameterized curve, we provide some error bounds that measure the closeness between the curves $\mathcal{C}$ and $\mathcal{D}$ (see Subsection 4.2).*

In Example 5, we show how to construct the ϵ–numerical reparametrization $\mathcal{Q}$. In particular, we illustrate Theorem 2 and Corollary 6. For this purpose, we consider the plane curve introduced in Example 2, and we use the rational function R constructed in Example 4.

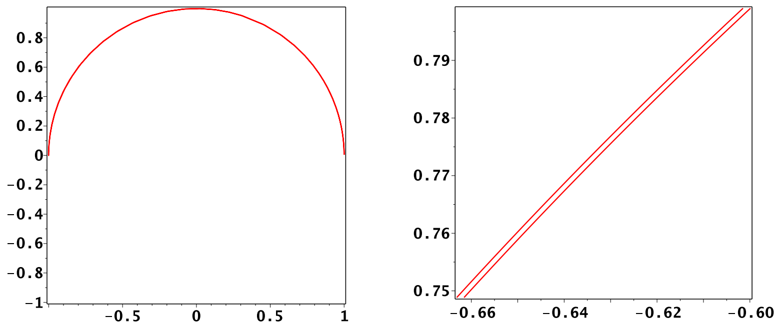
Example 5. *Continuing with Example 4, we obtain that*

$$R(t) = \frac{C_0(t)}{C_2(t)} = \frac{52160t^2 + 83t}{-52077}.$$

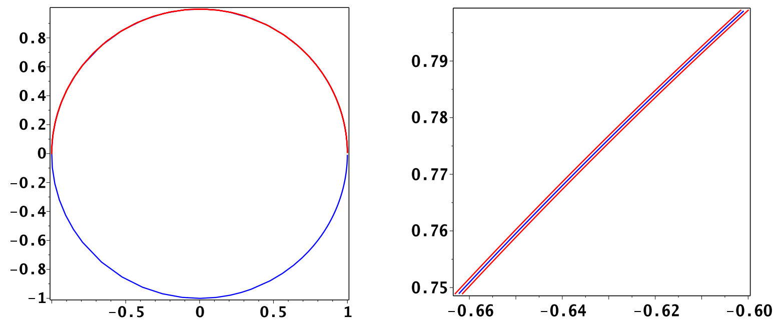
Note that $\deg(R) = \ell = 2$. Now, we show how to compute the ϵ–numerical reparametrization $\mathcal{Q}$. For this purpose, we determine the polynomials

$L_1(s, x_1) = \mathrm{Res}_t(G_1(t, x_1), sC_2(t) - C_0(t)) = .06216533631\, x_1 - .2494530177\, s^2 + .1389703128\, 10^{-5}\, s - .6226547971\, 10^{-4} s\, x_1 + .2480131435\, 10^{-3} s^2\, x_1 + .03105156714\, x_1^2 + .2492050357\, s^2\, x_1^2 - .6346614958\, 10^{-4} s\, x_1^2 + .5000000000\, s^4\, x_1^2 - 1.\, s^4\, x_1 + .2496021856\, 10^{-3} s^3\, x_1 - .2496021856\, 10^{-3} s^3\, x_1^2 + .5000000000\, s^4 + .03111380032$,

$L_2(s, x_2) = \mathrm{Res}_t(G_2(t, x_2), sC_2(t) - C_0(t)) = .6188610482\, 10^{-4} x_2 + .2492049406\, x_2\, s + .4992043712\, s^2 + .2492050042\, 10^{-3} s - .1268068308\, 10^{-5}\, x_2\, s^2 + .03110105693\, x_2^2 - .6356730153\, 10^{-4} x_2^2\, s + .2496022171\, x_2^2\, s^2 - .2500000000\, 10^{-3} x_2^2\, s^3 + x_2\, s^3 + .5007968969\, x_2^2\, s^4 + .3078504788\, 10^{-7}$,



Input curve $\mathcal{C}$ (left), partial enlarged view of $\mathcal{C}$ (right)



Curves $\mathcal{C}$ and $\widetilde{\mathcal{D}}$ (left), partial enlarged view of $\mathcal{C}$ and $\widetilde{\mathcal{D}}$ (right)

Figure 2: Curves $\mathcal{C}$ and $\widetilde{\mathcal{D}}$

where $G_k(t, x_k) = x_k p_{k2}(t) - p_{k1}(t),\ k = 1, 2$. *Now, we compute the root in the variable* x_k *of the polynomial* $\frac{\partial L_k}{\partial x_k}(s, x_k)$ *(see Corollary 6). We get the curve* $\widetilde{\mathcal{D}}$ *defined by the rational parametrization*

$$\widetilde{\mathcal{Q}}(t) = \left(\frac{-.06216533631 + .00006226547971t - .0002480131435t^2 + t^4 - .0002496021856t^3}{-.0001269322992t + t^4 + .06210313427 + .4984100713t^2 - .0004992043712t^3}, \right.$$

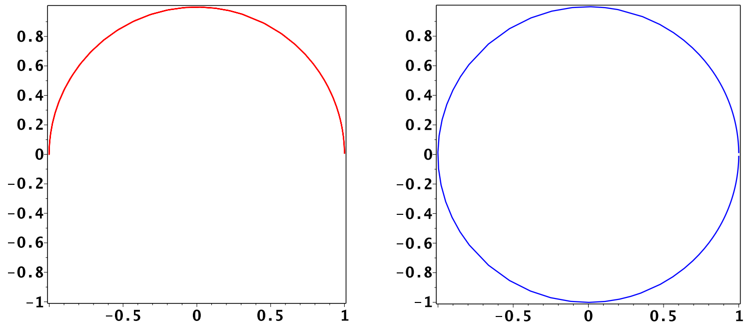
$$\left. \frac{-.00006178762808 - .2488083914t + .000001266050484t^2 - .9984087423t^3}{-.0001269322992t + t^4 + .06210313427 + .4984100713t^2 - .0004992043712t^3} \right).$$

In Figure 2, we plot the curves $\mathcal{C}$ *and* $\widetilde{\mathcal{D}}$*. We observe that the input curve,* $\mathcal{C}$*, and its approximate one,* $\widetilde{\mathcal{D}}$*, are almost overlapped in the view of big scalar, so we enlarge a small part to show the difference (note that for a better view, the coordinate scale may not be* $1:1$*).*

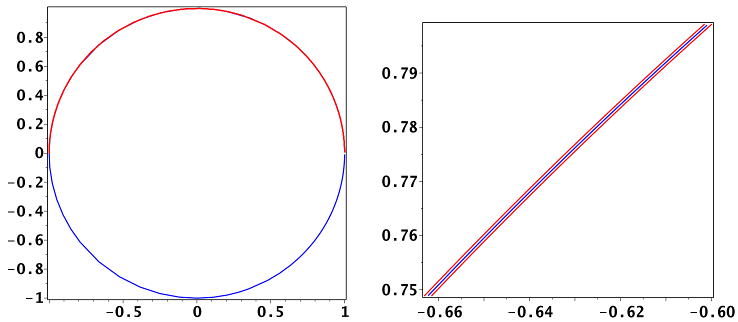
Finally, we simplify $\widetilde{\mathcal{Q}}$ *by removing the approximate gcd from the numerator and denominator of each component of* $\widetilde{\mathcal{Q}}$ *(see statement 1 in Remark 3 and Remark 9). We get the curve* $\mathcal{D}$ *defined by the* ϵ*–numerical reparametrization*

$$\mathcal{Q}(t) = \left(\frac{t^2 + .000005006649227t - .2494538109}{t^2 - .0002445955365t + .2492042101}, \frac{-.9984087427t - .0002529376363}{t^2 - .0002445955365t + .2492042101} \right).$$

In Figure 3, we plot the curves $\mathcal{C}$ and $\mathcal{D}$. Similarly as above, we observe that the input curve, $\mathcal{C}$, and its approximate one, $\mathcal{D}$, are almost overlapped in the view of big scalar, so we enlarge a small part to show the difference.



Input curve $\mathcal{C}$ (left), curve $\mathcal{D}$ (right)



Curves $\mathcal{C}$ and $\mathcal{D}$ (left), partial enlarged view of $\mathcal{C}$ and $\mathcal{D}$ (right)

Figure 3: Curves $\mathcal{C}$ and $\mathcal{D}$

One may check that the equality

$$L_k(s,x_k) = (x_k q_{k2}(s) - q_{k1}(s))^\ell + \epsilon^\ell W_k(s,x_k), \;\; \|\mathrm{num}(W_k(R,p_k))\| \leq \|H_k^{\mathcal{P}\mathcal{Q}}\|^\ell, \;\; k=1,2$$

holds. Then, $\mathcal{Q}$ is an ϵ–proper reparametrization of $\mathcal{P}$ (see Theorem 2 and Corollary 3). Finally, we observe that $\deg(\mathcal{P}) = \deg(\mathcal{Q})\deg(R)$ (see Corollary 4), and then the degree of the ϵ–proper parametrization $\mathcal{Q}$ is lower than the non ϵ–proper input parametrization $\mathcal{P}$.

4. Error Analysis

In this section, we show how the input curve and the output curve are related. For this purpose, we consider two different subsections. The first one (Subsection 4.1) shows the relationship between the input curve $\mathcal{C}$ and the curve $\widetilde{\mathcal{D}}$. More precisely, it is proved that $\deg(f) = \deg(h)$, where $f \in \mathbb{C}[x_1, x_2]$ is an irreducible polynomial defining implicitly the curve $\mathcal{C}$, and $h \in \mathbb{C}[x_1, x_2]$ is an irreducible polynomial defining implicitly the curve $\widetilde{\mathcal{D}}$ (see

Theorem 3). Moreover, we prove that the homogeneous form of maximum degree of both curves, $\mathcal{C}$ and $\widetilde{\mathcal{D}}$, is the same. Thus, in particular these two curves have the same points at infinity (see Theorem 4).

The parametrization $\widetilde{\mathcal{Q}}$ should be further simplified to obtain the searched parametrization $\mathcal{Q}$. However, when we simplify $\widetilde{\mathcal{Q}}$, the curve $\widetilde{\mathcal{D}}$ defined by $\widetilde{\mathcal{Q}}$ changes (the infinity points are not the same because the numerical simplification). Thus, in Subsection 4.2, we provide some error bounds that measure the closeness between the curves $\mathcal{C}$ and $\mathcal{D}$ (see Theorem 5).

4.1. The Curves $\mathcal{C}$ and $\widetilde{\mathcal{D}}$

In this subsection, we consider the input curve $\mathcal{C}$ defined by the rational parametrization $\mathcal{P} = (\frac{p_{11}}{p_{12}}, \frac{p_{21}}{p_{22}})$, $\gcd(p_{k1}, p_{k2}) = 1$, $k = 1, 2$, with $\mathrm{index}(\mathcal{P}) = 1$, and the output curve $\widetilde{\mathcal{D}}$ defined by the parametrization $\widetilde{\mathcal{Q}} = \left(\frac{\widetilde{q}_{11}}{\widetilde{q}_{12}}, \frac{\widetilde{q}_{21}}{\widetilde{q}_{22}}\right)$, $\gcd(\widetilde{q}_{k1}, \widetilde{q}_{k2}) = 1$, $k = 1, 2$. We may assume without loss of generality that $\mathrm{index}(\widetilde{\mathcal{Q}}) = 1$ (note that we are working with approximate mathematical objects and then, with probability almost one, $\deg_t(S) = 1$, where S is the polynomial introduced in Section 2).

Under these conditions, we first prove that these two curves have the same degree.

Theorem 3. *The curves $\mathcal{C}$ and $\widetilde{\mathcal{D}}$ have the same degree.*

Proof. First, we may write without loss of generality the parametrization $\mathcal{P}$ such that the denominators of both components are the same. That is, $\mathcal{P} = (\frac{p_{11}}{p_{12}}, \frac{p_{21}}{p_{12}})$. In addition, since $\mathcal{P}$ is expected to be given with perturbed float coefficients, we may assume that $\gcd(p_{k1}, p_{12}) = 1$, $k = 1, 2$. Furthermore, from Corollary 5,

$$\mathrm{Res}_t(p_{k2}(t), sC_j(t) - C_i(t)) = q_{k2}(s)^{\ell} + \epsilon^{\ell} b_k(s) = \widetilde{q}_{k2}(s), \quad k = 1, 2$$

which implies that if $p_{12} = p_{22}$, then $\widetilde{q}_{12} = \widetilde{q}_{22}$ (that is, the denominators of both components of the parametrization $\widetilde{\mathcal{Q}}$ are the same). In addition, we may assume that $\deg(p_{k1}) = \deg(p_{12})$ and $\deg(\widetilde{q}_{k1}) = \deg(\widetilde{q}_{12})$, for $k = 1, 2$ (otherwise, one may apply on both parametrizations a birational parameter transformation). Thus,

$$\deg(p_1) = \deg(p_2) = \deg(p_{11}) = \deg(p_{21}) = \deg(p_{12}), \quad \text{and}$$

$$\deg(\widetilde{q}_1) = \deg(\widetilde{q}_2) = \deg(\widetilde{q}_{11}) = \deg(\widetilde{q}_{21}) = \deg(\widetilde{q}_{12}).$$

Under these conditions, from Theorem 6.3.1 in [32], we have that all the infinity points of $\widetilde{\mathcal{D}}$ are reachable by the corresponding projective parametrization

$$\widetilde{\mathcal{Q}}^*(s, w) = (\widetilde{q}^*_{11}(s, w), \widetilde{q}^*_{21}(s, w), \widetilde{q}^*_{12}(s, w)).$$

Furthermore, since $\deg(\widetilde{q}_{11}) = \deg(\widetilde{q}_{21}) = \deg(\widetilde{q}_{12})$, it holds that $\widetilde{q}^*_{12}(s_0, w_0) = 0$ if and only if $\widetilde{q}_{12}(s_0) = 0$.

Now, taking into account that $\text{index}(\widetilde{\mathcal{Q}}) = 1$, we apply Theorem 4.3.8 in [32], and we have that $\deg_{x_1}(h) = \deg_{x_2}(h) = \deg(\widetilde{q}_1) = \deg(\widetilde{q}_2)$, where $h \in \mathbb{C}[x_1, x_2]$, $\deg(h) = r$, is an irreducible polynomial defining implicitly the curve $\widetilde{\mathcal{D}}$. In addition, we observe that x_k, $k = 1, 2$ does not divide (exactly) h_r, where h_r is the homogeneous form of maximum degree of the polynomial h. That is, $h_r(1, 0)h_r(0, 1) \neq 0$. Indeed, if $h_r(1, 0) = 0$ (similarly if $h_r(0, 1) = 0$), one gets that there exists $s_i \in \mathbb{C}$ such that $\widetilde{q}_{12}(s_i) = \widetilde{q}_{21}(s_i) = 0$. This is impossible, because we have assumed that $\gcd(\widetilde{q}_{12}, \widetilde{q}_{21}) = 1$.

Therefore, we conclude that

$$\deg(h) = r = \deg_{x_1}(h) = \deg_{x_2}(h) = \deg(\widetilde{q}_1) = \deg(\widetilde{q}_2) = \deg(\widetilde{q}_{11}) = \deg(\widetilde{q}_{21}) = \deg(\widetilde{q}_{12}).$$

In addition, since

$$\text{Res}_t(p_{12}(t), sC_j(t) - C_i(t)) = \widetilde{q}_{12}(s),$$

(see Corollary 5), we get that $\deg(\widetilde{q}_{12}) = \deg_s(sC_j(t) - C_i(t))\deg(p_{12}) = \deg(p_{12})$ (see Sections 5.8 and 5.9 in [36]), and hence $r = \deg(h) = \deg(p_{12})$.

Now, we reason with the input curve $\mathcal{C}$. For this purpose, we consider $f \in \mathbb{C}[x_1, x_2]$, $\deg(f) = d$, an irreducible polynomial defining implicitly the curve $\mathcal{C}$. Since $\text{index}(\mathcal{P}) = 1$, we reason as above and we get that $\deg_{x_1}(f) = \deg_{x_2}(f) = \deg(p_1) = \deg(p_2)$. We also get that $f_d(1, 0)f_d(0, 1) \neq 0$, where f_d is the homogeneous form of maximum degree of the polynomial f (taking into account that $\gcd(p_{12}, p_{k1}) = 1$, $k = 1, 2$ may reason similarly as above), and thus

$$\deg(f) = d = \deg_{x_1}(f) = \deg_{x_2}(f) = \deg(p_1) = \deg(p_2) = \deg(p_{11}) = \deg(p_{21}) = \deg(p_{12}).$$

Since $r = \deg(h) = \deg(p_{12})$, we conclude that $d = \deg(f) = \deg(h) = \deg(p_k) = \deg(\widetilde{q}_k)$. □

From Theorem 3, we can deduce that the curves $\mathcal{C}$ and $\widetilde{\mathcal{D}}$ have the same behavior at infinity. More precisely, in Theorem 4, we show that the homogeneous form of maximum degree of h is equal to the homogeneous form of maximum degree of f.

Theorem 4. *The implicit equations defining the curves $\mathcal{C}$ and $\widetilde{\mathcal{D}}$ have the same homogeneous form of maximum degree. Hence both curves have the same points at infinity.*

Proof. First, we assume that we are under the conditions stated in the proof of Theorem 3. Then, in particular $\mathcal{C}$ is defined parametrically by $\mathcal{P} = (p_{11}/p_{12}, p_{21}/p_{12})$ and implicitly by $f(x_1, x_2) \in \mathbb{C}[x_1, x_2]$, and $\widetilde{\mathcal{D}}$ is given parametrically by $\widetilde{\mathcal{Q}} = (\widetilde{q}_{11}/\widetilde{q}_{12}, \widetilde{q}_{21}/\widetilde{q}_{12})$ and implicitly by $h(x_1, x_2) \in \mathbb{C}[x_1, x_2]$. In addition, $\mathcal{P}$, $\widetilde{\mathcal{Q}}$ and f, h satisfy the properties stated in the proof of Theorem 3.

Under these conditions, and taking into account that $\text{index}(\mathcal{P}) = 1$, we apply Theorem 4.5.3 in [32], and one has that

$$f(x_1, x_2)^{\text{index}(\mathcal{P})} = \text{Res}_t(G_1(t, x_1), G_2(t, x_2)), \text{ where } \quad G_k(t, x_k) = x_k p_{12}(t) - p_{k1}(t), \ k = 1, 2.$$

Now, we consider the polynomials

$$L_k(s, x_k) = \mathrm{Res}_t(G_k(t, x_k), sC_j(t) - C_i(t)),\ k = 1, 2,$$

introduced in Theorem 2. By Corollary 6, we have that

$$L_k(s, x_k) = x_k^\ell \widetilde{q}_{12} - \ell x_k^{\ell-1} \widetilde{q}_{k1} + A_k(s, x_k), \quad \deg_{x_k}(A_k) \leq \ell - 2, \quad k = 1, 2.$$

Let us prove that there exists a non-empty open subset $\Omega \subset \mathbb{C}^2$, such that for every $q \in \Omega$ with $f(q) = 0$, it holds that $R(q) = 0$, where $R(x_1, x_2) := \mathrm{Res}_s(L_1, L_2)$. Thus, one would deduce that f divides R. Indeed, first we observe that $R \neq 0$, because there does not exist any factor depending on s that divides L_k (note that $\gcd(q_{12}, q_{k1}) = 1$). Now, let

$$\Omega := \{q \in \mathbb{C}^2 \mid \mathrm{lc}(G_1, t)(q)\mathrm{lc}(G_2, t)(q)D_2(q)C_j(\mathcal{P}^{-1}(q)) \neq 0\},$$

where $\mathcal{P}^{-1}(x_1, x_2) = D_1(x_1, x_2)/D_2(x_1, x_2)$ (note that $\mathrm{index}(\mathcal{P}) = 1$ and then, there exists the inverse of $\mathcal{P}$ in $\mathbb{C}(x_1, x_2) \setminus \mathbb{C}$). Observe that Ω is a non-empty open subset of $\mathbb{C}^2$ since

$$\mathrm{lc}(G_1, t)(x_1)\mathrm{lc}(G_2, t)(x_2)D_2(x_1, x_2)C_j(\mathcal{P}^{-1}(x_1, x_2)) \neq 0.$$

Now, let $q = (x_1^0, x_2^0) \in \Omega$ be such that $f(q) = 0$ (note that $\mathcal{C}$ and $\mathbb{C}^2 \setminus \Omega$ intersect at finitely many points). Since $\mathrm{lc}(G_j, t)(q) \neq 0$, $j = 1, 2$, by the resultant property (see Section 2), there exists $t_0 \in \mathbb{C}$ such that $G_k(t_0, x_k^0) = 0$, $k = 1, 2$. In addition, since $q \in \Omega$, one has that there exists $s_0 \in \mathbb{C}$ such that $s_0 C_j(t_0) - C_i(t_0) = 0$ (note that $t_0 = \mathcal{P}^{-1}(q)$, and $C_j(t_0) \neq 0$). Then, since $L_k(s, x_k) = \mathrm{Res}_t(G_k(t, x_k), sC_j(t) - C_i(t))$, we get that $L_k(s_0, x_k^0) = 0$, $k = 1, 2$. Hence, by the specialization of the resultant property (see Section 2), we deduce that

$$R(q) = \mathrm{Res}_s(L_1(s, x_1), L_2(s, x_2))(q) = \mathrm{Res}_s(L_1(s, x_1^0), L_2(s, x_2^0)) = 0.$$

Thus,

$$R(x_1, x_2) = f(x_1, x_2)m(x_1, x_2), \quad m \in \mathbb{C}[x_1, x_2].$$

Since we are working with approximate mathematical objects, we may assume without loss of generality that $\deg_{\{x_1,x_2\}}(R) = \deg_s(L_1)\deg_s(L_2)$ (see Sections 5.8 and 5.9 in [36]). Then, if we homogenize the above equation with respect to the variables x_1 and x_2, we get that

$$R^*(x_1, x_2, x_3) := \mathrm{Res}_s(L_1^*(s, x_1, x_3), L_2^*(s, x_2, x_3)) = F(x_1, x_2, x_3)M(x_1, x_2, x_3),$$

where $F, M \in \mathbb{C}[x_1, x_2, x_3]$ are the homogenization of f, m, respectively, with respect to the variables x_1 and x_2, and

$$L_k^*(s, x_k, x_3) = x_k^\ell \widetilde{q}_{12} - \ell x_k^{\ell-1} x_3 \widetilde{q}_{k1} + x_3^2 A_k(s, x_k, x_3), \quad \deg_{\{x_k,x_3\}}(A_k) = \ell - 2, \quad k = 1, 2,$$

is the homogenization of L_k with respect to x_1 and x_2. Observe that x_3 does not divide M, because $\deg_{\{x_1,x_2\}}(R) = \deg_s(L_1)\deg_s(L_2)$.

Now, we consider the system defined by the polynomials

$$L_1^* = (x_1^\ell \widetilde{q}_{12} + x_3^2 A_1(s, x_k, x_3)) + x_3(-\ell x_1^{\ell-1} \widetilde{q}_{11}),\ L_2^* = (x_2^\ell \widetilde{q}_{12} + x_3^2 A_2(s, x_k, x_3)) + x_3(-\ell x_2^{\ell-1} \widetilde{q}_{21}).$$

Observe that the two equations are independent. Thus, solving from $L_1^* = 0$, we have that $x_3 = (x_1^\ell q_{12} + x_3^2 A_1(s, x_1, x_3))/(\ell x_1^{\ell-1}\widetilde{q}_{11})$. Substituting it in L_2^*, we obtain the following equivalent system defined by the polynomials L_1^*, and L^*, where

$$L^*(s, x_1, x_2, x_3) := \widetilde{q}_{12}(s)x_1^{\ell-1}x_2^{\ell-1}(-\widetilde{q}_{21}(s)x_1 + \widetilde{q}_{11}(s)x_2) + x_3^2 B(s, x_1, x_2, x_3),\ B \in \mathbb{C}[s, x_1, x_2, x_3].$$

Thus,

$$R^*(x_1, x_2, x_3) = \mathrm{Res}_s(L_1^*(s, x_1, x_3), L^*(s, x_2, x_3)) = F(x_1, x_2, x_3)M(x_1, x_2, x_3).$$

Using the property of specialization of the resultants, we consider $x_3 = 0$ in the above equality, and we get that (we remind that x_3 does not divide M)

$$\mathrm{Res}_s(\widetilde{q}_{12}(s), x_1^{\ell-1}x_2^{\ell-1}(-\widetilde{q}_{21}(s)x_1 + \widetilde{q}_{11}(s)x_2)) =$$

$$x_1^{\deg(\widetilde{q}_{12})(\ell-1)}x_2^{\deg(\widetilde{q}_{12})(\ell-1)}\mathrm{Res}_s(\widetilde{q}_{12}(s), (-\widetilde{q}_{21}(s)x_1 + \widetilde{q}_{11}(s)x_2)) = f_d(x_1, x_2)m_\ell(x_1, x_2),$$

where f_d, m_ℓ are the homogeneous form of maximum degree of F, M, respectively.

On the other side, by applying Theorem 4.5.3 in [32], one also has that

$$h(x_1, x_2)^{\mathrm{index}(\widetilde{\mathcal{Q}})} = \mathrm{Res}_t(\widetilde{G}_1(t, x_1), \widetilde{G}_2(t, x_2)), \quad \text{where} \quad \widetilde{G}_k(t, x_k) = x_k\widetilde{q}_{12}(t) - \widetilde{q}_{k1}(t),\ k = 1, 2.$$

We recall that $\mathrm{index}(\widetilde{\mathcal{Q}}) = 1$. Since we are working with approximate mathematical objects, similarly as above we may assume that $\deg_{\{x_1,x_2\}}(h) = \deg_t(\widetilde{G}_1)\deg_t(\widetilde{G}_2)$. Then, if we homogenize the above equation with respect to the variables x_1 and x_2, we get that

$$H(x_1, x_2, x_3) = \mathrm{Res}_t(\widetilde{G}_1^*(t, x_1, x_3), \widetilde{G}_2^*(t, x_2, x_3)), \quad \text{where } \widetilde{G}_k^*(t, x_k, x_3) = x_k\widetilde{q}_{12}(t) - \widetilde{q}_{k1}(t)x_3,$$

and H is the homogenization of h with respect to the variables x_1 and x_2. Observe that x_3 does not divide H because $\deg_{\{x_1,x_2\}}(h) = \deg_t(\widetilde{G}_1)\deg_t(\widetilde{G}_1)$.

Now, reasoning as above, we have that the system defined by the polynomials $\widetilde{G}_1^*$ and $\widetilde{G}_2^*$ is equivalent to the system defined by $\widetilde{G}_1^*$ and the polynomial $\widetilde{G}^* = -\widetilde{q}_{21}(s)x_1 + \widetilde{q}_{11}(s)x_2$. Thus,

$$H(x_1, x_2, x_3) = \mathrm{Res}_t(\widetilde{G}_1^*(t, x_1, x_3), \widetilde{G}^*(t, x_1, x_2)).$$

Using the property of specialization of the resultants, we consider $x_3 = 0$ in the above equality, and we get that (observe that x_3 does not divide H)

$$\mathrm{Res}_s(\widetilde{q}_{12}(s), -\widetilde{q}_{21}(s)x_1 + \widetilde{q}_{11}(s)x_2) = h_d(x_1, x_2),$$

where h_d is the homogeneous form of maximum degree of H (we recall that $d = \deg(f) = \deg(h)$, see Theorem 3). Thus, since

$$f_d(x_1, x_2)m(x_1, x_2) = x_1^{\deg(\widetilde{q}_{12})(\ell-1)}x_2^{\deg(\widetilde{q}_{12})(\ell-1)}\mathrm{Res}_s(\widetilde{q}_{12}(s), (-\widetilde{q}_{21}(s)x_1 + \widetilde{q}_{11}(s)x_2)) =$$

$$x_1^{\deg(\widetilde{q}_{12})(\ell-1)}x_2^{\deg(\widetilde{q}_{12})(\ell-1)}h_d(x_1, x_2),$$

and $x_k,\ k = 1, 2$, does not divide f_d (see proof of Theorem 3), we conclude that $h_d = f_d$. Hence, $\mathcal{C}$ and $\widetilde{\mathcal{D}}$ have the same homogeneous form of maximum degree, and then both curves have the same degree and the same points at infinity. □

4.2. The Curves $\mathcal{C}$ and $\mathcal{D}$

As we know, the parametrization $\widetilde{\mathcal{Q}}$ should be further simplified to get the parametrization $\mathcal{Q}$ (see Remark 9). By Theorem 3, $\deg(\widetilde{q}_k) = \deg(p_k)$, and from Corollary 4, we need to look for $\mathcal{Q}$ such that $\ell \deg(q_k) = \deg(p_k)$. However, the curve $\widetilde{\mathcal{D}}$ defined by $\widetilde{\mathcal{Q}}$ changes with the simplification of $\widetilde{\mathcal{Q}}$ (the infinity points vary during the numerical simplification). This is due to the fact that the input parametrization $\mathcal{P}$ and the output parametrization $\mathcal{Q}$ have different degrees (see Corollary 4).

In order to analyze the behavior at affine points, we study the closeness of the curves $\mathcal{C}$ and $\mathcal{D}$, where $\mathcal{D}$ is the curve defined by the simplified parametrization $\mathcal{Q} = \left(\frac{q_{11}}{q_{12}}, \frac{q_{21}}{q_{22}}\right)$ (note that by Corollary 3, ϵ–index$(Q) = 1$), and $\mathcal{C}$ is the curve defined by $\mathcal{P} = (\frac{p_{11}}{p_{12}}, \frac{p_{21}}{p_{22}})$ (we remind that $\mathcal{P} \sim_\epsilon \mathcal{Q} \circ R$, where $R(t) \in \mathbb{C}(t) \setminus \mathbb{C}$, and $\deg(R) = \ell$). For this purpose, we first assume that $\deg(p_{i1}) = \deg(p_{i2})$, and $\deg(q_{i1}) = \deg(q_{i2})$, $i = 1, 2$ (otherwise, one can apply on both parametrizations a birational parameter transformation). In addition, let $\|p\| := \max\{\|p_{11}\|, \|p_{21}\|, \|p_{12}\|, \|p_{22}\|\}$, and $\|q\| := \max\{\|q_{11}\|, \|q_{21}\|, \|q_{12}\|, \|q_{22}\|\}$.

Finally, we also assume that Theorem 2 holds and then $\mathcal{Q}$ is an ϵ–proper reparametrization of $\mathcal{P}$ (see Corollary 3). If Theorem 2 does not hold, one applies Remark 7, and then $\mathcal{Q}$ an $\overline{\epsilon}$–proper reparametrization of $\mathcal{P}$. In this case, the formula obtained in Theorem 5 remains unchanged except that ϵ becomes $\overline{\epsilon}$.

Under these conditions, in order to analyze the behavior at affine points, we shall restrict to an interval where the parametrizations $\mathcal{P}$ and $\mathcal{Q}$ are both well defined. Thus, the general strategy is to show that almost all real affine points on $\mathcal{D}$ are at a small distance of an affine real point on $\mathcal{C}$, and reciprocally.

For this purpose, we consider the interval $I := (d_1, d_2) \subset \mathbb{R}$ satisfying that for all $t_0 \in I$, there exists $M \in \mathbb{N}$ such that $|q_{i2}(R(t_0))| \geq M$, and $|p_{i2}(t_0)| \geq M, i = 1, 2$. Note that we can decompose $\mathbb{R}$ to a union of finitely many intervals, $I_j, j = 1, \ldots, n$, satisfying the above condition (that is, the interval without any root of the denominators of the parametrizations; see [27]). Then we shall reason similarly as in Theorem 5 for each interval $I_j, j = 1, \ldots, n$.

Theorem 5. *The following statements hold:*

1. *Let $I := (d_1, d_2) \subset \mathbb{R}$, and $M \in \mathbb{N}$ be such that for every $t_0 \in I$, it holds that $|q_{i2}(R(t_0))| \geq M$, and $|p_{i2}(t_0)| \geq M$ for $i = 1, 2$. Let $d := \max\{|d_1|, |d_2|\}$. Then, for every $t_0 \in I$,*

$$|p_i(t_0) - q_i(R(t_0))| \leq 2/M^2 \epsilon\, \zeta \|p\| \|q\|, \quad i = 1, 2,$$

where

$$\zeta = \begin{cases} \dfrac{d^{\deg(\mathcal{P})+1}}{(d-1)^{1/\ell}} & \text{if } d > 1, \\ \dfrac{1}{(1-d)^{1/\ell}} & \text{if } d < 1, \\ \ell^{1/\ell} \deg(\mathcal{P})^{1/\ell} & \text{if } d = 1. \end{cases}$$

2. $\mathcal{C}_{t\in I}$ *is contained in the offset region of* $\mathcal{D}_{s\in J}$ *at distance* $4\sqrt{2}/M^2\epsilon\,\zeta\,\|p\|\|q\|$, *where* $J=R(I)$.
3. $\mathcal{D}_{s\in J}$ *is contained in the offset region of* $\mathcal{C}_{t\in I}$ *at distance* $4\sqrt{2}/M^2\epsilon\,\zeta\,\|p\|\|q\|$, *where* $J=R(I)$.

Proof. Firstly, statement (1) implies statements (2) and (3). To see this, we note that for almost all real affine points $Q\in\mathcal{D}$ there exists an affine real point $P\in\mathcal{C}$ such that

$$\|P-Q\|_2\leq 2\sqrt{2}/M^2\epsilon\,\zeta\,\|p\|\|q\|.$$

Indeed, using statement (1), we have

$$\|P-Q\|_2=\sqrt{(p_1(t_0)-q_1(R(t_0)))^2+(p_2(t_0)-q_2(R(t_0))^2}\leq$$
$$\sqrt{(2/M^2\epsilon\,\zeta\|p\|\|q\|)^2+(2/M^2\epsilon\,\zeta\|p\|\|q\|)^2}\leq 2\sqrt{2}/M^2\epsilon\,\zeta\,\|p\|\|q\|.$$

Now, reasoning as in Section 2.2 in [13], we deduce statements (2) and (3).

We next prove statement (1). By the proof of Theorem 2, we have

$$H_i^{\mathcal{P}\mathcal{Q}}(t,R(t))^\ell=(p_{i1}(t)q_{12}(R(t))-q_{i1}(R(t))p_{12}(t))^\ell=\epsilon^\ell e_i(t),\quad\text{where}$$
$$e_i(t)=-\mathrm{num}(W_i(R(t),p_i(t)))=e_{i,0}+e_{i,1}t+\ldots+e_{i,n_i}t^{n_i}\in\mathbb{C}[t],\quad\text{and}$$
$$\|e_i\|=\|\mathrm{num}(W_i(R,p_i))\|\leq\|H_i^{\mathcal{P}\mathcal{Q}}\|^\ell.$$

In addition, since $e_i(t)=-\mathrm{num}(W_i(R(t),p_i(t)))$, we have $n_i:=\deg(e_i)\leq\ell\deg(\mathcal{P})$ for $i=1,2$. Indeed, since $\deg_{x_i}(W_i)=\ell$ (see Remark 6), we deduce that

$$\deg(e_i)\leq\max\{\deg(R)\deg_t(W_i),\ell\deg(\mathcal{P})\}\leq\max\{\ell\deg(\mathcal{P}),\ell\deg(\mathcal{P})\}=\ell\deg(\mathcal{P}).$$

Under these conditions, for every $t_0\in I$, if $d\neq 1$, it holds that

$$|H_i^{\mathcal{P}\mathcal{Q}}(t_0,R(t_0))^\ell|=\epsilon^\ell|e_i(t_0)|\leq\epsilon^\ell\|H_i^{\mathcal{P}\mathcal{Q}}\|^\ell(|e_{i,0}|+|e_{i,1}||t_0|+\ldots+|e_{i,n_i}||t_0|^{n_i})\leq$$

$$\epsilon^\ell\|H_i^{\mathcal{P}\mathcal{Q}}\|^\ell(1+d+\ldots+d^{n_i})=\epsilon^\ell\|H_i^{\mathcal{P}\mathcal{Q}}\|^\ell\,\frac{d^{n_i+1}-1}{d-1},\quad i=1,2. \tag{6}$$

If $d=1$, then

$$|H_i^{\mathcal{P}\mathcal{Q}}(t_0,R(t_0))^\ell|\leq\epsilon^\ell\|H_i^{\mathcal{P}\mathcal{Q}}\|^\ell(1+|t_0|+\ldots+|t_0|^{n_i})\leq\epsilon^\ell\|H_i^{\mathcal{P}\mathcal{Q}}\|^\ell(1+1+\ldots+1)=\epsilon^\ell\|H_i^{\mathcal{P}\mathcal{Q}}\|^\ell\,n_i. \tag{7}$$

Therefore, we conclude that:

1. If $d>1$, by (6), and taking into account that $|q_{i2}(R(t_0))|\geq M$, and $|p_{i2}(t_0)|\geq M$ for $i=1,2$, we obtain that

$$|p_i(t_0)-q_i(R(t_0))|=\frac{|H_i^{\mathcal{P}\mathcal{Q}}(t_0,R(t_0))|}{|q_{i2}(R(t_0))p_{i2}(t_0)|}\leq 1/M^2\epsilon\|H_i^{\mathcal{P}\mathcal{Q}}\|\frac{d^{\deg(\mathcal{P})+1/\ell}}{(d-1)^{1/\ell}}\leq$$
$$1/M^2\,\epsilon\,\|H_i^{\mathcal{P}\mathcal{Q}}\|\,\frac{d^{\deg(\mathcal{P})+1}}{(d-1)^{1/\ell}}.$$

2. If $d<1$, from (6), and taking into account that $1-d^{n_i+1}<1$, and $|q_{i2}(R(t_0))|\geq M$, and $|p_{i2}(t_0)|\geq M$ for $i=1,2$, we obtain that

$$|p_i(t_0)-q_i(R(t_0))| = \frac{|H_i^{\mathcal{PQ}}(t_0,R(t_0))|}{|q_{i2}(R(t_0))p_{i2}(t_0)|} \leq 1/M^2\,\epsilon\,\|H_i^{\mathcal{PQ}}\|\frac{1}{(1-d)^{1/\ell}}.$$

3. If $d=1$, from (7), and taking into account that $|q_{i2}(R(t_0))|\geq M$, and $|p_{i2}(t_0)|\geq M$ for $i=1,2$, we obtain that

$$|p_i(t_0)-q_i(R(t_0))| = \frac{|H_i^{\mathcal{PQ}}(t_0,R(t_0))|}{|q_{i2}(R(t_0))p_{i2}(t_0)|} \leq 1/M^2\,\epsilon\,\|H_i^{\mathcal{PQ}}\|(\ell\deg(\mathcal{P}))^{1/\ell}.$$

Finally, we have

$$\|H_i^{\mathcal{PQ}}\| = \|p_{i1}(t)q_{i2}(s)-q_{i1}(s)p_{i2}(t)\| \leq 2\|p\|\|q\|.$$

□

From Theorem 5, we deduce the following corollary:

Corollary 7. *Under the conditions of Theorem 5, it holds that:*

1. *If $d\geq 2$, then $\zeta\leq d^{\deg(\mathcal{P})+1}$.*
2. *If $1<d<2$, then $\zeta\leq 2^{\deg(\mathcal{P})+1}$.*

Proof. If $d\geq 2$, by Theorem 5 we have $\zeta\leq \frac{d^{\deg(\mathcal{P})+1}}{(d-1)^{1/\ell}}\leq d^{\deg(\mathcal{P})+1}$.
If $1<d<2$,

$$|H_i^{\mathcal{PQ}}(t_0,R(t_0))^\ell| \leq \epsilon^\ell\|H_i^{\mathcal{PQ}}\|^\ell(1+|t_0|+\ldots+|t_0|^{n_i}) \leq$$
$$\epsilon^\ell\|H_i^{\mathcal{PQ}}\|^\ell(1+2+\ldots+2^{n_i}) \leq \epsilon^\ell\,\|H_i^{\mathcal{PQ}}\|^\ell 2^{n_i+1}, \quad i=1,2.$$

Thus,

$$|p_i(t_0)-q_i(R(t_0))| = \frac{|H_i^{\mathcal{PQ}}(t_0,R(t_0))|}{|q_{12}(R(t_0))p_{12}(t_0)|} \leq 1/M^2\,\epsilon\|H_i^{\mathcal{PQ}}\|\,2^{\deg(\mathcal{P})+1} \leq 2/M^2\,\epsilon\|p\|\|q\|\,2^{\deg(\mathcal{P})+1}.$$ □

The following example shows the error analysis in the computation corresponding to Example 5. In addition, it illustrates the application of the Theorem 5 and shows the resemblance between the curves $\mathcal{C}$ and $\mathcal{D}$.

Example 6. *We consider the rational curves $\mathcal{C}$ and $\mathcal{D}$ in Example 5 defined by the rational parametrizations*

$$\mathcal{P}(t) = \left(\frac{p_{11}(t)}{p_{12}(t)}, \frac{p_{21}(t)}{p_{22}(t)}\right) =$$
$$\left(\frac{t^4-.2502500000+.0005000000000\,t}{t^4+.2500000000+.0002500000000\,t^2}, \frac{t^2-.0002500000000}{t^4+.2500000000+.0002500000000\,t^2}\right)$$

and

$$\mathcal{Q}(t) = \left(\frac{q_{11}(t)}{q_{12}(t)}, \frac{q_{21}(t)}{q_{22}(t)}\right) =$$

$$\left(\frac{t^2 + .000005006649227t - .2494538109}{t^2 - .0002445955365t + .2492042101}, \frac{-.9984087427t - .0002529376363}{t^2 - .0002445955365t + .2492042101}\right),$$

respectively. We apply Theorem 5, and we consider $I = (-1, 1)$*. Thus,* $d = 1$*. Let* $M \in \mathbb{N}$ *be such that for every* $t_0 \in I$*, it holds that* $|q_{i2}(R(t_0))| \geq M$*, and* $|p_{i2}(t_0)| \geq M$*, for* $i = 1, 2$*. We have that* $M = .2492042100$*. Then, by Theorem 5, we get that*

$$\zeta = \ell^{1/\ell} \mathrm{deg}(\mathcal{P})^{1/\ell} = 2.828427125,$$

and for every $t_0 \in I$*, it holds that*

$$|p_i(t_0) - q_i(R(t_0))| < 2/M^2 \epsilon\, \zeta \|p\| \|q\| = 0.9108864449, \quad i = 1, 2,$$

where $\|p\| = \|q\| = 1$*. In Figure 4, we plot the curves* $\mathcal{C}$ *and* $\mathcal{D}$ *defined by* $\mathcal{P}(t)$ *and* $\mathcal{Q}(t)$*, respectively, for* $t \in I$*.*

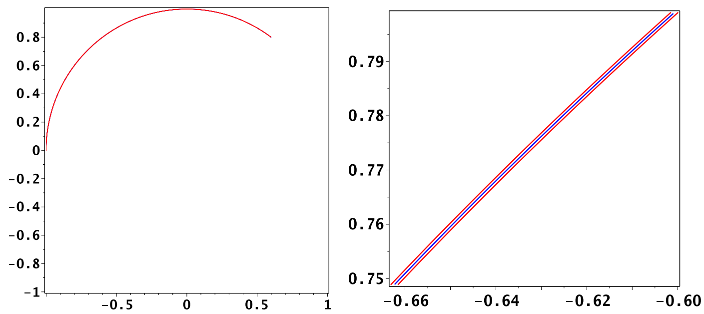


Figure 4: Curves $\mathcal{C}$ and $\mathcal{D}$ for $t \in I$ (left), partial enlarged view of $\mathcal{C}$ and $\mathcal{D}$ (right)

5. Numeric Algorithm of Reparametrization for Curves

In this section, we apply the results obtained in Section 3 to derive an algorithm that computes an ϵ–proper reparametrization of a given approximately improper parametrization of a plane curve. We outline this approach, and we illustrate it with some examples where we also show the error bound obtained by applying results in Subsection 4.2.

Numeric Algorithm Reparametrization for Curves.

INPUT: a tolerance $\epsilon > 0$, and a rational parametrization $\mathcal{P}(t) = \left(\frac{p_{11}(t)}{p_{12}(t)}, \frac{p_{21}(t)}{p_{22}(t)}\right) \in \mathbb{C}(t)^2$, ϵ–gcd$(p_{i1}, p_{i2}) = 1, i = 1, 2$, of an algebraic plane curve $\mathcal{C}$.

OUTPUT: a rational parametrization $\mathcal{Q}(t) = \left(\frac{q_{11}(t)}{q_{12}(t)}, \frac{q_{21}(t)}{q_{22}(t)}\right) \in \mathbb{C}(t)^2$, ϵ–gcd$(q_{i1}, q_{i2}) = 1$, $i = 1, 2$, such that ϵ–index$(\mathcal{Q}) = 1$ and $\mathcal{P} \sim_\epsilon \mathcal{Q} \circ R$, where $R(t) \in \mathbb{C}(t) \setminus \mathbb{C}$.

1. Compute the polynomials $H_k^{\mathcal{P}\mathcal{P}}(t, s) = p_{k1}(t)p_{k2}(s) - p_{k1}(s)p_{k2}(t),\ k = 1, 2$.
2. Compute
$$S_\epsilon^{\mathcal{P}\mathcal{P}}(t, s) = \epsilon\text{–gcd}(H_1^{\mathcal{P}\mathcal{P}}(t, s), H_2^{\mathcal{P}\mathcal{P}}(t, s)) \approx_\epsilon C_m(t)s^m + \cdots + C_0(t),$$
and ϵ–index$(\mathcal{P}) := \deg_t(S_\epsilon^{\mathcal{P}\mathcal{P}})$ (see Definition 1 in Subsection 3.1).
3. If ϵ–index$(\mathcal{P}) = 1$, RETURN $\mathcal{Q}(t) = \mathcal{P}(t)$, and $R(t) = t$. Otherwise go to step 4.
4. Consider $R(t) = \frac{C_i(t)}{C_j(t)} \in \mathbb{C}(t)$, such that $C_j(t)$, $C_i(t)$ are two of the polynomials obtained in step 2 satisfying that $C_jC_i \notin \mathbb{C}$, and ϵ–gcd$(C_j, C_i) = 1$ (see Subsection 3.3).
5. For $k = 1, 2$, compute the polynomials (see Theorem 2 in Subsection 3.4)
$$L_k(s, x_k) = \text{Res}_t(G_k(t, x_k), sC_j(t) - C_i(t)),\ \text{where } G_k(t, x_k) = x_kp_{k2}(t) - p_{k1}(t).$$
6. For $k = 1, 2$, compute the root in the variable x_k of the polynomial $\dfrac{\partial^{\ell-1}L_k}{\partial^{\ell-1}x_k}(s, x_k)$ (see Corollary 6 in Subsection 3.5), where $\ell := \deg(R) = \epsilon$–index$(\mathcal{P})$ (see Remark 8). Let $\widetilde{q}_k(t) = \widetilde{q}_{k1}(t)/\widetilde{q}_{k2}(t)$ be this root, and let $\widetilde{\mathcal{Q}}(t) = (\widetilde{q}_{11}(t)/\widetilde{q}_{12}(t), \widetilde{q}_{21}(t)/\widetilde{q}_{22}(t)) \in \mathbb{C}(t)^2$.
7. Simplify $\widetilde{\mathcal{Q}}(t)$ by removing the approximate gcd from the numerator and denominator of each component of $\widetilde{\mathcal{Q}}$ (see Remark 10). Let
$$\mathcal{Q}(t) = \left(\frac{q_{11}(t)}{q_{12}(t)}, \frac{q_{21}(t)}{q_{22}(t)}\right) \in \mathbb{C}(t)^2, \quad \epsilon\text{–gcd}(q_{k1}, q_{k2}) = 1,\ k = 1, 2,$$
be the obtained parametrization. Check whether the following equality holds
$$L_k(s, x_k) = (x_kq_{k2}(s) - q_{k1}(s))^\ell + \epsilon^\ell W_k(s, x_k), \qquad \|\text{num}(W_k(R, p_k))\| \leq \|H_k^{\mathcal{P}\mathcal{Q}}\|^\ell$$
(see Theorem 2). If it does not hold, use Remark 7 and compute $\overline{\epsilon}$.
8. RETURN $\mathcal{Q}$, R, and the message "`Q is an` ϵ`-proper reparametrization of` $\mathcal{P}$" (or "`Q is an` $\overline{\epsilon}$`-proper reparametrization of` $\mathcal{P}$", if Remark 7 is applied).

Remark 10. *For the simplification of $\widetilde{\mathcal{Q}}$ in step 7, we compute ϵ–gcd$(\widetilde{q}_{k1}(t), \widetilde{q}_{k2}(t))$, $k = 1, 2$ under the given tolerance ϵ, and we remove it from $\widetilde{q}_{k1}(t)$ and $\widetilde{q}_{k2}(t)$. For this purpose, one may apply well known ϵ–gcd algorithms proposed for inexact polynomials (see for instance,*

[3, 4, 10, 21, 40]). We use the SNAP package included in `Maple` *(see statement 1 in Remark 3).*
Also the ϵ–gcd computation in step 2 is done with the SNAP package included in `Maple`.

In the following, we illustrate Numeric Algorithm Reparametrization for Curves by one example in detail (Example 7) and two other examples (Examples 8 and 9). The algorithm is implemented with the computer algebra system Maple, and we work with floating point numbers with precision of 10 digits.

Example 7. *Let $\epsilon = 0.0001$, and the rational curve $\mathcal{C}$ defined by the parametrization*

$$\mathcal{P}(t) = \left(\frac{p_{11}(t)}{p_{12}(t)}, \frac{p_{21}(t)}{p_{22}(t)}\right) =$$

$$\left(\frac{.7498125469t^6 + t^3 + .4973756561}{1.749562609t^6 + 1.749812547t^3 + .2499375156}, \frac{.0002499375156t(10000t^5 + 1.)}{17.49562609t^6 + 17.49812547t^3 + 2.499375156}\right).$$

Using the SNAP package, one has that ϵ–gcd$(p_{j1}, p_{j2}) = 1$, $j = 1, 2$. In step 1 of the algorithm, we compute the polynomials

$H_1^{\mathcal{PP}}(t, s) = -7004000s^6t^3 - 10930000s^6 + 7004000s^3t^6 - 9930990s^3 + 10930000t^6 + 9930990t^3$,

$H_2^{\mathcal{PP}}(t, s) = 70010000s^6t^3 + 10000000s^6 + 7000st^6 + 7001st^3 + 1000s - 70010000s^3t^6 - 10000000t^6 - 7000ts^6 - 7001ts^3 - 1000t$.

Now, we compute the polynomial $S_\epsilon^{\mathcal{PP}}$. We have that

$$S_\epsilon^{\mathcal{PP}}(t, s) \approx_\epsilon C_0(t) + C_1(t)s + C_2(t)s^2 + C_3(t)s^3,$$

where

$$C_0(t) = t(69970939184 + 5354925982721008029 00t^2),$$

$$C_1(t) = t(-52478204388 + 535492598272100802900t) - 69970939184 - 535492598272100802900t^2,$$

$$C_2(t) = 63943722313t + 52478204388, \quad C_3 = -535492598336044525213.$$

Then, ϵ–index$(\mathcal{P}) = \deg_t(S_\epsilon^{\mathcal{PP}}) = 2$ (see Definition 1). Now, we apply step 4 of the algorithm, and we consider

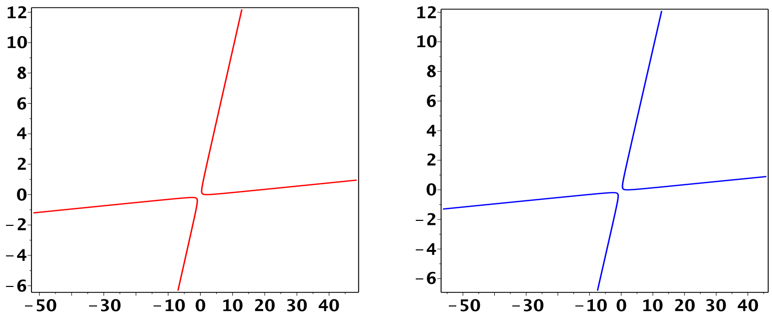
$$R(t) = \frac{C_0(t)}{C_3(t)} = \frac{-4t(17492734796 + 133873149568025200725t^2)}{535492598272100802900}.$$

In steps 5 and 6 of the algorithm, we determine the polynomials $L_k(s, x_k)$, and we compute the root in the variable x_k of the polynomial $\frac{\partial L_k}{\partial x_k}(s, x_k)$, $k = 1, 2$. We get the rational parametrization $\widetilde{\mathcal{Q}}(t)$. We simplify $\widetilde{\mathcal{Q}}$ by removing the approximate gcd from the numerator and denominator of each component of $\widetilde{\mathcal{Q}}$ (see Remark 10), and we return the curve $\mathcal{D}$ defined by the ϵ–numerical reparametrization

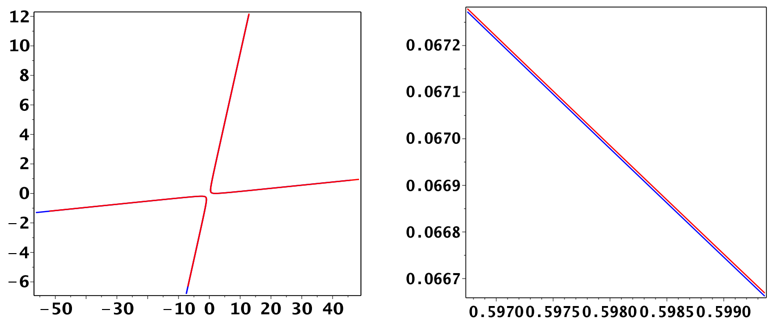
$$\mathcal{Q}(t) = \left(\frac{.7498125351t^2 + t + .4973756559}{1.749562581t^2 + 1.749812559t + .2499375114}, \right.$$

$$\left. \frac{.2499375117t^2 + .5551941368\,10^{-8}t - .5495954487\,10^{-8}}{1.749562581t^2 + 1.749812559t + .2499375114} \right).$$

One may check that the equality of Theorem 2 does not hold. However, Remark 7 holds under $\overline{\epsilon} = 0.0005$*. Then,* $\mathcal{Q}$ *is an* $\overline{\epsilon}$*–proper reparametrization of* $\mathcal{P}$*. In Figure 5, we plot the input curve* $\mathcal{C}$ *and the output curve* $\mathcal{D}$*.*



Input curve $\mathcal{C}$ (left), curve $\mathcal{D}$ (right)



Curves $\mathcal{C}$ and $\mathcal{D}$ (left), partial enlarged view of $\mathcal{C}$ and $\mathcal{D}$ (right)

Figure 5: Curves $\mathcal{C}$ and $\mathcal{D}$

We next perform error analysis by Theorem 5. Let $I = (3, 10)$*. Thus,* $d = 10$*. Let* $M \in \mathbb{N}$ *be such that for every* $t_0 \in I$*, it holds that* $|q_{i2}(R(t_0))| \geq M$ *and* $|p_{i2}(t_0)| \geq M$*, for* $i = 1, 2$*. We have that* $M = 1322.925998$*. Then, by Theorem 5 we deduce that*

$$\zeta = \frac{d^{\deg(\mathcal{P})+1}}{(d-1)^{1/\ell}} = 4807498.567,$$

and for every $t_0 \in I$, *it holds that*

$$|p_i(t_0) - q_i(R(t_0))| < 2/M^2\overline{\epsilon}\,\zeta\|p\|\|q\| = .08410680133, \quad i = 1, 2,$$

where $\|p\| = 17.49812547$, *and* $\|q\| = 1.749812559$. *In Figure 6, we plot the curves* $\mathcal{C}$ *and* $\mathcal{D}$ *defined by* $\mathcal{P}(t)$ *and* $\mathcal{Q}(t)$, *respectively, for* $t \in I$.

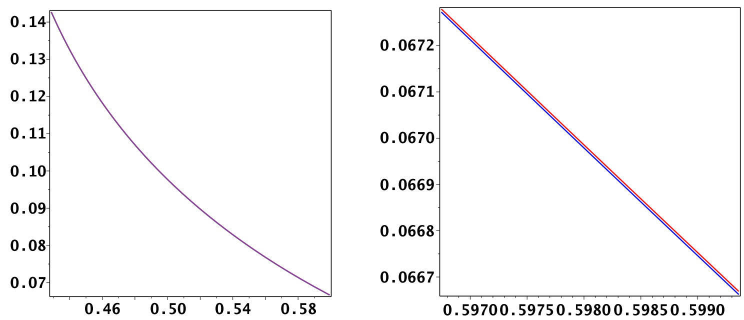


Figure 6: Curves $\mathcal{C}$ and $\mathcal{D}$ for $t \in I$ (left), partial enlarged view of $\mathcal{C}$ and $\mathcal{D}$ (right)

Example 8. *Let* $\epsilon = 0.02$, *and the rational curve* $\mathcal{C}$ *defined by the parametrization*

$$\mathcal{P}(t) = \left(\frac{t^6 - 3t^5 - 3.001t^4 + 11.001t^3 + 9t^2 - 15t - 9.002}{t^2 - t - 2.001}, \frac{t^4 - 2.001t^3 - 2t^2 + 3.002t + 3}{t^2 - t - 2.001}\right).$$

Numeric Algorithm Reparametrization for Curves *returns the curve* $\mathcal{D}$ *defined by the* ϵ*–numerical reparametrization*

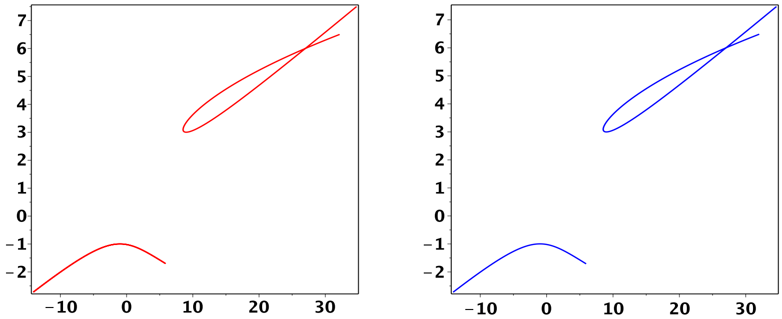
$$\mathcal{Q}(t) = \left(\frac{0.06667333664\,t^3 - 0.4000900188\,t^2 + t - 0.6001100173}{0.06667333664\,t - 0.1334077982},\right.$$

$$\left.\frac{0.06667333662\,t^2 - 0.2001089149\,t + 0.2000366790}{0.06667333664\,t - 0.1334077982}\right).$$

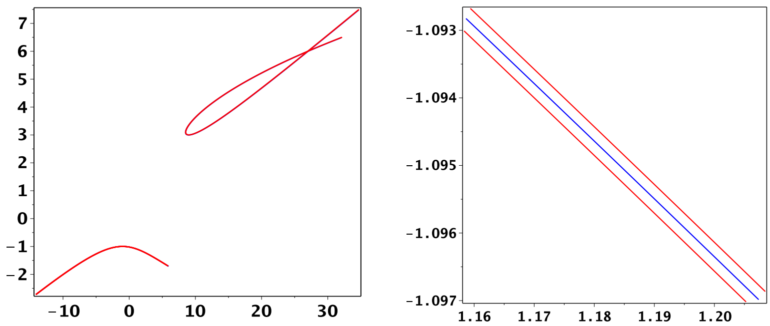
Using Theorem 5, we consider $I = (0, 0.5)$ *and for every* $t_0 \in I$, *it holds that*

$$|p_i(t_0) - q_i(R(t_0))| \leq 0.4582153762, \quad i = 1, 2.$$

In Figure 7, we plot the curves $\mathcal{C}$ *and* $\mathcal{D}$ *defined by* $\mathcal{P}(t)$ *and* $\mathcal{Q}(t)$, *respectively, for* $t \in I$.



Input curve $\mathcal{C}$ (left), curve $\mathcal{D}$ (right)



Curves $\mathcal{C}$ and $\mathcal{D}$ (left), partial enlarged view of $\mathcal{C}$ and $\mathcal{D}$ (right)

Figure 7: Curves $\mathcal{C}$ and $\mathcal{D}$

Example 9. *Let $\epsilon = 0.001$, and the rational curve $\mathcal{C}$ defined by the parametrization $\mathcal{P}(t) =$*

$$\left(\frac{20.001\,t^8 - 40\,t^5 + 20\,t^2 + 2\,t^7 - 2.001\,t^4 - t^6}{\left(t^3 - 1.001\right)^3}, \frac{-2\,t^4 - 6.002\,t^5 + 6\,t^2 + 6\,t^6 - 12.002\,t^3 + 6.002}{\left(-t^2 + t^3 - 1\right)\left(t^3 - 1.001\right)}\right).$$

Numeric Algorithm Reparametrization for Curves *returns the curve $\mathcal{D}$ defined by the ϵ–numerical reparametrization*

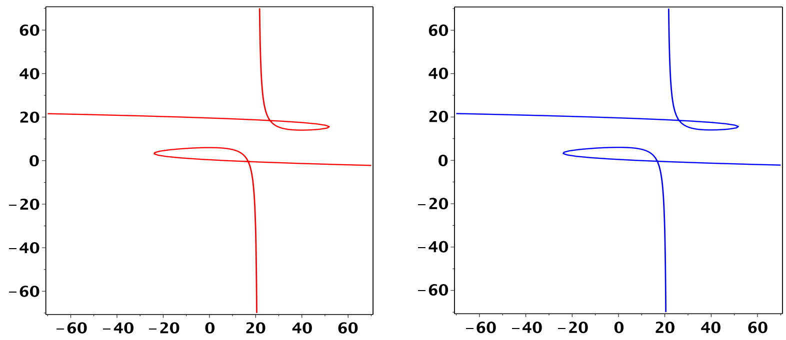
$$\mathcal{Q}(t) = \left(\frac{-t^2 + 0.1002417588\,t + 0.04999508896}{0.050095827\,t^3 - 1.520248235\,10^{-5}\,t^2 + 4.9502838\,10^{-8}\,t - 2.19451487\,10^{-10}},\right.$$

$$\left.\frac{9.046219880\,10^{-4}\,t^2 + 0.0009044005499\,t - 0.0003016044631}{0.0001508373032\,t^2 + 0.0001508171174\,t - 1.509849726\,10^{-7}}\right).$$

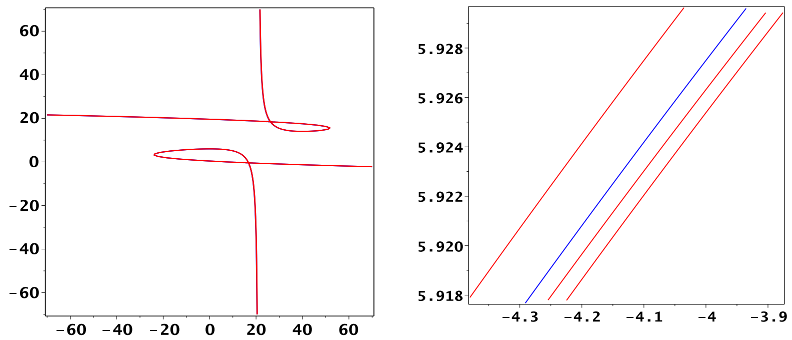
Using Theorem 5, we consider $I = (-5, 5)$ and for every $t_0 \in I$, it holds that

$$|p_i(t_0) - q_i(R(t_0))| \leq 0.1254659264, \quad i = 1, 2.$$

In Figure 8, we plot the curves $\mathcal{C}$ and $\mathcal{D}$ defined by $\mathcal{P}(t)$ and $\mathcal{Q}(t)$, respectively, for $t \in I$.



Input curve $\mathcal{C}$ (left), curve $\mathcal{D}$ (right)



Curves $\mathcal{C}$ and $\mathcal{D}$ (left), partial enlarged view of $\mathcal{C}$ and $\mathcal{D}$ (right)

Figure 8: Curves $\mathcal{C}$ and $\mathcal{D}$

6. Conclusion

The paper focuses on the problem of numerical proper reparametrization which has both theoretical and practical significance. Based on the existing results of the symbolic situation (see [22]), we build the corresponding parallel theory for the numerical situation. For a given numerical curve, we determine whether it is approximately improper under a given precision. For the affirmative case, an ϵ–proper reparametrization is computed and it is proved that the reparameterized curve always lies in a certain offset region of the input one (and reciprocally). A natural but difficult generalization of the work would be the numerical proper reparametrization for rational curves and surfaces.

Acknowledgements

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