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Computing the form of highest degree of the implicit equation of a rational surface

Juan Gerardo Alcázar and Sonia Pérez-Díaz
Dpto. de Física y Matemáticas
Universidad de Alcalá
E-28871 Madrid, Spain
sonia.perez@uah.es, juange.alcazar@uah.es

Abstract

A method is presented for computing the form of highest degree of the implicit equation of a rational surface, defined by means of a rational parametrization. Determining the form of highest degree is useful to study the asymptotic behavior of the surface, to perform surface recognition, or to study symmetries of surfaces, among other applications. The method is efficient, and works generally better than known algorithms for implicitizing the whole surface, in the absence of base points blowing up to a curve at infinity. Possibilities to compute the form of highest degree of the implicit equation under the presence of such base points are also discussed. We provide timings to compare our method with known methods for computing the whole implicit equation of the surface, both in absence and in presence of base points blowing up to a curve at infinity.

Keywords: highest order form; surface implicitization; rational surface.

AMS2000: 14Jxx, 14J29, 14J70

1 Introduction

Certain operations for rational surfaces are easier to perform when the implicit equation of the surface is known. For instance, curve-surface or surface-surface intersections, determining whether a point is on the surface, or, for compact surfaces, checking whether a point is inside or outside the surface, are easier when one works with the surface in implicit form. As a consequence, *implicitizing* is a classical problem, that has received much attention in the literature (see e.g. [4], [9], [10], [13], [16], [17]). The problem can be approached by means of elimination techniques like Gröbner bases and various types of resultants (see for example the paper [22], and the references therein). In recent years, alternative methods using μ-bases, moving lines and moving surfaces have been extensively studied (see [30, 31] and the references therein). These techniques have been successfully applied, first, to special types of surfaces, and then to more general surfaces. In particular, in [31] the authors combine different approaches (Dixon $\mathcal{A}$-resultant, moving planes, moving quadrics) for implicitizing rational tensor product surfaces. This algorithm has been implemented (see [34]) and is, up to our knowledge, the most recent, efficient and complete general implicitization method.

In this paper we do not consider the implicitization problem, but a weaker one: given a rational parametrization of a rational surface, we address the problem of computing the form of highest degree of the implicit equation of the surface. Therefore, we are not interested in the whole implicit equation, but only in the form of highest degree. Obviously, a first possibility to solve the problem is to compute the whole implicit equation, and then extract the form of highest degree from the implicit equation. Nevertheless, in this paper we consider alternative methods, that work faster in certain situations.

Knowing the form of highest degree of the implicit equation is useful for several purposes. Let us see some examples:

Asymptotic behaviour: asymptotes of planar and space curves are studied in [5, 6, 7]. A similar study for surfaces has, apparently, not been carried out so far. In order to study the behavior of planar and space curves at infinity, the points at infinity of the curve are needed. However, while curves have finitely many points at infinity, surfaces have a whole curve of points at infinity. This projective curve is defined by the form of highest degree of the implicit equation of the surface. Thus, computing the form of highest degree of the implicit equation is important in order to study the asymptotic behavior of the surface.

Recognition of special types of algebraic surfaces: special properties of a surface often lead to a special structure of the form of highest degree of the surface. In the case of *translation surfaces* (see [32]), and, as a consequence, of *minimal surfaces* (see also [32]), the form of highest degree is a product of linear factors. For *surfaces of revolution* (see [1]) as well as for *affine rotation surfaces*, which arise in the context of affine differential geometry as generalizations of surfaces of revolutions (see [2, 3]), the form of highest degree of the implicit equation factors into linear and quadratic factors. Furthermore, the nature of the quadrics defined by the quadratic factors appearing in the factorization depends on the type (elliptic -for surfaces of revolution-, hyperbolic or parabolic) of the affine rotation surface. Even more, the factorization provides information on the axis of affine rotation of the surface (see [1, 2, 3]). *D'Arboux cyclides* and *Dupin cyclides* (see [24]) also have a special structure in their form of highest degree: in an appropriate system of coordinates, the form of highest degree is $(x^2+y^2+z^2)^2$ for D'Arboux cyclides, and $(x^2+y^2+z^2+\alpha)$, with $\alpha \in \mathbb{R}$, for Dupin cyclides.

Symmetries of rational surfaces: A symmetry of a surface is an orthogonal transformation $h(\bar{x}) = \mathbf{Q}\bar{x} + \mathbf{b}$, $\mathbf{Q}^T\mathbf{Q} = \mathbf{Id}$, leaving the surface invariant. It is easy to see that if $h(\bar{x})$ is a symmetry of a surface S, then $\widehat{h}(\bar{x}) = \mathbf{Q}\bar{x}$ is a symmetry of the surface defined by the form of highest degree of the implicit equation of S. Thus, if we are able to compute the form of highest degree of the implicit equation of a rational surface, studying the symmetries of the surface defined by the form of highest degree provides information on the symmetries of the original surface.

Proper and surjective parametrizations: A rational parametrization is *proper* when it is generically injective. In a recent paper [11], the authors study the existence of *proper and surjective* parametrizations for a given rational surface. In Section 3 of [11], it is proven that if a regular rational surface admits a proper and surjective parametrization, then the curve at infinity of the surface has at least one rational component. Again, computing the form of highest degree of the implicit equation is useful in order to check whether this necessary condition is satisfied.

The algorithm we present here uses results of [20, 21, 22], has a similar flavour to these papers, and can work with completely general parametrizations, proper or not. The performance of the algorithm depends on the surjectivity *at infinity* of the given parametrization in projective form, i.e. on whether the parametrization completely covers the curve at infinity of the surface. We prove that this phenomenon is related to the existence of projective base points blowing up to a curve at infinity: intuitively, this happens by approaching a base point in the parameter space under different trajectories, we generate not a point, but a whole (rational) curve on the surface. If this bad case does not arise, our algorithm works efficiently, and generally better than the algorithms in [22, 31]. However, if this bad case arises, our algorithm can compute only those components at infinity which are reached by the parametrization. Although we provide a possible generalization of our algorithm to this bad situation, in the bad case the best timings are provided, in general, by the algorithm in [22]. In fact, in the bad case the algorithm in [31] may fail. In order to perform these comparisons, we have used an improved version of the implementation in [34], kindly provided by the authors of this implementation, which is based on the results of [31]. It is also worth mentioning that our algorithm is able to identify, without explicitly computing the base points, whether we are in a good or a bad case.

The structure of the paper is the following. Section 2 is preliminary and includes notions and algorithms to be used later in the paper. Our method is presented in Section 3: here we develop the main results of this paper, we provide the algorithm and we compare the performance of our algorithm with other approaches; the proofs of the results in this section are postponed to Section 5 to make the paper more easily readable. In Section 4 we discuss potential generalizations of the method to the case of base points blowing up to a curve at infinity. Section 5 is devoted to presenting the main proofs of the results appearing in Section 3. We close in Section 6 with a brief summary of our work.

2 Preliminaries

In this section we recall several notions, results and algorithms related to rational parametrizations of curves and surfaces. Throughout the paper, $\mathbb{K}$ denotes an algebraically closed field of zero characteristic. Furthermore, we consider curves defined over the affine plane $\mathbb{A}^2(\mathbb{K})$, and surfaces defined over the affine space $\mathbb{A}^3(\mathbb{K})$. Notice that these curves and surfaces can be naturally embedded into the projective plane $\mathbb{P}^2(\mathbb{K})$ and the projective space $\mathbb{P}^3(\mathbb{K})$, respectively. Elements of $\mathbb{A}^n(\mathbb{K})$ are denoted by $(x_1, x_2, \ldots, x_n)$, with $x_i \in \mathbb{K}$. Elements of $\mathbb{P}^n(\mathbb{K})$ are denoted by $(x_1 : x_2 : \cdots : x_n : x_{n+1})$, where points at infinity correspond to $x_{n+1} = 0$, and affine points correspond to $x_{n+1} \neq 0$.

Let

$$\mathcal{P}(\bar{t}) = (p_1(\bar{t}), p_2(\bar{t}), \ldots, p_n(\bar{t})) = \left(\frac{p_{11}(\bar{t})}{p(\bar{t})}, \frac{p_{21}(\bar{t})}{p(\bar{t})}, \ldots, \frac{p_{n1}(\bar{t})}{p(\bar{t})} \right), \tag{2.1}$$

where $\bar{t} = (t_1, t_2, \ldots, t_{n-1})$, be a rational parametrization of a variety $\mathbf{V} \subset \mathbb{A}^n(\mathbb{K})$ satisfying that

$$\gcd(p_{11}, p_{21}, \ldots, p_{n1}, p) = 1.$$

In this paper we only consider the cases $n = 2, 3$, and we assume that no $p_i(\bar{t})$ is constant; if $p_i(\bar{t}) = \mu$ then $\mathbf{V}$ is either the line $x_i = \mu$, when $n = 2$, or the plane $x_i = \mu$, when $n = 3$, so the problem is trivial. The variety $\mathbf{V}$ is a rational curve $\mathcal{C} \subset \mathbb{A}^2(\mathbb{K})$ for $n = 2$, and a rational surface $\mathcal{V} \subset \mathbb{A}^3(\mathbb{K})$ for $n = 3$. We will represent the projective closures of $\mathcal{C}, \mathcal{V}$ by $\mathcal{C}^*, \mathcal{V}^*$, i.e. $\mathcal{C}^* \subset \mathbb{P}^2(\mathbb{K})$, $\mathcal{V}^* \subset \mathbb{P}^3(\mathbb{K})$. These projective varieties are parametrized by

$$\mathcal{P}^*(\bar{t}) = (p_{11}(\bar{t}) : p_{21}(\bar{t}) : \cdots : p_{n1}(\bar{t}) : p(\bar{t})),$$

where $n = 2, 3$, respectively. Furthermore, we will represent by $\mathcal{P}_h^*(\bar{T})$ the parametrization obtained from $\mathcal{P}^*(\bar{t})$ by homogenizing the components of $\mathcal{P}^*(\bar{t})$ with a homogenization variable t_n. We denote the components of $\mathcal{P}_h^*(\bar{T})$ by

$$\mathcal{P}_h^*(\bar{T}) = (p_1^h(\bar{T}) : p_2^h(\bar{T}) : p_3^h(\bar{T}) : \cdots : p_{n+1}^h(\bar{T})) \tag{2.2}$$

where $\bar{T} = (t_1 : t_2 : \cdots : t_n)$.

Additionally, let $\bar{x} = (x_1, x_2, \ldots, x_n)$. We represent by $f(\bar{x})$ the implicit equation of the variety parametrized by Eq. (2.1), and we denote the total degree of f by d. Then we can write

$$f(\bar{x}) = f_d(\bar{x}) + f_{d-1}(\bar{x}) + \cdots + f_0(\bar{x}), \tag{2.3}$$

where $f_i(\bar{x})$ denotes the homogeneous form of degree $i = 0, 1, \ldots, d$. We refer to $f_d(\bar{x})$ as the *form of highest degree* of $f(\bar{x})$. *The goal of this paper is to provide algorithms* *for computing* $f_d(\bar{x})$.

By homogenizing Eq. (2.3) with a homogenizing variable x_{n+1}, we get a homogeneous polynomial

$$F(\bar{x}, x_{n+1}) = f_d(\bar{x}) + f_{d-1}(\bar{x})x_{n+1} + \cdots + f_0(\bar{x})x_{n+1}^d, \tag{2.4}$$

which implicitly defines the projective closure of the variety parametrized by Eq. (2.1).

Degree of a rational map

The parametrization in Eq. (2.1) has an associated rational map $\phi_{\mathcal{P}} : \mathbb{K}^{n-1} \to \mathbf{V}; \bar{t} \to \mathcal{P}(\bar{t})$, where $\phi_{\mathcal{P}}(\mathbb{K}^{n-1})$ is dense in $\mathbf{V}$. The *degree of the rational map* $\phi_{\mathcal{P}}$, which we denote by $\deg(\phi_{\mathcal{P}})$, is, intuitively speaking, the cardinality of a generic fiber of $\phi_{\mathcal{P}}$. A more algebraic definition can be found in [27] or in Lecture 7 of [15]. In particular, when $\deg(\phi_{\mathcal{P}}) = 1$ we say that $\mathcal{P}$ is *proper*, i.e. generically injective. In order to compute $\deg(\phi_{\mathcal{P}})$ it is useful to define, for $i = 1, \ldots, n$,

$$G_i(\bar{t}, \bar{s}) = \operatorname{numer}(p_i(\bar{t}) - p_i(\bar{s})), \tag{2.5}$$

where $\operatorname{numer}(p_i(\bar{t}) - p_i(\bar{s}))$ denotes the numerator of $p_i(\bar{t}) - p_i(\bar{s})$, $\bar{t} = (t_1, \ldots, t_{n-1})$ and $\bar{s} = (s_1, \ldots, s_{n-1})$.

Now for curves, i.e. when $n = 2$ (see Subsection 4.3 in [28]),

$$\deg(\phi_{\mathcal{P}}) = \deg_{\bar{t}}(\gcd(G_1(\bar{t}, \bar{s}),\, G_2(\bar{t}, \bar{s}))). \tag{2.6}$$

In particular, almost all points of the input curve defined by $\mathcal{P}(\bar{t})$ are generated via $\mathcal{P}(\bar{t})$ by the same number of parameter values, and this number is $\deg(\phi_{\mathcal{P}})$. Thus, for almost all $\bar{s}_0 \in \mathbb{K}$, $\deg(\phi_{\mathcal{P}}) = \deg(\gcd(G_1(\bar{t}, \bar{s}_0),\, G_2(\bar{t}, \bar{s}_0)))$ (see Chapter 4 in [28]). Hence, picking a random $\bar{s}_0 \in \mathbb{K}$ and computing $\deg(\phi_{\mathcal{P}}) = \deg(\gcd(G_1(\bar{t}, \bar{s}_0),\, G_2(\bar{t}, \bar{s}_0)))$ provides the value of $\deg(\phi_{\mathcal{P}})$ with probability almost one.

One can generalize the preceding idea to a mapping like

$$\mathcal{Q}(\bar{t}) = \left.\left(\frac{p_{11}(\bar{t})}{p_{13}(\bar{t})}, \frac{p_{12}(\bar{t})}{p_{13}(\bar{t})}\right)\right|_{q(\bar{t})=0},$$

where $\bar{t} = (t_1, t_2)$, and $\mathcal{Q}(\bar{t})$ is an affine, rational planar mapping $\mathbb{K}^2 \to \mathbb{K}^2$ restricted to the curve $\mathcal{D}$ defined by $q(\bar{t}) = 0$, with $q(\bar{t})$ an irreducible polynomial. Again, $\mathcal{Q}$ has an associated rational mapping $\phi_{\mathcal{Q}}$ from the planar curve $\mathcal{D}$ onto its image under $\mathcal{Q}$ (which is also a planar curve), and $\deg(\phi_{\mathcal{Q}})$ is, intuitively, the cardinality of a generic fiber of $\phi_{\mathcal{Q}}$. If the curve $\mathcal{D}$ defined by $q(\bar{t}) = 0$ is rational, then $\mathcal{Q}$ can be rewritten as a rational planar mapping taking values over the whole plane $\mathbb{K}^2$ and Eq. (2.6) can be applied. Otherwise, we notice again that $\deg(\phi_{\mathcal{Q}})$ is preserved for almost all points, so we can compute $\deg(\phi_{\mathcal{Q}})$ by picking a random point $\mathbf{q} \in \mathcal{D}$, and determining the number of elements in $\mathcal{Q}^{-1}(\mathcal{Q}(\mathbf{q}))$ belonging to $\mathcal{D}$.

For surfaces the computation of $\deg(\phi_{\mathcal{P}})$ is more elaborate (see [20]), and requires the use of resultants. We recall this computation in Algorithm 1. The notions of *primitive part* and *content* of a polynomial with respect to a given variable appear in this algorithm. Recall that the content of a polynomial $\mathbf{P}$ with respect to a variable ω is the gcd of the coefficients of $\mathbf{P}$, seen as a polynomial in ω, and the primitive part is the result of factoring out the content from $\mathbf{P}$.

Algorithm 1: Computation of $\deg(\phi_{\mathcal{P}})$ for surfaces

[Step 0] Check hypotheses:

[Step 0.1] If any of the projective curves defined by the components of $\mathcal{P}_h^*(\bar{T})$ (see Eq. (2.2)) passes through $(0:1:0)$, apply a random linear reparametrization $\bar{t} \to A\bar{t} + B$.
[Step 0.2] If the determinant of the Jacobian of $(p_2(\bar{t}), p_3(\bar{t}))$ is identically zero, exchange appropriately the affine coordinates of $\mathbb{K}^3$.

[Step 1] For $i = 1, 2, 3$, compute the $G_i(\bar{t}, \bar{s})$ in Eq. (2.5).
[Step 2] Determine $R(t_2, \bar{s}, Z) = \mathrm{Res}_{t_1}(G_1, G_2 + ZG_3)$ where $Z \in \mathbb{K}$ is a generic element.
[Step 3] Compute $S(t_2, \bar{s}, Z) = \mathrm{PrimPart}_{\{\bar{s}\}}(\mathrm{Content}_Z(R))$.
[Step 4] Return $\deg(\phi_{\mathcal{P}}) = \deg_{t_2}(S)$.

As in the case of curves, since $\deg(\phi_{\mathcal{P}})$ is preserved under almost all specializations of the variables s_1, s_2 (see [20]) one can compute $\deg(\phi_{\mathcal{P}})$, with probability almost one, by randomly taking a point $P \in \mathbb{K}^3$ on the input surface and determining the degree of the fiber of P.

Partial degrees of a polynomial

Let $f(\bar{x})$ in Eq. (2.3) implicitly define a variety $\mathbf{V} \subset \mathbb{A}^n(\mathbb{K})$. The *partial degree* of f in the variable x_i, denoted by $\deg_{x_i}(f)$, is the maximum power of x_i appearing in the polynomial $f(\bar{x})$. For curves, one has (see Chapter 4 in [28])

$$\deg_{x_2}(f) = \frac{\deg(p_1(t))}{\deg(\phi_{\mathcal{P}})}, \qquad \deg_{x_1}(f) = \frac{\deg(p_2(t))}{\deg(\phi_{\mathcal{P}})}. \tag{2.7}$$

The numerators $\deg(p_i(t))$ correspond to the degrees of rational functions, namely the $p_i(t)$. The degree of a rational function is defined as the maximum of the degrees of the numerator and denominator of the function. Notice that $\deg(p_i(t))$ coincides with the number of solutions of the equation $p_i(t) - a = 0$, for a generic $a \in \mathbb{K}$.

In certain cases (see the next section) we will be interested in computing the partial degrees of the polynomial defining the planar curve described by

$$\mathcal{Q}(\bar{t}) = \left(\frac{p_{11}(\bar{t})}{p_{13}(\bar{t})}, \frac{p_{12}(\bar{t})}{p_{13}(\bar{t})}\right)\Bigg|_{q(\bar{t})=0}$$

where $\bar{t} = (t_1, t_2)$ and $q(\bar{t})$ is irreducible, i.e. the partial degrees of the planar curve obtained as the image of the curve $\mathcal{D}$ defined by $q(\bar{t}) = 0$, under a rational planar mapping. If $\mathcal{D}$ is rational, then $\mathcal{Q}$ can be rewritten as a rational planar mapping taking values over the whole plane $\mathbb{K}^2$ and Eq. (2.7) can be applied. Otherwise, we adapt Eq. (2.7). In order to do this, in the denominator of Eq. (2.7) we replace $\deg(\phi_{\mathcal{P}})$ by $\deg(\phi_{\mathcal{Q}})$, which can be computed as we explained before. In the numerator of Eq. (2.7), now we must compute the degree a rational function, say $p(\bar{t}) = p_1(\bar{t})/p_3(\bar{t}) \in \mathbb{K}(\bar{t})$, $p_i \in \mathbb{K}[\bar{t}]$, restricted to the curve $q(\bar{t}) = 0$. This is equal to the number of solutions of $\{p(\bar{t}) - X = 0,\ q(\bar{t}) = 0\}$ for a generic $X \in \mathbb{K}$, i.e. the number of solutions of $\{p_1(\bar{t}) - Xp_3(\bar{t}) = 0,\ q(\bar{t}) = 0\}$. Using properties of resultants (see e.g. [28]), this number coincides with the degree with respect to t_2 of the polynomial $T(t_2, X)$, where

$$T(t_2, X) = \mathrm{PrimPart}_{\{t_2,X\}}(\mathrm{Res}_{t_1}(p_1(\bar{t}) - Xp_3(\bar{t}), q(\bar{t}))). \tag{2.8}$$

Again, for surfaces the procedure is more complicated. We recall this procedure in Algorithm 2, which follows from Theorem 6 in [22]. Previously we need to introduce some additional notation: for $i, j \in \{1, 2, 3\}$, with $i < j$, let π_{ij} be the (i, j)-projection mapping in 3-space, i.e. $\pi_{ij}(x_1, x_2, x_3) = (x_i, x_j)$. For $i, j \in \{1, 2, 3\}$, with $i < j$, let

$$\mathcal{P}_{ij}(\,\bar{t}\,) := \pi_{ij}(\mathcal{P}(\,\bar{t}\,)) = \left(\frac{p_{i1}(\,\bar{t}\,)}{p(\,\bar{t}\,)}, \frac{p_{j1}(\,\bar{t}\,)}{p(\,\bar{t}\,)}\right) \in \mathbb{K}(\,\bar{t}\,)^2,$$

and let $\phi_{\mathcal{P}_{ij}}$ be the rational map induced by $\mathcal{P}_{ij}(\,\bar{t}\,)$.

We present the algorithm for the computation of $\deg_{x_1}(f)$. For $\deg_{x_k}(f)$, $k = 2, 3$, one argues in a similar way.

Algorithm 2: Computation of $\deg_{x_1}(f)$ for surfaces

[Step 1] Apply Algorithm 1 to compute $\deg(\phi_{\mathcal{P}})$
[Step 2] Let $H_2(\,\bar{t}, Z_2) = p_{21}(\,\bar{t}\,) - Z_2p(\,\bar{t}\,)$ and $H_3(\,\bar{t}, Z_3) = p_{31}(\,\bar{t}\,) - Z_3p(\,\bar{t}\,)$, where $(Z_2, Z_3) \in \mathbb{K}^2$ is a generic element.
[Step 3] Let $\deg(\phi_{\mathcal{P}_{23}}) = \deg_{t_2}(\mathrm{PrimPart}_{\{Z_2,Z_3\}}(\mathrm{Res}_{t_1}(H_2(\,\bar{t}, Z_2), H_3(\,\bar{t}, Z_3)))$.
[Step 4] Return $\deg_{x_1}(f) := \dfrac{\deg(\phi_{\mathcal{P}_{23}})}{\deg(\phi_{\mathcal{P}})}$.

Note that the intuitive idea of Algorithm 2 is the following: in order to determine $\deg_{x_1}(f)$, one computes $a_k \in \overline{\mathbb{K}(Z_2, Z_3)}^2$ such that $\mathcal{P}_{23}(a_k) = (Z_2, Z_3)$, $k = 1, \ldots, n$. Then, $\mathcal{P}(a_k) = (p_1(a_k), Z_2, Z_3)$ which implies that the number of x_1–coordinates for every general point on the surface is n, i.e. $\deg_{x_1}(f) = n$. Observe that if $\mathcal{P}$ is not proper, i.e. if $\deg(\phi_{\mathcal{P}}) \neq 1$, then we have to eliminate redundant elements that are reached several times via $\mathcal{P}$. We have $\deg(\phi_{\mathcal{P}})$ elements of this kind, so we divide by this number.

Remark 1. *The following observations will be useful in the next section, when proving Theorem 2 and Corollary 1.*

1. *One can also work in a different chart and compute the degree with respect to x_1 of $F(x_1, x_2, 1, x_4)$ (similarly if one dehomogenizes with respect to another variable). In order to do this, one applies Algorithm 2 using the polynomials $H_2(\bar{t}, Z_2) = p_{21}(\bar{t}) - Z_2 p_{31}(\bar{t})$ and $H_4(\bar{t}, Z_4) = p_{41}(\bar{t}) - Z_4 p_{31}(\bar{t})$, where $(Z_2, Z_4) \in \mathbb{K}^2$ is a generic element.*

2. *Let us assume that $f_d(\bar{x}) = g(\bar{x})^n h(\bar{x})$, where g, h are irreducible polynomials with $\gcd(g, h) = 1$. Let us also assume that $p(\bar{t})$ is an irreducible polynomial such that $\mathcal{P}^*(\bar{t})|_{p(\bar{t})=0}$ parametrizes the curve defined by the irreducible polynomial $g(\bar{x})$. Using the previous observation, we notice that if one applies Algorithm 2 with Z_2 a generic element of $\mathbb{K}$ and $Z_4 = 0$, one gets the degree with respect to x_1 of $g(\bar{x})^n$.*

3 The method

Let $\mathcal{V} \subset \mathbb{K}^3$ be a surface defined by a rational parametrization

$$\mathcal{P}(\bar{t}) = \left(\frac{p_{11}(\bar{t})}{p(\bar{t})}, \frac{p_{21}(\bar{t})}{p(\bar{t})}, \frac{p_{31}(\bar{t})}{p(\bar{t})} \right) \tag{3.1}$$

(resp. by a projective parametrization $\mathcal{P}_h^*(\bar{T})$ like Eq. (2.2), with $n = 3$), different from a plane. Let us write $p(\bar{t}) = q_1(\bar{t})^{n_1} \cdots q_r(\bar{t})^{n_r}$, $r \geq 1$, where $q_i(\bar{t}) \in \mathbb{K}[\bar{t}]$, for $i = 1, \ldots, r$, is irreducible. Furthermore, let us denote by $\mathcal{D}_i$ the plane curve defined by $q_i(\bar{t}) = 0$ in the (t_1, t_2) affine plane. The projective closure of $\mathcal{D}_i$ is denoted by $\mathcal{D}_i^*$.

Also, let $f(\bar{x})$ in Eq. (2.3) represent the implicit equation of $\mathcal{V}$, and let $F(\bar{x}, x_4)$ in Eq. (2.4) (with $n = 3$) represent the implicit equation of the projective closure of $\mathcal{V}$, $\mathcal{V}^*$.

Now let $f_d(\bar{x}) = g_1(\bar{x})^{m_1} \cdots g_\ell(\bar{x})^{m_\ell}$, $\ell \geq 1$, and let $\mathcal{C}_j^*$ be the projective curve defined by $\{g_j(\bar{x}) = 0,\ x_4 = 0\}$, $j = 1, \ldots, \ell$. The $\mathcal{C}_j^*$ are the components of $\mathcal{V}^* \cap \{x_4 = 0\}$, i.e. the components of the *curve at infinity* of the surface $\mathcal{V}^*$. We refer to the m_j as the *multiplicities* of the $g_j(\bar{x})$; notice however that the m_j are not necessarily the multiplicities of the curves $\{g_j(\bar{x}) = 0,\ x_4 = 0\}$ seen as subsets of $\mathcal{V}$ (see [18]).

An important observation is that not all the $\mathcal{C}_j^*$ might be covered by $\mathcal{P}_h^*(\bar{T})$, i.e. $\mathcal{P}_h^*(\bar{T})$ might not be a surjective parametrization of $\mathcal{V}^*$, so some components of the curve at infinity of $\mathcal{V}^*$ might be missed. The components of the curve at infinity that are covered by the parametrization are the 1-dimensional varieties which are images of the $\mathcal{D}_i^*$ under the (projective) mapping defined by $\mathcal{P}_h^*(\bar{T})$. When we restrict to the hyperplane $x_4 = 0$, assuming that $\gcd(p_{13}, q_i) = 1$ such components are planar curves whose affine part can be described by

$$\mathcal{Q}_i(\bar{t}) = \left(\frac{p_{11}(\bar{t})}{p_{13}(\bar{t})}, \frac{p_{12}(\bar{t})}{p_{13}(\bar{t})} \right) \Bigg|_{q_i(\bar{t})=0} \tag{3.2}$$

where $\mathcal{Q}_i$ corresponds to an affine, rational planar mapping which is restricted to $q_i(\bar{t}) = 0$, i.e. to $\mathcal{D}_i$. If $\mathcal{D}_i$ is rational, then $\mathcal{Q}_i$ can be rewritten as a rational planar mapping taking values over the whole plane $\mathbb{K}^2$. If $\gcd(p_{13}, q_i) \neq 1$, we can consider instead either

$$\mathcal{Q}_i(\bar{t}) = \left(\frac{p_{11}(\bar{t})}{p_{12}(\bar{t})}, \frac{p_{13}(\bar{t})}{p_{12}(\bar{t})} \right) \Bigg|_{q_i(\bar{t})=0},$$

if $\gcd(p_{12}, q_i) = 1$, or

$$\mathcal{Q}_i(\bar{t}) = \left(\frac{p_{12}(\bar{t})}{p_{11}(\bar{t})}, \frac{p_{13}(\bar{t})}{p_{11}(\bar{t})} \right) \Bigg|_{q_i(\bar{t})=0},$$

if $\gcd(p_{11}, q_i) = 1$. Since $\gcd(p_{1k}, q_i) = 1$ for some $k = 1, 2, 3$, without loss of generality we can assume that $\mathcal{Q}_i(\bar{t})$ is written as in Eq. (3.2).

We wonder under what conditions we might not reach some component at infinity. To answer this question, one observes that in order to compute the implicit equation $F(\bar{x}, x_4) = 0$ from $\mathcal{P}_h^*(\bar{T})$ (see for instance Theorem 2.5 in [14]), one can consider the ideal

$$I = \langle x_1 - p_1^h(\bar{T}), x_2 - p_2^h(\bar{T}), x_3 - p_3^h(\bar{T}), x_4 - p_4^h(\bar{T}) \rangle,$$

and the ideal $I_4 = I \cap \mathbb{K}[\,\bar{x}\,, x_4]$. Then $F(\bar{x}, x_4) = 0$ defines the variety $V(I_4)$. Arguing in a similar way to Section 2 of [26], by the Projective Extension Theorem (see Theorem 6 in [12]) and whenever $\mathcal{P}_h^*(\bar{T})$ has no base points, every point of $F(\bar{x}, x_4) = 0$ is the image of some $\bar{T}$ via $\mathcal{P}_h^*(\bar{T})$. Additionally, since $\mathcal{P}(\bar{t})$ is rational, $\mathcal{P}(\bar{t})$ induces rational mappings between each $\mathcal{D}_i$ and its image. Furthermore, if $\mathcal{P}_h^*(\bar{T})$ has no projective base points, by Theorem 1.10 in [29] each of these images is a projective closed set. As a consequence, the following theorem follows.

Theorem 1. *If $\mathcal{P}_h^*(\bar{T})$ has no projective base points, then every $\mathcal{C}_j^*$ is the image of at least one $\mathcal{D}_i^*$ under the rational map defined by $\mathcal{P}_h^*(\bar{T})$.*

Thus, if $\mathcal{P}_h^*(\bar{T})$ has no projective base points we can be sure that the curve at infinity of $\mathcal{V}^*$ is completely reached by the parametrization $\mathcal{P}_h^*(\bar{T})$, and that each of the components of this curve is the image under $\mathcal{P}_h^*(\bar{T})$ of some $\mathcal{D}_i^*$. However, if $\mathcal{P}_h^*(\bar{T})$ has projective base points, some $\mathcal{C}_j^*$ might be missed. More precisely, we miss components at infinity when we have projective base points blowing up to a curve at infinity. Notice also that if a $\mathcal{C}_j^*$ is covered by the parametrization, $\mathcal{C}_j^*$ can be reached by several

$\mathcal{D}_i^*$: this can certainly happen when we work with a non-proper (i.e. non-injective) parametrization, but it can even happen when the parametrization is proper, when we have a self-intersection at infinity. Thus, in general $\ell \leq r$.

Example 1. *Consider the ruled surface $\mathcal{V}$ parametrized by*

$$\mathcal{P}(\bar{t}) = \left(\frac{t_1^2}{t_1^4 - 2} + t_2 t_1, \frac{1}{t_1^4 - 2} + t_2, \frac{t_1}{t_1^4 - 2} + t_2 t_1^2 \right).$$

The implicit equation of this surface is

$$f(\,\bar{x}\,) = x_1^6 + x_1^4 x_2 x_3 - 2x_1^2 x_2^4 + 2x_2^5 x_3 - x_1^2 x_2^3 + x_1 x_2^4,$$

and the form of highest degree of $f(\,\bar{x}\,)$ is $f_d(\,\bar{x}\,) = x_1^6 + x_1^4 x_2 x_3 - 2x_1^2 x_2^4 + 2x_2^5 x_3$. The polynomial $f_d(\,\bar{x}\,)$ defining the curve at infinity of $\mathcal{V}$ is reducible and consists of four linear factors, and a quadratic factor. The four linear factors of $f_d(\,\bar{x}\,)$ correspond to components of the curve at infinity that are covered by the parametrization. In fact, these components are rational curves corresponding to the four real roots of $t_1^4 - 2$. However, the quadratic factor defines a component not reached by the parametrization. Observe also that the parametrization has a projective base point, namely $(0 : 1 : 0)$.

In the rest of the section we first focus on finding, for each component $\mathcal{C}_j^*$ reached by the parametrization, the polynomial $g_j(\bar{x})$ defining $\mathcal{C}_j^*$, and its multiplicity m_j. Next, by computing the degree of the surface $\mathcal{V}$ (see for instance [23]), i.e. the degree of $f_d(\bar{x})$, we can know whether the parametrization covers all the components of the curve at infinity, i.e. whether or not we have missed some component. If we have not, then we have finished. If there are missed components at infinity then our method is not necessarily better than implicitizing the surface, and taking the highest degree form; however, we present some ideas for this case in the next section.

3.1 Components reached by the parametrization.

For simplicity, we start assuming that $p(\,\bar{t}\,) = q(\,\bar{t}\,)^n$, where $q(\,\bar{t}\,)$ is an irreducible polynomial defining a plane curve $\mathcal{D}$; then we generalize, as a corollary, to the case when $p(\,\bar{t}\,)$ has several irreducible factors. Now if $p(\,\bar{t}\,) = q(\,\bar{t}\,)^n$ then there is only at most one component $\mathcal{C}^*$ of the curve at infinity of $\mathcal{V}^*$ reached by the parametrization. Thus, $f_d(\,\bar{x}\,) = g(\,\bar{x}\,)^m h(\,\bar{x}\,)$, where $g(\,\bar{x}\,) = 0$ implicitly defines $\mathcal{C}^*$, $h(\,\bar{x}\,) = 0$ defines the components not covered by the parametrization, and $\gcd(g, h) = 1$.

If the curve $\mathcal{D}$ defined by $q(\bar{t}) = 0$ is rational, in order to compute $g(\,\bar{x}\,)$ one computes a parametrization of $\mathcal{D}$, substitutes the parametrization in $\mathcal{P}(\bar{t})$, and computes the implicit equation of the rational (projective, planar) curve obtained this way. If $\mathcal{D}$ is not rational, then we need to use elimination methods to compute $g(\,\bar{x}\,)$; we provide more details in Remark 2. However, we need the following theorem in order to compute the multiplicity m of the factor $g(\bar{x})$. Notice that this theorem does not require that $\mathcal{P}(\bar{t})$ is a proper parametrization.

Theorem 2. *Let $\mathcal{V}$ be a surface defined by a parametrization like Eq.* (3.1)*, with $p(\bar{t}) = q(\bar{t})^n$, where $q(\bar{t}) \in \mathbb{K}[\bar{t}]$ is an irreducible polynomial. Let*

$$\mathcal{Q}(\bar{t}) = \left(\frac{p_{11}(\bar{t})}{p_{31}(\bar{t})}, \frac{p_{21}(\bar{t})}{p_{31}(\bar{t})}\right)\Bigg|_{q(\bar{t})=0}.$$

Then $f_d(\bar{x}) = g(\bar{x})^m h(\bar{x})$, where $\gcd(g, h) = 1$, the polynomial $h(\bar{x})$ defines the missed components at infinity, and

$$m = n\,\frac{\deg(\phi_{\mathcal{Q}})}{\deg(\phi_{\mathcal{P}})}. \tag{3.3}$$

Remark 2. *In order to compute the polynomial $g(\bar{x})$, we can apply elimination techniques in the following way: If the curve $\mathcal{D}$ defined by $q(\bar{t}) = 0$ is not rational, one can proceed as follows (see [22]): first, one computes*

$$L_1(t_1, x_1, x_2) = \mathrm{resultant}_{t_2}(H_1, H_2), \qquad L_2(t_1, x_1, x_2) = \mathrm{resultant}_{t_2}(H_1, q)$$

(where $H_j = p_{1j} - x_j p_{13}$, $j = 1, 2$). Then $g(x_1, x_2, 1)$ divides $L_3(x_1, x_2) =$ $\mathrm{resultant}_{t_1}(L_1, \mathrm{Primpart}_{x_1}(L_2))$*. In order to detect the factor $g(x_1, x_2, 1)$ of the polynomial $L_3(x_1, x_2)$, one generates several values (a, b) via the given parametrization to check which factor of L_3 vanishes at the points (a, b).*

Remark 3. *We recall that* $\deg(\phi_{\mathcal{Q}})$ *(see Eq.* (5.1)*) is the degree of the rational mapping induced by $\mathcal{Q}$ restricted to $q(\bar{t}) = 0$. In order to compute it, one argues as in Section 2 (see the part corresponding to the degree of a rational map).*

Thus, by using Remark 2 and Theorem 2 we can compute $g(\bar{x})$ and m. Furthermore, we can check whether $h(\bar{x})$ is constant, and therefore whether we have computed the whole form of highest degree $f_d(\bar{x})$. In order to do this, first one computes $\deg(\mathcal{V})$ (applying for instance [23]) and then $\deg(h) = \deg(\mathcal{V}) - \deg(g^m)$: if $\deg(h) = 0$, then we have already found $f_d(\bar{x})$, otherwise we have not.

Now let us generalize the previous results to the case when $p(\bar{t})$ has several irreducible factors, i.e. when $p(\bar{t}) = q_1(\bar{t})^{n_1} \cdots q_\ell(\bar{t})^{n_\ell}$, $\ell > 1$.

Corollary 1. *Let $\mathcal{V}$ be a surface defined by a parametrization like Eq.* (3.1)*, where $p(\bar{t}) = q_1(\bar{t})^{n_1} \cdots q_\ell(\bar{t})^{n_\ell}$, $\ell \geq 1$, and $q_i(\bar{t}) \in \mathbb{K}[\bar{t}]$, $i = 1, \ldots, \ell$ are irreducible polynomials. For $i = 1, \ldots, \ell$, let $\mathcal{Q}_i(\bar{t})$ be like Eq.* (3.2)*. Then $f_d(\bar{x}) = g_1(\bar{x})^{m_1} \cdots g_\ell(\bar{x})^{m_\ell} \cdot h(\bar{x})$, where $\gcd(h, g_i) = 1$, the polynomial $h(\bar{x})$ defines the missed components at infinity, and*

$$m_i = n_i \frac{\deg(\phi_{\mathcal{Q}_i})}{\deg(\phi_{\mathcal{P}})}. \tag{3.4}$$

Remark 4. *If $q_{j_1}(\bar{t}), \ldots, q_{j_s}(\bar{t})$ give rise to the same $g_i(\bar{x})$, each one with multiplicity m_{j_k}, the multiplicity of the factor $g_i(\bar{x})$ is equal to $m_{j_1} + \cdots + m_{j_s}$.*

Remark 5. *In order to compute the polynomials $g_i(\bar{x})$ and $\deg(\phi_{\mathcal{Q}_i})$ for $i = 1, \ldots, \ell$, one may reason as in Remark 2.*

In the general case $p(\bar{t}) = q_1(\bar{t})^{n_1} \cdots q_\ell(\bar{t})^{n_\ell}$, $\ell > 1$, one can also check whether all the components at infinity are reached by the parametrization as in the case $p(\bar{t}) = q(\bar{t})^n$, i.e. by first determining $\deg(\mathcal{V})$, and then $\deg(h)$. The following algorithm summarizes the above method to find the form of highest degree of a rational surface defined by a parametrization like Eq. (3.1).

Algorithm 3: Computation of the highest-degree form

[Step 1] Compute $\deg(\phi_{\mathcal{P}})$ by applying Algorithm 1.

[Step 2] Factor the denominator of the input parametrization in the form $p(\bar{t}) = q_1(\bar{t})^{n_1} \cdots q_r(\bar{t})^{n_r}$, $r \geq 1$, where $q_i(\bar{t}) \in \mathbb{K}[\bar{t}]$, $i = 1, \ldots, r$ are irreducible polynomials.

[Step 3] For $i = 1, \ldots, r$, let $\mathcal{Q}_i(\bar{t})$ be like Eq. (3.2), and check whether $\mathcal{Q}_i(\bar{t})$ is a parametrization. In the affirmative case, if the curve $\mathcal{D}_i$ defined by $q_i(\bar{t})$ is rational go to Step 3.1. Otherwise, go to Step 3.2.

[Step 3.1] If the curve $\mathcal{D}_i$ defined by $q_i(\bar{t})$ is rational, rewrite $\mathcal{Q}_i(\bar{t})$ as a rational planar mapping taking values over the whole plane $\mathbb{K}^2$; that is, consider the rational parametrization $\mathcal{Q}_i(t) := \mathcal{Q}_i(R_i(t))$, where R_i is a parametrization of $\mathcal{D}_i$.

[Step 3.1.1] Apply statement (i) in Remark 2 to get $g_i(\bar{x})$.

[Step 3.1.2] Compute $\deg(\phi_{\mathcal{Q}_i})$ by applying Eq. (2.6), and $m_i = n_i \frac{\deg(\phi_{\mathcal{Q}_i})}{\deg(\phi_{\mathcal{P}})}$.

[Step 3.2] If the curve $\mathcal{D}_i$ defined by q_i is not rational, proceed as follows:

[Step 3.2.1] Apply statement (ii) in Remark 2 to get $g_i(\bar{x})$.

[Step 3.2.2] Compute $\deg(\phi_{\mathcal{Q}_i})$ by applying Remark 3, and $m_i = n_i \frac{\deg(\phi_{\mathcal{Q}_i})}{\deg(\phi_{\mathcal{P}})}$.

[Step 4] Let $g(\bar{x}) = g_1(\bar{x})^{m_1} \cdots g_\ell(\bar{x})^{m_\ell}$, and compute $\deg(\mathcal{V})$ (apply for instance [23]).

[Step 5] If $\deg(\mathcal{V}) = \deg(g)$, then **return** $f_d(\bar{x}) = g(\bar{x})$. Otherwise, **return** $f_d(\bar{x}) = g(\bar{x})h(\bar{x})$, where $h(\bar{x})$ `are missed components at infinity`.

Example 2. *Let*

$$\mathcal{P}(\bar{t}) = \left(\frac{t_1^4}{(t_2+t_1)\,(t_1^3 - t_2^2 + 1)}, \frac{t_2^4}{(t_2+t_1)\,(t_1^3 - t_2^2 + 1)}, \frac{t_2^5 - 1}{(t_2+t_1)\,(t_1^3 - t_2^2 + 1)} \right)$$

define a surface $\mathcal{V}$.

Step 1*: By applying Algorithm 1, we get that* $\deg(\phi_{\mathcal{P}}) = 1$.

Step 2*: Since $p(\bar{t}) = (t_2 + t_1)\,(t_1^3 - t_2^2 + 1)$, we have $q_1(\bar{t}) = t_1 + t_2$, $q_2(\bar{t}) = t_1^3 - t_2^2 + 1$ and $n_i = 1$, $i = 1, 2$.*

Step 3*: For $i = 1, 2$, let $\mathcal{Q}_i(\bar{t})$ be the rational parametrization in Eq. (3.2). Note that $\mathcal{Q}_i(\bar{t})$, $i = 1, 2$ are both rational parametrizations.*

Step 3.1*: The curve $\mathcal{D}_1$ defined by $q_1(\bar{t}) = 0$ is rational, and can be parametrized by $R_1(t) = (t, -t)$. Thus,*

$$\mathcal{Q}_1(t) = \left(\frac{p_{11}(R_1(t))}{p_{13}(R_1(t))}, \frac{p_{12}(R_1(t))}{p_{13}(R_1(t))}\right) =$$

$$\left(-\frac{t^4}{(t+1)\,(t^4 - t^3 + t^2 - t + 1)}, -\frac{t^4}{(t+1)\,(t^4 - t^3 + t^2 - t + 1)}\right),$$

*whose implicit equation is $g_1(\,\bar{x}\,) = x_1 - x_2$ (*Step 3.1.1*). Additionally, $\deg(\phi_{\mathcal{Q}_1}) = 5$ (*Step 3.1.2*). Hence, $m_1 = 1 \cdot 5/1 = 5$. Thus, $g_1^5(\,\bar{x}\,) = (x_1 - x_2)^5$ is a factor appearing in $f_d(\,\bar{x}\,)$.*

Step 3.2*: The curve $\mathcal{D}_2$ defined by $q_2(\bar{t}) = 0$ is not rational. Thus, we consider*

$$\mathcal{Q}_2(\bar{t}) = \left(\frac{p_{11}(\bar{t})}{p_{13}(\bar{t})}, \frac{p_{12}(\bar{t})}{p_{13}(\bar{t})}\right)\Bigg|_{q_2(\bar{t})=0} =$$

$$= \left(\frac{t_1^4}{(t_2^4 + t_2^3 + t_2^2 + t_2 + 1)\,(t_2 - 1)}, \frac{t_2^4}{(t_2^4 + t_2^3 + t_2^2 + t_2 + 1)(t_2 - 1)}\right)\Bigg|_{q_2(\bar{t})=0}.$$

*The implicit equation of the projective curve defined by $\mathcal{Q}_2$ is computed by applying statement (ii) of Remark 2 (*Step 3.2.1*). We get that*

$g_2(\,\bar{x}\,) = -125x_1^6x_2^9 + 625x_1^3x_2^{12} + 150x_1^9x_2^6 + x_1^{15} + 20x_2^3x_1^{12} + 356x_2^6x_1^3x_3^6 + 212x_2^5x_1^3x_3^7 + 28x_1^3x_2^2x_3^{10} - 24x_1^{12}x_2x_3^2 - 40x_1^9x_2x_3^5 - 8x_1^6x_2x_3^8 + 580x_2^7x_1^3x_3^5 + 122x_2^4x_1^3x_3^8 - 41x_2^5x_1^6x_3^4 + 1650x_2^{10}x_1^3x_3^2 + 332x_1^6x_2^6x_3^3 - 220x_2^4x_1^6x_3^5 + 200x_1^6x_2^8x_3 + 550x_1^6x_2^7x_3^2 - 300x_1^9x_2^5x_3 + 1500x_1^3x_2^{11}x_3 + 64x_1^3x_2^3x_3^9 + 282x_1^9x_2^4x_3^2 + 164x_1^9x_2^2x_3^4 + 4x_1^{12}x_2^2x_3 - 68x_1^6x_2^2x_3^7 - 136x_1^9x_2^3x_3^3 - 220x_1^6x_2^3x_3^6 + 8x_2x_1^3x_3^{11} + 1260x_2^9x_1^3x_3^3 + 875x_2^8x_1^3x_3^4 - x_2^7x_3^8 + 4x_1^6x_3^9 + 6x_1^9x_3^6 + 4x_1^{12}x_3^3 - 32x_2^{10}x_3^5 - 16x_2^{11}x_3^4 - 24x_2^9x_3^6 - 8x_2^8x_3^7 + x_1^3x_3^{12}$.

*Additionally, we have $\deg(\phi_{\mathcal{Q}_2}) = 1$ (*Step 3.2.2*). In order to compute $deg(\phi_{\mathcal{Q}_2})$, we pick $\mathbf{q} \in \mathcal{D}_2$ and observe that the number of elements in $\mathcal{Q}_2^{-1}(\mathcal{Q}_2(\mathbf{q}))$ belonging to $\mathcal{D}_2$ is 1 (see Remark 3). Hence, $m_2 = 1 \cdot 1/1 = 1$. Thus, $g_2(\,\bar{x}\,)$ is a factor appearing in $f_d(\,\bar{x}\,)$.*

Step 4*: Let $g(\,\bar{x}\,) = g_1(\,\bar{x}\,)^{m_1} g_2(\,\bar{x}\,)^{m_2}$. In addition, we compute $\deg(\mathcal{V})$ by applying [23], and we get $\deg(\mathcal{V}) = 20$.*

Step 5*: Since $\deg(\mathcal{V}) = \deg(g)$, the algorithm returns the highest order form of the implicit equation of $\mathcal{V}$:*

$$f_d(\,\bar{x}\,) = g_1(\,\bar{x}\,)^{m_1} g_2(\,\bar{x}\,)^{m_2}.$$

$\mathcal{P}(\bar{t})$	Time 1	Time 2	Time 3
$\mathcal{P}_1(\bar{t})$	5.813	223.906	17.469
$\mathcal{P}_2(\bar{t})$	3.969	154.625	6.391
$\mathcal{P}_3(\bar{t})$	3.235	65.281	2.140
$\mathcal{P}_4(\bar{t})$	1.906	39.312	3.860
$\mathcal{P}_5(\bar{t})$	2.078	39.641	34.765
$\mathcal{P}_6(\bar{t})$	1.438	8.234	3.329
$\mathcal{P}_7(\bar{t})$	1.046	4.828	0.312
$\mathcal{P}_8(\bar{t})$	9.547	> 500	> 500
$\mathcal{P}_9(\bar{t})$	0.500	0.078	1.562

Table 1: Times of Implementations

3.2 Experimentation, and comparison with other approaches

We have implemented our algorithm (Algorithm 3) using Maple 2016 on a Lenovo ThinkPad Intel(R) Core(TM) i7-7500U CPU @ 2.70 GHz 2.90 GHz and 16 GB of RAM, OS-Windows 10 Pro. In Table 1, we provide timings, in CPU seconds, for our method (*Time 1*), the implicitization algorithm developed in [22] (*Time 2*) and the method recently implemented in [34], based on the results of [31] (*Time 3*). The method in [31] is, up to our knowledge, the most complete and up-to-date implicitization algorithm, and combines the use of the Dixon resultant, moving planes and moving surfaces. Notice that in [22] and [31] the algorithm outputs not the homogeneous form of maximum degree but the whole implicit equation of the surface.

The timings in Table 1 correspond to the nine parametrizations of the surfaces used in [31] to illustrate the performance of their algorithm. These parametrizations involve various types of affine base points, simple and multiple. For details on these parametrizations, we refer the interested reader to [31]. In each case, we highlight in blue the best timing.

The results in Table 1 show that in general, the method presented in this section works efficiently. Furthermore, in most of the examples in Table 1 our method works better than the method in [31, 34]. In all these cases, the parametrizations completely cover the curve at infinity. In fact, the approach presented in [31, 34] may fail when the parametrization does not reach completely the curve at infinity (see Section 9 in [31], as well as Table 2 in Section 4). Recall, from Theorem 1, that this situation appears when there are base points blowing up to a curve at infinity.

4 Potential generalizations

The ideas in the previous subsection allow us to compute $f_d(\bar{x})$ when all the components of the curve at infinity of $\mathcal{V}$ are covered by the parametrization, but we fail to compute the factors, and their multiplicities, corresponding to the components non-covered by the parametrization, if any. From Theorem 1, this situation arises precisely when there are projective base points blowing up to a curve at infinity. Certainly, an alternative possibility in this case is using known implicitization methods to compute the whole implicit equation $f(\bar{x})$ of $\mathcal{V}$, and then extract $f_d(\bar{x})$ from $f(\bar{x})$. In this section we will discuss some additional possibilities.

An obvious first possibility to overcome this difficulty is to reparametrize the surface so that the new parametrization does not have any base points blowing up to a curve at infinity. For instance, consider the surface $\mathcal{V}$ implicitly defined by $f(\bar{x}) = x_3 - x_1x_2$, whose projective closure $\mathcal{V}^*$ is $F(\bar{x}, x_4) = x_3x_4 - x_1x_2 = 0$. A parametrization of $\mathcal{V}$ is $\mathcal{P}(\bar{t}) = (t_1, t_2, t_1t_2)$, which, written in projective form, corresponds to

$$\mathcal{P}_h^*(\bar{T}) = (t_1t_3 : t_2t_3 : t_1t_2 : t_3^2).$$

The curve at infinity of $\mathcal{V}$ is the union of two projective lines, $\{x_1 = 0, x_4 = 0\}$ and $\{x_2 = 0, x_4 = 0\}$, not covered by the parametrization. In fact, $\mathcal{P}_h^*(\bar{T})$ has two projective base points, namely $(1 : 0 : 0)$ and $(0 : 1 : 0)$. However, another parametrization of $\mathcal{V}$ is

$$\mathcal{M}(\bar{t}) = \left(\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_1t_2}\right),$$

which covers completely the curve at infinity of $\mathcal{V}$. In projective form, this new parametrization is

$$\mathcal{M}_h^*(\bar{T}) = (t_2t_3 : t_1t_3 : t_3^2 : t_1t_2).$$

Notice that $\mathcal{M}_h^*(\bar{T})$ also has two projective base points, which are again the points $(1 : 0 : 0)$, $(0 : 1 : 0)$. However, while in the case of $\mathcal{P}_h^*$ the base points blow up to a curve at infinity, in the case of $\mathcal{M}_h^*$ they do not. Unfortunately, up to our knowledge there is no algorithm available at the moment for computing a reparametrization with the desired property.

Another possibility is using not one, but several, parametrizations to completely cover the surface. For instance, in [25] it is shown that every *affine* rational surface without projective base points can be fully covered by at most three rational parametrizations. For *affine* ruled surfaces, in [26] it is shown that two rational parametrizations suffice. Furthermore, in [26] an algorithm to find these parametrizations is provided. However, again unfortunately, up to date there is no general algorithm to compute a set of rational parametrizations that completely covers a projective rational surface.

So let us discuss a third possibility, which allows us to reduce the case of missed components at infinity to the case addressed in the previous section, i.e. the case when all the components at infinity are reached. As we show in Table 2 at the end of this section, the method we are about to present can be costly, and not necessarily better than directly implicitizing the surface, and extracting the form of highest degree from the whole implicit equation. However, we believe that the description of this method, which completes the picture of the method presented in the previous section, can be of some interest.

In order to describe the method, let us write the implicit equation of the projective closure of $\mathcal{V}$, $\mathcal{V}^*$ (see Eq. (2.3) and Eq. (2.4)) as

$$F(\bar{x}, x_4) = f_d(\bar{x}) + x_4 f_{d-1}(\bar{x}) + \cdots + x_4^d f_0(\bar{x}),$$

where x_4 is a homogenization variable. Recall that $\mathcal{V}^*$ is parametrized (omitting, for simplicity, the homogenization variable) by

$$\mathcal{P}_h^*(\bar{T}) = (p_{11}^h(\bar{T}) : p_{12}^h(\bar{T}) : p_{13}^h(\bar{T}) : p^h(\bar{T})),$$

where $\gcd(p_{11}, p_{12}, p_{13}, p) = 1$ (see Eq. (2.2)). Now let $\mathcal{W}_\lambda^*$ be the result of applying to $\mathcal{V}^*$ the projective transformation $\mathbf{T}_\lambda$ defined by

$$x_1 := x_1,\ x_2 := x_2,\ x_3 := x_3,\ x_4 := x_4 - \lambda x_3, \tag{4.1}$$

where λ is regarded as a nonzero parameter. Then $\mathcal{W}_\lambda^*$ is rational, and is parametrized by

$$\mathcal{R}_\lambda^*(\bar{T}) = (p_{11}^h(\bar{T}) : p_{12}^h(\bar{T}) : p_{13}^h(\bar{T}) : p^h(\bar{T}) + \lambda p_{13}^h(\bar{T})). \tag{4.2}$$

Observe that the implicit equation of $\mathcal{W}_\lambda^*$, which has the same degree as $\mathcal{V}^*$, is

$$G_\lambda(\bar{x}, x_4) = g_d^\lambda(\bar{x}) + x_4 g_{d-1}^\lambda(\bar{x}) + \cdots + x_4^d g_0^\lambda(\bar{x}).$$

Additionally, $G_\lambda(\bar{x}, x_4)$ can be written as

$$(f_d - \lambda x_3 f_{d-1} + \lambda^2 x_3^2 f_{d-2} + \cdots + (-1)^d x_3^d \lambda^d f_0) + x_4(f_{d-1} + \lambda x_3 f_{d-2} + \cdots) + \cdots.$$

Thus, the highest order form of G_λ is

$$g_d^\lambda = f_d - \lambda x_3 f_{d-1} + \lambda^2 x_3^2 f_{d-2} + \cdots + (-1)^d \lambda^d x_3^d f_0,$$

so evaluating g_d^λ at $\lambda = 0$ we get f_d. The next lemma shows that g_d^λ can be computed from the results in the previous subsection (including λ in the computations as a parameter).

Lemma 1. *The curve at infinity of $\mathcal{W}_\lambda^*$ is completely reached by the parametrization $\mathcal{R}_\lambda^*$.*

Proof. Because $\mathcal{W}^*_\lambda$ is the image of $\mathcal{V}^*$ under the transformation $\mathbf{T}_\lambda$ in Eq. (4.1), the curve at infinity of $\mathcal{W}^*_\lambda$ is the image under $\mathbf{T}_\lambda$ of the intersection curve $\mathcal{D}^*_\lambda$ of $\mathcal{V}^*$ with the hyperplane $x_3 - \frac{1}{\lambda}x_4 = 0$. Now if a curve $\mathcal{D}^* \subset \mathcal{V}^*$ is covered by the parametrization $\mathcal{P}^*(\bar{t})$, then $\mathbf{T}_\lambda(\mathcal{D}^*) \subset \mathcal{W}^*$ is covered by the parametrization $\mathcal{R}^*_\lambda(\bar{t})$. Since the set of points of $\mathcal{V}^*$ not covered by $\mathcal{P}^*(\bar{t})$ is at most 1-dimensional, there are only finitely many values of λ, if any, such that $\mathcal{D}^*_\lambda$ is not reached by $\mathcal{P}^*(\bar{t})$. In other words, for a generic value $\lambda \in \mathbb{K}$, the curve $\mathcal{D}^*_\lambda$ is covered by $\mathcal{P}^*(\bar{t})$. Since the curve at infinity of $\mathcal{W}^*$ is $\mathbf{T}_\lambda(\mathcal{D}^*_\lambda)$, the result follows. $\square$

Remark 6. *Notice that we might also pick other transformations $\mathbf{T}_\lambda$: for instance, replacing $x_4 := x_4 - \lambda x_3$ by $x_4 := x_4 - \lambda x_i$, with $i \neq 4$, does as well.*

Thus, in order to compute f_d we proceed as follows: (1) we compute the highest order form g^λ_d by using Algorithm 3; (2) evaluating g^λ_d at $\lambda = 0$, we get f_d. However, the computations in (1) include one parameter, which makes the method costly in the generic case.

Example 3. *Let*

$$\mathcal{P}(\bar{t}) = \left(\frac{t_1t_2 - 1}{t_1^2 + t_2^2 - 1}, \frac{t_1t_2}{t_1^2 + t_2^2 - 1}, \frac{t_2 - 1}{t_1(t_1^2 + t_2^2 - 1)}\right)$$

define a surface $\mathcal{V}$. We apply Algorithm 3 *to compute the components of the curve at infinity covered by the parametrization:*

Step 1*: By applying Algorithm 1, we get that* $\deg(\phi_{\mathcal{P}}) = 1$.

Step 2*: In this case $p(\bar{t}) = t_1(t_1^2 + t_2^2 - 1)$, so $q_1(\bar{t}) = t_1^2 + t_2^2 - 1$, $q_2(\bar{t}) = t_1$, and $n_i = 1$, $i = 1, 2$.*

Step 3*: For $i = 1, 2$, let $\mathcal{Q}_i(\bar{t})$ be the rational mapping in Eq. (3.2). Notice that $\mathcal{Q}_2(\bar{t})$ does not correspond to a component $\mathcal{C}^*_2$ of the curve at infinity: although the curve $\mathcal{D}_2$ defined by $q_2(\bar{t}) = t_1 = 0$ is rational, and can be parametrized by $R_2(t) = (0, t)$, we get $\mathcal{P}^*(0, t) = (0 : 0 : t - 1 : 0)$, which does not parametrize a projective curve.*

Step 3.1*: The curve $\mathcal{D}_1$ defined by $q_1(\bar{t}) = 0$ is rational, and can be parametrized by $R_1(t) = ((t^2 - 1)/(t^2 + 1), 2t/(t^2 + 1))$. Thus,*

$$\mathcal{Q}_1(t) = \left(\frac{p_{11}(R_1(t))}{p_{13}(R_1(t))}, \frac{p_{12}(R_1(t))}{p_{13}(R_1(t))}\right) =$$

$$\left(\frac{(-2t^3 + 2t + t^4 + 2t^2 + 1)(t + 1)}{(t - 1)(t^2 + 1)^2}, -\frac{2t(t + 1)^2}{(t^2 + 1)^2}\right),$$

whose implicit equation is

$g_1(\bar{x}) = 2x_1^2x_2x_3^2 - 8x_2^3x_1x_3 + 4x_1x_2x_3^3 - 4x_2^4x_1 + 2x_2^4x_3 + x_2x_3^4 + 2x_3^2x_2^3 + 6x_1^2x_2^3 + x_2^5 - 8x_2x_1^3x_3 - 4x_1^3x_2^2 + x_1^4x_2 - 2x_1^2x_3^3 - 4x_2^2x_1x_3^2 + 12x_2^2x_1^2x_3 - 2x_2^2x_3^3 + 2x_1^4x_3$

*(*Step 3.1.1*). Additionally, we have* $\deg(\phi_{\mathcal{Q}_1}) = 1$ *(*Step 3.1.2*). Hence,* $m_1 = 1 \cdot 1/1 = 1$ *and thus,* $g_1(\bar{x})$ *is a factor appearing in* $f_d(\bar{x})$*.*

Step 4*: Let* $g(\bar{x}) = g_1(\bar{x})^{m_1}$*. We compute* $\deg(\mathcal{V})$ *by applying [23], and we get that* $\deg(\mathcal{V}) = 6$*.*

Step 5*: Since* $\deg(\mathcal{V}) \neq \deg(g)$ *the algorithm* **returns** $f_d(\bar{x}) = g(\bar{x})h(\bar{x})$*, where* $h(\bar{x})$ `are missed components at infinity`*. In fact, since* $\deg(\mathcal{V}) - \deg(g) = 1$*, we deduce that we are missing a linear factor.*

Therefore, in order to compute the whole $f_d(\bar{x})$*, we consider the parametrization* Eq. (4.2) *of the surface* $\mathcal{W}_\lambda^*$*. Notice that*

$$q_\lambda(\bar{t}) = p(\bar{t}) + \lambda p_{13}(\bar{t}) = t_1(t_2^2 + t_1^2 - 1) + \lambda(t_2 - 1).$$

The curve $\mathcal{D}_\lambda$ *defined by* $q_\lambda(\bar{t}) = 0$ *is not rational. Thus, we consider*

$$\mathcal{Q}_\lambda(\bar{t}) = \left(\frac{p_{11}(\bar{t})}{p_{13}(\bar{t})}, \frac{p_{12}(\bar{t})}{p_{13}(\bar{t})}\right)\bigg|_{q_\lambda(\bar{t})=0} =$$

$$\left(\frac{(t_2t_1 - 1)t_1}{t_2 - 1}, \frac{t_2t_1^2}{t_2 - 1}\right)\bigg|_{q_\lambda(\bar{t})=0}.$$

Arguing as in Example 2, we compute the implicit equation, $g_d^\lambda(\bar{x})$*, of the projective curve defined by* $\mathcal{Q}_\lambda$ *(see Remark 2), which is*

$g_d^\lambda(\bar{x}) = 2x_2^5x_3 + x_2^2x_3^4 + 2x_2^4x_3^2 - 2x_2^3x_3^3 - 3\lambda x_1^2x_2x_3^3 - 2\lambda^2x_1x_2x_3^4 + 3\lambda x_1x_2^2x_3^3 - 2\lambda x_1^3x_2x_3^2 + 10x_2^3\lambda x_1^2x_3 - 10\lambda x_1^3x_2^2x_3 + 6x_2^2\lambda x_1^2x_3^2 - 5x_2^4\lambda x_1x_3 - 2\lambda x_1x_2x_3^4 - 6x_2^3\lambda x_1x_3^2 + 5\lambda x_1^4x_2x_3 + \lambda^2x_1^2x_3^4 + \lambda x_1^3x_3^3 - \lambda x_1^5x_3 - 8x_1^3x_2^2x_3 - 8x_2^4x_1x_3 + 12x_1^2x_2^3x_3 + 2x_1^4x_2x_3 + 2x_1^2x_2^2x_3^2 - 4x_2^3x_1x_3^2 + 4x_2^2x_1x_3^3 - 2x_1^2x_2x_3^3 + x_2^5\lambda x_3 + 2\lambda x_2^2x_3^4 - \lambda x_2^3x_3^3 + 2x_2^4\lambda x_3^2 + \lambda^2x_2^2x_3^4 + x_2^6 + x_1^4x_2^2 - 4x_2^5x_1 + 6x_2^4x_1^2 - 4x_1^3x_2^3.$

Evaluating g_d^λ *at* $\lambda = 0$*, we get*

$f_d(\bar{x}) = x_2(2x_1^2x_2x_3^2 - 8x_2^3x_1x_3 + 4x_1x_2x_3^3 - 4x_2^4x_1 + 2x_2^4x_3 + x_2x_3^4 + 2x_3^2x_2^3 + 6x_1^2x_2^3 + x_2^5 - 8x_2x_1^3x_3 - 4x_1^3x_2^2 + x_1^4x_2 - 2x_1^2x_3^3 - 4x_2^2x_1x_3^2 + 12x_2^2x_1^2x_3 - 2x_2^2x_3^3 + 2x_1^4x_3).$

Observe that $f_d(\bar{x}) = g_1(\bar{x})^{m_1}h(\bar{x})$*, where* $h(\bar{x}) = x_2$ *is the missed component.*

We have implemented this last method in the same machine of Section 3.2, and compared, as in Section 3.2, the performance of this method with the algorithms

$\mathcal{P}(\bar{t})$	Time 1	Time 2	Time 3
$\mathcal{P}_1(\bar{t})$	> 500	0.641	36.406
$\mathcal{P}_2(\bar{t})$	> 500	73.765	50.188
$\mathcal{P}_3(\bar{t})$	> 500	483.797	Fail
$\mathcal{P}_4(\bar{t})$	> 500	105.531	Fail
$\mathcal{P}_5(\bar{t})$	0.438	2.407	Fail
$\mathcal{P}_6(\bar{t})$	20.343	27.531	Fail
$\mathcal{P}_7(\bar{t})$	0.250	0.031	0.594
$\mathcal{P}_8(\bar{t})$	0.735	0.047	1.063
$\mathcal{P}_9(\bar{t})$	0.656	0.641	3.078

Table 2: Times of Implementations for the Generalizations

in [22] and [31]. In order to do this, we tested the three methods on nine rational surfaces, of degrees 4, 5 and 6, where the curve at infinity is not completely reached by the parametrization; the parametrizations were randomly generated, although respecting the condition of not reaching all the components at infinity. The timings are given in Table 2: *Time 1* is the timing for our method, *Time 2* corresponds to [22], and *Time 3* corresponds to [31]. In general, it is the method presented in [22] (*Times 2*) that provides the best results. In most cases the timing for our method is higher than the timing for the method in [22]. The algorithm in [31] provides an answer in five of the surfaces, and fails to give an answer in the rest of the surfaces. In some cases, this phenomenon happens because the algorithm needs to expand the determinant of a large matrix (of dimension greater than 25). In other cases, because this matrix is identically zero; this can happen when there are base points blowing up to a curve, which is the case discussed in this section.

Remark 7. *It is well known that the method based on Gröbner basis computations is not very good for computing implicit equation of parameterized surfaces. In fact, did not finish in an reasonable amount of time, in any of the considered the parametrizations used in this paper.*

5 Main proofs

This section is devoted to show the proofs of the main results of this paper, Theorem 2 and Corollary 1 in Subsection 3.1.

Theorem 2 Let $\mathcal{V}$ be a surface defined by a parametrization like Eq. (3.1), with

$p(\bar{t}) = q(\bar{t})^n$, where $q(\bar{t}) \in \mathbb{K}[\bar{t}]$ is an irreducible polynomial. Let

$$\mathcal{Q}(\bar{t}) = \left(\frac{p_{11}(\bar{t})}{p_{31}(\bar{t})}, \frac{p_{21}(\bar{t})}{p_{31}(\bar{t})}\right)\Bigg|_{q(\bar{t})=0}.$$

Then $f_d(\bar{x}) = g(\bar{x})^m h(\bar{x})$, where $\gcd(g,h) = 1$, the polynomial $h(\bar{x})$ defines the missed components at infinity, and

$$m = n\,\frac{deg(\phi_{\mathcal{Q}})}{deg(\phi_{\mathcal{P}})}. \tag{5.1}$$

Proof. First, we assume without loss of generality that $\deg(f) = \deg_{x_1}(f)$, which is equivalent to $(1:0:0:0) \notin \mathcal{V}^*$. We can always achieve this by applying a linear birational transformation $\mathcal{T}(\bar{x}) = (a_1x_1 + a_2x_2 + a_3x_3, b_1x_1 + b_2x_2 + b_3x_3, c_1x_1 + c_2x_2 + c_3x_3) \in \mathbb{K}[x_1, x_2, x_3]$ on the input parametrization $\mathcal{P}$, that is, we consider $\mathcal{T}(\mathcal{P})$ and obtain a new surface $\mathcal{V}_{\mathcal{T}}$ defined parametrically by $\mathcal{P}_{\mathcal{T}} := \mathcal{T}(\mathcal{P})$, and implicitly by $f^{(\mathcal{T})} = f(\mathcal{T}^{-1})$. Observe that for almost all linear transformations $\mathcal{T}$, we get $\deg(f^{(\mathcal{T})}) = \deg_{x_1}(f^{(\mathcal{T})})$. Furthermore, if the form of highest degree of $f(\bar{x})$ is $f_d(\bar{x}) = g(\bar{x})^m h(\bar{x})$, then the form of highest degree of $f^{(\mathcal{T})}(\bar{x})$ is $f_d^{(\mathcal{T})}(\bar{x}) = g_{\mathcal{T}}(\bar{x})^m h_{\mathcal{T}}(\bar{x})$, where $g = g_{\mathcal{T}}(\mathcal{T})$, $h = h_{\mathcal{T}}(\mathcal{T})$, $\gcd(g_{\mathcal{T}}, h_{\mathcal{T}}) = 1$. In particular, the multiplicity m of g and $g_{\mathcal{T}}$ is the same.

Under the above conditions, we consider the homogeneous polynomials

$$H_2(\bar{T}, x_2, x_4) = x_4 t_3^{k_2} p_{21}(\bar{T}) - x_2 t_3^k q^n(\bar{T}), \quad H_3(\bar{T}, x_3, x_4) = x_4 t_3^{k_3} p_{31}(\bar{T}) - x_3 t_3^k q^n(\bar{T}),$$

where $\bar{T} := (t_1, t_2, t_3)$, and $k, k_2, k_3 \geq 0$. Note that $\gcd(H_2, H_3) = 1$ since we have assumed that $\gcd(p_{31}, q) = 1$. Using some properties of resultants (see Lemma 4 in [8]),

$$\begin{aligned}\mathrm{Res}_{t_1}(x_3H_2 - x_2H_3, H_3) &= \mathrm{Res}_{t_1}(x_4(x_3 t_3^{k_2} p_{21}(\bar{T}) - x_2 t_3^{k_3} p_{31}(\bar{T})), H_3) = \\ &= x_4^{\deg_{t_1}(H_3)} \mathrm{Res}_{t_1}(x_3 t_3^{k_2} p_{21}(\bar{T}) - x_2 t_3^{k_3} p_{31}(\bar{T}), H_3)\end{aligned}$$

(note that the above resultant is not identically zero since $\gcd(H_2, H_3) = 1$). Additionally, and denoting the leading coefficient of H_3 with respect to t_1 by $\mathrm{lc}_{t_1}(H_3)$,

$$\mathrm{Res}_{t_1}(x_3H_2 - x_2H_3, H_3) = \mathrm{lc}_{t_1}(H_3)^r \mathrm{Res}_{t_1}(x_3H_2, H_3) = \mathrm{lc}_{t_1}(H_3)^r x_3^{\deg_{t_1}(H_3)} \mathrm{Res}_{t_1}(H_2, H_3),$$

where $r = \deg_{t_1}(x_3H_2 - x_2H_3) - \deg_{t_1}(x_3H_2)$. Let $\ell(t_2, t_3, x_2, x_3, x_4) := \mathrm{lc}_{t_1}(H_3)^r \in \mathbb{K}[t_2, x_2, x_3, x_4]$, and let us define

$$R(t_2, t_3, x_2, x_3, x_4) := \mathrm{Res}_{t_1}(x_3 p_{21}(\bar{t}) - x_2 p_{31}(\bar{t}), H_3) =$$

$$= \frac{x_3^{\deg_{t_1}(H_3)}}{x_4^{\deg_{t_1}(H_3)}} \cdot \ell(t_2, t_3, x_2, x_3, x_4) \cdot \mathrm{Res}_{t_1}(H_2, H_3).$$

Since $R(t_2, t_3, x_2, x_3, x_4)$ is a polynomial, $x_4^{\deg_{t_1}(H_3)}$ divides $\ell(t_2, t_3, x_2, x_3, x_4) \cdot \mathrm{Res}_{t_1}(H_2, H_3)$. Next we write

$$\ell(t_2, t_3, x_2, x_3, x_4) \cdot \mathrm{Res}_{t_1}(H_2, H_3) = x_4^{\deg_{t_1}(H_3)} M(t_2, t_3) U(x_2, x_3, x_4) S(t_2, t_3, x_2, x_3, x_4),$$

where $\mathrm{Content}_{\{t_2,t_3\}}(S) = \mathrm{Content}_{\{x_2,x_3,x_4\}}(S) = 1$. Therefore,

$$R(t_2, t_3, x_2, x_3, x_4) = M(t_2, t_3) x_3^{\deg_{t_1}(H_3)} U(x_2, x_3, x_4) S(t_2, t_3, x_2, x_3, x_4).$$

Now, by applying the properties of specialization of resultants (see Lemma 4.3.1 of [33]), we get

$$\begin{aligned} R(t_2, t_3, x_2, x_3, 0) &= M(t_2, t_3) x_3^{\deg_{t_1}(H_3)} U(x_2, x_3, 0) S(t_2, t_3, x_2, x_3, 0) = \\ &= \ell(t_2, t_3, x_2, x_3, 0) \cdot \mathrm{Res}_{t_1}(x_3 t_3^{k_2} p_{21}(\bar{t}) - x_2 t_3^{k_3} p_{31}(\bar{t}), H_3)\big|_{x_4=0} = \\ &= \ell(t_2, t_3, x_2, x_3, 0) \cdot \mathrm{Res}_{t_1}(x_3 t_3^{k_2} p_{21}(\bar{T}) - x_2 t_3^{k_3} p_{31}(\bar{T}), x_3 t_3^{k} q(\bar{T})^n) = \\ &= x_3^{\deg_{t_1}(H_3)} t_3^{k \cdot \deg_{t_1}(H_3)} \ell(t_2, t_3, x_2, x_3, 0) \cdot \mathrm{Res}_{t_1}(x_3 t_3^{k_2} p_{21}(\bar{T}) - x_2 t_3^{k_3} p_{31}(\bar{T}), q(\bar{T}))^n \end{aligned}$$

(in the last equality we have applied properties of resultants; see again Lemma 4 in [8]). Thus, from the above equality there exist $\overline{M}(t_2, t_3)$, $\overline{U}(x_2, x_3)$, $\overline{S}(t_2, t_3, x_2, x_3)$ such that

$$M(t_2, t_3) = t_3^{k \cdot \deg_{t_1}(H_3)} \overline{M}(t_2, t_3),$$
$$U(x_2, x_3, 0) S(t_2, t_3, x_2, x_3, 0) = \ell(t_2, t_3, x_2, x_3, 0) \overline{U}(x_2, x_3) \overline{S}(t_2, t_3, x_2, x_3).$$

Then,

$$\mathrm{Res}_{t_1}(x_3 t_3^{k_2} p_{21}(\bar{T}) - x_2 t_3^{k_3} p_{31}(\bar{T}), q(\bar{T})) = \overline{M}(t_2, t_3)^{1/n} \overline{U}(x_2, x_3)^{1/n} \overline{S}(t_2, t_3, x_2, x_3)^{1/n}.$$

Since $\mathrm{Res}_{t_1}(x_3 t_3^{k_2} p_{21}(\bar{t}) - x_2 t_3^{k_3} p_{31}(\bar{t}), q(\bar{T}))$ is a polynomial, the last equality implies that

$$\overline{S}(t_2, t_3, x_2, x_3) = N(t_2, t_3)^n W(t_2, t_3, x_2, x_3)^n, \tag{5.2}$$

where $\mathrm{Content}_{t_2}(W) = 1$. Therefore, we get

$$\mathrm{PrimPart}_{\{t_2,t_3,x_2,x_3\}}(\mathrm{Res}_{t_1}(x_3 t_3^{k_2} p_{21}(\bar{T}) - x_2 t_3^{k_3} p_{31}(\bar{T}), q(\bar{T}))) = W(t_2, t_3, x_2, x_3).$$

Since $\deg(f) = \deg_{x_1}(f)$ we deduce that $\deg(g) = \deg_{x_1}(g)$. Using results in Section 2 and in particular Eq. (2.8), we get

$$\deg(g) = \frac{\deg_{t_2}(W)}{\deg(\phi_{\mathcal{Q}})}. \tag{5.3}$$

Furthermore, since $f_d = g^m h$,

$$d = \deg(f_d) = \deg(g^m h) = m\deg(g) + \deg(h).$$

Thus, taking this into account, Eq. (5.2), and since the curve implicitly defined by g is reached via $\mathcal{P}$ (when $q(\bar{t})=0$), we deduce that

$$m\mathrm{deg}(g) = \frac{\mathrm{deg}_{t_2}(W^n)}{\mathrm{deg}(\phi_{\mathcal{P}})} = n\frac{\mathrm{deg}_{t_2}(W)}{\mathrm{deg}(\phi_{\mathcal{P}})}$$

(see Remark 1). Now, using Eq. (5.3) we conclude that

$$m\frac{\mathrm{deg}_{t_2}(W)}{\mathrm{deg}(\phi_{\mathcal{Q}})} = n\frac{\mathrm{deg}_{t_2}(W)}{\mathrm{deg}(\phi_{\mathcal{P}})}$$

and thus

$$m = n\,\frac{\mathrm{deg}(\phi_{\mathcal{Q}})}{\mathrm{deg}(\phi_{\mathcal{P}})}.$$

□

Corollary 1 Let $\mathcal{V}$ be a surface defined by a parametrization like Eq. (3.1), where $p(\,\bar{t}\,) = q_1(\,\bar{t}\,)^{n_1}\cdots q_\ell(\,\bar{t}\,)^{n_\ell}$, $\ell \geq 1$, and $q_i(\,\bar{t}\,) \in \mathbb{K}[\,\bar{t}\,]$, $i=1,\ldots,\ell$ are irreducible polynomials. For $i=1,\ldots,\ell$, let $\mathcal{Q}_i(\bar{t})$ be like Eq. (3.2). Then $f_d(\,\bar{x}\,) = g_1(\,\bar{x}\,)^{m_1}\cdots g_\ell(\,\bar{x}\,)^{m_\ell}\cdot h(\,\bar{x}\,)$, where $\mathrm{gcd}(h,g_i)=1$, the polynomial $h(\,\bar{x}\,)$ defines the missed components at infinity, and

$$m_i = n_i\frac{\mathrm{deg}(\phi_{\mathcal{Q}_i})}{\mathrm{deg}(\phi_{\mathcal{P}})}. \tag{5.4}$$

Proof. The proof is similar to the proof of Theorem 2, taking into account that now

$$\mathrm{PrimPart}_{\{t_2,t_3,x_2,x_3\}}(\mathrm{Res}_{t_1}(x_3t_3^{k_2}p_{21}(\bar{T})-x_2t_3^{k_2}p_{31}(\bar{T}),q_1(\bar{T})\cdots q_\ell(\bar{T}))) = W(t_2,t_3,x_2,x_3),$$

with

$$W(t_2,t_3,x_2,x_3) = W_1(t_2,t_3,x_2,x_3)\cdots W_\ell(t_2,t_3,x_2,x_3),$$

where for $i=1,\ldots,\ell$,

$$W_i(t_2,t_3,x_2,x_3) = \mathrm{PrimPart}_{\{t_2,t_3,x_2,x_3\}}(\mathrm{Res}_{t_1}(x_3t_3^{k_2}p_{21}(\bar{T})-x_2t_3^{k_3}p_{31}(\bar{T}),q_i(\bar{T})).$$

Thus

$$\mathrm{deg}(g_i) = \mathrm{deg}_{x_1}(g_i) = \frac{\mathrm{deg}_{t_2}(W_i)}{\mathrm{deg}(\phi_{\mathcal{Q}_i})},\ \ i=1,\ldots,\ell,$$

and additionally,

$$d = \mathrm{deg}(f_d) = \mathrm{deg}(g_1(\,\bar{x}\,)^{m_1}\cdots g_\ell(\,\bar{x}\,)^{m_\ell}\cdot h) = \sum_{i=1}^{\ell} m_i\mathrm{deg}(g_i) + \mathrm{deg}(h),$$

where $h(\,\bar{x}\,)=0$ implicitly defines the component of the curve at infinity of the given surface that is not reached by $\mathcal{P}$. Thus, taking this equality into account, Eq. (5.2), and since the curves implicitly defined by the g_i are reached via $\mathcal{P}$ (when $q_i(\bar{t}) = 0$), we argue as in Theorem 2 (see Remark 1), and we deduce

$$m_i \deg(g_i) = \frac{\deg_{t_2}(W_i^n)}{\deg(\phi_{\mathcal{P}})} = n\frac{\deg_{t_2}(W_i)}{\deg(\phi_{\mathcal{P}})}.$$

Since

$$\deg(g_i) = \frac{\deg_{t_2}(W_i)}{\deg(\phi_{\mathcal{Q}_i})},$$

we conclude that

$$m_i \frac{\deg_{t_2}(W_i)}{\deg(\phi_{\mathcal{Q}_i})} = n\frac{\deg_{t_2}(W_i)}{\deg(\phi_{\mathcal{P}})}$$

and thus

$$m_i = n\frac{\deg(\phi_{\mathcal{Q}_i})}{\deg(\phi_{\mathcal{P}})}.$$

□

6 Conclusion

A method has been presented to compute the form of highest degree of the implicit equation of a rational surface, defined by a rational parametrization. The method is suitable both for proper and non-proper parametrizations, and works efficiently whenever the given parametrization completely covers the curve at infinity. This phenomenon is related to the existence of base points blowing up to a curve at infinity. In absence of this bad case, our experiments show that our algorithm works generally better than the algorithms for computing the whole implicit equation in [22] and [31], with which we have compared our results. We provide a potential generalization of our method for the bad case. However, in the bad case the algorithm that seems to be more effective is the algorithm in [22]; in fact, in the bad case the algorithm in [31] fails, in some cases, to give an answer.

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