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Characterizing the finiteness of the Hausdorff distance between two algebraic curves[☆]

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ABSTRACT

In this paper, we present a characterization for the Hausdorff distance between two given algebraic curves in the n-dimensional space (parametrically or implicitly defined) to be finite. The characterization is related with the asymptotic behavior of the two curves and it can be easily checked. More precisely, the Hausdorff distance between two curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite if and only if for each infinity branch of $\mathcal{C}$ there exists an infinity branch of $\overline{\mathcal{C}}$ such that the terms with positive exponent in the corresponding series are the same, and reciprocally.

1. Introduction

The Hausdorff distance is one of the most used measures in geometric pattern matching algorithms, computer aided design or computer graphics (see e.g. [1–4]).

Given a metric space (E, d) and two arbitrary subsets $A, B \subset E$, the Hausdorff distance assigns to each point of one set the distance to its closest point on the other and takes the maximum over all these values (see [5]). More precisely, the *Hausdorff distance* between A and B is defined as:

$$d_H(A, B) = \max\{\sup_{x\in A}\inf_{y\in B} d(x, y), \sup_{y\in B}\inf_{x\in A} d(x, y)\}.$$

In this paper, we deal with the particular case where $E = \mathbb{C}^n$ or $E = \mathbb{R}^n$, and d is the usual unitary or Euclidean distance (see Chapter 5 in [6]). In addition, the two arbitrary subsets are two real algebraic curves $\mathcal{C}$ and $\overline{\mathcal{C}}$. In this case, the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ is given by

$$d_H(\mathcal{C}, \overline{\mathcal{C}}) = \max\{\sup_{p\in\mathcal{C}} d(p, \overline{\mathcal{C}}), \sup_{\overline{p}\in\overline{\mathcal{C}}} d(\overline{p}, \mathcal{C})\}, \quad \text{where } d(p, \mathcal{C}) = \min\{d(p, q) : q \in \mathcal{C}\}.$$

In general, $d_H(A, B)$ may be infinite, and some restrictions have to be imposed to guarantee its finiteness (see e.g. [7] or [8]).

As far as the authors know, there is no efficient algorithms for the exact computation of the Hausdorff distance between algebraic varieties (in fact, if both varieties are given in implicit form, the computation of the Hausdorff distance is even harder). Only some results for bounding or estimating the Hausdorff distance as well as computing it for some special cases

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can be found. For instance, in [9], the Hausdorff distance between planar free-form curves using a polyline approach is estimated. More precisely, the input curves are approximated with polylines and the precise Hausdorff distance between polylines is computed. It is shown that the approximation error can be totally controlled. In [10], a method for computing the Hausdorff distance between two B-spline curves is developed. An estimation of the upper bound of the Hausdorff distance in an sub-interval is given, which is used to eliminate the sub-intervals whose upper bounds are smaller than the given lower bound. The conditions whether the Hausdorff distance occurs at an end point of the two curves are also provided, and these conditions are used to turn the Hausdorff distance computation problem between two curves into a minimum or maximum distance computation problem between a point and a curve, which can be solved as well. [11] defines and discusses the Hausdorff metric on the space of nonempty, closed, and bounded subsets of a given metric space. Two important topological properties are considered, completeness and boundedness. It is proved that each of these properties is possessed by a Hausdorff metric space if the property is possessed by the underlying metric space. The paper [12] is devoted to computational techniques for generating upper bounds on the Hausdorff distance between two planar curves (implicitly or parametrically defined). The bounds are computed directly from the control points (spline coefficients of the curves). Potential applications include error bounds for the approximate implicitization of spline curves, for the approximate parametrization of (piecewise) algebraic curves, and for algebraic curve fitting. This approach assumes that the two curves are fairly close to each other. In [2], a real-time algorithm for computing the precise Hausdorff distance between two planar freeform curves is presented. The algorithm is based on an effective technique that approximates each curve with a sequence of G^1 biarcs within an arbitrary error bound. In [8], authors consider real space algebraic curves, not necessarily bounded, whose Hausdorff distance is finite, and bounds of their distance are provided. These bounds are related to the distance between the projections of the space curves onto a plane. Thus, authors provide a theoretical result that reduces the estimation and bounding of the Hausdorff distance of algebraic curves from the spatial to the planar case.

In this paper, we deal with a different aspect concerning the Hausdorff distance. We do not deal with the computation or estimation of the Hausdorff distance as in the papers we mentioned above, but with a characterization on whether the Hausdorff distance between two given algebraic curves (parametrically or implicitly defined) in the n-dimensional space and that are not necessarily bounded, is finite. The characterization is based on the concept of *similar asymptotic behavior* introduced in this paper, and it improves Proposition 5.4 presented in [11] (see also [13]). The notion of similar asymptotic behavior has to deal with the convergence/divergence of the infinity branches of two given curves $\mathcal{C}$ and $\overline{\mathcal{C}}$. More precisely, we say that $\mathcal{C}$ and $\overline{\mathcal{C}}$ have a *similar asymptotic behavior* if there are no infinity branches in $\mathcal{C}$ which diverge from all the infinity branches in $\overline{\mathcal{C}}$, and reciprocally. In fact, we show that the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite if and only if both curves have *a similar asymptotic behavior*. This condition is very easy to formulate from the computational point of view and thus, we present an effective algorithm that checks if it holds.

Although the curves used in computer aided geometric design (CAGD) are usually bounded and there is no need to decide whether or not the Hausdorff distance is bounded, the characterization presented in this paper plays an important role in some other applications to CAGD as for instance in the approximate parametrization problem (see e.g. [7,14–17]). In that problem, given an affine curve $\mathcal{C}$ (say that it is a perturbation of a rational curve), the goal is to compute a rational proper parametrization of a rational affine curve $\overline{\mathcal{C}}$ near $\mathcal{C}$ (one may state the problem also for surfaces; see [18]). As one can check in the papers mentioned, the effectiveness of the algorithm depends on the closeness of $\mathcal{C}$ and $\overline{\mathcal{C}}$ and, at least, the finiteness of the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ has to be guaranteed (which is equivalent to ensure a similar behavior of both curves at infinity). The potential applications of the results presented in this paper also include the approximate implicitization problem for curves and surfaces (see [19,20]).

Moreover, since this characterization is based on the notion of infinity branch which reflects the status of a curve at the points with sufficiently large coordinates, one may think in applying the results presented to the analysis of the behavior at infinity of an algebraic curve, which implies a wide applicability in many active research fields. For instance, the following problems could be considered: sketch the graph of a given algebraic curve as well as to study its topology (see e.g. [21–24]), compute the shapes in a family of space curves (see [25]), determine the symmetries of a given curve (see [26]), etc.

The structure of the paper is as follows: in Section 2, we present the terminology that will be used throughout the paper as well as some previous results. In this section, we assume that the given curve is implicitly defined but one can easily check that the results obtained are independent on the representation of the curve. Only computational aspects change, and thus, in Section 3, we present the necessary computational techniques to deal with curves parametrically defined. Section 4 is devoted to present the main theorem where the finiteness of the Hausdorff distance is characterized. For this purpose, some technical lemmas are proved. In Section 5, we derive an algorithm that decides whether the Hausdorff distance between two given algebraic curves is finite and we show how this algorithm can be adapted to be applied only to the real parts of the given curves. We illustrate the method with some examples in detail. Moreover, some practical examples are also shown where one can check the applicability of our results to problems in CAGD. We finish with a section of conclusions (Section 6) where we summarize the results obtained, we emphasize the new contributions of this paper, and we propose topics for further study.

2. Previous results for implicit space curves

In this section, we present some previous definitions and results concerning curves in the n-dimensional space. We assume that the curves are defined by a finite set of real polynomials but we will see that all the results and concepts introduced do not depend on the representation of the given curve. Only the practical computation differs from the implicit to the parametric case. This moves us to deal with parametric curves in Section 3 where we show the computational techniques one has to apply to this case.

The results obtained in this section will provide very important tools that allow to analyze whether the Hausdorff distance between two given algebraic curves is finite (see Sections 4 and 5).

We start with some briefs notions and terminology that will be used throughout the paper. In particular, we need some previous results concerning local parametrizations and Puiseux series. For further details see [27–29], Section 2.5 in [30], and Chapter 4 (Section 2) in [31].

We denote by $\mathbb{C}[[t]]$ the domain of *formal power series* in the indeterminate t with coefficients in the field $\mathbb{C}$, i.e. the set of all sums of the form $\sum_{i=0}^{\infty} a_i t^i$, $a_i \in \mathbb{C}$. The quotient field of $\mathbb{C}[[t]]$ is called the field of *formal Laurent series*, and it is denoted by $\mathbb{C}((t))$. It is well known that every non-zero formal Laurent series $A \in \mathbb{C}((t))$ can be written in the form $A(t) = t^k \cdot (a_0 + a_1 t + a_2 t^2 + \cdots)$, where $a_0 \neq 0$ and $k \in \mathbb{Z}$. In addition, the field $\mathbb{C} \ll t \gg := \bigcup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$ is called the field of *formal Puiseux series*. Note that Puiseux series are power series of the form

$$\varphi(t) = m + a_1 t^{N_1/N} + a_2 t^{N_2/N} + a_3 t^{N_3/N} + \cdots \in \mathbb{C} \ll t \gg, \quad a_i \neq 0, \quad \forall i \in \mathbb{N},$$

where $N, N_i \in \mathbb{N}$, $i \geq 1$, $0 < N_1 < N_2 < \cdots$. The natural number N is known as *the ramification index* of the series. We denote it as $\nu(\varphi)$ (see [29]).

The most important property of Puiseux series is given by Puiseux's Theorem, which states that if $\mathbb{K}$ is an algebraically closed field, then the field $\mathbb{K} \ll x \gg$ is algebraically closed (see Theorems 2.77 and 2.78 in [30]). A proof of Puiseux's Theorem can be given constructively by the Newton Polygon Method (see e.g. Section 2.5 in [30]).

Under these conditions, we consider $\mathcal{C} \subset \mathbb{C}^n$ a curve in the n-dimensional space defined by a finite set of real polynomials $f_1(\overline{x}), \ldots, f_s(\overline{x}) \in \mathbb{R}[\overline{x}]$, $s \geq n-1$, where $\overline{x} = (x_1, \ldots, x_n)$.

The assumption of reality of the curve $\mathcal{C}$ is included because of the nature of the problem, but the theory developed in this paper can be applied for the case of complex non-real curves.

Let $\mathcal{C}^*$ be the corresponding projective curve defined by the homogeneous polynomials $F_i(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}[x_1, \ldots, x_n, x_{n+1}]$, $i = 1, \ldots, s$. Furthermore, let $P = (1 : m_2 : \ldots : m_n : 0)$, $m_j \in \mathbb{C}$, $j = 2, \ldots, n$, be an infinity point of $\mathcal{C}^*$.

In addition, we consider the curve implicitly defined by the polynomials $g_i(x_2, \ldots, x_n, x_{n+1}) := F_i(1, x_2, \ldots, x_n, x_{n+1}) \in \mathbb{R}[x_2, \ldots, x_n, x_{n+1}]$ for $i = 1, \ldots, s$. Observe that $g_i(p) = 0$, where $p = (m_2, \ldots, m_n, 0)$. Let $I \in \mathbb{R}(x_{n+1})[x_2, \ldots, x_n]$ be the ideal generated by $g_i(x_2, \ldots, x_n, x_{n+1})$, $i = 1, \ldots, s$, in the ring $\mathbb{R}(x_{n+1})[x_2, \ldots, x_n]$. We assume that $\mathcal{C}$ is not contained in some hyperplane $x_{n+1} = c$, $c \in \mathbb{C}$ (otherwise, one can consider $\mathcal{C}$ as a curve in the $(n-1)$-dimensional space), and thus we have that x_{n+1} is not algebraic over $\mathbb{R}$. Under this assumption, the ideal I (i.e. the system of equations $g_1 = \cdots = g_s = 0$) has only finitely many solutions in the n-dimensional affine space over the algebraic closure of $\mathbb{R}(x_{n+1})$ (which is contained in $\mathbb{C} \ll x_{n+1} \gg$). Then, there are finitely many $(n-1)$-tuples $(\varphi_2(t), \ldots, \varphi_n(t))$ where $\varphi_j(t) \in \mathbb{C} \ll t \gg$, $j \in \{2, \ldots, n\}$, such that $g_i(\varphi_2(t), \ldots, \varphi_n(t), t) = 0$, $i = 1, \ldots, s$, and $\varphi_j(0) = m_j$, $j = 2, \ldots, n$. Each of these $(n-1)$-tuples is a solution of the system associated with the infinity point $(1 : m_2 : \ldots : m_n : 0)$, and each $\varphi_j(t)$ converges in a neighborhood of $t = 0$. Moreover, since $\varphi_j(0) = m_j$, $j = 2, \ldots, n$, these series do not have terms with negative exponents; in fact, they have the form $\varphi_j(t) = m_j + \sum_{i \geq 1} a_{i,j} t^{N_{i,j}/N_j}$ where $N_j, N_{i,j} \in \mathbb{N}$, $0 < N_{1,j} < N_{2,j} < \cdots$.

It is important to remark that if $\varphi(t) := (\varphi_2(t), \ldots, \varphi_n(t))$ is a solution of the system, then $\sigma_\epsilon(\varphi)(t) := (\sigma_\epsilon(\varphi_2)(t), \ldots, \sigma_\epsilon(\varphi_n)(t))$ is another solution of the system, where

$$\sigma_\epsilon(\varphi_j)(t) = m_j + \sum_{i \geq 1} a_{i,j} \epsilon^{\lambda_{i,j}} t^{N_{i,j}/N_j}, \quad N_j, N_{i,j} \in \mathbb{N}, \ 0 < N_{1,j} < N_{2,j} < \cdots,$$

$N := \mathrm{lcm}(N_2, \ldots, N_n)$, $\lambda_{i,j} := N_{i,j} N / N_j \in \mathbb{N}$, and $\epsilon^N = 1$ (see [27]). We refer to these solutions as the *conjugates* of φ. The set of all (distinct) conjugates of φ is called the *conjugacy class* of φ, and the number of different conjugates is N.

Under these conditions and reasoning as in [28, Section 3], we get that there exists $M \in \mathbb{R}^+$ such that for $i = 1, \ldots, s$, it holds that $F_i(1 : \varphi_2(t) : \ldots : \varphi_n(t) : t) = g_i(\varphi_2(t), \ldots, \varphi_n(t), t) = 0$ for $t \in \mathbb{C}$ and $|t| < M$. This implies that

$$F_i(t^{-1} : t^{-1}\varphi_2(t) : \ldots : t^{-1}\varphi_n(t) : 1) = f_i(t^{-1}, t^{-1}\varphi_2(t), \ldots, t^{-1}\varphi_n(t)) = 0,$$

for $t \in \mathbb{C}$ and $0 < |t| < M$. Now, we set $t^{-1} = z$, and we obtain that for $i = 1, \ldots, s$,

$$f_i(z, r_2(z), \ldots, r_n(z)) = 0, \quad z \in \mathbb{C} \text{ and } |z| > M^{-1}, \text{ where}$$

$$r_j(z) = z\varphi_j(z^{-1}) = m_j z + a_{1,j} z^{1-N_{1,j}/N_j} + a_{2,j} z^{1-N_{2,j}/N_j} + a_{3,j} z^{1-N_{3,j}/N_j} + \cdots, \tag{1}$$

$a_{i,j} \neq 0$, $N_j, N_{i,j} \in \mathbb{N}$, $i = 1, \ldots,$ and $0 < N_{1,j} < N_{2,j} < \cdots$.

Since $\nu(\varphi) = N$, we get that there are N different series in its conjugacy class. Let $\varphi_{\alpha,j}$, $\alpha = 1, \ldots, N$ be these series, and

$$r_{\alpha,j}(z) = z\varphi_{\alpha,j}(z^{-1}) = m_j z + a_{1,j} c_\alpha^{\lambda_{1,j}} z^{1-N_{1,j}/N_j} + a_{2,j} c_\alpha^{\lambda_{2,j}} z^{1-N_{2,j}/N_j} + a_{3,j} c_\alpha^{\lambda_{3,j}} z^{1-N_{3,j}/N_j} + \cdots \tag{2}$$

where $N = \mathrm{lcm}(N_2, \ldots, N_n)$, $\lambda_{i,j} = N_{i,j} N / N_j \in \mathbb{N}$, and $c_1, \ldots, c_N$ are the N complex roots of $x^N = 1$.

Now we are ready to define *infinity branch*. This concept was introduced in [28, Section 3] for algebraic plane curves (see also [13]), and in [32, Section 2] for algebraic space curves.

Definition 2.1. An *infinity branch of a n-dimensional space curve* $\mathcal{C}$ associated to the infinity point $P=(1:m_2:\dots:m_n:0)$, $m_j\in\mathbb{C}$, $j=2,\dots,n$, is a set $B=\bigcup_{\alpha=1}^{N}L_\alpha$, where $L_\alpha=\{(z,r_{\alpha,2}(z),\dots,r_{\alpha,n}(z))\in\mathbb{C}^n : z\in\mathbb{C},\ |z|>M\}$, $M\in\mathbb{R}^+$, and the series $r_{\alpha,j}$, $j=2,\dots,n$, are given by (2). The subsets $L_1,\dots,L_N$ are called the *leaves* of the infinity branch B.

Remark 2.2. An infinity branch is uniquely determined from one leaf, up to conjugation. More precisely, let B be an infinity branch and let $L=\{(z,r_2(z),\dots,r_n(z))\in\mathbb{C}^n : z\in\mathbb{C},\ |z|>M\}$ be one of its leaves, where r_j is of the form given in Eq. (1). Then, any other leaf L_α has the form $L_\alpha=\{(z,r_{\alpha,2}(z),\dots,r_{\alpha,n}(z))\in\mathbb{C}^n : z\in\mathbb{C},\ |z|>M\}$, where $r_{\alpha,j}=r_j$, $j=2,\dots,N$, up to conjugation. That is, $r_{\alpha,j}$ is of the form given in Eq. (2).

Remark 2.3. Observe that the above approach is presented for infinity points of the form $(1:m_2:\dots:m_n:0)$. For the infinity points $(0:m_2:\dots:m_n:0)$, with $m_j\neq 0$ for some $j=2,\dots,n$, we reason similarly but we dehomogenize w.r.t x_j. More precisely, let us assume that $m_2\neq 0$. Then, we consider the curve defined by the polynomials $g_i(x_1,x_3,\dots,x_{n+1}) := F_i(x_1,1,x_3,\dots,x_{n+1})\in\mathbb{R}[x_1,x_3,\dots,x_{n+1}]$, $i=1,\dots,s$, and we reason as above. We get that an infinity branch of the space curve $\mathcal{C}$ associated to $(0:m_2:\dots:m_n:0)$, $m_2\neq 0$, is a set $B=\bigcup_{\alpha=1}^{N}L_\alpha$, where $L_\alpha=\{(r_{\alpha,1}(z),z,r_{\alpha,3}(z),\dots,r_{\alpha,n}(z))\in\mathbb{C}^n : z\in\mathbb{C},\ |z|>M\}$, $M\in\mathbb{R}^+$.

Additionally, instead of working with this type of branches, if $\mathcal{C}$ has infinity points of the form $(0:m_2:\dots:m_n:0)$, one may consider a linear change of coordinates. Thus, in the following, we assume w.l.o.g that $\mathcal{C}$ only has infinity points of the form $(1:m_2:\dots:m_n:0)$. More details on these type of branches are given in [28, see Definition 3.3 in Section 3] and [32, see Remark 2.3 in Section 2].

In the following, we introduce the notions of convergent and divergent leaves. Intuitively speaking, two leaves converge (diverge) if they get closer (get away) as they tend to infinity. The notion of convergence was introduced in [28, Section 4] for algebraic plane curves, and in [32, Section 2] for algebraic space curves.

Definition 2.4. Let $L=\{(z,r_2(z),\dots,r_n(z))\in\mathbb{C}^n : z\in\mathbb{C},\ |z|>M\}$ and $\overline{L}=\{(z,\overline{r}_2(z),\dots,\overline{r}_n(z))\in\mathbb{C}^n : z\in\mathbb{C},\ |z|>\overline{M}\}$ be two leaves that belong to two infinity branches B and $\overline{B}$, respectively (see Definition 2.1 and Remark 2.2). We say that

1. L and $\overline{L}$ converge if $\lim_{z\to\infty} d((r_2(z),\dots,r_n(z)),(\overline{r}_2(z),\dots,\overline{r}_n(z)))=0$.
2. L and $\overline{L}$ diverge if $\lim_{z\to\infty} d((r_2(z),\dots,r_n(z)),(\overline{r}_2(z),\dots,\overline{r}_n(z)))=\infty$.

Remark 2.5. Taking into account that the distance d is derived from the Euclidean inner product and that all the norms are equivalent in $\mathbb{C}^{n-1}$ (see Chapter 5 in [6]), we get that:

1. $\lim_{z\to\infty} d((r_2(z),\dots,r_n(z)),(\overline{r}_2(z),\dots,\overline{r}_n(z)))=0$ if and only if $\lim_{z\to\infty}(\overline{r}_j(z)-r_j(z))=0$ for every $j=2,\dots,n$.
2. $\lim_{z\to\infty} d((r_2(z),\dots,r_n(z)),(\overline{r}_2(z),\dots,\overline{r}_n(z)))=\infty$ if and only if there exists some $j=2,\dots,n$ such that $\lim_{z\to\infty}(\overline{r}_j(z)-r_j(z))=\infty$.

Remark 2.6. Observe that it may happen that

$$\lim_{z\to\infty} d((r_2(z),\dots,r_n(z)),(\overline{r}_2(z),\dots,\overline{r}_n(z)))=c\in\mathbb{R}^+\setminus\{0\}$$

which is equivalent to $\lim_{z\to\infty}(\overline{r}_j(z)-r_j(z))=c_j\in\mathbb{C}$ for every $j=2,\dots,n$ and $c_j\neq 0$ for some $j=2,\dots,n$. In this case, L and $\overline{L}$ do not converge neither diverge (compare with Definition 2.4).

The following lemma provides a method to determine whether two leaves converge or diverge without the need of computing limits.

Lemma 2.7. *Let $L=\{(z,r_2(z),\dots,r_n(z))\in\mathbb{C}^n : z\in\mathbb{C},\ |z|>M\}$ and $\overline{L}=\{(z,\overline{r}_2(z),\dots,\overline{r}_n(z))\in\mathbb{C}^n : z\in\mathbb{C},\ |z|>\overline{M}\}$ be two leaves that belong to two infinity branches B and $\overline{B}$, respectively. It holds that:*

1. *L and $\overline{L}$ converge if and only if the terms with non-negative exponent in the series $r_j(z)$ and $\overline{r}_j(z)$ are the same, for every $j=2,\dots,n$.*
2. *L and $\overline{L}$ diverge if and only if the terms with positive exponent in the series $r_j(z)$ and $\overline{r}_j(z)$ are not the same, for some $j=2,\dots,n$.*

Proof. Let

$$r_j(z)=m_jz+a_{1,j}z^{1-N_{1,j}/N_j}+a_{2,j}z^{1-N_{2,j}/N_j}+a_{3,j}z^{1-N_{3,j}/N_j}+\cdots,$$

$a_{i,j}\neq 0,\ \forall i\in\mathbb{N},\ i\geq 1$, $N_j, N_{i,j}\in\mathbb{N}$, and $0<N_{1,j}<N_{2,j}<\cdots$ for $j=2,\dots,n$ and

$$\overline{r}_j(z)=\overline{m}_jz+\overline{a}_{1,j}z^{1-\overline{N}_{1,j}/\overline{N}_j}+\overline{a}_{2,j}z^{1-\overline{N}_{2,j}/\overline{N}_j}+\overline{a}_{3,j}z^{1-\overline{N}_{3,j}/\overline{N}_j}+\cdots,$$

$\overline{a}_{i,j}\neq 0,\ \forall i\in\mathbb{N},\ i\geq 1$, $\overline{N}_j, \overline{N}_{i,j}\in\mathbb{N}$, and $0<\overline{N}_{1,j}<\overline{N}_{2,j}<\cdots$ for $j=2,\dots,n$. Then,

$$r_j(z)-\overline{r}_j(z)=m_jz-\overline{m}_jz+a_{1,j}z^{\frac{N-N_1}{N}}-\overline{a}_{1,j}z^{\frac{\overline{N}-\overline{N}_1}{\overline{N}}}+a_{2,j}z^{\frac{N-N_2}{N}}-\overline{a}_{2,j}z^{\frac{\overline{N}-\overline{N}_2}{\overline{N}}}+\cdots.$$

Under these conditions, it holds that:

1. $\lim_{z\to\infty}(r_j(z)-\overline{r}_j(z))=0$ for every $j=2,\dots,n$, if and only if all the exponents in the series $r_j(z)-\overline{r}_j(z)$ are negative. This situation holds if the terms with non-negative exponent in the series $r_j(z)$ and $\overline{r}_j(z)$ are the same for every $j=2,\dots,n$.
2. $\lim_{z\to\infty}(r_j(z)-\overline{r}_j(z))=\infty$ for some $j=2,\dots,n$, if and only if $r_j(z)-\overline{r}_j(z)$ has some term with positive exponent. This situation holds if the terms with positive exponent in the series, $r_j(z)$ and $\overline{r}_j(z)$, are not the same for some $j=2,\dots,n$. $\square$

Remark 2.8. If the terms with positive exponent in the series $r_j(z)$ and $\overline{r}_j(z)$ are the same for every $j=2,\dots,n$, but the independent terms (the terms with exponent zero) are different for some $j=2,\dots,n$, we have that L and $\overline{L}$ do not diverge neither converge.

In the following, we introduce the notions of convergent and divergent branches. These concepts are obtained from Definition 2.4, and they are an indispensable tool for comparing the asymptotic behavior of two curves.

Definition 2.9. Let $B=\bigcup_{\alpha=1}^{N}L_\alpha$ and $\overline{B}=\bigcup_{\beta=1}^{\overline{N}}\overline{L}_\beta$ be two infinity branches of two algebraic curves $\mathcal{C}$ and $\overline{\mathcal{C}}$, respectively.

1. B and $\overline{B}$ converge if there are two convergent leaves $L_\alpha\subseteq B$, $\alpha=1,\dots,N$ and $\overline{L}_\beta\subseteq\overline{B}$, $\beta=1,\dots,\overline{N}$.
2. B and $\overline{B}$ diverge if any two leaves $L_\alpha\subseteq B$, $\alpha=1,\dots,N$ and $\overline{L}_\beta\subseteq\overline{B}$, $\beta=1,\dots,\overline{N}$ diverge.

From Definition 2.9 we get that two infinity branches B and $\overline{B}$ do not diverge if there are two leaves, $L\subseteq B$ and $\overline{L}\subseteq\overline{B}$, that do not diverge. Furthermore, the next lemma states that, in this case, every leaf of B is non-divergent with some leaf of $\overline{B}$, and reciprocally.

Lemma 2.10. *Let $B=\bigcup_{\alpha=1}^{N}L_\alpha$ and $\overline{B}=\bigcup_{\beta=1}^{\overline{N}}\overline{L}_\beta$ be two non-divergent infinity branches. Then, for each leaf $L_\alpha\subseteq B$ there exists a leaf $\overline{L}_\beta\subseteq\overline{B}$ that does not diverge with L_α, and reciprocally.*

Proof. Let B and $\overline{B}$ be two non-divergent branches. Let us prove that for any leaf $L_\alpha\subseteq B$ there exist one or more leaves $\overline{L}_\beta\subseteq\overline{B}$ non-divergent with L_α, and reciprocally. From the discussion above, we know that there exist two leaves $\{(z,r_2(z),\dots,r_n(z))\in\mathbb{C}^n : z\in\mathbb{C},\ |z|>M\}\subset B$ and $\{(z,\overline{r}_2(z),\dots,\overline{r}_n(z))\in\mathbb{C}^n : z\in\mathbb{C},\ |z|>\overline{M}\}\subset\overline{B}$ that do not diverge. Let

$$r_j(z)=z\varphi_j(z^{-1})=m_jz+u_{1,j}z^{1-\frac{N_{1,j}}{N}}+\cdots+u_{k,j}z^{1-\frac{N_{k,j}}{N}}+u_{k+1,j}z^{1-\frac{N_{k+1,j}}{N}}+\cdots,$$
$$\overline{r}_j(z)=z\overline{\varphi}_j(z^{-1})=\overline{m}_jz+\overline{u}_{1,j}z^{1-\frac{\overline{N}_{1,j}}{\overline{N}}}+\cdots+\overline{u}_{k,j}z^{1-\frac{\overline{N}_{k,j}}{\overline{N}}}+\overline{u}_{k+1,j}z^{1-\frac{\overline{N}_{k+1,j}}{\overline{N}}}+\cdots,$$

where $\overline{u}_{i,j}u_{i,j}\neq 0$, $N=\nu(B)=\operatorname{lcm}(N_2,\dots,N_n)$, $\overline{N}=\nu(\overline{B})=\operatorname{lcm}(\overline{N}_2,\dots,\overline{N}_n)$, $N_{k,j}<N\le N_{k+1,j}$ and $\overline{N}_{k,j}<\overline{N}\le\overline{N}_{k+1,j}$ for some $k\in\mathbb{N}$ (note that k may depend on j). Note also that the expression above differs slightly from that of (1), since we are using N and $\overline{N}$ as the common denominators for the exponents of the series r_j and $\overline{r}_j$ respectively.

From Lemma 2.7, we deduce that the terms with positive exponent in r_j and $\overline{r}_j$ are the same. Thus, $\overline{m}_j=m_j$, $\overline{u}_{i,j}=u_{i,j}$, for $i=1,\dots,k$, $j=2,\dots,n$, and

$$r_j(z)=m_jz+u_{1,j}z^{1-\frac{n_{1,j}}{n}}+\cdots+u_{k,j}z^{1-\frac{n_{k,j}}{n}}+u_{k+1,j}z^{1-\frac{N_{k+1,j}}{N}}+\cdots,$$
$$\overline{r}_j(z)=m_jz+u_{1,j}z^{1-\frac{n_{1,j}}{n}}+\cdots+u_{k,j}z^{1-\frac{n_{k,j}}{n}}+\overline{u}_{k+1,j}z^{1-\frac{\overline{N}_{k+1,j}}{\overline{N}}}+\cdots,$$

where $\overline{u}_{i,j},u_{i,j}\neq 0$, $n,n_{i,j}\in\mathbb{N}$ and $0<n_{1,j}<\cdots<n_{k,j}<n$. Observe that we have simplified the non negative exponents such that $\gcd(n,n_{1,j},\dots,n_{k,j})=1$,for $j=2,\dots,n$. Hence, there are $b,\overline{b}\in\mathbb{N}$ such that $N_{i,j}=bn_{i,j}$, $N=bn$, $\overline{N}_{i,j}=\overline{b}n_{i,j}$, and $\overline{N}=\overline{b}n$ for $i=1,\dots,k$ and $j=2,\dots,n$.

Under these conditions, we observe that the different leaves of B and $\overline{B}$ are obtained by conjugation on $r_j(z)$ and $\overline{r}_j(z)$, $j=2,\dots,n$. That is, any two leaves $L_\alpha\subseteq B$, $\alpha=1,\dots,N$ and $\overline{L}_\beta\subseteq\overline{B}$, $\beta=1,\dots,\overline{N}$ will have the form $L_\alpha=\{(z,r_{\alpha,2}(z),\dots,r_{\alpha,n}(z))\in\mathbb{C}^n : z\in\mathbb{C},\ |z|>M\}$ and $\overline{L}_\beta=\{(z,\overline{r}_{\beta,2}(z),\dots,\overline{r}_{\beta,n}(z))\in\mathbb{C}^n : z\in\mathbb{C},\ |z|>\overline{M}\}$, where

$$r_{\alpha,j}(z)=m_jz+u_{1,j}c_\alpha^{N_{1,j}}z^{1-\frac{N_{1,j}}{N}}+\cdots+u_{k,j}c_\alpha^{N_{k,j}}z^{1-\frac{N_{k,j}}{N}}+u_{k+1,j}c_\alpha^{N_{k+1,j}}z^{1-\frac{N_{k+1,j}}{N}}+\cdots,$$

and

$$\overline{r}_{\beta,j}(z)=\overline{m}_jz+\overline{u}_{1,j}d_\beta^{\overline{N}_{1,j}}z^{1-\frac{\overline{N}_{1,j}}{\overline{N}}}+\cdots+\overline{u}_{k,j}d_\beta^{\overline{N}_{k,j}}z^{1-\frac{\overline{N}_{k,j}}{\overline{N}}}+\overline{u}_{k+1,j}d_\beta^{\overline{N}_{k+1,j}}z^{1-\frac{\overline{N}_{k+1,j}}{\overline{N}}}+\cdots,$$

$c_1,\dots,c_N$ are the N complex roots of $x^N=1$, and $d_1,\dots,d_{\overline{N}}$ are the $\overline{N}$ complex roots of $x^{\overline{N}}=1$ (see Eq. (2)).

We simplify the exponents and, using that $\overline{u}_{i,j} = u_{i,j},\ i = 1, \ldots, k$, we get that:

$$r_{\alpha,j}(z) = m_j z + u_{1,j} c_\alpha^{N_{1,j}} z^{1-\frac{n_{1,j}}{n}} + \cdots + u_{k,j} c_\alpha^{N_{k,j}} z^{1-\frac{n_{k,j}}{n}} + u_{k+1,j} c_\alpha^{N_{k+1,j}} z^{1-\frac{N_{k+1,j}}{N}} + \cdots$$

$$\overline{r}_{\beta,j}(z) = m_j z + u_{1,j} d_\beta^{\overline{N}_{1,j}} z^{1-\frac{n_{1,j}}{n}} + \cdots + u_{k,j} d_\beta^{\overline{N}_{k,j}} z^{1-\frac{n_{k,j}}{n}} + \overline{u}_{k+1,j} d_\beta^{\overline{N}_{k+1,j}} z^{1-\frac{\overline{N}_{k+1,j}}{\overline{N}}} + \cdots.$$

Now, we prove that for any leaf L_α there exist one or more leaves $\overline{L}_\beta$ non-divergent with L_α. For this purpose, we just need to show that, given any value of $\alpha = 1, \ldots, N$, there exist one or more values of $\beta = 1, \ldots, \overline{N}$ such that $c_\alpha^{N_{i,j}} = d_\beta^{\overline{N}_{i,j}}, i = 1, \ldots, k, j = 2, \ldots, n$.

Indeed, since the coefficients $c_\alpha,\ \alpha = 1, \ldots, N$ are the N complex roots of $x^N = 1$, we have that $c_\alpha = e^{\frac{2(\alpha-1)\pi I}{N}}$, where I is the imaginary unit. Taking into account that $N = bn$, we deduce that $c_\alpha^b = e^{\frac{2(\alpha-1)\pi I}{n}}$ for each $\alpha = 1, \ldots, N$ and $c_\alpha^b = c_{\alpha+(m-1)n}^b$ for each $\alpha = 1, \ldots, n$ and $m = 1, \ldots, b$. That is, $c_\alpha^b,\ \alpha = 1, \ldots, n$ are the n complex roots of $x^n = 1$. Reasoning similarly, we have that $d_\beta^{\overline{b}} = e^{\frac{2(\beta-1)\pi I}{n}}$ for each $\beta = 1, \ldots, \overline{N}$ and $d_\beta^{\overline{b}} = d_{\beta+(m-1)n}^{\overline{b}}$ for each $\beta = 1, \ldots, n$ and $m = 1, \ldots, \overline{b}$. That is, $d_\beta^{\overline{b}},\ \beta = 1, \ldots, n$ are the n complex roots of $x^n = 1$. Hence, for each $\alpha = 1, \ldots, N$ there are one or more $\beta = 1, \ldots, \overline{N}$ such that $c_\alpha^b = d_\beta^{\overline{b}}$, and reciprocally. Finally, the result follows taking into account that $c_\alpha^{N_{i,j}} = \left(c_\alpha^b\right)^{n_{i,j}} = \left(d_\beta^{\overline{b}}\right)^{n_{i,j}} = d_\beta^{\overline{N}_{i,j}}$. □

Remark 2.11. Let B and $\overline{B}$ be two infinity branches associated with two infinity points $P = (1 : m_2 : \cdots : m_n)$ and $\overline{P} = (1 : \overline{m}_2 : \cdots : \overline{m}_n)$, respectively. From the proof of Lemma 2.10, if B and $\overline{B}$ do not diverge, then $m_j = \overline{m}_j$ for every $j = 2, \ldots, n$ which implies that two non-divergent infinity branches are associated with the same infinity point (see Remark 4.5 in [28]).

For the sake of simplicity, and taking into account that an infinity branch B is uniquely determined from one leaf, up to conjugation (see Remark 2.2), we identify an infinity branch by just one of its leaves. Hence, in the following

$$B = \{(z, r_2(z), \ldots, r_n(z)) \in \mathbb{C}^n : z \in \mathbb{C},\ |z| > M\}, \quad M \in \mathbb{R}^+$$

will stand for the infinity branch whose leaves are obtained by conjugation on

$$r_j(z) = m_j z + a_{1,j} z^{1-N_{1,j}/N_j} + a_{2,j} z^{1-N_{2,j}/N_j} + a_{3,j} z^{1-N_{3,j}/N_j} + \cdots,$$

$a_{i,j} \neq 0,\ \forall i \in \mathbb{N},\ i \geq 1, N_j, N_{i,j} \in \mathbb{N}$, and $0 < N_{1,j} < N_{2,j} < \cdots$ for $j = 2, \ldots, n$. Observe that the results stated above hold for any leaf of B.

Finally, we remark that there exists well known algorithms that allow to compute the series $\varphi_j(t) \in \mathbb{C} \ll t \gg, j = 2, \ldots, n$, and then the branch $B = \{(z, r_2(z), \ldots, r_n(z)) \in \mathbb{C}^n : z \in \mathbb{C},\ |z| > M\}$ (see e.g. [27]). In addition, in [32], a procedure for computing the branches for $n = 3$ is presented. This method is based on projections over the plane, and it can be generalized for a given curve in the n-dimensional space by successively eliminating variables and reducing the problem to the computation of infinity branches for plane curves (a method for successively eliminating the variables, by means of univariate resultants, is presented in [33]). For the plane case ($n = 2$) methods are well known (see e.g. Section 3 in [28]).

In the following example, we compute the infinity branches for a given algebraic curve in the 4-dimensional space implicitly defined by the polynomials $f_i(x_1, x_2, x_3, x_4) \in \mathbb{R}[x_1, x_2, x_3, x_4],\ i = 1, 2, 3$.

Example 2.12. Let $\mathcal{C}$ be the irreducible curve defined over $\mathbb{C}$ by

$$f_1(x_1, x_2, x_3, x_4) = x_1 - x_2^2 + 2x_3, \qquad f_2(x_1, x_2, x_3, x_4) = x_1 + x_2 - x_4^2, \quad \text{and} \quad f_3(x_1, x_2, x_3, x_4) = 2x_2 - x_3^2 + x_4.$$

The projection along the x_4-axis, $\mathcal{C}^p$, is defined by the polynomials

$$f_1^p(x_1, x_2, x_3) = x_1 - x_2^2 + 2x_3, \quad \text{and} \quad f_2^p(x_1, x_2, x_3) = x_1 + x_2 - 4x_2^2 + 4x_2x_3^2 - x_3^4$$

(these polynomials can be obtained by computing univariate resultants).

By applying the method described in Section 3 in [32] (we use the algcurves package included in the computer algebra system Maple), we compute the infinity branches of $\mathcal{C}^p$. We obtain the branch $B_1^p = \{(z, r_{1,2}(z), r_{1,3}(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M_1^p\}$, where

$$r_{1,2}(z) = z^{1/2} + \sqrt{3}z^{-1/4} + \frac{\sqrt{3}z^{-3/4}}{12} - \frac{z^{-1}}{2} - \frac{7\sqrt{3}z^{-5/4}}{288} + \cdots$$

$$r_{1,3}(z) = \sqrt{3}z^{1/4} + \frac{\sqrt{3}z^{-1/4}}{12} + z^{-1/2} - \frac{7\sqrt{3}z^{-3/4}}{288} + \frac{z^{-1}}{4} + \cdots,$$

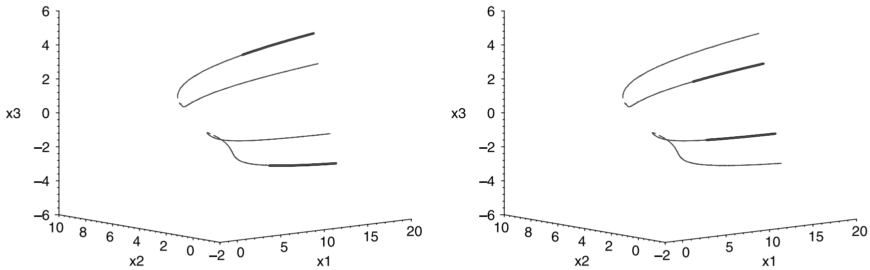

Fig. 1. Curve $\mathcal{C}^p$ and infinity branches B_1^p (left) and B_2^p (right).

and the branch $B_2^p = \{(z, r_{2,2}(z), r_{2,3}(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M_2^p\}$, where

$$r_{2,2}(z) = z^{1/2} + z^{-1/4} - \frac{z^{-3/4}}{4} + \frac{z^{-1}}{2} - \frac{z^{-5/4}}{32} + \cdots,$$
$$r_{2,3}(z) = z^{1/4} - \frac{z^{-1/4}}{4} + z^{-1/2} + \frac{z^{-3/4}}{32} - \frac{z^{-1}}{4} + \cdots.$$

Note that both branches are associated to the infinity point $P_1 = (1 : 0 : 0 : 0)$. Moreover, $\nu(B_1^p) = \nu(B_2^p) = 4$, and thus each branch has 4 (conjugated) leaves. That is, $B_1^p = \bigcup_{\alpha=1}^4 L_{1,\alpha}$, where $L_{1,\alpha}$ are obtained by conjugation in the above series $r_{1,2}$ and $r_{1,3}$ (similarly for B_2^p).

Once we have the infinity branches of the projected curve $\mathcal{C}^p$, we compute the infinity branches of the curve $\mathcal{C}$. We use the lift function $h(x_1, x_2, x_3) = -2x_2 + x_3^2$ to get the fourth component of these branches (we apply the results in [34] to compute h). Thus, the infinity branches of $\mathcal{C}$ are $B_1 = \{(z, r_{1,2}(z), r_{1,3}(z), r_{1,4}(z)) \in \mathbb{C}^4 : z \in \mathbb{C},\ |z| > M_1\}$, where

$$r_{1,4}(z) = h(z, r_{1,2}(z), r_{1,3}(z)) = z^{1/2} + \frac{1}{2} - \frac{z^{-1/2}}{8} + \frac{\sqrt{3}z^{-3/4}}{2} + \cdots$$

and $B_2 = \{(z, r_{2,2}(z), r_{2,3}(z), r_{2,4}(z)) \in \mathbb{C}^4 : z \in \mathbb{C},\ |z| > M_2\}$, where

$$r_{2,4}(z) = h(z, r_{2,2}(z), r_{2,3}(z)) = -z^{1/2} - \frac{1}{2} + \frac{z^{-1/2}}{8} - \frac{z^{-3/4}}{2} + \cdots.$$

In Fig. 1, we plot the curve $\mathcal{C}^p$ and some points of the infinity branches B_1^p and B_2^p.

3. Parametric space curves: computation of infinity branches

In Section 2, we have assumed that the given real algebraic curve in the n-dimensional space is implicitly defined. In this section, we deal with algebraic curves defined by a rational real parametrization.

Note that the definitions introduced in Section 2, and the obtained results, are independent on whether the curve is parametrically or implicitly defined. However, the method to compute the infinity branches has to be different (of course, one may implicitize and reason as in Section 2, but we are interested in computing the infinity branches from the given parametrization without implicitizing).

Thus, in this section, we present a method to compute infinity branches of a rational curve in the n-dimensional space from their parametric representation (without implicitizing). Similarly as above, we work over $\mathbb{C}$, but we assume that the curve has infinitely many points in the affine plane over $\mathbb{R}$ and then, the curve has a real parametrization. The method presented generalize the results in [32, see Section 5].

The computation of infinity branches of the given curve will be an essential tool for checking whether the Hausdorff distance between two algebraic curves is finite or not (see Sections 4 and 5).

Therefore, in the following, we consider a real space curve $\mathcal{C}$ in the n-dimensional space $\mathbb{C}^n$, defined by the parametrization

$$\mathcal{P}(s) = (p_1(s), \ldots, p_n(s)) \in \mathbb{R}(s)^n \setminus \mathbb{R}^n, \qquad p_i(s) = p_{i1}(s)/p(s), \quad i = 1, \ldots, n.$$

We assume that we have prepared the input curve $\mathcal{C}$, by means of a suitable linear change of coordinates (if necessary) such that $(0 : m_2 : \ldots : m_n : 0)$ ($m_j \neq 0$ for some $j = 2, \ldots, n$) is not an infinity point (see Remark 2.3). Note that, hence, $\deg(p_1) \geq 1$.

Now, let $\mathcal{C}^*$ denote the projective curve associated to $\mathcal{C}$. We have that a parametrization of $\mathcal{C}^*$ is given by $\mathcal{P}^*(s) = (p_{11}(s) : \cdots : p_{n1}(s) : p(s))$ or, equivalently,

$$\mathcal{P}^*(s) = \left(1 : \frac{p_{21}(s)}{p_{11}(s)} : \cdots : \frac{p_{n1}(s)}{p_{11}(s)} : \frac{p(s)}{p_{11}(s)}\right).$$

Under these conditions, we show how to compute the infinity branches of $\mathcal{C}$. That is, the sets $B = \{(z : r_2(z) : \dots : r_n(z)) : z \in \mathbb{C},\ |z| > M\}$, where $r_j(z) = z\varphi_j(z^{-1}) \in \mathbb{C} \ll z \gg$, $j = 2, \dots, n$. We recall that these series must verify $F_i(1 : \varphi_2(t) : \dots : \varphi_n(t) : t) = 0$ around $t = 0$, where $F_i,\ i = 1, \dots, s$ are the polynomials defining implicitly $\mathcal{C}^*$ (see Section 2). Observe that in this section, we are given the parametrization $\mathcal{P}^*$ of $\mathcal{C}^*$ and then, $F_i(\mathcal{P}^*(s)) = F_i\left(1 : \frac{p_{21}(s)}{p_{11}(s)} : \dots : \frac{p_{n1}(s)}{p_{11}(s)} : \frac{p(s)}{p_{11}(s)}\right) = 0$. Thus, intuitively speaking, in order to compute the infinity branches of $\mathcal{C}$, and in particular the series φ_j, $j = 2, \dots, n$, one needs to "reparametrize" the parametrization $\mathcal{P}^*(s) = \left(1 : \frac{p_{21}(s)}{p_{11}(s)} : \dots : \frac{p_{n1}(s)}{p_{11}(s)} : \frac{p(s)}{p_{11}(s)}\right)$ in the form $(1 : \varphi_2(t) : \dots : \varphi_n(t) : t)$ around $t = 0$. For this purpose, the idea is to look for a value of the parameter s, say $\ell(t) \in \mathbb{C} \ll t \gg$, such that $\mathcal{P}^*(\ell(t)) = (1 : \varphi_2(t) : \dots : \varphi_n(t) : t)$ around $t = 0$.

Hence, from the above reasoning, we deduce that first, we have to consider the equation $p(s)/p_{11}(s) = t$ (or equivalently, $p(s) - tp_{11}(s) = 0$), and we have to solve it in the variable s around $t = 0$ (note that $\deg(p_1) \geq 1$). From Puiseux's Theorem, there exist solutions $\ell_1(t), \ell_2(t), \dots, \ell_k(t) \in \mathbb{C} \ll t \gg$, where $k = \deg(p_1)$, such that, $p(\ell_i(t)) - tp_{11}(\ell_i(t)) = 0$, $i = 1, \dots, k$, in a neighborhood of $t = 0$. For this purpose, we may use for instance the command puiseux included in the package algcurves of the computer algebra system Maple.

Thus, for each $i = 1, \dots, k$, there exists $M_i \in \mathbb{R}^+$ such that the points $(1 : \varphi_{i,2}(t) : \dots : \varphi_{i,n}(t) : t)$ or equivalently, the points $(t^{-1} : t^{-1}\varphi_{i,2}(t) : \dots : t^{-1}\varphi_{i,n}(t) : 1)$, where

$$\varphi_{i,j}(t) = \frac{p_{j,1}(\ell_i(t))}{p_{11}(\ell_i(t))}, \quad j = 2, \dots, n, \tag{3}$$

are in $\mathcal{C}^*$ for $|t| < M_i$. Observe that $\varphi_{i,j}(t)$, $j = 2, \dots, n$, are Puiseux series, since $p_{j,1}(\ell_i(t))$, $j = 2, \dots, n$, and $p_{11}(\ell_i(t))$ can be written as Puiseux series (around $t = 0$) and $\mathbb{C} \ll t \gg$ is a field.

Finally, we set $z = t^{-1}$. Then, we have that the points $(z, r_{i,2}(z), \dots, r_{i,n}(z))$, where $r_{i,j}(z) = z\varphi_{i,j}(z^{-1})$, $j = 2, \dots, n$, are in $\mathcal{C}$ for $|z| > M_i^{-1}$. Hence, the infinity branches of $\mathcal{C}$ are the sets

$$B_i = \{(z, r_{i,2}(z), \dots, r_{i,n}(z)) \in \mathbb{C}^n : z \in \mathbb{C},\ |z| > M_i^{-1}\}, \quad i = 1, \dots, k.$$

Remark 3.1. 1. The series $\ell_i(t)$ satisfies that $p(\ell_i(t))/p_{11}(\ell_i(t)) = t$, for $i = 1, \dots, k$. Then, from equality (3), we have that for $j = 2, \dots, n$

$$\varphi_{i,j}(t) = \frac{p_{j,1}(\ell_i(t))}{p(\ell_i(t))} t = p_j(\ell_i(t))t, \quad \text{and} \quad r_{i,j}(z) = z\varphi_{i,j}(z^{-1}) = p_j(\ell_i(z^{-1})).$$

2. In order to compute $r_{i,j}(z)$, we first write $p_j(\ell_i(t))$ as Puiseux series around $t = 0$, and then we set $t = z^{-1}$. For this purpose, we may use for instance the command series included in the computer algebra system Maple.
3. When we compute the series ℓ_i, we cannot handle its infinite terms so it must be truncated, which may distort the computation of the series $r_{i,j}$. The number of affected terms in $r_{i,j}$ depends on the number of terms computed in ℓ_i. That is, as more terms we compute in ℓ_i, as more accurate the computation of $r_{i,j}$ is. More details on this question are analyzed in Proposition 5.4 in [32].

In the following example, we illustrate the above procedure and we compute the infinity branches for a given curve defined by a parametrization $\mathcal{P}(s) \in \mathbb{R}(s)^4$.

Example 3.2. Let $\mathcal{C}$ be the curve defined by the parametrization

$$\begin{aligned}\mathcal{P}(s) &= (p_1(s), p_2(s), p_3(s), p_4(s)) = \left(\frac{p_{11}(s)}{p(s)}, \frac{p_{21}(s)}{p(s)}, \frac{p_{31}(s)}{p(s)}, \frac{p_{41}(s)}{p(s)}\right) \\ &= \left(\frac{-1 + 2s^3 - s}{s}, \frac{s+1}{s}, \frac{-1}{s}, \frac{s^2 + 3s - 5}{s}\right) \in \mathbb{R}(s)^4.\end{aligned}$$

We compute the solutions of the equation $p(s) - tp_{11}(s) = 0$ in the variable s around $t = 0$. For this purpose, we use the algcurves package included in the computer algebra system Maple; in particular, the command puiseux is used. We get the Puiseux series

$$\begin{aligned}\ell_1(t) &= -t + t^2 - t^3 - t^4 + 7t^5 + \cdots, \\ \ell_2(t) &= \frac{1}{2}\sqrt{2}t^{-1/2} + \frac{1}{4}\sqrt{2}t^{1/2} + \frac{1}{2}t - \frac{1}{16}\sqrt{2}t^{3/2} - \frac{1}{2}t^2 - \frac{11}{32}\sqrt{2}t^{5/2} + \frac{1}{2}t^3 + \frac{235}{256}\sqrt{2}t^{7/2} + \cdots\end{aligned}$$

(note that $\ell_2(t)$ represents a conjugation class composed by two conjugated series).

Now, we determine the series $r_{i,j}(z)$, $i = 1, 2$, $j = 2, 3, 4$. We get

$$\begin{aligned}r_{1,2}(z) &= p_2(\ell_1(z^{-1})) = -z + 2z^{-2} - 4z^{-3} - 13z^{-4} - 11z^{-5} + \cdots \\ r_{1,3}(z) &= p_3(\ell_1(z^{-1})) = z + 1 - 2z^{-2} + 4z^{-3} + 13z^{-4} + 11z^{-5} + \cdots \\ r_{1,4}(z) &= p_4(\ell_1(z^{-1})) = 5z + 8 - z^{-1} - 9z^{-2} + 19z^{-3} + 64z^{-4} + 62z^{-5} + \cdots,\end{aligned}$$

and

$$r_{2,2}(z) = p_2(\ell_2(z^{-1})) = 1 + \sqrt{2}z^{-1/2} - \frac{1}{2}\sqrt{2}z^{-3/2} - z^{-2} + \frac{3}{8}\sqrt{2}z^{-5/2} + 2z^{-3} + \cdots$$
$$r_{2,3}(z) = p_3(\ell_2(z^{-1})) = -\sqrt{2}z^{-1/2} + \frac{1}{2}\sqrt{2}z^{-3/2} + z^{-2} - \frac{3}{8}\sqrt{2}z^{-5/2} - 2z^{-3} + \cdots$$
$$r_{2,4}(z) = p_4(\ell_2(z^{-1})) = \frac{1}{2}\sqrt{2}z^{1/2} + 3 - \frac{19}{4}\sqrt{2}z^{-1/2} + \frac{1}{2}z^{-1} + \frac{39}{16}\sqrt{2}z^{-3/2} + \frac{9}{2}z^{-2} - \frac{71}{32}\sqrt{2}z^{-5/2} - \frac{19}{2}z^{-3} + \cdots$$

Therefore, the curve has two infinity branches given by

$$B_1 = \{(z, r_{1,2}(z), r_{1,3}(z), r_{1,4}(z)) \in \mathbb{C}^4 : z \in \mathbb{C},\ |z| > M_1\}, \quad \text{and}$$
$$B_2 = \{(z, r_{2,2}(z), r_{2,3}(z), r_{2,4}(z)) \in \mathbb{C}^4 : z \in \mathbb{C},\ |z| > M_2\}$$

for some $M_1, M_2 \in \mathbb{R}^+$. Note that B_1 is associated to the infinity point $(1 : -1 : 1 : 5 : 0)$, and B_2 is associated to the infinity point $(1 : 0 : 0 : 0 : 0)$. In addition, we observe that $\nu(B_1) = 1$ and $\nu(B_2) = 2$, and thus B_1 has one leaf, and B_2 has two (conjugated) leaves.

4. Asymptotic behavior and Hausdorff distance

In this section, we consider two algebraic curves, $\mathcal{C}$ and $\overline{\mathcal{C}}$, in the n-dimensional space defined by a finite set of real polynomials or by a rational real parametrization, and we prove the main theorem where the finiteness of the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ is characterized. For this purpose, we need to compute the infinity branches of $\mathcal{C}$ and $\overline{\mathcal{C}}$. Depending on whether they are parametrically or implicitly defined, one proceeds as in Section 2 or as in Section 3, respectively.

We remind that $\mathcal{C}$ and $\overline{\mathcal{C}}$ are prepared such that $(0 : m_2 : \ldots : m_n : 0)$ ($m_j \neq 0$ for some $j = 2, \ldots, n$) is not an infinity point of their corresponding projective curves (see Remark 2.3).

The main result of the section (Theorem 4.5) states that the Hausdorff distance between two algebraic curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite if and only if their asymptotic behaviors are similar (we say that two algebraic curves have similar asymptotic behaviors if their infinity branches are pair-wise non-divergent; see Definition 4.1). From this result an effective and fast algorithm is derived, and it can be stated from two different points of view: for characterizing the finiteness of the Hausdorff distance between the real parts of $\mathcal{C}$ and $\overline{\mathcal{C}}$, or for characterizing the finiteness of the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ considering also the complex part of both curves (see Section 5).

The computation of the Hausdorff distance and in particular, the characterization of the finiteness of the Hausdorff distance will play an important role in the frame of practical applications in CAGD such as approximate parametrization problems where the curves are in general not bounded (see Section 1 for more details). In particular, the characterization presented in this section is specially interesting since it is an appropriate tool for measuring the closeness between two given curves (we will see that the finiteness of the Hausdorff distance will imply a similar asymptotic behavior at infinity). Many authors have addressed some problems in this frame (see e.g. [2,3,7–10], etc.) but all of them assume that the given curves are bounded or that the Hausdorff distance between them is finite. In this paper, we go one step further, and we characterize whether the Hausdorff distance between two given algebraic curves is finite without assuming any additional condition (as boundedness of the given curves). In Section 5, we illustrate the applications of the method derived in this section to particular and practical problems.

To start with, we first introduce the following definition.

Definition 4.1. We say that two algebraic curves, $\mathcal{C}$ and $\overline{\mathcal{C}}$, have a similar asymptotic behavior if, for every infinity branch $B \subset \mathcal{C}$ there exist an infinity branch $\overline{B} \subset \overline{\mathcal{C}}$ non-divergent with B, and reciprocally.

Now, we introduce the notion of *Hausdorff distance*. For this purpose, we recall that, given an algebraic space curve $\mathcal{C}$ over $\mathbb{C}$ and a point $p \in \mathbb{C}^n$, *the distance from p to* $\mathcal{C}$ is defined as $d(p, \mathcal{C}) = \min\{d(p, q) : q \in \mathcal{C}\}$.

Definition 4.2. Given a metric space (E, d) and two subsets $A, B \subset E \setminus \{\emptyset\}$, the *Hausdorff distance* between them is defined as:

$$d_H(A, B) = \max\{\sup_{x\in A} \inf_{y\in B} d(x, y), \sup_{y\in B} \inf_{x\in A} d(x, y)\}$$

(see [5]). If $E = \mathbb{C}^n$ (or $E = \mathbb{R}^n$), and d is the unitary (or euclidean) distance (see Chapter 5 in [6]), the Hausdorff distance between two curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ can be expressed as:

$$d_H(\mathcal{C}, \overline{\mathcal{C}}) = \max\{\sup_{p\in\mathcal{C}} d(p, \overline{\mathcal{C}}), \sup_{\overline{p}\in\overline{\mathcal{C}}} d(\overline{p}, \mathcal{C})\}.$$

In order to prove the main theorem (see Theorem 4.5), we first need to prove some technical lemmas. The first one (Lemma 4.3) states that any point of the curve with sufficiently large coordinates belongs to some infinity branch (see also Lemma 3.6 and Remark 3.7 in [28]).

Lemma 4.3. *Let $\mathcal{C}$ be an algebraic space curve. There exists $K \in \mathbb{R}^+$ such that every $p = (a_1, \ldots, a_n) \in \mathcal{C}$ with $|a_i| > K$ (for some $i \in \{1, \ldots, n\}$) belongs to some infinity branch of $\mathcal{C}$.*

Proof. First, let us prove that there exists $K^1 \in \mathbb{R}^+$ such that every point $p = (a_1, \ldots, a_n) \in \mathcal{C}$ with $|a_1| > K^1$ belongs to some infinity branch.

Let us assume that this is not true and let us consider a sequence $\{K_\kappa\}_{\kappa\in\mathbb{N}} \in \mathbb{R}^+$ such that $\lim_{\kappa\to\infty} K_\kappa = \infty$. Then, for every $\kappa \in \mathbb{N}$ there exists a point $p_\kappa = (a_{1,\kappa}, \ldots, a_{n,\kappa}) \in \mathcal{C}$ such that $|a_{1,\kappa}| > K_\kappa$, and p_κ does not belong to any infinity branch of $\mathcal{C}$. The corresponding projective point is $P_\kappa = (a_{1,\kappa} : \ldots : a_{n,\kappa} : 1)$, and it holds that $F(P_\kappa) = f(p_\kappa) = 0$. Thus, we have a sequence $\{P_\kappa\}_{\kappa\in\mathbb{N}}$ of points in the projective curve $\mathcal{C}^*$ such that $\lim_{\kappa\to\infty} |a_{1,\kappa}| = \infty$. Note that these projective points can be expressed as

$$P_\kappa = (1 : a_{2,\kappa}/a_{1,\kappa} : \ldots : a_{n,\kappa}/a_{1,\kappa} : 1/a_{1,\kappa}).$$

Under these conditions, we extract a subsequence $\{P_{\kappa_l}\}_{l\in\mathbb{N}}$ for the sequences $\{a_{i,\kappa_l}/a_{1,\kappa_l}\}_{l\in\mathbb{N}}$, $i = 2, \ldots, n$ to be monotone. In order to simplify the notation, we also denote it as $\{P_\kappa\}_{\kappa\in\mathbb{N}}$. Now, we distinguish two cases:

1. Let us assume that all these monotone sequences are bounded. Then, $\lim_{\kappa\to\infty} a_{i,\kappa}/a_{1,\kappa} = m_i \in \mathbb{C}$, $i = 2, \ldots, n$ and $\lim_{\kappa\to\infty} 1/a_{1,\kappa} = 0$. Furthermore, since $F(P_\kappa) = 0$ for every $\kappa \in \mathbb{N}$, we get that $\lim_{\kappa\to\infty} F(P_\kappa) = F(\lim_{\kappa\to\infty} P_\kappa) = F(1 : m_2 : \cdots : m_n : 0) = 0$. We conclude that the sequence $\{P_\kappa\}_{\kappa\in\mathbb{N}}$ converges to the infinity point $P = (1 : m_2 : \cdots : m_n : 0)$ as κ tends to infinity; that is, there exists $M \in \mathbb{R}^+$ such that $\|P_\kappa - P\| \le \epsilon$, for $\kappa \ge M$. Thus, we deduce that the points $\{P_\kappa\}_{\kappa\in\mathbb{N},\,\kappa\ge M}$ can be obtained by a place centered at P. Hence, the points $\{p_\kappa\}_{\kappa\in\mathbb{N},\,\kappa\ge M}$ belong to some infinity branch of $\mathcal{C}$, which contradicts the hypothesis.
2. If not all the sequences are bounded, then there is some $i = 2, \ldots, n$ such that $\lim_{l\to\infty} a_{i,\kappa}/a_{1,\kappa} = \pm\infty$. We assume without lost of generality that $\lim_{l\to\infty} a_{2,\kappa}/a_{1,\kappa} = \pm\infty$. Then, we write

   $$P_\kappa = (a_{1,\kappa}/a_{2,\kappa} : 1 : a_{3,\kappa}/a_{2,\kappa} : \ldots : a_{n,\kappa}/a_{2,\kappa} : 1/a_{2,\kappa}),$$

   and we extract a subsequence $\{P_{\kappa_l}\}_{l\in\mathbb{N}}$ for the sequences $\{a_{i,\kappa_l}/a_{2,\kappa_l}\}_{l\in\mathbb{N}}$, $i = 3, \ldots, n$ to be monotone. For the sake of simplicity, we denote it by $\{P_\kappa\}_{\kappa\in\mathbb{N}}$.

   At this point, we consider two different situations:

   - If all these monotone sequences are bounded, we get that
     $$\lim_{\kappa\to\infty} a_{i,\kappa}/a_{1,\kappa} = m_i \in \mathbb{C}, \quad i = 3, \ldots, n.$$
     Furthermore, $\lim_{\kappa\to\infty} a_{1,\kappa}/a_{2,\kappa} = \lim_{\kappa\to\infty} 1/a_{2,\kappa} = 0$ and thus, reasoning as above, we deduce that the sequence $\{P_\kappa\}_{\kappa\in\mathbb{N}}$ converges to an infinity point $P = (0 : 1 : m_3 : \cdots : m_n : 0)$.
   - If some of the sequences $\{a_{i,\kappa_l}/a_{2,\kappa_l}\}_{l\in\mathbb{N}}$, $i = 3, \ldots, n$ are not bounded, we can assume w.l.o.g. that $\lim_{l\to\infty} a_{3,\kappa}/a_{2,\kappa} = \pm\infty$ and we reason as above. Finally, we obtain a subsequence that converges to an infinity point of the form $(0 : m_2 : m_3 : \cdots : m_n : 0)$.

   In both cases, we find a contradiction, since we have prepared the input curve such that it does not have infinity points of the form $(0 : m_2 : m_3 : \cdots : m_n : 0)$.

From the above discussion, the initial assumption leads us to a contradiction. Therefore, there exists $K^1 \in \mathbb{R}^+$ such that every point of the curve $p = (a_1, \ldots, a_n)$ with $|a_1| > K^1$ belongs to some infinity branch. Reasoning similarly, we deduce that for each $i = 2, \ldots, n$, there exists $K^i \in \mathbb{R}^+$ such that every point of the curve $p = (a_1, \ldots, a_n)$ with $|a_i| > K^i$ belongs to some infinity branch. Finally, the result follows by taking $K = \min\{K^1, \ldots, K^n\}$. □

The following technical lemma states that, given two divergent branches B and $\overline{B}$, we can find points in B as far as we want from any point in $\overline{B}$ (and reciprocally).

Lemma 4.4. *Let $B = \{(z, r_2(z), \ldots, r_n(z)) \in \mathbb{C}^n : z \in \mathbb{C},\ |z| > M\}$ and $\overline{B} = \{(z, \overline{r}_2(z), \ldots, \overline{r}_n(z)) \in \mathbb{C}^n : z \in \mathbb{C},\ |z| > \overline{M}\}$ be two divergent infinity branches. For each $K > 0$, there exists $\delta > 0$ such that if $|x| > \delta$ then $d((x, r_2(x), \ldots, r_n(x)), (y, \overline{r}_2(y), \ldots, \overline{r}_n(y))) > K$ for any point $(y, \overline{r}_2(y), \ldots, \overline{r}_n(y)) \in \overline{B}$.*

Proof. We assume w.l.o.g. that B is associated to the infinity point $(1 : 0 : \ldots : 0)$ (otherwise we can apply a linear change of coordinates). Note that since all the norms in $\mathbb{C}^n$ are equivalent, there exists some $\lambda > 0$ such that

$$d((x, r_2(x), \ldots, r_n(x)), (y, \overline{r}_2(y), \ldots, \overline{r}_n(y))) > \lambda(|x - y| + |r_2(x) - \overline{r}_2(y)| + \cdots + |r_n(x) - \overline{r}_n(y)|).$$

Thus, we only need to prove that, for each $K > 0$ there exists $\delta > 0$ such that if $|x| > \delta$ then

$$\phi(x, y) := |x - y| + |r_2(x) - \overline{r}_2(y)| + \cdots + |r_n(x) - \overline{r}_n(y)| > K.$$

First of all, if $|x - y| > K$ the result follows, so we assume that $|x - y| \le K$. Hence, $|y| > |x| - K$ since $|x - y| > |x| - |y|$.

On the other hand, note that

$$|r_i(x) - \overline{r}_i(y)| = |\overline{r}_i(y) - r_i(x)| > |\overline{r}_i(y) - r_i(y) + r_i(y) - r_i(x)|$$
$$> |\overline{r}_i(y) - r_i(y)| - |r_i(y) - r_i(x)|, \quad i = 2, \dots, n. \tag{4}$$

From the proof of Theorem 4.11 in [28], we get that $r_i(z)$ is derivable for $|z| > M$ and $\mathrm{limit}_{z\to\infty} r_i'(z) = m_i$, where $(1 : m_2 : \dots : m_n : 0)$ is the infinity point associated to B. In this case $m_i = 0$, so there is $\delta_0 > 0$ such that for $|z| > \delta_0$, it holds that $|r_i'(z)| < 1/\sqrt{2}$. Hence, applying the Mean Value Theorem (see [35]), we have that if $|x|, |y| > \delta_0$, then

$$|r_i(x) - r_i(y)|^2 = (\mathrm{Re}({r_i}'(c_1))^2 + \mathrm{Im}({r_i}'(c_2))^2)|x - y|^2, \quad i = 2, \dots, n$$

where $\mathrm{Re}(q)$ and $\mathrm{Im}(q)$ denote the real part and the imaginary part of $q(z) \in \mathbb{C} \ll z \gg$, respectively, and $c_1, c_2 \in]x, y[$, where $]x, y[:= \{z \in \mathbb{C} : z = x + (x - y)t,\ t \in (0, 1)\}$. Since $|r_i'(z)| < 1/\sqrt{2}$ for $|z| > \delta_0$, we get that $|r_i(y) - r_i(x)| < |x - y|$, for $i = 2, \dots, n$. In addition, since $|y| > |x| - K$, we deduce that $|r_i(y) - r_i(x)| < |x - y|$ for $|x| > \delta_0 + K$, and $i = 2, \dots, n$.

Now, substituting in (4), we get that

$$|r_i(x) - \overline{r}_i(y)| > |\overline{r}_i(y) - r_i(y)| - |x - y|$$

which implies that $\phi(x, y) > |\overline{r}_i(y) - r_i(y)|$ for $i = 2, \dots, n$. Note that, since B and $\overline{B}$ are divergent branches, there exists $i_0 \in \{1, \dots, n\}$ such that $|\overline{r}_{i_0}(y) - r_{i_0}(y)|$ may be as large as we want by choosing $|x|$ (and thus $|y|$) large enough (see Remark 2.5, statement 2). Then, for each $K > 0$, there exists $\delta > 0$ such that if $|x| > \delta$, it holds that $\phi(x, y) > |\overline{r}_{i_0}(y) - r_{i_0}(y)| > K$. □

Under these conditions, we obtain Theorem 4.5 that characterizes whether the Hausdorff distance between two curves is finite.

Theorem 4.5. *Let $\mathcal{C}$ and $\overline{\mathcal{C}}$ be two algebraic space curves. It holds that $\mathcal{C}$ and $\overline{\mathcal{C}}$ have a similar asymptotic behavior if and only if the Hausdorff distance between them is finite.*

Proof. First, let us prove that if $\mathcal{C}$ and $\overline{\mathcal{C}}$ have a similar asymptotic behavior then, the Hausdorff distance between them is finite.

Let κ be the number of infinity branches of $\mathcal{C}$. Then, $\mathcal{C} = B_1 \cup \dots \cup B_\kappa \cup \widehat{B}$, where $\widehat{B}$ is the set of points of $\mathcal{C}$ that do not belong to any infinity branch. Thus,

$$\sup_{p\in\mathcal{C}} d(p, \overline{\mathcal{C}}) = \max\{\sup_{p\in B_1} d(p, \overline{\mathcal{C}}), \dots, \sup_{p\in B_\kappa} d(p, \overline{\mathcal{C}}), \sup_{p\in\widehat{B}} d(p, \overline{\mathcal{C}})\}.$$

For each $i = 1, \dots, \kappa$, let $B_i = \bigcup_{j=1}^{N_i} L_{i,j}$, where $L_{i,j} = \{(z, r_{i,j,2}(z), \dots, r_{i,j,n}(z)) \in \mathbb{C}^n : z \in \mathbb{C},\ |z| > M_i\}$, and $N_i = \nu(B_i)$. Then,

$$\sup_{p\in B_i} d(p, \overline{\mathcal{C}}) = \max_{j=1,\dots,N_i} \left\{ \sup_{|z|>M_i} d((z, r_{i,j,2}(z), \dots, r_{i,j,n}(z)), \overline{\mathcal{C}}) \right\}.$$

Moreover, since $\mathcal{C}$ and $\overline{\mathcal{C}}$ have a similar asymptotic behavior then there exists an infinity branch $\overline{B}_i \subset \overline{\mathcal{C}}$ non-divergent with B_i (see Definition 4.1). This implies that there is a leaf

$$\overline{L}_{i,j} = \{(z, \overline{r}_{i,j,2}(z), \dots, \overline{r}_{i,j,n}(z)) \in \mathbb{C}^n : z \in \mathbb{C},\ |z| > \overline{M}_i\} \subset \overline{B}_i$$

such that

$$\lim_{z\to\infty} d((r_{i,j,2}(z), \dots, r_{i,j,n}(z)), (\overline{r}_{i,j,2}(z), \dots, \overline{r}_{i,j,n}(z))) = c_{i,j} < \infty$$

(see Lemma 2.10 and Remark 2.6). Then

$$\lim_{z\to\infty} d((z, r_{i,j,2}(z), \dots, r_{i,j,n}(z)), \overline{\mathcal{C}}) \le \lim_{z\to\infty} d((z, r_{i,j,2}(z), \dots, r_{i,j,n}(z)), (z, \overline{r}_{i,j,2}(z), \dots, \overline{r}_{i,j,n}(z))) = c_{i,j} < \infty.$$

Hence, given $\eta > 0$ there exists $\delta > 0$ such that for $|z| > \delta$ it holds that

$$d((z, r_{i,j,2}(z), \dots, r_{i,j,n}(z)), \overline{\mathcal{C}}) < \eta$$

for every $i = 1, \dots, \kappa$ and $j = 1, \dots, N_i$.

On the other hand, since $r_{i,j,2}, \dots, r_{i,j,n}$ are continuous functions, and $\{z \in \mathbb{C} : M_i \le |z| \le \delta\}$ is a compact set, there exists $\xi > 0$ such that

$$\sup_{M_i\le|z|\le\delta} d((z, r_{i,j,2}(z), \dots, r_{i,j,n}(z)), \overline{\mathcal{C}}) < \xi$$

for every $i = 1, \dots, \kappa$ and $j = 1, \dots, N_i$.

As a consequence, we have that

$$\sup_{p\in B_i} d(p, \overline{\mathcal{C}}) \leq \max\{\xi, \eta\} < \infty.$$

Now, let $p = (a_1, \ldots, a_n) \in \widehat{B}$. From Lemma 4.3, we have that there exists $K \in \mathbb{R}^+$ such that $|a_i| \leq K$, for $i = 1, \ldots, n$. Thus, $d(p, \mathcal{O}) \leq K$, where $\mathcal{O}$ is the origin and,

$$d(p, \overline{\mathcal{C}}) \leq d(p, \mathcal{O}) + d(\mathcal{O}, \overline{\mathcal{C}}) \leq K + d(\mathcal{O}, \overline{\mathcal{C}}).$$

Note that $K < \infty$, and $d(\mathcal{O}, \overline{\mathcal{C}}) < \infty$, which implies that $\sup_{p\in\widehat{B}} d(p, \overline{\mathcal{C}}) < \infty$.

Therefore, we conclude that $\sup_{p\in\mathcal{C}} d(p, \overline{\mathcal{C}}) < \infty$. Reasoning similarly, we deduce that $\sup_{\overline{p}\in\overline{\mathcal{C}}} d(\overline{p}, \mathcal{C}) < \infty$, which implies that $d_H(\mathcal{C}, \overline{\mathcal{C}}) < \infty$.

Reciprocally, let us assume that the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite (that is, $d_H(\mathcal{C}, \overline{\mathcal{C}}) = K < \infty$), and let us prove that the asymptotic behavior of both curves is similar (i.e. for any infinity branch $B \subset \mathcal{C}$ there exists an infinity branch $\overline{B} \subset \overline{\mathcal{C}}$ that does not diverge with B).

For this purpose, we assume that this statement does not hold and let $B = \{(z, r_{i,j,2}(z), \ldots, r_{i,j,n}(z)) \in \mathbb{C}^n : z \in \mathbb{C},\ |z| > M\} \subset \mathcal{C}$ be such that every infinity branch of $\overline{\mathcal{C}}$ diverges from B. Then, according to Lemma 4.4, for each infinity branch $\overline{B}_i = \{(z, \overline{r}_{i,j,2}(z), \ldots, \overline{r}_{i,j,n}(z)) \in \mathbb{C}^n : z \in \mathbb{C},\ |z| > \overline{M}_i\} \subset \overline{\mathcal{C}}$ $(i = 1, \ldots, \kappa)$, there exists $\delta_i > 0$ such that if $|x| > \delta_i$, then

$$d((x, r_{i,j,2}(x), \ldots, r_{i,j,n}(x)), (\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n)) > K$$

for every $(\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n) \in \overline{B}_i$. In addition, from Lemma 4.3, there exists $\delta_0 > 0$ such that any point $(\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n) \in \overline{\mathcal{C}}$ with $|\overline{a}_j| > \delta_0$ for some $j = 1, \ldots, n$, belongs to some infinity branch $\overline{B}_i \subset \overline{\mathcal{C}}$.

Under these conditions, let $\delta := \max\{\delta_0, \delta_1, \ldots, \delta_\kappa\}$, and we consider a point $(x, r_{i,j,2}(x), \ldots, r_{i,j,n}(x)) \in B$ such that $|x| > \delta + K$. Since $d_H(\mathcal{C}, \overline{\mathcal{C}}) = K$, there should exist some point $(\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n) \in \overline{\mathcal{C}}$ such that

$$d((x, r_{i,j,2}(x), \ldots, r_{i,j,n}(x)), (\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n)) \leq K.$$

However, this implies that $|\overline{a}_1| > |x| - K$ (see the proof of Lemma 4.4) and, hence, $|\overline{a}_1| > \delta$. Now, Lemma 4.3 states that this point must belong to some infinity branch $\overline{B}_i \subset \overline{\mathcal{C}}$ and then, Lemma 4.4 claims that

$$d((x, r_{i,j,2}(x), \ldots, r_{i,j,n}(x)), (\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n)) > K,$$

which is a contradiction. $\square$

5. Algorithm and examples

In this section, we present an algorithm that allows us to decide whether the Hausdorff distance between two given curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ (implicitly or parametrically defined) is finite. For this purpose, we use the results obtained in Section 4 (in particular, Theorem 4.5), and the computational techniques developed in Section 2 (for implicitly defined space curves) and in Section 3 (for parametrically defined space curves). Moreover, the deal with the particular case of characterizing the finiteness of the Hausdorff distance between the real parts of $\mathcal{C}$ and $\overline{\mathcal{C}}$, and an algorithm is also presented for this situation.

In addition, we illustrate the method with two examples in detail (see Examples 5.1 and 5.2), and three more examples where we show the application to practical problems in CAGD (see Examples 5.4–5.6). In particular, we show how the Hausdorff distance between two algebraic curves obtained by applying approximate parametrization problems is finite (we consider examples presented in [7,8,17]). Once one has ensured that the Hausdorff distance between the two curves is finite, some additional existing methods can be applied to estimate the Hausdorff distance between two not bounded curves (see [8]), or for a chosen bounded frame of the input curves (see e.g. [2,3,9,10], etc.).

The goal of this section is to illustrate the performance and the application to practical problems of the algorithm presented. We do not intend to compare our results with some other existing methods that allow to compute or estimate the Hausdorff distance between two sets. The results obtained in this paper do not provide an alternative method to this question, but a way of applying important and effective existing methods to the case in which some assumptions on the input curves are not satisfied, as for instance the boundedness (see e.g. [2,3,9,10], etc.) or the finiteness of the Hausdorff distance between them (see [7,8]). Thus, in particular, the algorithm presented will allow to explore further the existing techniques for analyzing the Hausdorff distance between curves in the n-dimensional space that are not bounded. These curves are especially important as for instance in approximate parametrization or implicitization methods where, in general, curves are not bounded.

In order to apply the algorithm, we first should assume that we have prepared the input curves, $\mathcal{C}$ and $\overline{\mathcal{C}}$, by means of a suitable linear change of coordinates (the same change applied to both curves), such that $(0 : a_2 : \ldots : a_n : 0)$ ($a_i \neq 0$ for some $i = 2, \ldots, n$) is not an infinity point of $\mathcal{C}^*$ and $\overline{\mathcal{C}}^*$ (see Remark 2.3).

Algorithm Hausdorff Distance.

Given two algebraic curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ in the n-dimensional space, the algorithm decides whether the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite.

1. Compute the infinity points of $\mathcal{C}$ and $\overline{\mathcal{C}}$. If they are not the same, RETURN *the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ is not finite*. Otherwise, let $P_1, \dots, P_\kappa$ be these infinity points.
2. For each $P_\ell, \ell = 1, \dots, \kappa$ do:
   2.1. Compute the infinity branches of $\mathcal{C}$ associated to P_ℓ (see Sections 2 and 3). Let $B_1, \dots, B_{n_\ell}$ be these branches. For each $i = 1, \dots, n_\ell$, let $B_i = \{(z, r_{i,2}(z), \dots, r_{i,n}(z)) \in \mathbb{C}^n : z \in \mathbb{C}, |z| > M_i\}$.
   2.2. Compute the infinity branches of $\overline{\mathcal{C}}$ associated to P_ℓ (see Sections 2 and 3). Let $\overline{B}_1, \dots, \overline{B}_{l_\ell}$ be these branches. For each $j = 1, \dots, l_\ell$, let $\overline{B}_j = \{(z, \overline{r}_{j,2}(z), \dots, \overline{r}_{j,n}(z)) \in \mathbb{C}^n : z \in \mathbb{C}, |z| > M_j\}$.
   2.3. For each $i = 1, \dots, n_\ell$, find $j = 1, \dots, l_\ell$ such that the terms with positive exponent in $r_{i,k}(z)$ and $\overline{r}_{j,k}(z)$ for $k = 2, \dots, n$, are the same up to conjugation. If there isn't such $j = 1, \dots, l_\ell$, RETURN *the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ is not finite* (see Lemmas 2.7 and 2.10, and Theorem 4.5).
   2.4. For each $j = 1, \dots, l_\ell$, find $i = 1, \dots, n_\ell$ such that the terms with positive exponent in $r_{i,k}(z)$ and $\overline{r}_{j,k}(z)$ for $k = 2, \dots, n$, are the same up to conjugation. If there isn't such $i = 1, \dots, n_\ell$, RETURN *the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ is not finite* (see Lemmas 2.7 and 2.10, and Theorem 4.5).
3. RETURN *the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite.*

If one is interested in characterizing the finiteness between the real parts of the input curves, some steps of the algorithm have to be subtly modified. In particular, we observe that if we restrict to $\mathbb{R}$, a problem appears with Lemma 2.10. More precisely, Lemma 2.10 states that if two branches $B = \bigcup_{\alpha=1}^{N} L_\alpha \subset \mathcal{C}$ and $\overline{B} = \bigcup_{\beta=1}^{\overline{N}} \overline{L}_\beta \subset \overline{\mathcal{C}}$ are two non-divergent infinity branches, for each (complex) leaf $L_\alpha \subseteq B$ there should exists a (complex) leaf $\overline{L}_\beta \subseteq \overline{B}$ that does not diverge with L_α, and reciprocally. Thus, L_α and $\overline{L}_\beta$ can be complex or real leaves. However, if we restrict to the real part, necessarily the leaves L_α and $\overline{L}_\beta$ have to be both real leaves.

Taking into account this question, the algorithm Hausdorff Distance can be easily adapted to characterize whether the Hausdorff distance between the real parts of the input curves is finite. For this purpose, one has to check if for each real leaf of each branch of $\mathcal{C}$ there exists a real leaf of $\overline{\mathcal{C}}$ satisfying that the terms with positive exponent in the corresponding series that determine the leaves of the two curves are the same.

Under these conditions, we note that the Hausdorff distance could go from being finite (if we consider the input curves over $\mathbb{C}$) to be infinite (if we consider the real parts of the input curves). We will analyze this situation in Example 5.2 (see also Remark 5.3).

In the following, we present the algorithm that decides whether the Hausdorff distance between the real parts of two given algebraic curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ in the n-dimensional space is finite. We will illustrate this algorithm with three examples (see Examples 5.4–5.6).

Algorithm Hausdorff Distance (over $\mathbb{R}$).

1. Compute the real infinity points of $\mathcal{C}$ and $\overline{\mathcal{C}}$. If they are not the same, RETURN *the Hausdorff distance between the real parts of $\mathcal{C}$ and $\overline{\mathcal{C}}$ is not finite*. Otherwise, let $P_1, \dots, P_\kappa$ be these real infinity points.
2. For each $P_\ell,\ \ell = 1, \dots, \kappa$ do:
   2.1 Compute the infinity branches of $\mathcal{C}$ associated to P_ℓ. Let $B_1, \dots, B_{n_\ell}$ be these branches, and for each $i = 1, \dots, n_\ell$, let
   $$L_{\alpha,i} = \{(z, r_{\alpha,i,2}(z), \dots, r_{\alpha,i,n}(z)) \in \mathbb{C}^n : z \in \mathbb{C}, |z| > M_{\alpha,i}\}, \quad \alpha = 1, \dots, N_i$$
   be the real leaves of B_i.
   2.2 Compute the infinity branches of $\overline{\mathcal{C}}$ associated to P_ℓ. Let $\overline{B}_1, \dots, \overline{B}_{l_\ell}$ be these branches, and for each $j = 1, \dots, l_\ell$, let
   $$\overline{L}_{\beta,j} = \{(z, \overline{r}_{\beta,j,2}(z), \dots, \overline{r}_{\beta,j,n}(z)) \in \mathbb{C}^n : z \in \mathbb{C}, |z| > M_{\beta,j}\}, \quad \beta = 1, \dots, \overline{N}_j$$
   be the real leaves of $\overline{B}_j$.
   2.3 For each $i = 1, \dots, n_\ell$ and each $\alpha = 1, \dots, N_i$, find $j = 1, \dots, l_\ell$ and $\beta = 1, \dots, \overline{N}_j$ such that the terms with positive exponent in $r_{\alpha,i,k}(z)$ and $\overline{r}_{\beta,j,k}(z)$ for $k = 2, \dots, n$, are the same up to conjugation. If there isn't such $j = 1, \dots, l_\ell$ and $\beta = 1, \dots, \overline{N}_j$, RETURN *the Hausdorff distance between the real parts of $\mathcal{C}$ and $\overline{\mathcal{C}}$ is not finite*.
   2.4 For each $j = 1, \dots, l_\ell$ and each $\beta = 1, \dots, \overline{N}_j$, find $i = 1, \dots, n_\ell$ and $\alpha = 1, \dots, N_i$ such that the terms with positive exponent in $r_{\alpha,i,k}(z)$ and $\overline{r}_{\beta,j,k}(z)$ for $k = 2, \dots, n$, are the same up to conjugation. If there isn't such $i = 1, \dots, n_\ell$ and $\alpha = 1, \dots, N_i$, RETURN *the Hausdorff distance between the real parts of $\mathcal{C}$ and $\overline{\mathcal{C}}$ is not finite*.
3. RETURN *the Hausdorff distance between the real parts of $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite*.

In the following, we illustrate with two examples the performance in detail of the algorithm Hausdorff Distance. In the first one, we compare two rational curves parametrically defined. In the second one, the curves are implicitly defined.

Example 5.1. Let $\mathcal{C}$ and $\overline{\mathcal{C}}$ be two rational space curves in the 4-dimensional space defined by the parametrizations

$$\mathcal{P}(s)=\left(\frac{-1+2s^3-s}{s},\ \frac{s+1}{s},\ \frac{-1}{s},\ \frac{s^2+3s-5}{s}\right),\quad \text{and}$$
$$\overline{\mathcal{P}}(s)=\left(\frac{-1+2s^3-s^2}{s},\ \frac{s+1}{s},\ \frac{-1}{s},\ \frac{s^2+3s-5}{s}\right),$$

respectively. We apply the algorithm Hausdorff Distance to decide whether the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite.

Step 1: Compute the infinity points of $\mathcal{C}$ and $\overline{\mathcal{C}}$. We obtain that $\mathcal{C}$ and $\overline{\mathcal{C}}$ have the same infinity points: $P_1=(1:-1:1:5:0)$ and $P_2=(1:0:0:0:0)$.

We start by analyzing the infinity branches associated to P_1:

Step 2.1: Reasoning as in Example 3.2, we get only one infinity branch associated to P_1 in $\mathcal{C}$. It is given by $B_1=\{(z, r_{1,2}(z), r_{1,3}(z), r_{1,4}(z))\in\mathbb{C}^4 : z\in\mathbb{C},\ |z|>M_1\}$, where

$$r_{1,2}(z)=-z+2z^{-2}-4z^{-3}-13z^{-4}-11z^{-5}+\cdots,$$
$$r_{1,3}(z)=z+1-2z^{-2}+4z^{-3}+13z^{-4}+11z^{-5}+\cdots,$$
$$r_{1,4}(z)=5z+8-z^{-1}-9z^{-2}+19z^{-3}+64z^{-4}+62z^{-5}+\cdots.$$

Step 2.2: We also have that there exists only one infinity branch associated to P_1 in $\overline{\mathcal{C}}$. It is given by $\overline{B}_1=\{(z, \overline{r}_{1,2}(z), \overline{r}_{1,3}(z), \overline{r}_{1,4}(z))\in\mathbb{C}^4 : z\in\mathbb{C},\ |z|>\overline{M}_1\}$, where

$$\overline{r}_{1,2}(z)=-z+1+z^{-1}+2z^{-2}+z^{-3}-4z^{-4}-7z^{-5}+\cdots,$$
$$\overline{r}_{1,3}(z)=z-z^{-1}-2z^{-2}-z^{-3}+4z^{-4}+7z^{-5}+\cdots,$$
$$\overline{r}_{1,4}(z)=5z+3-6z^{-1}-10z^{-2}-6z^{-3}+18z^{-4}+33z^{-5}+\cdots.$$

Step 2.3 and Step 2.4: $r_{1,j}$ and $\overline{r}_{1,j}$, $j=2,3,4$ have the same terms with positive exponent. Thus, the branches B_1 and $\overline{B}_1$ do not diverge.

Now we analyze the infinity branches associated to P_2:

Step 2.1: Reasoning as in Example 3.2, we get that the only infinity branch associated to P_2 in $\mathcal{C}$ is given by $B_2=\{(z, r_{2,2}(z), r_{2,3}(z), r_{2,4}(z))\in\mathbb{C}^4 : z\in\mathbb{C},\ |z|>M_2\}$, where

$$r_{2,2}(z)=1+\sqrt{2}z^{-1/2}-\frac{\sqrt{2}z^{-3/2}}{2}-z^{-2}+\frac{3\sqrt{2}z^{-5/2}}{8}+2z^{-3}+\cdots,$$
$$r_{2,3}(z)=-\sqrt{2}z^{-1/2}+\frac{\sqrt{2}z^{-3/2}}{2}+z^{-2}-\frac{3\sqrt{2}z^{-5/2}}{8}-2z^{-3}+\cdots,$$
$$r_{2,4}(z)=\frac{\sqrt{2}z^{1/2}}{2}+3-\frac{19\sqrt{2}z^{-1/2}}{4}+\frac{z^{-1}}{2}-\frac{39\sqrt{2}z^{-3/2}}{16}+\frac{9z^{-2}}{2}+\cdots.$$

We note that $\nu(B_2)=2$, and thus B_2 has 2 (conjugated) leaves. That is, $B_2=L_{2,1}\cup L_{2,2}$, where $L_{2,i}$ are obtained by conjugation in the series $r_{2,2}$, $r_{2,3}$ and $r_{2,4}$.

Step 2.2: We also have that there exists only one infinity branch associated to P_2 in $\overline{\mathcal{C}}$. It is given by $\overline{B}_2=\{(z, \overline{r}_{2,2}(z), \overline{r}_{2,3}(z), \overline{r}_{2,i4}(z))\in\mathbb{C}^4 : z\in\mathbb{C},\ |z|>\overline{M}_2\}$, $i=1,2$, where

$$\overline{r}_{2,2}(z)=1+\sqrt{2}z^{-1/2}-\frac{z^{-1}}{2}-\frac{\sqrt{2}z^{-3/2}}{16}-z^{-2}+\frac{383\sqrt{2}z^{-5/2}}{512}-\frac{z^{-3}}{2}+\cdots,$$
$$\overline{r}_{2,3}(z)=-\sqrt{2}z^{-1/2}+\frac{z^{-1}}{2}+\frac{\sqrt{2}z^{-3/2}}{16}+z^{-2}-\frac{383\sqrt{2}z^{-5/2}}{512}+\frac{z^{-3}}{2}+\cdots,$$
$$\overline{r}_{2,4}(z)=\frac{\sqrt{2}z^{1/2}}{2}+\frac{13}{4}-\frac{159\sqrt{2}z^{-1/2}}{32}+3z^{-1}-\frac{449\sqrt{2}z^{-3/2}}{1024}+5z^{-2}+\cdots.$$

We note that $\nu(\overline{B}_2)=2$, and thus $\overline{B}_2$ has 2 (conjugated) leaves. That is, $\overline{B}_2=\overline{L}_{2,1}\cup\overline{L}_{2,2}$, where $\overline{L}_{2,i}$ are obtained by conjugation in the series $\overline{r}_{2,2}$, $\overline{r}_{2,3}$ and $\overline{r}_{2,4}$.

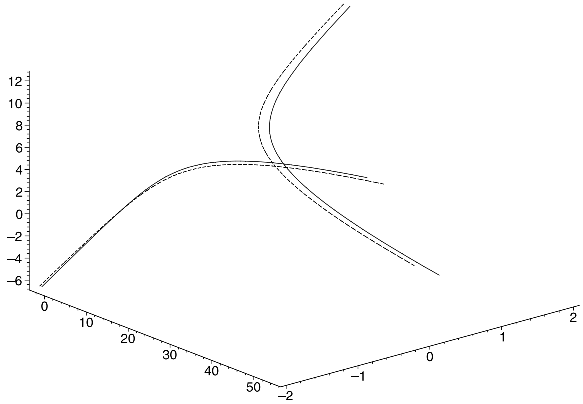

Fig. 2. Projections of $\mathcal{C}$ and $\overline{\mathcal{C}}$ along the axis x_2.

Step 2.3 and Step 2.4: $r_{2,j}$ and $\overline{r}_{2,j}$, $j = 2, 3, 4$ have the same terms with positive exponent. Thus, the branches B_2 and $\overline{B}_2$ do not diverge.

Step 3: The algorithm returns that the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite.

We observe that, in this case, the infinity branches of $\mathcal{C}$ and $\overline{\mathcal{C}}$ do not converge neither diverge (see Fig. 2).

Example 5.2. Let $\mathcal{C}$ and $\overline{\mathcal{C}}$ be two space curves in the 3-dimensional space implicitly defined by the polynomials

$$f_1(x_1, x_2, x_3) = -x_2 + x_1^2 - 2x_1x_2^2 + x_2^4, \qquad f_2(x_1, x_2, x_3) = x_1 + x_2^2 - x_3x_2^2 - x_3$$

and

$$\overline{f}_1(x_1, x_2, x_3) = x_2^2 - x_1, \qquad \overline{f}_2(x_1, x_2, x_3) = 2x_1 - x_3x_2^2 - x_3,$$

respectively. We apply the algorithm Hausdorff Distance to decide whether the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite:

1. Step 1: Compute the infinity points of $\mathcal{C}$ and $\overline{\mathcal{C}}$. We obtain that $\mathcal{C}$ and $\overline{\mathcal{C}}$ have $P = (1 : 0 : 0 : 0)$ as their unique infinity point.

   We analyze the infinity branches associated to P:
2. Step 2.1: Reasoning as in Example 2.12, we get that the only infinity branch associated to P in $\mathcal{C}$ is given by $B = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M\}$, where

   $$r_2(z) = z^{1/2} + \frac{z^{-1/4}}{2} - \frac{z^{-7/4}}{64} + \frac{z^{-10/4}}{128} + \cdots,$$
   $$r_3(z) = 2 - z^{-3/4} - 2z^{-1} + \frac{3z^{-3/2}}{4} + 3z^{-7/4} + \cdots.$$

   We note that $\nu(B) = 4$, and thus B has 4 (conjugated) leaves. That is, $B = \bigcup_{\alpha=1}^{4} L_\alpha$, where L_α are obtained by conjugation in the series r_2 and r_3.
3. Step 2.2: We also have that there exists only one infinity branch associated to P in $\overline{\mathcal{C}}$. It is given by $\overline{B} = \{(z, \overline{r}_2(z), \overline{r}_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > \overline{M}\}$, where

   $$\overline{r}_2(z) = z^{1/2},$$
   $$\overline{r}_3(z) = 2 - 2z^{-1} + 2z^{-2} - 2z^{-3} + 2z^{-4} - 2z^{-5} + \cdots.$$

   We note that $\nu(\overline{B}) = 2$, and thus $\overline{B}$ has 2 (conjugated) leaves. That is, $\overline{B} = \bigcup_{\beta=1}^{2} \overline{L}_\beta$, where $\overline{L}_\beta$ are obtained by conjugation in the series $\overline{r}_2$ and $\overline{r}_3$.
4. Step 2.3 and Step 2.4: r_j and $\overline{r}_j$, $j = 2, 3$, have the same terms with positive exponent. Thus, the infinity branches B and $\overline{B}$ do not diverge.
5. Step 3: The algorithm returns that the Hausdorff distance between the curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite (see Fig. 3).

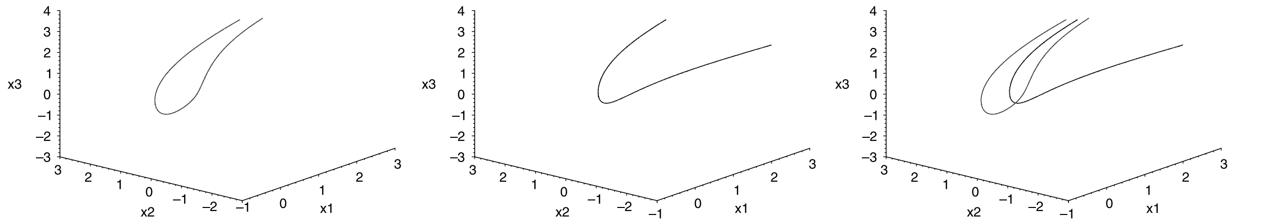

Fig. 3. $\mathcal{C}$ (left), $\overline{\mathcal{C}}$ (center), and the asymptotic behavior of $\mathcal{C}$ and $\overline{\mathcal{C}}$ (right).

Remark 5.3. In Example 5.2, we show that the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite. However, as one can deduce from Fig. 3, the Hausdorff distance between the real parts of $\mathcal{C}$ and $\overline{\mathcal{C}}$ is not finite. Indeed, the infinity branch $\overline{B} \subset \overline{\mathcal{C}}$ is such that $\nu(\overline{B}) = 2$ and thus, $\overline{B} = \overline{L}_1 \cup \overline{L}_2$, where $\overline{L}_j = \{(z, \overline{r}_{j,2}(z), \overline{r}_{j,3}(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > \overline{M}\}$, $j = 1, 2$, and

$$\overline{r}_{1,2}(z) = z^{1/2}, \qquad \overline{r}_{1,3}(z) = 2 - 2z^{-1} + 2z^{-2} - 2z^{-3} + 2z^{-4} - 2z^{-5} + \cdots,$$
$$\overline{r}_{2,2}(z) = -z^{1/2}, \qquad \overline{r}_{2,3}(z) = 2 - 2z^{-1} + 2z^{-2} - 2z^{-3} + 2z^{-4} - 2z^{-5} + \cdots.$$

On the other side, $\nu(B) = 4$ and then, $B = \bigcup_{i=1}^{4} L_i \subset \mathcal{C}$, where $L_i = \{(z, r_{i,2}(z), r_{i,3}(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M\}$, $i = 1, \ldots, 4$, and

$$r_{1,2}(z) = z^{1/2} + \frac{z^{-1/4}}{2} - \frac{z^{-7/4}}{64} + \cdots, \qquad r_{1,3}(z) = 2 - z^{-3/4} - 2z^{-1} + \frac{3z^{-3/2}}{4} + \cdots,$$
$$r_{2,2}(z) = z^{1/2} - \frac{z^{-1/4}}{2} + \frac{z^{-7/4}}{64} + \cdots, \qquad r_{2,3}(z) = 2 + z^{-3/4} - 2z^{-1} + \frac{3z^{-3/2}}{4} + \cdots,$$
$$r_{3,2}(z) = -z^{1/2} + \frac{Iz^{-1/4}}{2} + \frac{Iz^{-7/4}}{64} + \cdots, \qquad r_{3,3}(z) = 2 + Iz^{-3/4} - 2z^{-1} - \frac{3z^{-3/2}}{4} + \cdots,$$
$$r_{4,2}(z) = -z^{1/2} - \frac{Iz^{-1/4}}{2} - \frac{Iz^{-7/4}}{64} + \cdots, \qquad r_{4,3}(z) = 2 - Iz^{-3/4} - 2z^{-1} - \frac{3z^{-3/2}}{4} + \cdots.$$

Note that the real leaf $\overline{L}_1$ converges to the real leaves L_1 and L_2, and the real leaf $\overline{L}_2$ converges to the complex leaves L_3 and L_4. However, if we restrict to the real parts of $\mathcal{C}$ and $\overline{\mathcal{C}}$, Step 2.4 of the algorithm Hausdorff Distance (over $\mathbb{R}$) outputs that the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ is not finite since $\overline{L}_2$ does not converge to any real leaf of B.

Practical applications

In the following, we illustrate the performance of the algorithm Hausdorff Distance (over $\mathbb{R}$) with three examples. We note that, in general, in practical applications, one is interested in analyzing the Hausdorff distance between the real parts of the input curves.

The first example is obtained from [8, see Example 5.2], and we prove that the Hausdorff distance between the real parts of the input (not bounded) space curves is finite. We recall that results in [8] (where a method for estimating the Hausdorff distance between two given space curves is presented) can be applied for curves satisfying that the Hausdorff distance between them is finite.

Example 5.4. Let $\mathcal{C}$ and $\overline{\mathcal{C}}$ be two space curves implicitly defined by

$$f_1(x_1, x_2, x_3) = x_2^3 - x_2x_1 - x_1^3, \qquad f_2(x_1, x_2, x_3) = x_1x_3 - x_2^2$$
$$f_3(x_1, x_2, x_3) = x_3x_2 - x_2 - x_1^2, \qquad f_4(x_1, x_2, x_3) = x_3^2 - x_3 - x_2x_1,$$

and

$$\overline{f}_1(x_1, x_2, x_3) = -9 - x_1 + 13x_2 + 3x_1^2 - x_2x_1 - 6x_2^2 - x_1^3 + x_2^3,$$
$$\overline{f}_2(x_1, x_2, x_3) = -3 - x_3 - x_1 + 4x_2 + x_1x_3 - x_2^2,$$
$$\overline{f}_3(x_1, x_2, x_3) = 3 + x_3x_2 - 2x_2 + 2x_1 - 2x_3 - x_1^2,$$
$$\overline{f}_4(x_1, x_2, x_3) = -x_2x_1 + 2x_1 + x_2 - 3x_3 + x_3^2,$$

respectively. In this particular example, $\mathcal{C}$ and $\overline{\mathcal{C}}$ are rational curves and they can be parametrized by

$$\mathcal{P}(s) = \left(\frac{s}{s^3 - 1}, \frac{s^2}{s^3 - 1}, \frac{s^3}{s^3 - 1}\right), \qquad \overline{\mathcal{P}}(s) = \left(\frac{s + s^3 - 1}{s^3 - 1}, \frac{s^2 + 2s^3 - 2}{s^3 - 1}, \frac{2s^3 - 1}{s^3 - 1}\right),$$

respectively. In order to apply the results in [8], one has to ensure that the Hausdorff distance between the real parts of the input curves is finite. We apply the algorithm Hausdorff Distance (over $\mathbb{R}$).

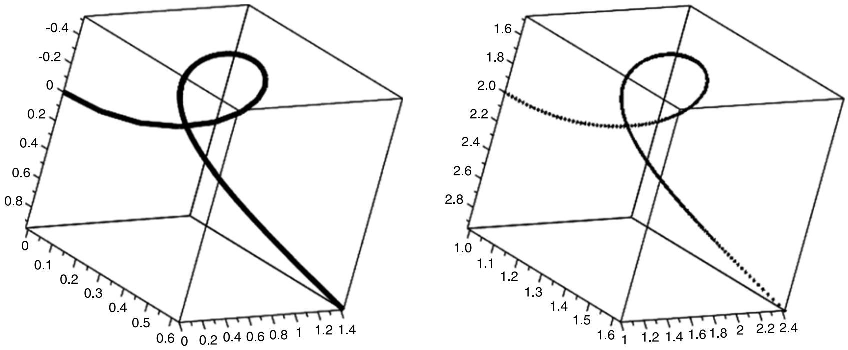

Fig. 4. $\mathcal{C}$ (left) and $\overline{\mathcal{C}}$ (right).

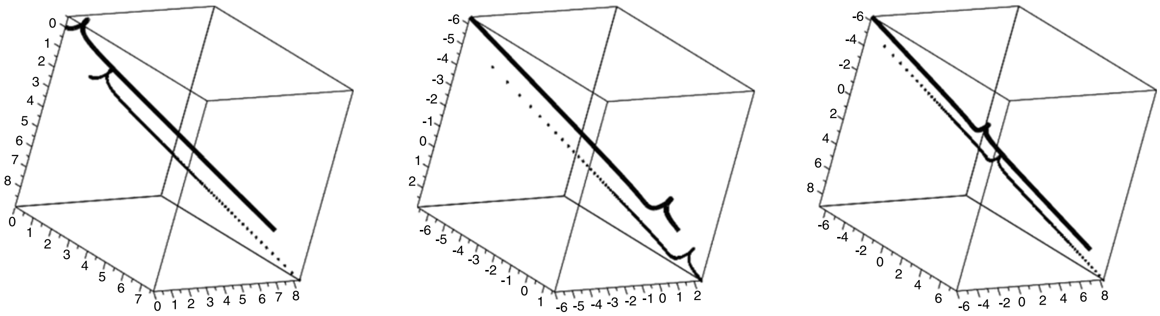

Fig. 5. Asymptotic behavior of $\mathcal{C}$ and $\overline{\mathcal{C}}$ from different perspectives.

1. Step 1: Compute the real infinity points of $\mathcal{C}$ and $\overline{\mathcal{C}}$. We obtain that $\mathcal{C}$ and $\overline{\mathcal{C}}$ have $P = (1 : 1 : 1 : 0)$ as their unique (real) infinity point.

   We analyze the infinity branches associated to P:
2. Step 2.1: Reasoning similarly to Example 2.12 or 3.2, we get that $\mathcal{C}$ has only an infinity branch B associated to P. We have that $B = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M\}$, where
$$r_2(z) = z + 1/3 - 1/81z^{-2} + 1/243z^{-3} - 4/2187z^{-4} + \cdots,$$
$$r_3(z) = z + 2/3 + 1/9z^{-1} - 2/81z^{-2} + 2/2187z^{-4} + \cdots.$$
   Note that $\nu(B) = 1$ (that is, B has one leaf), and the leaf is real.
3. Step 2.2: We also get that there exists only one infinity branch $\overline{B}$ associated to P in $\overline{\mathcal{C}}$. We have that $\overline{B} = \{(z, \overline{r}_2(z), \overline{r}_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > \overline{M}\}$, where
$$\overline{r}_2(z) = z + 4/3 - 1/81z^{-2} + 196/243z^{-3} - 2029/2187z^{-4} + \cdots,$$
$$\overline{r}_3(z) = z + 2/3 + 1/9z^{-1} + 7/81z^{-2} + 8/9z^{-3} - 1888/2187z^{-4} + \cdots.$$
   Note that $\nu(\overline{B}) = 1$, and the only leaf of $\overline{B}$ is also real.
4. Step 2.3 and Step 2.4: r_j and $\overline{r}_j$, $j = 2, 3$, have the same terms with positive exponent. Thus, the infinity branches B and $\overline{B}$ do not diverge.
5. Step 3: The algorithm returns that the Hausdorff distance between the real parts of $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite (see Figs. 4 and 5).

Once the finiteness of the Hausdorff distance is guaranteed, one may apply the method in [8] to estimate the Hausdorff distance between the real parts of $\mathcal{C}$ and $\overline{\mathcal{C}}$ (see Example 5.2 in [8]).

The second example is obtained from [7, see Example 5.4]. In [7], given a non-rational irreducible real space curve $\mathcal{C}$, satisfying certain additional conditions, a rational parametrization of a space curve, $\overline{\mathcal{C}}$, near $\mathcal{C}$ is computed. It is proved that $\overline{\mathcal{C}}$ is of the same degree as $\mathcal{C}$, both curves have the same structure at infinity, and the Hausdorff distance between the real parts of $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite. The finiteness of the Hausdorff distance is proved from the initial conditions imposed to the input curve $\mathcal{C}$. We observe that, in general, the curves considered in this type of problems are not bounded.

Taking into account the results presented in this paper, the initial conditions imposed to $\mathcal{C}$ can be avoided and the method in [7] could be applied to a more general space curves (not satisfying the initial assumptions).

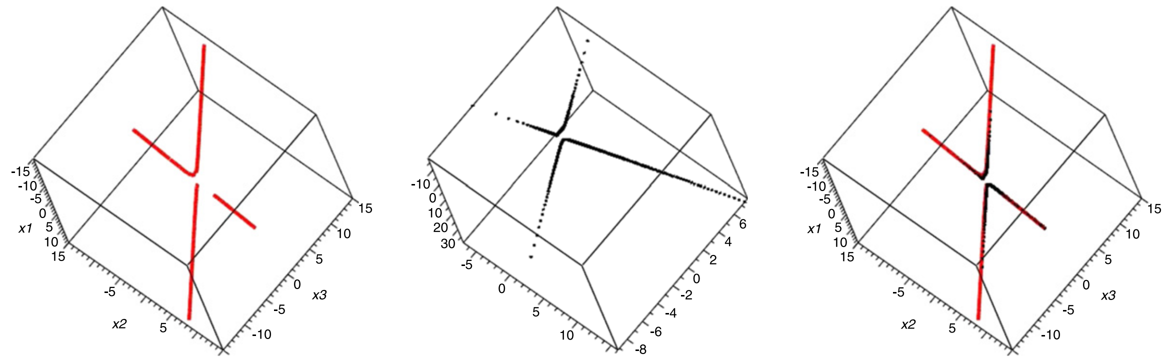

Fig. 6. $\mathcal{C}$ (left) and $\overline{\mathcal{C}}$ (center), and asymptotic behavior of $\mathcal{C}$ and $\overline{\mathcal{C}}$ (right).

Example 5.5. Let $\mathcal{C}$ be the input space curve implicitly defined by

$$\begin{aligned} f_1(x_1,x_2,x_3) &= 20052827033x_1x_2 + 2850904342x_2x_3 - 7155364672x_3x_1 + 1610946062x_3^2 \\ &\quad - 215763180597/100x_1 - 7869010116x_3 + 1743412651801/100x_2 - 43102722226x_2^2, \\ f_2(x_1,x_2,x_3) &= -18330943984x_3x_2 + 33857630124x_1x_3 - 390188402999/25x_1 - 56921602320x_3 \\ &\quad + 12611223036001/100x_2 - 166608514760x_2^2 + 57179742076x_3^2 + 20052827033x_1^2. \end{aligned}$$

In [7, see Example 5.4], an approximate rational space curve $\overline{\mathcal{C}}$ defined by the parametrization $\overline{\mathcal{P}}(s) = \big(p_1(s)/q(s), p_2(s)/q(s), p_3(s)/q(s)\big)$, where

$$\begin{aligned} p_1 &= -17673/21055s - 134689/69893s^2 - 199091/1476253 - 18405/25661s^4 - 53236/27529s^3, \\ p_2 &= 24795/6974s^3 + s^4 + 39160/8471s^2 + 24359/9278s + 27655/50016, \\ p_3 &= 8442/921241s - 10679/122442s^3 + 2769/593578 - 1721/32378s^4 - 11925/389597s^2, \\ q &= 24795/6974s^3 + s^4 + 39160/8471s^2 + 24359/9278s + 27655/50016 \end{aligned}$$

is computed.

In order to prove the effectiveness of the method, one has to check that both curves are close. For this purpose, the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ has to be estimated (see Example 5.4 in [7]). However, previously one has to ensure that the Hausdorff distance between the real parts of the both curves is finite. By applying the algorithm Hausdorff Distance (over $\mathbb{R}$), we compute the branches associated to the two different (real) infinity points that both curves share, and the algorithm returns that the Hausdorff distance between the real parts of $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite (see Fig. 6).

Finally, the last example is obtained from [17, see Example 9]. In [17] given a rational plane curve, $\mathcal{C}$, defined by a parametrization $\mathcal{P}$ that is "almost" improper (numerically speaking), a method for computing a new rational plane curve, $\overline{\mathcal{C}}$, defined by a proper parametrization, $\overline{\mathcal{P}}$, is presented. In [17], some bounds for measuring the closeness between $\mathcal{C}$ and $\overline{\mathcal{C}}$ are presented but it can be applied for bounded frames of the curves (see Section 4 in [17]). In order to have a total analysis of both curves and to ensure the effectiveness of the method presented, the behavior at infinity has to be studied, and the finiteness of the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ has to be guaranteed.

Example 5.6. Let $\mathcal{C}$ be the plane curve defined by the rational parametrization

$$\mathcal{P}(s) = \left(\frac{s^2(20s^6 - 40s^3 + 20 + 2s^5 - 2s^2 - s^4)}{(s^3-1)^3}, \frac{2(-s^4 - 3s^5 + 3s^2 + 3s^6 - 6s^3 + 3)}{(-s^2 + s^3 - 1)(s^3 - 1)} \right).$$

This parametrization is "almost" improper (see Section 3 in [17]). The algorithm presented in [17] returns the curve $\overline{\mathcal{C}}$ defined by the parametrization $\overline{\mathcal{P}}(s) = (p_1(s)/q_1(s), p_2(s)/q_2(s))$, where

$$\begin{aligned} p_1(s) &= -6068815952453355020 1(20s^2 - 2s - 1), \\ q_1(s) &= 60688159524533550201s^3 - 18449333330658180s^2 + 60081278530101s - 265814138756, \\ p_2(s) &= 10975164641(1105s^2 + 1104s - 368), \\ q_2(s) &= 92(21953640540s^2 + 21950329282s - 21975135). \end{aligned}$$

Using Theorem 5 in [17], the closeness between $\mathcal{C}$ and $\overline{\mathcal{C}}$ is measured for every $s_0 \in (-5, 5)$. More precisely, the bounds presented in [17] can be applied but these bounds work for each bounded frame of the curves (see Section 4 in [17]). Thus, since both curves are (in general) not bounded, as a previous step, one has to check whether the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite. For this purpose, we apply the algorithm Hausdorff Distance (over $\mathbb{R}$), we compute the branches concerning

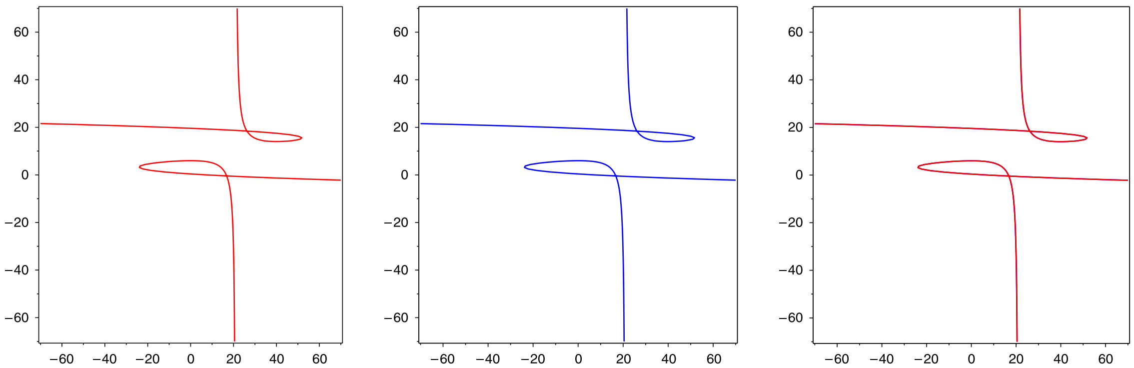

Fig. 7. $\mathcal{C}$ (left) and $\overline{\mathcal{C}}$ (center), and asymptotic behavior of $\mathcal{C}$ and $\overline{\mathcal{C}}$ (right).

the two real infinity points that both curves share, and the algorithm returns that the Hausdorff distance between the real parts of $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite (see Fig. 7).

6. Conclusions

Given two real algebraic curves in the n-dimensional space, we provide a method for checking whether the Hausdorff distance between them is finite or not. The algorithm is derived by checking the terms with positive exponent in the corresponding series that determine the infinity branches of the two given curves.

The method presented is very useful for measuring the performance of approximate parametrization and approximate implicitization methods where, in general, the curves are not bounded (see e.g. [7,14–17,20]). More precisely, in approximate parametrization problems, the Hausdorff distance is an essential tool for measuring the resemblance between the input and the output curves (which could not be bounded). In fact, the effectiveness of the method will depend on whether one may ensure that both curves have a similar behavior at infinity (that is, the finiteness of the Hausdorff distance), and estimate the Hausdorff distance between them.

In order to estimate the Hausdorff distance between two curves, only a chosen bounded frame of the curves can be analyzed since most of the existing results applied to bounded curves (see e.g. [2,3,9,10], etc.). Only a general method for estimating the distance between two space curves (not necessarily bounded) is presented in [8]. In [8], bounds for the Hausdorff distance between space curves are provided and they are related to the distance between the projections of the space curves onto a plane. Thus, the algorithm in [8] allows the use of every method developed so far to estimate the Hausdorff distance between plane curves to achieve estimations of the distance for space curves. However, in [8], an important assumption has to be imposed: the finiteness of the Hausdorff distance between the input space curves.

In this paper, we go one step further, and we characterize whether the Hausdorff distance between two given algebraic curves (parametrically or implicitly defined) in the n-dimensional space is finite. Hence, the results in this paper allow (for instance) the application of [8] without any additional assumption and then, the profit obtained with the results in [8] is ensured. In addition, we provide an effective and accurate algorithm that can be easily applied to measure the resemblance between two curves at infinity (which is equivalent to ensure the finiteness of the Hausdorff distance).

We think that the characterization presented in this paper could allow the use of many methods developed so far to achieve estimations of the distance for space curves not necessarily bounded. In this sense, more analysis is necessary but the idea that previous result and existing techniques could be adapted for computing the Hausdorff distance for any pair of given curves (not necessarily bounded) is promising.

Moreover, since this characterization is based on the notion of infinity branch (which reflects the status of a curve at the points with sufficiently large coordinates), one may think in applying the results presented to the analysis of the behavior at infinity of a given algebraic curve not necessarily bounded. This would imply a wide applicability in many active research fields as for instance in the study of the topology (see e.g. [21–24]), the determination of the symmetries (see [26]), etc.

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