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The T–function of a Parametric Curve

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Abstract

In this paper, we introduce the *T–function*, $T(s)$, which is a polynomial defined by means of a univariate resultant constructed from a given parametrization $\mathcal{P}(t) \in \mathbb{K}(t)^n$, $n \geq 2$ of an algebraic space curve \mathcal{C} . It is shown that $T(s) = \prod_{i=1}^n H_{P_i}(s)^{m_i-1}$, where $H_{P_i}(s)$, $i = 1, \dots, n$ are polynomials (*the fibre functions*) whose roots are the fibre of the ordinary singularities $P_i \in \mathcal{C}$ of multiplicity m_i , $i = 1, \dots, n$ of \mathcal{C} . Therefore, a complete classification of the singularities of \mathcal{C} , via the factorization of a resultant, is obtained.

Keywords: Singularities of an algebraic curve; Multiplicity of a point; Resultant; T–function; Fibre function; Rational curve parametrization

1 Introduction

Parametrizations of rational algebraic curves play an important role in many practical applications in computer aided geometric design (CAGD) where objects are often given and manipulated parametrically (see e.g. [10], [11], [12]). In the last years, important advances have been made concerning the information one may obtain from a given rational parametrization defining an algebraic variety. For instance, a complete analysis of the asymptotic behavior of a given curve has been carried out in [3]; efficient algorithms for computing the implicit equations that define the curve are provided in [5] and [19] and the study and computation of the fibre of a point via the

parametrization can be found in [19]. In addition, some aspects concerning the singularities of the curve and their multiplicities are studied in the vast literature on CAGD, e.g. [1], [2], [6], [7], [14], [18], [21] and [22]. Similar problems, for the case of a given rational parametric surface, are being analyzed. For instance, the computation of the singularities and their multiplicities from the input parametrization is presented in [17], a univariate resultant-based implicitization algorithm for surfaces is provided in [16], and the computation of the fibre of rational surface parametrizations is developed in [15].

In this paper, we show how to relate the fibre and the singularities of a given curve defined parametrically, by means of a univariate resultant which is constructed directly from the parametrization. For this purpose, we consider $\mathcal{P}(t) \in \mathbb{P}^n(\mathbb{K}(t))$ a rational projective parametrization of an algebraic curve \mathcal{C} over an algebraically closed field of characteristic zero, \mathbb{K} . Associated with $\mathcal{P}(t)$, we consider the induced rational map $\psi_{\mathcal{P}} : \mathbb{K} \rightarrow \mathcal{C} \subset \mathbb{P}^n(\mathbb{K}); t \mapsto \mathcal{P}(t)$. We denote by $\deg(\psi_{\mathcal{P}})$ the degree of the rational map $\psi_{\mathcal{P}}$. The birationality of $\psi_{\mathcal{P}}$, i.e. the properness of $\mathcal{P}(t)$, is characterized by $\deg(\psi_{\mathcal{P}}) = 1$ (see [9] and [20]). Intuitively speaking, $\mathcal{P}(t)$ proper means that $\mathcal{P}(t)$ traces the curve once, except for at most a finite number of points. We will see that, in fact, these points are the singularities of \mathcal{C} .

We recall that the degree of a rational map can be seen as the cardinality of the fibre of a generic element (see [20]). We use this characterization in our reasoning and thus, we denote by $\mathcal{F}_{\mathcal{P}}(P)$ the fibre of a point $P \in \mathcal{C}$ via the parametrization $\mathcal{P}(t)$; that is $\mathcal{F}_{\mathcal{P}}(P) = \mathcal{P}^{-1}(P) = \{t \in \mathbb{K} \mid \mathcal{P}(t) = P\}$.

In order to make the paper more reader-friendly, we first consider the case of a given plane curve \mathcal{C} defined parametrically by $\mathcal{P}(t) \in \mathbb{P}^2(\mathbb{K}(t))$ (see Sections 2 and 3) to, afterwards, generalize the results obtained to rational space curves in any dimension (see Section 4). We also assume that \mathcal{C} has only ordinary singularities (otherwise, one may apply quadratic transformations for birationally transforming the curve into a curve with only ordinary singularities). Non-ordinary singularities have to be treated specially since a non-ordinary singularity might have other singularities in its “neighborhood”. This specific case will be addressed in a future work and in fact, we will show that similar results to those presented in this paper can be stated for curves with non-ordinary singularities.

Under these conditions, the main goal of the paper is to prove that a

univariate resultant constructed directly from $\mathcal{P}(t)$, which we will call the *T-function*, $T(t)$, describes totally the singularities of \mathcal{C} . It will be proved that the factorization of $T(t)$ provides the fibre functions of the different singularities of \mathcal{C} as well as their corresponding multiplicities. The *fibre function* of a point $P \in \mathcal{C}$ via $\mathcal{P}(t)$ is given by a polynomial $H_P(t)$ which satisfies that $t_0 \in \mathcal{F}_{\mathcal{P}}(P)$ if and only if $H_P(t_0) = 0$. In [14], it is proved that if $H_P(t) = \prod_{i=1}^n (t - s_i)^{k_i}$ then, \mathcal{C} has n tangents at P of multiplicities k_1, \dots, k_n , respectively. In addition, these tangents can be computed using $\mathcal{P}(t)$ and the roots of each corresponding fibre function. Furthermore, it is shown that $\text{mult}_P(\mathcal{C}) = \deg(H_P(t))$.

Taking into account these previous results, in this paper we prove that the T-function can be factorized as $T(t) = \prod_{i=1}^n H_{P_i}(s)^{m_i-1}$, where $H_{P_i}(t)$ is the fibre function of the ordinary singularity $P_i \in \mathcal{C}$ and m_i is its multiplicity (for $i = 1, \dots, n$). Thus, a complete classification of the singularities of a given rational curve, via the factorization of a univariate resultant, is obtained.

On finishing this work, we just found a paper by Abhyankar (see [1]) that proves the factorization of the T-function for a given polynomial parametrization. In addition, Busé et al., in [6], provide a generalization of Abhyankar's formula for the case of rational parametrizations (not necessarily polynomial). This approach is based on the concept of singular factors introduced in [7], and it involves the construction of μ -basis. Our approach is totally different, since we generalize Abhyankar's formula by using the methods and techniques presented in [14]. This allows us to group the factors of the T-function to easily obtain the fibre functions of the different singularities and their multiplicity. Furthermore, we show how to deal with singularities that are reached by algebraic values of the parameter (see Definition 4 of family of conjugate parametric points).

As we mentioned above, the results obtained can be stated similarly for the case of rational space curves in any dimension. The methods developed in this paper generalize some previous results that partially approach the computation and analysis of singularities for rational parametrized space curves (see e.g. [5], [14], [18], [21] or [22]), and it provides, as in the case of plane curves, a complete classification of the singularities of a given rational space curve, via the factorization of a univariate resultant. More precisely, we introduce an extension of the T-function which allows us to perfectly generalize the results of plane curves to the space case. Any factor of the new T-function ($T_E(s)$) corresponds to a singularity of the space curve.

Hence, this paper provides some new achievements on the study of singularities of parametric curves. In particular, the results presented allow, not only to compute the singularities of the curve, but also to collect the fibre functions of all the singularities in a single polynomial (the T–function) and they hold for planar curves and for space curves as well. In addition, we show how to handle a family of conjugate parametric points to deal with algebraic numbers.

The ideas presented open several important ways that may be used to obtain significant results concerning rational parametrizations of surfaces. In a future work, this problem will be developed in more detail and some important results are expected to be provided.

The structure of the paper is as follows. Sections 2 and 3 are devoted to the study of plane curves. In particular, in Section 2, we introduce the terminology that will be used throughout this paper as well as some previous results. In Section 3, we introduce the T–function and we present the main result of the paper. It claims that the factorization of the T–function provides the fibre functions of the different singularities of the curve. The proof of this result as well as some previous technical lemmas appear in Section 5. Section 4 is devoted to generalize the results in Section 3 to parametric space curves in any dimension. Throughout the whole paper, we outline all the results obtained with illustrative examples.

2 Analysis and computation of the fibre

Let \mathcal{C} be a rational (projective) plane curve defined by the projective parametrization

$$\mathcal{P}(t) = (p_1(t) : p_2(t) : p(t)) \in \mathbb{P}^2(\mathbb{K}(t)),$$

where $\gcd(p_1, p_2, p) = 1$, and \mathbb{K} is an algebraically closed field of characteristic zero. We assume that \mathcal{C} is not a line (a line does not have multiple points). Let $d_1 = \deg(p_1)$, $d_2 = \deg(p_2)$, $d_3 = \deg(p)$, and $d = \max\{d_1, d_2, d_3\}$. Thus, we may write p_1 , p_2 and p as

$$\begin{cases} p_1(t) = a_0 + a_1t + a_2t^2 + \cdots + a_d t^d \\ p_2(t) = b_0 + b_1t + b_2t^2 + \cdots + b_d t^d \\ p(t) = c_0 + c_1t + c_2t^2 + \cdots + c_d t^d. \end{cases}$$

Associated with $\mathcal{P}(t)$, we consider the induced rational map $\psi_{\mathcal{P}} : \mathbb{K} \rightarrow \mathcal{C} \subset \mathbb{P}^2(\mathbb{K}); t \mapsto \mathcal{P}(t)$. We denote by $\deg(\psi_{\mathcal{P}})$ the degree of the rational map

$\psi_{\mathcal{P}}$ (for further details see e.g. [20] pp.143, or [9] pp.80). As an important result, we recall that the birationality of $\psi_{\mathcal{P}}$, i.e. the properness of $\mathcal{P}(t)$, is characterized by $\deg(\psi_{\mathcal{P}}) = 1$ (see [9] and [20]). Also, we recall that the degree of a rational map can be seen as the cardinality of the fibre of a generic element (see Theorem 7, pp. 76 in [20]). We will use this characterization in our reasoning. For this purpose, we denote by $\mathcal{F}_{\mathcal{P}}(P)$ the fibre of a point $P \in \mathcal{C}$ via the parametrization $\mathcal{P}(t)$; that is

$$\mathcal{F}_{\mathcal{P}}(P) = \mathcal{P}^{-1}(P) = \{t \in \mathbb{K} \mid \mathcal{P}(t) = P\}.$$

In general, it holds that $P \in \mathcal{C}$ if and only if $\mathcal{F}_{\mathcal{P}}(P) \neq \emptyset$, although an exception can be found for *the limit point* of the parametrization.

Definition 1. *We define the limit point of the parametrization $\mathcal{P}(t)$ as*

$$P_L = \lim_{t \rightarrow \infty} \mathcal{P}(t)/t^d = (a_d : b_d : c_d).$$

Note that $P_L \in \mathcal{C}$ since $\mathcal{P}(t)/t^d = \mathcal{P}(t) \in \mathcal{C}$, for $t \in \mathbb{K}$, and \mathcal{C} is a closed set. Furthermore, we observe that, given a parametrization $\mathcal{P}(t)$, there always exists an associated limit point, and it is unique.

The limit point is reachable via the parametrization $\mathcal{P}(t)$, if there exists $t_0 \in \mathbb{K}$ such that $\mathcal{P}(t_0) = P_L$. However, the value $t_0 \in \mathbb{K}$ could not exist, and then $\mathcal{F}_{\mathcal{P}}(P_L) = \emptyset$. Taking into account this statement, if P_L is not an affine point or it is a reachable affine point, we have that $\mathcal{P}(t)$ is a *normal parametrization*. Otherwise, we say that $\mathcal{P}(t)$ is not normal and P_L is the *critical point* (see Subsection 6.3 in [19]). Further properties of the limit point are stated and proved in [4].

In Subsection 2.2. in [19], it is stated that the degree of a dominant rational map between two varieties of the same dimension is the cardinality of the fiber of a generic element. Therefore, in the case of the mapping $\psi_{\mathcal{P}}$, this implies that almost all points of \mathcal{C} (except at most a finite number of points) are generated via $\mathcal{P}(t)$ by the same number of parameter values, and this number is the degree of $\psi_{\mathcal{P}}$. Thus, intuitively speaking, the degree measures the number of times the parametrization traces the curve when the parameter takes values in \mathbb{K} . Taking into account this intuitive notion, the degree of the mapping $\psi_{\mathcal{P}}$ is also called *the tracing index* of $\mathcal{P}(t)$. In order

to compute the tracing index, the following polynomials are considered,

$$\begin{cases} G_1(s, t) := p_1(s)p(t) - p(s)p_1(t) \\ G_2(s, t) := p_2(s)p(t) - p(s)p_2(t) \\ G_3(s, t) := p_1(s)p_2(t) - p_2(s)p_1(t) \end{cases} \quad (1)$$

and $G(s, t) = \gcd(G_1(s, t), G_2(s, t), G_3(s, t))$. In the following theorem, we compute the tracing index of $\mathcal{P}(t)$ using the polynomial $G(s, t)$ (see Subsection 4.3 in [19]).

Theorem 1. *It holds that $\deg(\psi_{\mathcal{P}}) = \deg_t(G)$.*

Remark 1. *We observe that:*

1. *The polynomials G_1, G_2 and G_3 satisfy that $G_i(s, t) = -G_i(t, s)$. Clearly, $G(s, t)$ also has this property.*
2. *Taking into account the above statement, it holds that $\deg_s(G_i) = \deg_t(G_i)$ for $i = 1, 2, 3$, and $\deg_s(G) = \deg_t(G)$.*
3. *It holds that $\deg_t(G_1) = \max\{d_1, d_3\}$. Indeed: if $d_1 \neq d_3$, the statement trivially holds. If $d_1 = d_3$, $\deg_t(G_1)$ may decrease if $p_1(s)c_d - p(s)a_d = 0$. But this would imply that \mathcal{C} is a line, which is impossible by the assumption. Similarly, it holds that $\deg_t(G_2) = \max\{d_2, d_3\}$, and $\deg_t(G_3) = \max\{d_1, d_2\}$.*
4. *It holds that*

$$G(s, t) = \gcd(G_1(s, t), G_2(s, t)).$$

Indeed: since $p(t)G_3(s, t) = p_2(t)G_1(s, t) - p_1(t)G_2(s, t)$, if $h(s, t) \in \mathbb{K}[s, t]$ divides to $G_1(s, t)$ and $G_2(s, t)$, then $h(s, t)$ divides to $G_3(s, t)$ or $p(t)$. However, if $h(s, t)$ divides $p(t)$, then $h(s, t) = h(t)$ which would imply that there exists $t_0 \in \mathbb{K}$ such that $G_1(s, t_0) = G_2(s, t_0) = p(t_0) = 0$. Hence, $p_i(s)/p(s) \in \mathbb{K}$, $i = 1, 2$, and \mathcal{C} would be a line, which is impossible by the assumption. Similarly, it holds that

$$G(s, t) = \gcd(G_1(s, t), G_3(s, t)) = \gcd(G_2(s, t), G_3(s, t)).$$

Throughout this paper, we assume that $\mathcal{P}(t)$ is proper, that is $\deg(\psi_{\mathcal{P}}) = 1$. Otherwise, we can reparametrize the curve using, for instance, the results

in [13]. Under these conditions, it holds that the degree of \mathcal{C} is d (see Theorem 6 in [14]). In addition, $G(t, s) = t - s$ (see Theorem 1) and the cardinality of the fibre for a generic point of \mathcal{C} is 1, although for a particular point it can be different.

In order to analyze these special points, in the following, we consider a particular point $P = (a, b, c) \in \mathcal{C}$. The fibre of P consists of the values $t \in \mathbb{K}$ such that $\mathcal{P}(t) = P$, that is, those which satisfy the *fibre equations*, defined as

$$\begin{cases} \phi_1(t) := ap(t) - cp_1(t) = 0 \\ \phi_2(t) := bp(t) - cp_2(t) = 0 \\ \phi_3(t) := ap_2(t) - bp_1(t) = 0. \end{cases} \quad (2)$$

Hence, the fibre of P is given by the common roots of these equations, which motivates the following definition:

Definition 2. *Given $P \in \mathbb{P}^2(\mathbb{K})$ and the rational parametrization $\mathcal{P}(t) \in \mathbb{P}^2(\mathbb{K}(t))$, we define the fibre function of P at $\mathcal{P}(t)$ as*

$$H_P(t) := \gcd(\phi_1, \phi_2, \phi_3).$$

Thus, $t_0 \in \mathcal{F}_{\mathcal{P}}(P)$ if and only if $H_P(t_0) = 0$.

Remark 2. *Depending on whether P is an affine point or an infinity point, the fibre function can be expressed as follows:*

- *If P is an affine point, then $c \neq 0$. Thus, ϕ_3 can be obtained from ϕ_1 and ϕ_2 and, therefore, $H_P(t) = \gcd(\phi_1(t), \phi_2(t))$.*
- *If P is an infinity point, then $c = 0$. Thus, ϕ_1 and ϕ_2 are equivalent to $p(t) = 0$ (note that $a \neq 0$ or $b \neq 0$) and, therefore, $H_P(t) = \gcd(p(t), \phi_3(t))$.*

Note that the functions ϕ_1 , ϕ_2 and ϕ_3 depend on P and $\mathcal{P}(t)$. However, for the sake of simplicity, we do not represent this fact in the notation.

In the following, we show how the fibre function of P is related with the tangents of \mathcal{C} at P , and with the multiplicity of P . For this purpose, we first recall that P is a *point of multiplicity ℓ* on \mathcal{C} if and only if all the derivatives of F (where F denotes the implicit polynomial defining \mathcal{C}) up to and including

those of $(\ell - 1)$ -th order, vanish at P but at least one ℓ -th derivative does not vanish at P . We denote it by $\text{mult}_P(\mathcal{C})$. The point P is called a *simple point* on \mathcal{C} if and only if $\text{mult}_P(\mathcal{C}) = 1$. If $\text{mult}_P(\mathcal{C}) = \ell > 1$, then we say that P is a multiple or singular point (or singularity) of multiplicity ℓ on \mathcal{C} or an ℓ -fold point. Clearly $P \notin \mathcal{C}$ if and only if $\text{mult}_P(\mathcal{C}) = 0$.

Observe that the multiplicity of \mathcal{C} at P is given as the order of the Taylor expansion of F at P . The tangents to \mathcal{C} at P are the irreducible factors of the first non-vanishing form in the Taylor expansion of F at P , and the multiplicity of a tangent is the multiplicity of the corresponding factor. If all the ℓ tangents at the ℓ -fold point P are different, then this singularity is called *ordinary*, and *non-ordinary* otherwise. Thus, we say that the character of P is either ordinary or non-ordinary.

In [14], it is shown how to compute the singularities and its corresponding multiplicities from a given parametrization defining a rational plane curve. Furthermore, it is provided a method for computing the tangents and for analyzing the non-ordinary singularities. In particular, the following theorem and corollary are proved.

Theorem 2. *Let \mathcal{C} be a rational algebraic curve defined by a proper parametrization $\mathcal{P}(t)$, with limit point P_L . Let $P \neq P_L$ be a point of \mathcal{C} and let $H_P(t) = \prod_{i=1}^n (t - s_i)^{k_i}$ be its fibre function (under $\mathcal{P}(t)$). Then, \mathcal{C} has n tangents at P of multiplicities k_1, \dots, k_n , respectively.*

Remark 3. *It can not be ensured that two different values of t , namely s_{i_0} and s_{i_1} , provide different tangents. Thus, we could have a same tangent (at $\mathcal{P}(s_{i_0}) = \mathcal{P}(s_{i_1})$) of multiplicity $k_{i_0} + k_{i_1}$.*

Corollary 1. *Let \mathcal{C} be a rational algebraic curve defined by a proper parametrization $\mathcal{P}(t)$, with limit point P_L . Let $P \neq P_L$ be a point of \mathcal{C} and let $H_P(t)$ be its fibre function (under $\mathcal{P}(t)$). Then, $\text{mult}_P(\mathcal{C}) = \deg(H_P(t))$.*

Example 1. *Let \mathcal{C} be the rational plane curve defined by the projective parametrization*

$$\mathcal{P}(t) = (-t^3 - 5t^2 - 7t - 3 : t^4 + 7t^3 + 17t^2 + 17t + 6 : t^4 + 1) \in \mathbb{P}^2(\mathbb{C}(t)).$$

Let us compute $H_P(t)$, where $P = (0 : 0 : 1)$. Since P is an affine point, we can obtain $H_P(t)$ from ϕ_1 and ϕ_2 (see Remark 2). Since $\phi_1(t) = -2t^3 - 10t^2 - 14t - 6$ and $\phi_2(t) = 2t^4 + 14t^3 + 34t^2 + 34t + 12$, we get that

$$H_P(t) = \gcd(\phi_1, \phi_2) = 2(t + 3)(t + 1)^2.$$

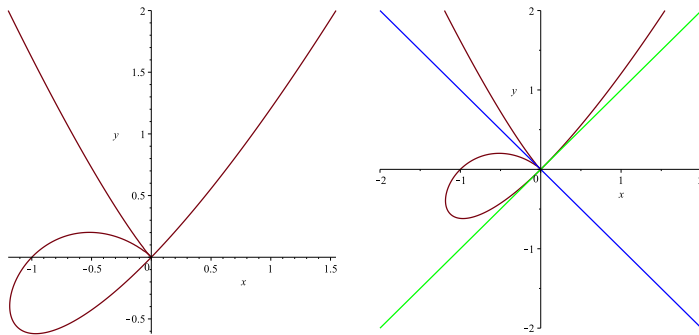


Figure 1: Triple point with two tangents

Therefore, $\mathcal{P}(-1) = \mathcal{P}(-3) = P$, and applying Theorem 2, we deduce that \mathcal{C} has at P two different tangents, one of multiplicity 1 and the other one of multiplicity 2. The parametrizations defining these tangents are given as

$$\tau_1(t) = \mathcal{P}(-3) + \mathcal{P}'(-3)t \quad \text{and} \quad \tau_2(t) = \mathcal{P}(-1) + \frac{\mathcal{P}''(-1)}{2}t^2,$$

respectively (see [14]). Note that these tangents are the lines $y = x$ and $y = -x$ (see Figure 1). Finally, we conclude that P is a non-ordinary point of multiplicity 3 (see Corollary 1).

3 Resultants and singularities

In Section 2, we show that, given a rational proper parametrization, $\mathcal{P}(t)$, the multiplicity of a given point, $P \neq P_L$, is the cardinality of the fibre of $\mathcal{P}(t)$ at P (see Corollary 1). That is, the multiplicity of $P = \mathcal{P}(s_0)$, $s_0 \in \mathbb{K}$ is given by the cardinality of the set

$$\mathcal{F}_{\mathcal{P}}(\mathcal{P}(s_0)) = \{t \in \mathbb{K} : \mathcal{P}(t) = \mathcal{P}(s_0)\}$$

(note that we are assuming that $P \neq P_L$). Observe that $s_0 \in \mathcal{F}_{\mathcal{P}}(\mathcal{P}(s_0))$ and hence, the cardinality of $\mathcal{F}_{\mathcal{P}}(\mathcal{P}(s_0))$ is greater than or equal to 1. Thus, $\mathcal{P}(s_0)$ is a singular point if and only if the cardinality of $\mathcal{F}_{\mathcal{P}}(\mathcal{P}(s_0))$ is greater than 1.

Taking into account the above statement, in this section, we show how the different factors of a univariate resultant computed from the polynomials $G_i(s, t)$, $i = 1, 2, 3$, are exactly the fibre functions of the singularities of \mathcal{C} . Thus, in particular, the singularities of \mathcal{C} and its corresponding multiplicities are determined. The idea for the construction of the resultant is that a point $\mathcal{P}(s_0) \in \mathcal{C}$, $s_0 \in \mathbb{K}$, is a singularity if and only if $\deg(H_{\mathcal{P}(s_0)}(t)) > 1$ (i.e. the fibre equations of $\mathcal{P}(s_0)$ have more than one common solution).

For this purpose, we first assume that $\mathcal{P}(s_0)$ is an affine point. Thus, Remark 2 implies that the fibre equations are given by

$$\begin{cases} p_1(t)p(s_0) - p_1(s_0)p(t) = 0 \\ p_2(t)p(s_0) - p_2(s_0)p(t) = 0. \end{cases}$$

Note that this is equivalent to $G_1(s_0, t) = G_2(s_0, t) = 0$, where $G_1(s, t)$ and $G_2(s, t)$ are the polynomials introduced in (1).

Then, $\mathcal{P}(s_0)$ is a singular point if and only if $G_1(s_0, t)$ and $G_2(s_0, t)$ have more than one common root or, equivalently, if and only if the polynomials $G_1(s_0, t)/(t - s_0)$ and $G_2(s_0, t)/(t - s_0)$ have a common root (we note that s_0 is already a root of $G_1(s_0, t)$ and $G_2(s_0, t)$). This implies that

$$\text{Res}_t \left(\frac{G_1(s_0, t)}{t - s_0}, \frac{G_2(s_0, t)}{t - s_0} \right) = 0.$$

Hence, given the polynomial

$$R(s) = \text{Res}_t \left(\frac{G_1(s, t)}{t - s}, \frac{G_2(s, t)}{t - s} \right),$$

if the point $\mathcal{P}(s_0)$ is singular, then $R(s_0) = 0$. In fact, in [1], it is proved that this resultant provides the product of the fibre functions of the singularities of the curve, in the case that $\mathcal{P}(t)$ is a polynomial parametrization. A generalization for the case of a given rational parametrization (not necessarily polynomial) is presented in [6].

Thus, $R(s)$ can be used to compute the singularities of the curve, but some problems could appear. First, the values $s \in \mathbb{K}$ that provide singular

points are roots of the polynomial R but the reciprocal is not true; i.e. a root of R may not provide a singular point. In addition, we are assuming that the singularity is an affine point, but also singularities at infinity have to be detected.

The T -function, that we introduce below, improves the properties of $R(s)$ and characterizes the singular points of \mathcal{C} (affine and at infinity). In order to introduce it, we need to consider

$$\delta_i := \deg_t(G_i), \quad \lambda_{ij} := \min\{\delta_i, \delta_j\}, \quad G_i^*(s, t) := \frac{G_i(s, t)}{t - s} \in \mathbb{K}[s, t]$$

and

$$R_{ij}(s) := \text{Res}_t(G_i^*, G_j^*) \in \mathbb{K}[s] \quad \text{for } i, j = 1, 2, 3, \quad i < j.$$

Definition 3. We define the T -function of the parametrization $\mathcal{P}(t)$ as

$$T(s) = R_{12}(s)/p(s)^{\lambda_{12}-1}.$$

In the following we show that this function provides essential information concerning the singularities of the given curve \mathcal{C} (see Theorem 3). To start with, the following proposition claims that the T -function can be defined similarly from $R_{13}(s)$ or $R_{23}(s)$. In addition, in Corollary 2, we prove that $T(s)$ is a polynomial.

Proposition 1. It holds that

$$T(s) = \frac{R_{12}(s)}{p(s)^{\lambda_{12}-1}} = \frac{R_{13}(s)}{p_1(s)^{\lambda_{13}-1}} = \frac{R_{23}(s)}{p_2(s)^{\lambda_{23}-1}}.$$

Proof: We distinguish two steps to prove the proposition:

Step 1

First, we show that

$$\frac{R_{12}(s)}{p(s)^{\lambda_{12}-1}} = \frac{R_{13}(s)}{p_1(s)^{\lambda_{13}-1}}.$$

For this purpose, we see the polynomial $G_1(s, t) \in \mathbb{K}[s, t]$ as a polynomial in the variable t that is, $G_1(s, t) \in (\mathbb{K}[s])[t]$. Since $\deg_t(G_1) = \deg_s(G_1) = \delta_1$

(see Remark 1), then G_1 has δ_1 roots (in the variable t), and one of them is $t = s$. Thus, we may write

$$G_1(s, t) = \text{lc}_t(G_1)(t - s)(t - \alpha_1(s)) \cdots (t - \alpha_{\delta_1-1}(s))$$

and

$$G_1^*(s, t) = \text{lc}_t(G_1)(t - \alpha_1(s)) \cdots (t - \alpha_{\delta_1-1}(s)), \quad (3)$$

where $\text{lc}_t(\cdot)$ denotes the leader coefficient with respect to the variable t of a polynomial (\cdot) . Now, taking into account the properties of the resultants (see e.g. [8], [19], [23]), we get that

$$R_{12}(s) := \text{Res}_t(G_1^*, G_2^*) = \text{lc}_t(G_1^*)^{\delta_2-1} \prod_{i=1}^{\delta_1-1} G_2^*(s, \alpha_i(s)). \quad (4)$$

Note that $G_2(s, t) = p_2(s)p(t) - p(s)p_2(t)$, and thus $G_2(s, \alpha_i(s)) = p_2(s)p(\alpha_i(s)) - p(s)p_2(\alpha_i(s))$. Furthermore, since $G_1(s, \alpha_i(s)) = p_1(s)p(\alpha_i(s)) - p(s)p_1(\alpha_i(s)) = 0$, we get that

$$p(\alpha_i(s)) = \frac{p_1(\alpha_i(s))}{p_1(s)} p(s).$$

Therefore,

$$\begin{aligned} G_2(s, \alpha_i(s)) &= p_2(s) \frac{p_1(\alpha_i(s))}{p_1(s)} p(s) - p(s) p_2(\alpha_i(s)) = \\ &= (p_1(\alpha_i(s)) p_2(s) - p_2(\alpha_i(s)) p_1(s)) \frac{p(s)}{p_1(s)} = G_3(s, \alpha_i(s)) \frac{p(s)}{p_1(s)} \end{aligned}$$

which implies that

$$G_2^*(s, \alpha_i(s)) = G_3^*(s, \alpha_i(s)) \frac{p(s)}{p_1(s)}.$$

Now we substitute in (4) and we get

$$R_{12}(s) = \left(\text{lc}_t(G_1^*)^{\delta_2-1} \prod_{i=1}^{\delta_1-1} G_3^*(s, \alpha_i(s)) \right) \left(\frac{p(s)}{p_1(s)} \right)^{\delta_1-1},$$

which can be expressed as

$$R_{12}(s) = R_{13}(s) \text{lc}_t(G_1^*)^{\delta_2-\delta_3} \left(\frac{p(s)}{p_1(s)} \right)^{\delta_1-1}.$$

Hence, we only have to prove that

$$\text{lc}_t(G_1^*)^{\delta_2-\delta_3} \left(\frac{p(s)}{p_1(s)} \right)^{\delta_1-1} = \frac{p(s)^{\lambda_{12}-1}}{p_1(s)^{\lambda_{13}-1}}, \quad (5)$$

and, for this purpose, we consider different cases depending on d_1 , d_2 and d_3 (that is, on the degrees of p_1 , p_2 and p). We remind that $\delta_1 = \max\{d_1, d_3\}$, $\delta_2 = \max\{d_2, d_3\}$ and $\delta_3 = \max\{d_1, d_2\}$ (see Remark 1).

- Case 1: Let $d_1 < d_3$. Then, $\delta_1 = d_3 \leq \delta_2$, and $\lambda_{12} = \delta_1$. In addition, it holds that $\text{lc}_t(G_1^*) = \text{lc}_t(G_1) = p_1(s)$ since $G_1(s, t) = p_1(s)p(t) - p(s)p_1(t)$ and $d_1 < d_3$. Thus,

$$\text{lc}_t(G_1^*)^{\delta_2-\delta_3} \left(\frac{p(s)}{p_1(s)} \right)^{\delta_1-1} = \frac{p(s)^{\delta_1-1}}{p_1(s)^{\delta_1-\delta_2+\delta_3-1}} = \frac{p(s)^{\lambda_{12}-1}}{p_1(s)^{\delta_1-\delta_2+\delta_3-1}}.$$

In order to check that (5) holds, we only have to prove that $\delta_1 - \delta_2 + \delta_3 = \lambda_{13}$. Let us see that this equality holds in the following situations:

- $\delta_1 < \delta_3$: then, $d_2 > d_1, d_3$ which implies that $\delta_2 = \delta_3 = d_2$ and $\lambda_{13} = \delta_1$.
 - $\delta_1 > \delta_3$: then, $d_3 > d_1, d_2$ which implies that $\delta_1 = \delta_2 = d_3$ and $\lambda_{13} = \delta_3$.
 - $\delta_1 = \delta_3$: then, $d_3 = \max\{d_1, d_2\}$ which implies that $d_1 < d_2 = d_3$ and $\delta_1 = \delta_2 = \delta_3$.
- Case 2: Let $d_1 > d_3$. Then, $\delta_1 = d_1 \leq \delta_3$, which implies that $\lambda_{13} = \delta_1$. In addition, $\text{lc}_t(G_1^*) = p(s)$, and then

$$\text{lc}_t(G_1^*)^{\delta_2-\delta_3} \left(\frac{p(s)}{p_1(s)} \right)^{\delta_1-1} = \frac{p(s)^{\delta_1+\delta_2-\delta_3-1}}{p_1(s)^{\delta_1-1}} = \frac{p(s)^{\delta_1+\delta_2-\delta_3-1}}{p_1(s)^{\lambda_{13}-1}}.$$

Thus, we only have to prove that $\delta_1 + \delta_2 - \delta_3 = \lambda_{12}$. For this purpose, we reason similarly as in Case 1 by considering the following situations: $\delta_1 < \delta_2$, $\delta_1 > \delta_2$ and $\delta_1 = \delta_2$.

- Case 3: Let $d_1 = d_3 < d_2$. Then, $\delta_2 = \delta_3 = d_2$, and thus $\text{lc}_t(G_1^*)^{\delta_2-\delta_3} = 1$. In addition, $\delta_1 \leq \delta_2 = \delta_3$, which implies that $\lambda_{12} = \lambda_{13} = \delta_1$.
- Case 4: Let $d_2 < d_1 = d_3$. In this case, we have that $\delta_1 = \delta_2 = \delta_3$ and then, (5) trivially holds.

- Case 5: Let $d_1 = d_2 = d_3$. This case is similar to Case 4.

Step 2

Let us prove that

$$\frac{R_{12}(s)}{p(s)^{\lambda_{12}-1}} = \frac{R_{23}(s)}{p_2(s)^{\lambda_{23}-1}}.$$

For this purpose, we observe that, up to constants in $\mathbb{K} \setminus \{0\}$, it holds that $R_{12}(s) = R_{21}(s)$. Thus, we may write

$$R_{12}(s) = R_{21}(s) = \text{lc}_t(G_2^*)^{\delta_1-1} \prod_{i=1}^{\delta_2-1} G_1^*(s, \beta_i(s))$$

where

$$G_2^*(s, t) = \text{lc}_t(G_2)(t - \beta_1(s)) \cdots (t - \beta_{\delta_2-1}(s)).$$

Now, we observe that these equalities are equivalent to (4) and (3), respectively. Thus, reasoning similarly as above, we obtain that

$$R_{12}(s) = R_{23}(s) \text{lc}_t(G_2^*)^{\delta_1-\delta_3} \left(\frac{p(s)}{p_2(s)} \right)^{\delta_2-1}$$

and that

$$\text{lc}_t(G_2^*)^{\delta_1-\delta_3} \left(\frac{p(s)}{p_2(s)} \right)^{\delta_2-1} = \frac{p(s)^{\lambda_{12}-1}}{p_2(s)^{\lambda_{23}-1}}.$$

□

Corollary 2. *It holds that $T(s) \in \mathbb{K}[s]$.*

Proof: Let us assume that $T(s)$ is not a polynomial. Then, we simplify the rational function and we write

$$\frac{R_{12}(s)}{p(s)^{\lambda_{12}-1}} = \frac{M_{12}(s)}{\bar{p}(s)},$$

where $M_{12}(s) \in \mathbb{K}[s]$, $\bar{p}(s) \in \mathbb{K}[s] \setminus \mathbb{K}$ and $\text{gcd}(M_{12}, \bar{p}) = 1$. Note that \bar{p} divides $p^{\lambda_{12}-1}$.

Similarly, from Proposition 1, we have that

$$\left\{ \begin{array}{l} \frac{R_{13}(s)}{p_1(s)^{\lambda_{13}-1}} = \frac{M_{13}(s)}{\bar{p}_1(s)} \quad \text{where } \bar{p}_1 \text{ divides } p_1^{\lambda_{13}-1}, \text{ and } \gcd(M_{13}, \bar{p}_1) = 1 \\ \frac{R_{23}(s)}{p_2(s)^{\lambda_{23}-1}} = \frac{M_{23}(s)}{\bar{p}_2(s)} \quad \text{where } \bar{p}_2 \text{ divides } p_2^{\lambda_{23}-1}, \text{ and } \gcd(M_{23}, \bar{p}_2) = 1. \end{array} \right.$$

Furthermore, we have that (see Proposition 1)

$$\frac{M_{12}(s)}{\bar{p}(s)} = \frac{M_{13}(s)}{\bar{p}_1(s)} = \frac{M_{23}(s)}{\bar{p}_2(s)}$$

which implies that

$$M_{12}(s)\bar{p}_1(s) = M_{13}\bar{p}(s) \text{ and } M_{23}(s)\bar{p}_1(s) = M_{13}\bar{p}_2(s).$$

Taking into account that $\gcd(M_{12}, \bar{p}) = \gcd(M_{13}, \bar{p}_1) = \gcd(M_{23}, \bar{p}_2) = 1$, and the above equalities, we get that $\bar{p}_1 = \bar{p}_2 = \bar{p}$. Then, we deduce that \bar{p} divides p , p_1 and p_2 , which is impossible since $\gcd(p_1, p_2, p) = 1$. \square

In the following theorem, we show how the ordinary singularities of \mathcal{C} can be determined from the T–function. In fact, $T(s)$ describes totally the singularities of the curve, since its factorization provides the fibre functions of each singularity as well as its corresponding multiplicity. From the fibre function, $H_P(t)$, of a point P , one obtains the multiplicity of P , its fibre, and the tangent lines at P (see Section 2).

An alternative approach for computing this factorization, based on the construction of μ –basis, can be found in [6].

In Theorem 3, we assume that \mathcal{C} has only ordinary singularities. Otherwise, for applying this theorem, we should apply quadratic transformations (blow-ups) for birationally transforming \mathcal{C} into a curve with only ordinary singularities (see Chapter 2 in [19]). For such a curve the following theorem holds.

Theorem 3. (Main theorem) *Let \mathcal{C} be a rational algebraic curve defined by a parametrization $\mathcal{P}(t)$, with limit point P_L . Let P_1, \dots, P_n be the singular points of \mathcal{C} , with multiplicities m_1, \dots, m_n respectively. Let us assume that they are ordinary singularities and that $P_i \neq P_L$ for $i = 1, \dots, n$. Then, it holds that*

$$T(s) = \prod_{i=1}^n H_{P_i}(s)^{m_i-1}.$$

This theorem will be proved in Section 5 and a generalization for the case of space curves of any dimension will be presented in Section 4. Moreover, in [4], we prove that the theorem holds also if P_L is a singularity. Finally, an analogous result which admits the existence of non-ordinary singularities in the curve will be developed in a future work.

Corollary 3. *Let \mathcal{C} be a rational plane curve of degree d such that all its singularities are ordinary. Let $\mathcal{P}(t)$ be a parametrization of \mathcal{C} such that P_L is regular. It holds that $\deg(T) = (d - 1)(d - 2)$.*

Proof: From Theorem 3 and Corollary 1, we have that $\deg(T) = \sum_{i=1}^n m_i(m_i - 1)$, where P_1, \dots, P_n are the singular points, and m_1, \dots, m_n its corresponding multiplicities. Since \mathcal{C} is a rational curve, its genus is 0 and thus $\sum_{i=1}^n m_i(m_i - 1) = (d - 1)(d - 2)$ (see Chapter 3 in [19]). \square

Example 2. *Let \mathcal{C} be the rational plane curve defined by the projective parametrization*

$$\mathcal{P}(t) = (t^5 - 5t^4 + 5t^3 + 5t^2 - 6t : t^5 + 5t^4 + 5t^3 - 5t^2 - 6t : t^4 - 13t^2 + 36) \in \mathbb{P}^2(\mathbb{C}(t)).$$

We compute the T -function and we get that, up to constants in \mathbb{C} ,

$$T(s) = (t - 2)(t - 3)(t + 3)(t + 2)t^2(t^2 + 6)(t - 1)^2(t + 1)^2.$$

Thus, from Theorem 3, we deduce that the fibre function of each singularity of \mathcal{C} appears in the polynomial T . Let us analyze the different factors of T :

- *The factors with power 2 correspond to triple points. Indeed: these factors are t , $(t - 1)$ and $(t + 1)$, and we have that $P_1 = \mathcal{P}(0) = \mathcal{P}(1) = \mathcal{P}(-1) = (0 : 0 : 1)$. Then, P_1 is a triple point whose fibre function is*

$$H_{P_1}(t) = (t - 1)(t + 1)t.$$

- *The factors with power 1 correspond to different double points. In order to determine the associated factors, we should compute the corresponding fibre functions. For the factors $(t - 2)$ and $(t - 3)$, we have that $P_2 = \mathcal{P}(2) = \mathcal{P}(3) = (0 : 1 : 0)$ and thus*

$$H_{P_2}(t) = (t - 2)(t - 3).$$

This implies that P_2 is a double point at infinity.

- Similarly, if we consider the factors $(t + 2)$ and $(t + 3)$, we get that $P_3 = \mathcal{P}(-2) = \mathcal{P}(-3) = (1 : 0 : 0)$, and its fibre function is

$$H_{P_3}(t) = (t + 2)(t + 3),$$

which implies that P_3 is a double point at infinity.

- Finally, the factor $(t^2 + 6)$ provides the point $P_4 = \mathcal{P}(-I\sqrt{6}) = \mathcal{P}(+I\sqrt{6}) = (-7/5 : 7/5 : 1)$. Hence, P_4 is an affine double point and its fibre function is

$$H_{P_4}(t) = (t^2 + 6).$$

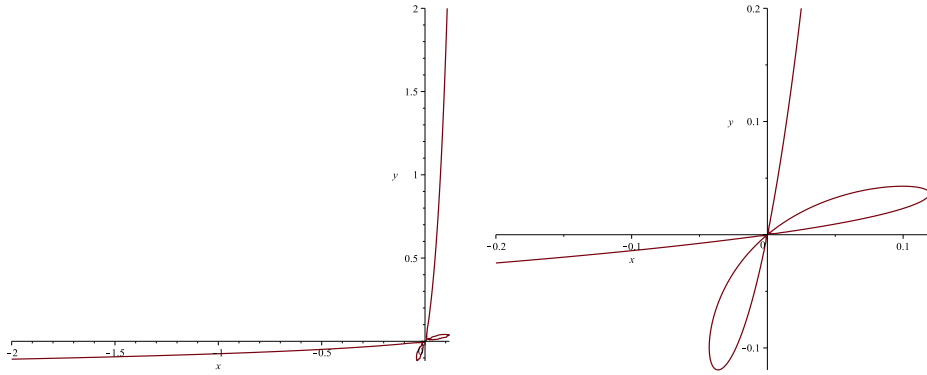


Figure 2: Curve \mathcal{C} (left) and a neighborhood of the triple point P_1 (right)

Note that,

$$T(s) = H_{P_1}(s)^{m_1-1} H_{P_2}(s)^{m_2-1} H_{P_3}(s)^{m_3-1} H_{P_4}(s)^{m_4-1}.$$

Furthermore, we observe that Corollary 3 is verified. Indeed, we have that $d = 5$ and $\deg_t(T) = 12 = (d - 1)(d - 2)$.

In Figure 2, we plot the curve \mathcal{C} , and a neighborhood of the triple point P_1 . Observe that P_4 is an isolated point.

In Example 2, we have been able to determine the singularities of \mathcal{C} and its corresponding multiplicities, by computing the factors of the polynomial

$T(s)$. However, in general, one needs to introduce algebraic numbers during the computations. In the following, we present a method that allows us to determine the factors of the polynomial $T(s)$ (and thus, the singularities of a curve) without directly using algebraic numbers. For this purpose, we introduce the notion of *family of conjugate parametric points* (see [14]), which generalizes the concept of family of conjugate points (see e.g. Chapter 2 in [19]). The idea is to collect points whose coordinates depend algebraically on all the conjugate roots of the same irreducible polynomial $m(t)$. The computations on such a family of points can be carried out by using the polynomial $m(t)$.

Definition 4. *Let*

$$\mathcal{G} = \{(p_1(\alpha) : p_2(\alpha) : p(\alpha)) \mid m(\alpha) = 0\} \subset \mathbb{P}^2(\mathbb{K}).$$

The set \mathcal{G} is called a family of conjugate parametric points over \mathbb{K} if the following conditions are satisfied:

1. $p_1, p_2, p, m \in \mathbb{K}[t]$ and $\gcd(p_1, p_2, p) = 1$.
2. m is irreducible.
3. $\deg(p_1), \deg(p_2), \deg(p) < \deg(m)$.

We denote such a family by $\mathcal{G} = \{\mathcal{P}(s)\}_{m(s)} = \{(p_1(s) : p_2(s) : p(s))\}_{m(s)}$.

Condition 2 in Definition 4 can be stated considering that m is only square-free (see Definition 12 in [14]). However, in order to prove Theorem 4, one needs m to be an irreducible polynomial (see Theorem 16 in [14]). Hence, using the above definition, in [14] it is proved the following theorem.

Theorem 4. *The singularities of the curve \mathcal{C} can be decomposed as a finite union of families of conjugate parametric points over \mathbb{K} such that all points in the same family have the same multiplicity and character.*

If some singularities of the given curve are in a family $\mathcal{G} = \{\mathcal{P}(s)\}_{m(s)} = \{(p_1(s) : p_2(s) : p(s))\}_{m(s)}$, then the polynomial $m(s)$ will be an irreducible factor of the T–function. In this case, Theorem 5 allows us to determine the singularities provided by \mathcal{G} and their corresponding multiplicities.

Theorem 5. *Let $m(s)$ be an irreducible polynomial such that $m(s)^{k-1}, k \in \mathbb{N}, k \geq 1$, divides $T(s)$. Then, the roots of $m(s)$ determine the fibre of some singularities of multiplicity k that are defined by a family of conjugate parametric points. The number of singularities in such a family is $n = \deg(m(s))/k$.*

Proof: From Theorem 3, we get that the points in $\mathcal{G} = \{\mathcal{P}(s)\}_{m(s)}$ are singularities of multiplicity k . In addition, if $\mathcal{G} = \{P_1, \dots, P_n\}$, we have that

$$m(s) = \prod_{i=1}^n H_{P_i}(s),$$

where H_{P_1}, \dots, H_{P_n} are the fibre functions of the points P_1, \dots, P_n , respectively. From Corollary 1, we get that $\deg(H_{P_i}) = \text{mult}_{P_i}(\mathcal{C}), i = 1, \dots, n$, and since, in this case, $\text{mult}_{P_i}(\mathcal{C}) = k$, we conclude that $\deg(m(s)) = nk$. \square

Example 3. *Let \mathcal{C} be the rational curve defined by $\mathcal{P}(t) = (p_1(t) : p_2(t) : p(t)) \in \mathbb{P}^2(\mathbb{C}(t))$, where*

$$\begin{aligned} p_1(t) &= t^5 - 13t^4 + 63t^3 - 143t^2 + 152t - 60 \\ p_2(t) &= t^5 - 21t^4 + 157t^3 - 507t^2 + 706t - 336 \\ p(t) &= t^5 + 7t^4 - 2t^3 + t - 1. \end{aligned}$$

The T -function is given by

$$T(s) = 28161216(968585964 - 2319881360s + 2070988203s^2 - 904722886s^3 + 208513387s^4 - 24407436s^5 + 1145528s^6)(s-1)^2(s-2)^2(s-3)^2.$$

From the polynomial $T(s)$, we deduce that the singularities of \mathcal{C} are:

- *The triple point $P_1 = \mathcal{P}(1) = \mathcal{P}(2) = \mathcal{P}(3) = (0 : 0 : 1)$. The fibre function of P_1 is $H_{P_1}(s) = (s-1)(s-2)(s-3)$, and these factors appear with power 2 in the polynomial $T(s)$.*
- *Three double points associated to the irreducible factor*
 $m(s) = 968585964 - 2319881360s + 2070988203s^2 - 904722886s^3 + 208513387s^4 - 24407436s^5 + 1145528s^6$.

Since $m(s)$ appears with power 1 in the polynomial $T(s)$, we conclude that it is associated to double points. In addition, using Theorem 5,

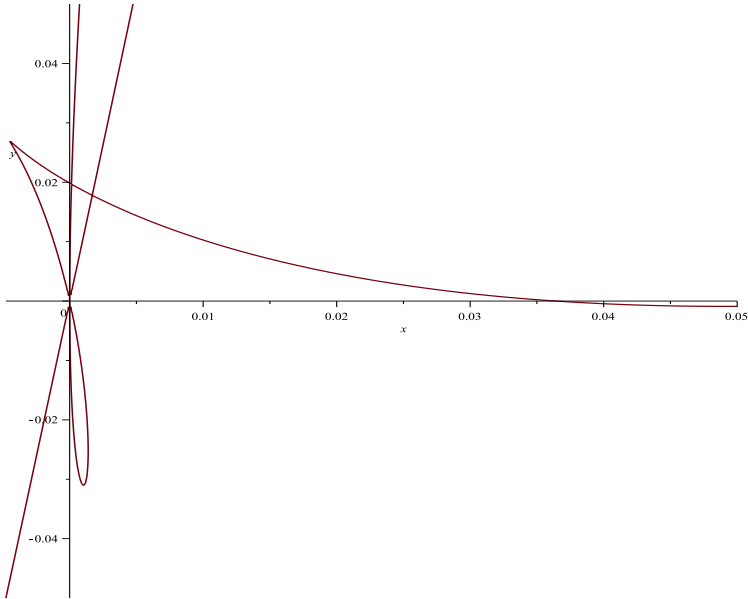


Figure 3: The curve \mathcal{C} that has a triple point and three double points

we get that this factor provides three different points (each point has a fibre function of degree 2 and the three fibre functions have power 1; multiplying these fibre functions we obtain the polynomial m).

In Figure 3, we plot the given curve \mathcal{C} , and we can see the singularities obtained (note that each singularity is real and affine).

4 The general case for rational space curves

In this section, we show that Theorem 3 can be applied for the case that the given curve \mathcal{C} is a rational space curve. In this case, we construct an equivalent polynomial to the T-function introduced for plane curves (see Definition 3), and we prove that this polynomial, which will be denoted as $T_E(s)$, describes totally the singularities of \mathcal{C} , since each factor of $T_E(s)$ is a power of the fibre function of one singularity of the given curve. This power is, in fact, the multiplicity of the singularity minus 1. We recall that from the fibre function of a point P , one may determine the multiplicity of P as well as its fibre $\mathcal{F}_{\mathcal{P}}(P)$ and the tangent lines of \mathcal{C} at P (see Section 2). The

method presented generalizes the results obtained in [18], since a complete classification of the singularities of a given space curve, via the factorization of a univariate resultant, is obtained.

In the following, we consider

$$\mathcal{P}(t) = (p_1(t) : \dots : p_n(t) : p(t)) \in \mathbb{P}^n(\mathbb{K}(t)), \quad \gcd(p_1, \dots, p_n, p) = 1,$$

a proper parametrization of a given rational space curve \mathcal{C} . In addition, we define the associated rational parametrization over $\mathbb{K}(Z)$, where $Z = (Z_1, \dots, Z_{n-2})$ and Z_1, \dots, Z_{n-2} are new variables, given by

$$\begin{aligned} \widehat{\mathcal{P}}(t) &= (\widehat{p}_1(t) : \widehat{p}_2(t) : \widehat{p}(t)) = \\ &= (p_1(t) : p_2(t) + Z_1 p_3(t) + \dots + Z_{n-2} p_n(t) : p(t)) \in \mathbb{P}^2((\mathbb{K}(Z))(t)). \end{aligned}$$

This notation is used for the sake of simplicity, but we note that $\widehat{\mathcal{P}}(t)$ depends on Z . Observe that $\widehat{\mathcal{P}}(t)$ is a proper parametrization of a rational plane curve $\widehat{\mathcal{C}}$ defined over the algebraic closure of $\mathbb{K}(Z)$.

We can establish a correspondence between the points of \mathcal{C} and the points of $\widehat{\mathcal{C}}$. More precisely, for each point $P = (a_1 : a_2 : a_3 : \dots : a_n : a_{n+1}) \in \mathcal{C}$ we have another point $\widehat{P} = (a_1 : a_2 + Z_1 a_3 + \dots + Z_{n-2} a_n : a_{n+1}) \in \widehat{\mathcal{C}}$. Moreover, this correspondence is bijective for the points satisfying that $a_1 \neq 0$ or $a_{n+1} \neq 0$. For these points, it holds that $\mathcal{F}_{\mathcal{P}}(P) = \mathcal{F}_{\widehat{\mathcal{P}}}(\widehat{P})$, which implies that $H_P(s) = H_{\widehat{P}}(s)$. Note that the polynomial H_P represents the fibre function of a point P in the space curve \mathcal{C} computed from $\mathcal{P}(t)$; i.e. the roots of H_P are the fibre of $P \in \mathcal{C}$ (this notion was introduced in Definition 2 for a given plane curve but it can be easily generalized for space curves).

Observe that the above correspondence may also be established between the places of \mathcal{C} and $\widehat{\mathcal{C}}$ centered at \widehat{P} and P , respectively. That is, for each place $\varphi(t) = (\varphi_1(t) : \varphi_2(t) : \varphi_3(t) : \dots : \varphi_n(t) : \varphi_{n+1}(t))$ of \mathcal{C} centered at P we have the place $\widehat{\varphi}(t) = (\varphi_1(t) : \varphi_2(t) + Z_1 \varphi_3(t) + \dots + Z_{n-2} \varphi_n(t) : \varphi_{n+1}(t))$ of $\widehat{\mathcal{C}}$ centered at \widehat{P} . Hence, the number of tangents of \mathcal{C} at P is the same that the number of tangents of $\widehat{\mathcal{C}}$ at \widehat{P} and, as a consequence, $\text{mult}_P(\mathcal{C}) = \text{mult}_{\widehat{P}}(\widehat{\mathcal{C}})$.

The correspondence above introduced is not bijective if $a_1 = a_{n+1} = 0$. In this case, we have that $\widehat{P} = (0 : 1 : 0) \in \widehat{\mathcal{C}}$ and the corresponding points in \mathcal{C} are all the points of the form $(0 : a_2 : a_3 : \dots : a_n : 0)$. Thus, if we have exactly ℓ points $P_1, \dots, P_\ell \in \mathcal{C}$ with $P_i = (0 : a_{2,i} : a_{3,i} : \dots : a_{n,i} : 0)$, $i =$

$1, \dots, \ell$, then $\mathcal{F}_{\widehat{\mathcal{P}}}(\widehat{P}) = \cup_{i=1}^{\ell} \mathcal{F}_{\mathcal{P}}(P_i)$. Hence, $H_{\widehat{\mathcal{P}}}(s) = \prod_{i=1}^{\ell} H_{P_i}(s)$ (note that $\mathcal{F}_{\mathcal{P}}(P_i) \cap \mathcal{F}_{\mathcal{P}}(P_j) = \emptyset$ if $i \neq j$) and $\text{mult}_{\widehat{\mathcal{P}}}(\widehat{\mathcal{C}}) = \sum_{i=1}^{\ell} \text{mult}_{P_i}(\mathcal{C})$.

Thus, in order to study the singularities of \mathcal{C} through those of $\widehat{\mathcal{C}}$, an additional difficulty arises when \mathcal{C} contains two or more points of the form $(0 : a_2 : a_3 : \dots : a_n : 0)$. Let us call them *bad points*. In the following, we may assume w.l.o.g. that we are not in this case, i.e. \mathcal{C} does not have two or more bad points. Otherwise, we apply a change of coordinates, and we consider the new parametrization $\mathcal{P}^*(t) = (p_1^*(t) : p_2(t) : \dots : p_n(t) : p(t))$ of the transformed curve \mathcal{C}^* , where $p_1^* = \sum_{i=1}^n \lambda_i p_i$, $\lambda_i \in \mathbb{K}$. By appropriately choosing $\lambda_1, \dots, \lambda_n \in \mathbb{K}$, we have that $\text{gcd}(p_1^*, p) = 1$ (note that $\text{gcd}(p_1, \dots, p_n, p) = 1$) and thus, \mathcal{C}^* does not have bad points.

Finally, we also note that additional points, which can not be written in the form $(a_1 : a_2 + Z_1 a_3 + \dots + Z_{n-2} a_n : a_{n+1})$, $a_i \in \mathbb{K}$, $i = 1, \dots, n+1$, may appear in the curve $\widehat{\mathcal{C}}$. Such points are obtained as $\widehat{\mathcal{P}}(t)$ for $t \in \mathbb{K}(Z) \setminus \mathbb{K}$ and they do not have a correspondence with any point of \mathcal{C} .

Under these conditions, let $\widehat{G}_1, \widehat{G}_2$ and \widehat{G}_3 be the equivalent polynomials to G_1, G_2 and G_3 (defined in (1)), constructed from the parametrization $\widehat{\mathcal{P}}(t)$. In addition, let $\widehat{\delta}_i := \text{deg}_t(\widehat{G}_i)$ and $\widehat{\lambda}_{ij} := \min\{\widehat{\delta}_i, \widehat{\delta}_j\}$, $i, j = 1, 2, 3$, $i < j$,

$$\widehat{G}_i^*(s, t) := \frac{\widehat{G}_i(s, t)}{t - s} \in (\mathbb{K}[Z])[s, t], \quad i = 1, 2, 3,$$

and

$$\widehat{R}_{ij}(s) := \text{Res}_t(\widehat{G}_i^*, \widehat{G}_j^*) \in (\mathbb{K}[Z])[s], \quad i, j = 1, 2, 3, \quad i < j.$$

Then, the T–function of the parametrization $\widehat{\mathcal{P}}(t)$ is given by

$$\widehat{T}(s) = \widehat{R}_{12}(s) / \widehat{p}(s)^{\widehat{\lambda}_{12}-1}.$$

It holds that $\widehat{T}(s) \in (\mathbb{K}[Z])[s]$ (see Corollary 2), and by Proposition 1 we get that

$$\widehat{T}(s) = \frac{\widehat{R}_{12}(s)}{\widehat{p}(s)^{\widehat{\lambda}_{12}-1}} = \frac{\widehat{R}_{13}(s)}{\widehat{p}_1(s)^{\widehat{\lambda}_{13}-1}} = \frac{\widehat{R}_{23}(s)}{\widehat{p}_2(s)^{\widehat{\lambda}_{23}-1}}.$$

The following theorem is obtained as a consequence of Theorem 3 (see Section 3), and it shows how the function $\widehat{T}(s)$ can be used to define an equivalent polynomial to the T–function introduced for plane curves (see Definition 3). This polynomial, will be denoted as $T_E(s)$.

Similarly as in the case of plane curves, we assume that the space curve, \mathcal{C} , has only ordinary singularities. The case of space curves with non-ordinary singularities will be analyzed in a future work and an equivalent theorem to Theorem 6 will be obtained for this case.

Finally we remind, that if \mathcal{C} has two or more bad points, we consider the transformed curve \mathcal{C}^* introduced above. Note that $H_P(s) = H_{P^*}(s)$, where $P \in \mathcal{C}$ is moved to the point $P^* \in \mathcal{C}^*$ when the change of coordinates is applied. Undoing this change of coordinates, one recovers the initial singularities $P \in \mathcal{C}$.

Theorem 6. *Let \mathcal{C} be a rational algebraic space curve defined by a parametrization $\mathcal{P}(t)$, with limit point P_L . Let P_1, \dots, P_n be the singular points of \mathcal{C} , with multiplicities m_1, \dots, m_n respectively. Let us assume that they are ordinary singularities and that $P_i \neq P_L$, for $i = 1, \dots, n$. Then, it holds that*

$$T_E(s) = \prod_{i=1}^n H_{P_i}(s)^{m_i-1},$$

where $T_E(s) = \text{Content}_Z(\widehat{T}(s)) \in \mathbb{K}[s]$, and $\text{Content}_Z(\widehat{T})$ represents the content of the polynomial \widehat{T} w.r.t Z .

Proof: From the above statements, we observe that there exists a bijective correspondence between the points $\widehat{P} = (a_1 : a_2 + Z_1 a_3 + \dots + Z_{n-2} a_n : a_{n+1})$, $a_i \in \mathbb{K}$, $i = 1, \dots, n+1$, of $\widehat{\mathcal{C}}$ and the points $P = (a_1 : a_2 : a_3 : \dots : a_n : a_{n+1})$ of \mathcal{C} . Consequently, we have that $\text{mult}_{\widehat{P}}(\widehat{\mathcal{C}}) = \text{mult}_P(\mathcal{C})$, which implies that \widehat{P} is a singularity of $\widehat{\mathcal{C}}$ of multiplicity m if and only if P is a singularity of \mathcal{C} of multiplicity m . Hence, using Theorem 3, we deduce that

$$\widehat{T}(s) = \prod_{i=1}^n H_{P_i}(s)^{m_i-1} L(s, Z).$$

We observe that the factor $L(s, Z) \in \mathbb{K}[s, Z] \setminus \mathbb{K}[s]$ is a product of the fibre functions corresponding to the singularities of $\widehat{\mathcal{C}}$ that can not be written as $(a_1 : a_2 + Z_1 a_3 + \dots + Z_{n-2} a_n : a_{n+1})$, $a_i \in \mathbb{K}$, $i = 1, \dots, n+1$ (these singularities do not have an equivalent singularity in \mathcal{C} , and its fibre function necessarily is a polynomial in $\mathbb{K}[s, Z] \setminus \mathbb{K}[s]$). Then, we conclude that

$$T_E(s) = \text{Content}_Z(\widehat{T}(s)) = \prod_{i=1}^n H_{P_i}(s)^{m_i-1}.$$

□

Example 4. Let \mathcal{C} be the rational space curve defined by the projective parametrization $\mathcal{P}(t) = (p_1(t) : p_2(t) : p_3(t) : p(t)) \in \mathbb{P}^3(\mathbb{C}(t))$, where

$$\begin{aligned} p_1(t) &= t^5 - 5t^4 + 5t^3 + 5t^2 - 6t \\ p_2(t) &= t^5 + 5t^4 + 5t^3 - 5t^2 - 6t \\ p_3(t) &= t^5 + 7t^4 + 17t^3 + 17t^2 + 6t \\ p(t) &= t^4 - 13t^2 + 36. \end{aligned}$$

We consider the plane curve

$$\widehat{\mathcal{P}}(t) = (p_1(t) : p_2(t) + Zp_3(t) : p(t)) \in \mathbb{P}^2((\mathbb{C}(Z))(t))$$

and compute the corresponding T -function:

$$\widehat{T}(s) = 298598400(s-2)(s-3)(s+3)(s+2)s(s+1)L(s, Z),$$

where $L(s, Z) = (3Z+1)(2Z+1)(25s^6 + 25Z^2s^6 + 50Zs^6 + 60Z^2s^5 + 35Zs^5 - 25s^5 + 125s^4 + 322Z^2s^4 + 375Zs^4 + 360Z^2s^3 - 65Zs^3 - 125s^3 - 150s^2 - 185Zs^2 + 229Z^2s^2 + 150Zs + 300Z^2s + 150s - 360Z + 432Z^2)$. Removing $L(s, Z)$ (which depends on Z), we get

$$T_E(s) = 298598400(s-2)(s-3)(s+3)(s+2)s(s+1).$$

Now, reasoning as in Example 2, we deduce that \mathcal{C} has three singularities:

- The infinity point $P_1 = (0 : 1 : 3 : 0)$, with fibre function $H_{P_1}(t) = (t-2)(t-3)$ (let us remark that P_1 is a bad point; however, it does not represent a problem in this case since there are no more bad points in \mathcal{C}).
- The infinity point $P_2 = (1 : 0 : 0 : 0)$, with fibre function $H_{P_2}(t) = (t+2)(t+3)$.
- The affine point $P_3 = (0 : 0 : 0 : 1)$, with fibre function $H_{P_3}(t) = t(t+1)$.

Note that P_1 , P_2 and P_3 are double points of \mathcal{C} (see Figure 4).

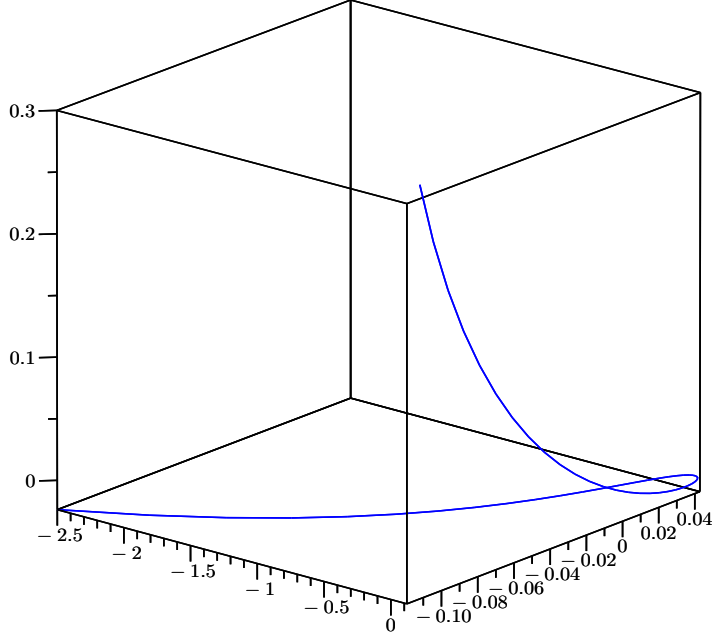


Figure 4: Curve \mathcal{C} and the double point P_3

5 Proof of the main theorem

This section is devoted to show the main result of this paper, Theorem 3 in Section 3. For this purpose, we first prove some previous results. In particular, the following lemma is obtained using the main properties of the resultants (see e.g. [8], [19], [23]).

Lemma 1. *Let $A(s, t), B(s, t), C(s, t) \in \mathbb{K}[s, t]$, and $K(s) \in \mathbb{K}[s]$. The following properties hold:*

1. $\text{Res}_t(A, K) = K^{\deg_t(A)}$.
2. $\text{Res}_t(A, B \cdot C) = \text{Res}_t(A, B) \cdot \text{Res}_t(A, C)$.
3. *If B divides A , it holds that $\text{Res}_t(A/B, C) = \text{Res}_t(A, C)/\text{Res}_t(B, C)$.*

4. $\text{Res}_t(A, B+CA) = \text{lc}(A)^k \text{Res}_t(A, B)$, where $k = \deg_t(B + CA) - \deg_t B$.

Proof: First, we remind that if $A(t), B(t) \in \mathbb{K}[t]$, it holds that

$$\text{Res}_t(A, B) = \text{lc}(A)^{\deg(B)} \prod_{A(\alpha_i)=0} B(\alpha_i).$$

Now, let $A(s, t) \in (\mathbb{K}[s])[t]$. The leader coefficient of $A(s, t)$ w.r.t the variable t is $\text{lc}_t(A) \in \mathbb{K}[s]$ and, for each $s \in \mathbb{K}$, the polynomial $A(s, t)$ has $\deg_t(A)$ roots, $\alpha_1(s), \dots, \alpha_{\deg_t(A)}(s)$, in the algebraic closure of $\mathbb{K}(s)$, such that $A(s, \alpha_i(s)) = 0$, for $i = 1, \dots, \deg_t(A)$. Then,

$$\text{Res}_t(A, B) = \text{lc}_t(A)^{\deg_t(B)} \prod_{A(s, \alpha_i(s))=0} B(s, \alpha_i(s)). \quad (6)$$

Now, we prove the four statements of the lemma.

1. The first statement follows using (6) for the case that $B(s, t) \in \mathbb{K}[s]$. Then $\deg_t(B) = 0$ and $B(s, \alpha_i(s)) = B(s)$ for each $i = 1, \dots, \deg_t(A)$.
2. In order to prove statement 2, we use (6), and we get that

$$\text{Res}_t(A, B \cdot C) = \text{lc}_t(A)^{\deg_t(B \cdot C)} \prod_{A(s, \alpha_i(s))=0} B(s, \alpha_i(s)) \cdot C(s, \alpha_i(s)).$$

Since $\deg_t(B \cdot C) = \deg_t(B) + \deg_t(C)$, we have that

$$\begin{aligned} \text{Res}_t(A, B \cdot C) &= \\ &= \left(\text{lc}_t(A)^{\deg_t(B)} \prod_{A(s, \alpha_i(s))=0} B(s, \alpha_i(s)) \right) \left(\text{lc}_t(A)^{\deg_t(C)} \prod_{A(s, \alpha_i(s))=0} C(s, \alpha_i(s)) \right) \\ &= \text{Res}_t(A, B) \cdot \text{Res}_t(A, C). \end{aligned}$$

3. Reasoning similarly as in statement 2, we get that

$$\text{Res}_t(A/B, C) = \text{lc}_t(A/B)^{\deg_t(C)} \prod_{A(s, \alpha_i(s))=0, B(s, \alpha_i(s)) \neq 0} C(s, \alpha_i(s)).$$

Since B divides A , we obtain that

$$\begin{aligned} \text{lc}_t(A/B)^{\deg_t(C)} \prod_{A(s, \alpha_i(s))=0, B(s, \alpha_i(s)) \neq 0} C(s, \alpha_i(s)) &= \\ \frac{\text{lc}_t(A)^{\deg_t(C)} \prod_{A(s, \alpha_i(s))=0} C(s, \alpha_i(s))}{\text{lc}_t(B)^{\deg_t(C)} \prod_{B(s, \alpha_i(s))=0} C(s, \alpha_i(s))} &= \frac{\text{Res}_t(A, C)}{\text{Res}_t(B, C)}. \end{aligned}$$

4. We reason similarly as above, and we get that

$$\begin{aligned}
\text{Res}_t(A, B + CA) &= \text{lc}_t(A)^{\deg_t(B+CA)}. \\
\prod_{A(s, \alpha_i(s)=0} (B(s, \alpha_i(s)) + C(s, \alpha_i(s))A(s, \alpha_i(s))) &= \\
= \text{lc}(A)^{\deg_t(B+CA) - \deg_t B} \left(\text{lc}(A)^{\deg_t(B)} \prod_{A(s, \alpha_i(s)=0} B(s, \alpha_i(s)) \right) &= \\
= \text{lc}(A)^k \text{Res}_t(A, B), \quad \text{where } k = \deg_t(B + CA) - \deg_t B. &
\end{aligned}$$

The following lemma provides a first approach to the proof of the main result presented in this paper (Theorem 3 in Section 3). In particular, it is shown that each factor $H_P(s)^{m-1}$, where P is an ordinary singular point of multiplicity m , divides the T–function.

Lemma 2. *Let \mathcal{C} be a rational algebraic curve defined by a parametrization $\mathcal{P}(t)$, with limit point P_L . Let $P \neq P_L$ be an ordinary singular point of multiplicity m . It holds that*

$$T(s) = H_P(s)^{m-1} T^*(s),$$

where $T^*(s) \in \mathbb{K}[s]$ and $\gcd(H_P(s), T^*(s)) = 1$.

Proof: In order to prove this lemma, we distinguish three different steps. In Step 1, we prove that the lemma holds if $P = (0 : 0 : 1)$. In Step 2, we show that the lemma holds for any affine singularity. Finally, in Step 3, we prove the lemma for P being a singular point at infinity.

Step 1

Let us assume that the given singularity is the point $P = (0 : 0 : 1)$. Note that $H_P(t) = \gcd(\phi_1, \phi_2) = \gcd(p_1, p_2)$ since, in this case, $a = b = 0$ (see (2)). Thus, we may write

$$\begin{cases} p_1(t) = H_P(t) \bar{p}_1(t) \\ p_2(t) = H_P(t) \bar{p}_2(t), \end{cases}$$

where $\bar{p}_1(t)$ and $\bar{p}_2(t)$ are polynomials satisfying that $\gcd(\bar{p}_1, \bar{p}_2) = 1$. In addition, it holds that $\gcd(H_P(t), p(t)) = 1$, since $\gcd(p_1, p_2, p) = 1$. Hence,

from (1), we may write $G_3(s, t) = H_P(s)H_P(t)(\bar{p}_1(s)\bar{p}_2(t) - \bar{p}_2(s)\bar{p}_1(t))$ that is,

$$G_3(s, t) = H_P(s)H_P(t)\bar{G}_3(s, t), \quad (7)$$

where $\bar{G}_3(s, t) = \bar{p}_1(s)\bar{p}_2(t) - \bar{p}_2(s)\bar{p}_1(t)$.

Observe that since $P \neq P_L$, Corollary 1 holds and then $\deg(H_P(t)) = m \geq 2$. Hence, there exist at least two values $s_0, s_1 \in \mathbb{K}$ such that $H_P(s_0) = H_P(s_1) = 0$ and then, since these roots belong to the fibre of P , we deduce that

$$\left(\left(\frac{p_1}{p} \right) (s_0), \left(\frac{p_2}{p} \right) (s_0) \right) = \left(\left(\frac{p_1}{p} \right) (s_1), \left(\frac{p_2}{p} \right) (s_1) \right) = (0, 0).$$

Since P is an ordinary singularity, we have that there can not exist $K_1, K_2 \in \mathbb{K}$ such that

$$K_1 \left(\left(\frac{p_1}{p} \right)' (s_0), \left(\frac{p_2}{p} \right)' (s_0) \right) = K_2 \left(\left(\frac{p_1}{p} \right)' (s_1), \left(\frac{p_2}{p} \right)' (s_1) \right). \quad (8)$$

We also note that, for $i = 1, 2$ and $j = 0, 1$, it holds that

$$\left(\frac{p_i}{p} \right)' (s_j) = \frac{p_i'(s_j)p(s_j) - p_i(s_j)p'(s_j)}{p(s_j)^2} = \frac{p_i'(s_j)}{p(s_j)},$$

(note that $p_i(s_j) = 0$). In addition, since $p_i(t) = H_P(t)\bar{p}_i(t)$, we get that

$$\frac{p_i'(s_j)}{p(s_j)} = \frac{H_P'(s_j)\bar{p}_i(s_j) + H_P(s_j)\bar{p}_i'(s_j)}{p(s_j)} = \frac{H_P'(s_j)\bar{p}_i(s_j)}{p(s_j)},$$

(note that $H_P(s_j) = 0$). By substituting in (8), we obtain that

$$K_1 \left(\frac{H_P'(s_0)\bar{p}_1(s_0)}{p(s_0)}, \frac{H_P'(s_0)\bar{p}_2(s_0)}{p(s_0)} \right) \neq K_2 \left(\frac{H_P'(s_1)\bar{p}_1(s_1)}{p(s_1)}, \frac{H_P'(s_1)\bar{p}_2(s_1)}{p(s_1)} \right).$$

Hence, we remark that:

- $H_P'(s_i) \neq 0$, $i = 0, 1$. That is, s_0 and s_1 are simple roots of H_P .
- None of the following equalities may be verified:

$$\begin{cases} \bar{p}_1(s_0) = \bar{p}_2(s_0) = 0 \\ \bar{p}_1(s_1) = \bar{p}_2(s_1) = 0 \\ \bar{p}_1(s_0) = \bar{p}_1(s_1) = 0 \\ \bar{p}_2(s_0) = \bar{p}_2(s_1) = 0 \end{cases} \quad (9)$$

- If $\bar{p}_1(s_1) \neq 0$ and $\bar{p}_2(s_1) \neq 0$, then

$$\frac{\bar{p}_1(s_0)}{\bar{p}_1(s_1)} \neq \frac{\bar{p}_2(s_0)}{\bar{p}_2(s_1)}. \quad (10)$$

Taking into account these remarks, we prove that $T(s) = H_P(s)^{m-1}T^*(s)$. For this purpose, we first write the T-function as

$$T(s) = R_{13}(s)/p_1(s)^{\lambda_{13}-1}$$

(see Proposition 1). From (7), we get that

$$R_{13}(s) = \text{Res}_t \left(G_1^*(s, t), H_P(s)H_P(t)\bar{G}_3^*(s, t) \right),$$

where $\bar{G}_3^*(s, t) = \bar{G}_3(s, t)/(t - s)$ (note that $\bar{G}_3^*(s, t) \in \mathbb{K}[s, t]$ since $(t - s)$ divides $\bar{G}_3(s, t)$). By applying statement 2 of Lemma 1, we have that

$$R_{13}(s) =$$

$$\text{Res}_t (G_1^*(s, t), H_P(s)) \text{Res}_t (G_1^*(s, t), H_P(t)) \text{Res}_t \left(G_1^*(s, t), \bar{G}_3^*(s, t) \right). \quad (11)$$

Let us analyse the first two factors:

- From statement 1 of Lemma 1, we have that

$$\text{Res}_t (G_1^*(s, t), H_P(s)) = H_P(s)^{\deg_t(G_1^*)} = H_P(s)^{\delta_1-1}.$$

- On the other side,

$$\text{Res}_t (G_1^*(s, t), H_P(t)) = \text{Res}_t \left(\frac{p_1(s)p(t) - p(s)p_1(t)}{t - s}, H_P(t) \right)$$

and, by applying statement 3 of Lemma 1, we get that

$$\frac{\text{Res}_t (p_1(s)p(t) - p(s)p_1(t), H_P(t))}{\text{Res}_t (t - s, H_P(t))}.$$

Note that $\text{Res}_t (t - s, H_P(t)) = H_P(s)$ and, since $p_1(t) = H_P(t)\bar{p}_1(t)$, the above expression can be written as

$$\frac{\text{Res}_t (p_1(s)p(t) - p(s)H_P(t)\bar{p}_1(t), H_P(t))}{H_P(s)}.$$

Now, from statements 1 and 4 of Lemma 1, we get that

$$\operatorname{Res}_t(G_1^*(s, t), H_P(t)) = \frac{p_1(s)^m \operatorname{Res}_t(p(t), H_P(t)) \operatorname{lc}(H_P(t))^k}{H_P(s)},$$

where $k \in \mathbb{K}$. Note that, $\operatorname{lc}(H_P(t))^k \in \mathbb{K} \setminus \{0\}$ and $\operatorname{Res}_t(p(t), H_P(t)) \in \mathbb{K} \setminus \{0\}$ (since $\gcd(p, H_P) = 1$). Furthermore, we have that $p_1(s) = H_P(s)\bar{p}_1(s)$. Thus, up to constants in $\mathbb{K} \setminus \{0\}$, we deduce that

$$\operatorname{Res}_t(G_1^*(s, t), H_P(t)) = H_P(s)^{m-1}\bar{p}_1(s)^m.$$

Substituting in (11), we obtain that

$$R_{13}(s) = H_P(s)^{\delta_1-1} H_P(s)^{m-1} \bar{p}_1(s)^m \operatorname{Res}_t(G_1^*(s, t), \bar{G}_3^*(s, t)),$$

and thus

$$T(s) = \frac{R_{13}(s)}{p_1(s)^{\lambda_{13}-1}} = \frac{R_{13}(s)}{H_P(s)^{\lambda_{13}-1} \bar{p}_1(s)^{\lambda_{13}-1}} = H_P(s)^{m-1} T^*(s),$$

where

$$T^*(s) = \frac{\operatorname{Res}_t(G_1^*(s, t), \bar{G}_3^*(s, t)) H_P(s)^{\delta_1-\lambda_{13}}}{\bar{p}_1(s)^{\lambda_{13}-1-m}}.$$

Note that $\lambda_{13} = \delta_1$ since $\delta_1 \leq \delta_3$. Otherwise, if $\delta_1 > \delta_3$, we would have that $\max\{d_1, d_3\} > \max\{d_1, d_2\}$ and then, $d_3 > d_1, d_2$. However, this would imply that $P = P_L$ (see Definition 1), which contradicts the assumption. Therefore,

$$T^*(s) = \frac{\operatorname{Res}_t(G_1^*(s, t), \bar{G}_3^*(s, t))}{\bar{p}_1(s)^{\delta_1-1-m}}. \quad (12)$$

Now, we prove that $T^*(s) \in \mathbb{K}[s]$. We can assume that $\delta_1 - 1 - m \geq 0$, since $\delta_1 = \max\{d_1, d_3\} \geq d_1$ and $d_1 = \deg(p_1) = \deg(H_P \cdot \bar{p}_1) = m + \deg(\bar{p}_1)$. Hence, $\delta_1 \geq m + 1$ except for the case that $\deg(\bar{p}_1) = 0$, but in this situation we would have that

$$T^*(s) = \operatorname{Res}_t(G_1^*(s, t), \bar{G}_3^*(s, t)) \in \mathbb{K}[s].$$

So, let $\delta_1 - 1 - m \geq 0$. Now, we reason similarly as in the proof of Corollary 2. Indeed: let us assume that $T^*(s)$ is not a polynomial. Then, by taking the simplified rational function, we may write

$$T^*(s) = \frac{M_{13}(s)}{\widehat{p}_1(s)}$$

where $M_{13}(s) \in \mathbb{K}[s]$, $\widehat{p}_1(s) \in \mathbb{K}[s] \setminus \mathbb{K}$ and $\gcd(M_{13}(s), \widehat{p}_1(s)) = 1$. Note that $\widehat{p}_1(s)$ divides $\bar{p}_1(s)^{\delta_1-1-m}$.

We observe that (12) is obtained from $T(s) = R_{13}(s)/p_1(s)^{\lambda_{13}-1}$. However, taking into account Proposition 1, we could have considered the expression

$$T(s) = R_{23}(s)/p_2(s)^{\lambda_{23}-1}$$

concluding that $T(s) = H_P(s)^{m-1}T^*(s)$, where

$$T^*(s) = \frac{\text{Res}_t \left(G_2^*(s, t), \overline{G}_3^*(s, t) \right)}{\bar{p}_2(s)^{\delta_2-1-m}}. \quad (13)$$

Reasoning similarly as above, we get that there would exist $M_{23}(s) \in \mathbb{K}[s]$, $\widehat{p}_2(s) \in \mathbb{K}[s] \setminus \mathbb{K}$ with $\gcd(M_{23}(s), \widehat{p}_2(s)) = 1$, such that

$$T^*(s) = \frac{M_{23}(s)}{\widehat{p}_2(s)}.$$

In addition, $\widehat{p}_2(s)$ would divide $\bar{p}_2(s)^{\delta_2-1-m}$. Thus, we have that

$$\frac{M_{13}(s)}{\widehat{p}_1(s)} = \frac{M_{23}(s)}{\widehat{p}_2(s)}$$

where $\gcd(M_{13}(s), \widehat{p}_1(s)) = \gcd(M_{23}(s), \widehat{p}_2(s)) = 1$, which implies that $\widehat{p}_1(s) = \widehat{p}_2(s)$. Hence $\gcd(\bar{p}_1, \bar{p}_2) \neq 1$, which contradicts the definition of these functions.

Thus, we have proved that T^* is a polynomial. Finally, we show that $\gcd(H_P(s), T^*(s)) = 1$. Indeed: if $\gcd(H_P(s), T^*(s)) \neq 1$, there exists $s_0 \in \mathbb{K}$ such that $H_P(s_0) = 0$ and $T^*(s_0) = 0$, which implies that

$$\text{Res}_t \left(G_1^*(s, t), \overline{G}_3^*(s, t) \right) (s_0) = 0.$$

Then, by the properties of the resultants, one of the following statements hold:

1. There exists $s_1 \in \mathbb{K}$ such that $G_1^*(s_0, s_1) = \overline{G}_3^*(s_0, s_1) = 0$. This would imply that $G_3^*(s_0, s_1) = 0$, and, then, s_0 and s_1 are elements of the fibre of P . Note that $s_0 \neq s_1$ since H_P only has simple roots (P is an ordinary point). On the other side,

$$\overline{G}_3(s_0, s_1) = \bar{p}_1(s_0)\bar{p}_2(s_1) - \bar{p}_2(s_0)\bar{p}_1(s_1) = 0$$

and thus,

$$\frac{\bar{p}_1(s_0)}{\bar{p}_1(s_1)} = \frac{\bar{p}_2(s_0)}{\bar{p}_2(s_1)}$$

(note that $\bar{p}_i(s_1) \neq 0$, $i = 1, 2$, since the equalities in (9) can not be verified). This implies that P is a non-ordinary singular point (see (10)), which contradicts the assumptions.

2. It holds that $\gcd(\text{lc}_t(G_1^*), \text{lc}_t(\bar{G}_3^*))(s_0) = 0$. Then, in particular,

$$\text{lc}_t(G_1^*)(s_0) = \text{lc}_t(G_1)(s_0) = p_1(s_0)c_d - p(s_0)a_d = 0 \Rightarrow a_d = 0.$$

Now we reason similarly with the equality $T(s) = R_{23}(s)/p_2(s)^{\lambda_{23}-1}$ and we get (13). From this expression, and reasoning similarly as above, we obtain that $\gcd(\text{lc}_t(G_2^*), \text{lc}_t(\bar{G}_3^*))(s_0) = 0$, which implies that $b_d = 0$. However, if $a_d = b_d = 0$ we deduce that $P = P_L$, which contradicts the assumptions.

Therefore, we conclude that $\gcd(H_P(s), T^*(s)) = 1$.

Step 2

Let $P = (a : b : 1)$ be a singularity of multiplicity m . In this case, we consider the translation of the curve \mathcal{C} defined by the parametrization

$$\tilde{\mathcal{P}}(t) = (p_1(t) - ap(t) : p_2(t) - bp(t) : p(t)).$$

We have that the point $P = (a : b : 1)$ moves to the point $\tilde{P} = (0 : 0 : 1)$, and then $H_P(t) = H_{\tilde{P}}(t)$ (note that the polynomial $H_P(t)$ is computed from $\mathcal{P}(t)$, and the polynomial $H_{\tilde{P}}(t)$ is computed from $\tilde{\mathcal{P}}(t)$).

On the other side, if we compute the polynomial equivalent to $G_1(s, t)$ with the new parametrization $\tilde{\mathcal{P}}(t)$, we get that

$$\begin{aligned} \tilde{G}_1(s, t) &= \tilde{p}_1(s)\tilde{p}(t) - \tilde{p}(s)\tilde{p}_1(t) = \\ &= (p_1(s) - ap(s))p(t) - p(s)(p_1(t) - ap(t)) = p_1(s)p(t) - p(s)p_1(t) = G_1(s, t). \end{aligned}$$

Similarly, one obtains that $\tilde{G}_2(s, t) = G_2(s, t)$. Thus,

$$\tilde{R}_{12}(s) = \text{Res}_t \left(\frac{\tilde{G}_1(s, t)}{t - s}, \frac{\tilde{G}_2(s, t)}{t - s} \right) = \text{Res}_t \left(\frac{G_1(s, t)}{t - s}, \frac{G_2(s, t)}{t - s} \right) = R_{12}(s),$$

and then

$$\tilde{T}(s) = \tilde{R}_{12}(s)/\tilde{p}(s)^{\lambda_{12}-1} = R_{12}(s)/p(s)^{\lambda_{12}-1} = T(s).$$

Thus, it holds that $T(s) = H_P(s)^{m-1}T^*(s)$ and $\gcd(H_P, T^*) = 1$, since from Step 1, these equalities hold for $H_{\tilde{P}}(s)$ and $\tilde{T}(s)$.

Step 3

Let us prove that the lemma holds for a singularity at infinity. For this purpose, we assume that $P = (1 : 0 : 0)$. Note that we can reason similarly as in Step 1 taking into account that for this case, $H_P(t) = \gcd(p(t), \phi_3(t)) = \gcd(p(t), p_2(t))$ (see Remark 2). Hence,

$$G_2(s, t) = H_P(s)H_P(t)(\bar{p}_2(s)\bar{p}(t) - \bar{p}(s)\bar{p}_2(t)).$$

and

$$R_{12}(s) = \text{Res}_t(G_1^*, G_2^*) = \text{Res}_t\left(G_1^*(s, t), H_P(s)H_P(t)\overline{G}_2^*(s, t)\right)$$

where

$$\overline{G}_2^*(s, t) = \frac{\bar{p}_2(s)\bar{p}(t) - \bar{p}(s)\bar{p}_2(t)}{t - s}.$$

Thus, using the expression $T(s) = R_{12}(s)/p(s)^{\lambda_{12}-1}$, we deduce that the lemma also holds if the singularity is the point $(1 : 0 : 0)$. A similar reasoning with the expression $T(s) = R_{23}(s)/p_2(s)^{\lambda_{23}-1}$ shows that the lemma holds for the point $(1 : 0 : 0)$.

Finally, let us assume that $P = (a : b : 0)$. Then, we reason similarly as in Step 2 and we apply a translation such that the point P is moved to the point $\tilde{P} = (1 : 0 : 0)$. This translation can be defined parametrically by

$$\tilde{\mathcal{P}}(t) = (p_1(t) : p_2(t) - (b/a)p_1(t) : p(t)).$$

We assume that $a \neq 0$; otherwise, it should be $b \neq 0$ and we would use a translation that would move P to $(0 : 1 : 0)$.

Under these conditions, we have that $H_P(t) = H_{\tilde{P}}(t)$. In addition, if we compute the equivalent polynomials to $G_1(s, t)$ and $G_3(s, t)$ with the new parametrization $\tilde{\mathcal{P}}(t)$, we get that $\tilde{G}_1(s, t) = G_1(s, t)$ and $\tilde{G}_3(s, t) = G_3(s, t)$. Thus, from Proposition 1,

$$\tilde{T}(s) = \tilde{R}_{13}(s)/\tilde{p}_1(s)^{\lambda_{13}-1} = R_{13}(s)/p_1(s)^{\lambda_{13}-1} = T(s).$$

Therefore

$$T(s) = H_P(s)^{m-1}T^*(s) \quad \text{and} \quad \gcd(H_P, T^*) = 1,$$

since both equalities hold for $H_{\tilde{P}}(s)$ and $\tilde{T}(s)$. \square

Proof of the Main Formula (Theorem 3 in Section 3)

Taking into account Lemma 2, we have that for each singular point P_i , it holds that $T(s) = H_{P_i}(s)^{m_i-1}T_i^*(s)$, where $T_i^*(s) \in \mathbb{K}[s]$ and $\gcd(H_{P_i}, T_i^*) = 1$. In addition, $\gcd(H_{P_i}, H_{P_j}) = 1$ for $i \neq j$ (otherwise, there would exist $s_1 \in \mathbb{K}$ such that $\mathcal{P}(s_1) = P_i = P_j$). Then, we get that

$$T(s) = \prod_{i=1}^n H_{P_i}(s)^{m_i-1}V(s),$$

where $V(s) \in \mathbb{K}[s]$ and $\gcd(H_{P_i}, V) = 1$ for $i = 1, \dots, n$.

Note that if $V(s_0) = 0$, then $T(s_0) = 0$ and thus, $R_{12}(s_0) = R_{13}(s_0) = R_{23}(s_0) = 0$. From $R_{13}(s_0) = \text{Res}_t(G_1^*(s, t), G_3^*(s, t))(s_0) = 0$ and using the properties of the resultant, we deduce that one of the following two statements hold:

1. There exists $s_1 \in \mathbb{K}$ such that $G_1^*(s_0, s_1) = G_3^*(s_0, s_1) = 0$. Thus, $H_P(s_1) = 0$, where $P = \mathcal{P}(s_0)$, which is impossible since $\gcd(V, H_P) = 1$.
2. It holds that $\gcd(\text{lc}_t(G_1^*), \text{lc}_t(G_3^*))(s_0) = 0$. However, this is also a contraction since we would have that

$$\text{lc}_t(G_1^*)(s_0) = \text{lc}_t(G_1)(s_0) = p_1(s_0)c_d - p(s_0)a_d = 0 \Rightarrow \frac{p_1(s_0)}{p(s_0)} = \frac{a_d}{c_d}$$

and

$$\text{lc}_t(G_3^*)(s_0) = \text{lc}_t(G_3)(s_0) = p_1(s_0)b_d - p_2(s_0)a_d = 0 \Rightarrow \frac{p_1(s_0)}{p_2(s_0)} = \frac{a_d}{b_d}.$$

From both equalities, we deduce that

$$\frac{p_2(s_0)}{p(s_0)} = \frac{b_d}{c_d},$$

and then $\mathcal{P}(s_0) = P_L$. This would imply that P_L can be reached by the parametrization $\mathcal{P}(t)$ but this implies that it is a singularity (see Proposition 3.4 in [4]), which contradicts the assumption of the theorem.

Thus, we have that $V \in \mathbb{K}$ and, hence, we conclude that, up to constants in $\mathbb{K} \setminus \{0\}$,

$$T(s) = \prod_{i=1}^n H_{P_i}(s)^{m_i-1}.$$

□

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