On Gevrey asymptotics for linear singularly perturbed equations with linear fractional transforms

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Abstract

A family of linear singularly perturbed Cauchy problems is studied. The equations defining the problem combine both partial differential operators together with the action of linear fractional transforms. The exotic geometry of the problem in the Borel plane, involving both sectorial regions and strip-like sets, gives rise to asymptotic results relating the analytic solution and the formal one through Gevrey asymptotic expansions. The main results lean on the appearance of domains in the complex plane which remain intimately related to Lambert W function, which turns out to be crucial in the construction of the analytic solutions.

On the way, an accurate description of the deformation of the integration paths defining the analytic solutions and the knowledge of Lambert W function are needed in order to provide the asymptotic behavior of the solution near the origin, regarding the perturbation parameter. Such deformation varies depending on the analytic solution considered, which lies in two families with different geometric features.

Key words: asymptotic expansion, Lambert W function, Borel-Laplace transform, Fourier transform, initial value problem, formal power series, singular perturbation. 2010 MSC: 35C10, 35C20.

1 Introduction

This work is devoted to the study of a family of linear singularly perturbed Cauchy problems combining partial differential operators together with the action of linear fractional transforms. More precisely, we deal with equations of the form

$$(1)\qquad \begin{aligned} Q(\partial_z)u(t,z,\epsilon) &= \epsilon^{\delta_D}(t^2\partial_t)^{\delta_D}R_D(\partial_z)u(t,z,\epsilon) + \epsilon^{\delta_0}\left((t^2\partial_t)^{\delta_0}R_0(\partial_z)u\right)\left(\frac{t}{1+k_0\epsilon t},z,\epsilon\right) \\ &+ \sum_{\ell\in I}\epsilon^{\Delta_\ell}t^{\delta_\ell}\partial_t^{d_\ell}c_\ell(z,\epsilon)R_\ell(\partial_z)u\,(t,z,\epsilon) + f(t,z,\epsilon), \end{aligned}$$

under null initial data $u(0,z,\epsilon)\equiv 0$, and where ϵ is a small perturbation complex parameter. $I\subseteq\mathbb{N}$ stands for a finite set of indices, $Q(X),R_0(X),R_\ell(X),R_D(X)\in\mathbb{C}[X]$, $\delta_0,\delta_\ell,\Delta_\ell,d_\ell,\delta_D$

and k_0 are positive integers, and the coefficients $c_\ell(z,\epsilon)$ and $f(t,z,\epsilon)$ are holomorphic functions defined on $H_{\beta'} \times D(0,\epsilon_0)$ and $D(0,r_f) \times H_{\beta'} \times D(0,\epsilon_0)$, respectively, for some $r_f, \beta', \epsilon_0 > 0$, where

$$H_{\beta'} = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \beta'\}.$$

The precise nature of each of the elements involved in the equation are detailed in Section 2.

It is worth mentioning at this point that we are dealing with the case $\delta_0 > 0$ in the present work. However, there is no additional theoretical difficulty on considering the case $\delta_0 = 0$, which only entangles a different geometry, explained in Section 3.2. However, we have decided to omit further duplicated computations in this particular case for the sake of clarity.

The point of depart of the present study is the recent work [16] by the authors, where a family of nonlinear singularly perturbed equations combining linear fractional transforms, partial derivatives and differential operators of infinite order, was studied. More precisely, in that previous study we considered equations of the form

$$(2) \quad Q(\partial_z)u(t,z,\epsilon) = \exp(\alpha\epsilon^k t^{k+1}\partial_t)R(\partial_z)u(t,z,\epsilon) + P(t,\epsilon,\{m_{k,t,\epsilon}\}_{k\in I},\partial_t,\partial_z)u(t,z,\epsilon) \\ Q_1(\partial_z)u(t,z,\epsilon)Q_2(\partial_z)u(t,z,\epsilon) + f(t,z,\epsilon),$$

where $Q \in \mathbb{C}[X]$ and the polynomial P admits holomorphic coefficients in some neighborhood of the origin with respect to the perturbation parameter ϵ. We write $m_{k,t,\epsilon}$ for the operator

$$m_{k,t,\epsilon}u(t,z,\epsilon) = u\left(\frac{t}{1+k\epsilon t}, z, \epsilon\right).$$

The term $\exp(\alpha\epsilon^k t^{k+1}\partial_t)$ is the exponential formal differential operator of infinite order with respect to t

$$\exp(\alpha\epsilon^k t^{k+1}\partial_t) = \sum_{p\geq 0} \frac{(\alpha\epsilon^k)^p}{p!}(t^{k+1}\partial_t)^{(p)}.$$

Here, $(t^{k+1}\partial_t)^{(p)}$ represents the $p-$th iterate of $t^{k+1}\partial_t$.

The main result in that work establishes the existence of a formal power series $\hat{u}(t,z,\epsilon) = \sum_{m\geq 0} h_m(t,z)\epsilon^m \in \mathbb{E}[[\epsilon]]$, where $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ is certain Banach space of functions, which is the common asymptotic expansion of a family of sectorial solutions $(\epsilon \mapsto u_p(t,z,\epsilon))_{0\leq p\leq \varsigma-1}$, defined on finite sectors which conform a good covering $(\mathcal{E}_p)_{0\leq p\leq\varsigma-1}$ (see Definition 4), and with coefficients in $\mathbb{E}$. The previous asymptotic expansion is of Gevrey order $1/k$, i.e. for every $0 \leq p \leq \varsigma - 1$ there exist $C_p, M_p > 0$ such that

$$\left\| u_p(t,z,\epsilon) - \sum_{m=0}^{n-1} h_m(t,z)\epsilon^m \right\|_{\mathbb{E}} \leq C_p(M_p)^n \Gamma\left(1 + \frac{n}{k}\right)|\epsilon|^n,$$

for all $n \geq 1$ and $\epsilon \in \mathcal{E}_p$. In the case that there exists $0 \leq p_0 \leq \varsigma - 1$ such that the aperture of the corresponding sector of the good covering is larger than π/k, then the map $\epsilon \mapsto u_{p_0}(t,z,\epsilon)$ is indeed the $k-$sum of $\hat{u}(t,z,\epsilon)$ on such sector. We refer to the reference [1] for further details on the classical theory of Gevrey asymptotic expansions in sectors of the complex plane.

In that previous study, k was assumed to be a parameter smaller than 1. The techniques used did not succeed when applied to the limit case $k = 1$, and were postponed to a future study. As a matter of fact, our first hypotheses (see the introduction of [16]) based on previous experiences [20, 14] pointed to the existence of double scale structures involving 1 and 1+ Gevrey estimates. Contrary to our expectations, this work reveals that this setting has a limit behavior which is reflected in its exotic geometry, rather than the asymptotic expansions involved. Therefore, the present study stands as a limit setup in [16].

For the sake of clarity, we have decided to deal with the case of a single shift operator, whereas the general case can be treated in a similar manner, carrying cumbersome and heavy calculations which may avoid the reader to have a clear idea of the main purpose of the study.

The statements of the main problem under study are displayed in Section 2. The first step in the research is to search for solutions of the problem in the form of inverse Fourier and Laplace-like transformations of an unknown function

$$u(t,z,\epsilon)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\int_{\mathcal{L}}\omega(u,m,\epsilon)e^{-\frac{u}{\epsilon t}}e^{izm}\frac{du}{u}dm, \tag{3}$$

where $\mathcal{L}$ is an infinite path which becomes effective in different forms through the two alternative directions followed in the work. The problem turns out to be in some sense symbolic, before establishing the description of the convergence conditions on appropriate domains. An auxiliary problem satisfied by $\omega(\tau,m,\epsilon)$ (see(17)) allows to distinguish the two independent options in order to give rise to analytic solutions to the problem.

Section 3 is devoted to construct a first family of analytic solutions of (17). The assumptions made on the elements involved in the main problem and on its geometry give rise to certain geometric conditions on the domains where the function $\omega(\tau,m,\epsilon)$ is well defined. The precise knowledge of the behavior of Lambert W function is essential in order to describe such domains. We have decided to include a particular case of this study, displayed in Section 3.2, in which the details of this geometry can be easily illustrated. Once the adequate domains of $\omega(\tau,m,\epsilon)$ are established, we prove the existence of the solution of the auxiliary problem within a Banach space of function (see Section 3.3). The elements belonging to that Banach space satisfy exponential growth/decay at infinity with respect to certain variables, giving an analytic meaning to (3).

A parallel path is traced in Section 4 where a second framework for the main problem is established. The auxiliary equation (17) is rewritten in order to search for analytic solutions of the main problem in this novel situation. This second setting is closer to that of the classical results obtained on the solutions of analytic solutions to singularly perturbed differential equations in the complex domain. More precisely, the solution of the auxiliary problem, $\omega(\tau,m,\epsilon)$, turns out to be holomorphic in a neighborhood of the origin, and can be extended to an infinite sector with exponential growth, w.r.t. τ. The path $\mathcal{L}$ in (3) takes the form of a half-line with endpoint at the origin, contained in such infinite sector. Analyticity of $u(t,z,\epsilon)$, as defined in (3) makes sense, describing a second family of analytic solutions to the main problem.

The two families of analytic solutions of the main problem, independently acquired, turn out to be holomorphic functions on sets of the form $\mathcal{T}\times H_{\beta'}\times\mathcal{E}$, where $\mathcal{T},\mathcal{E}$ are certain bounded sectors in the complex plane, and $0<\beta'<\beta$ (see Proposition 5 and Proposition 7).

The existence of asymptotic results at the origin with respect to the perturbation parameter, i.e. relating each of the analytic solutions to some formal power series in the perturbation parameter ϵ, needs that both approaches converge into one. As a matter of fact, it is hopeless to cover a full punctured neighborhood of the origin with a finite number of domains $(\mathcal{E}_p)_{0\le p\le\varsigma-1}$ obtained from ς analytic solutions coming from just one of the approaches described above. A finite family of solutions of the main problem $(u_p(t,z,\epsilon))_{0\le p\le\varsigma-1}$, with $u_p(t,z,\epsilon)$ being analytic on $\mathcal{T}\times H_{\beta'}\times\mathcal{E}_p$, for every $0\le p\le\varsigma-1$, and where $(\mathcal{E}_p)_{0\le p\le\varsigma-1}$ defines a good covering of the origin (see Definition 4), is constructed. We observe that a set of such functions comprises solutions to the main problem coming from both approaches.

The first main result of the work (Theorem 1) states that for every pair of indices $0\le p,q\le\varsigma-1$ with $p\neq q$ and such that $\mathcal{E}_p\cap\mathcal{E}_q\neq\emptyset$, then the difference of the corresponding analytic solutions $u_p(t,z,\epsilon)$ and $u_q(t,z,\epsilon)$ is exponentially small with respect to $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_q$, uniformly with respect to the rest of the variables. More precisely, there exist $C, D > 0$ such that

$$\sup_{t\in\mathcal{T}, z\in H_{\beta'}} |u_p(t,z,\epsilon) - u_q(t,z,\epsilon)| \leq C\exp\left(-\frac{D}{|\epsilon|}\right),$$

for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_q$. The proof of this result faces three cases, depending on the nature of u_p and u_q. The work concludes with the application of the classical Ramis-Sibuya Theorem (Theorem (RS)), achieving the existence of a common formal power series in ϵ, say $\hat{u}(t,z,\epsilon)$, with coefficients in the Banach space of holomorphic and bounded functions in $\mathcal{T} \times H_{\beta'}$, say $\mathbb{E}$. This formal power series is the asymptotic expansion of $u_p(t,z,\epsilon)$ of Gevrey order 1 in $\mathcal{E}_p$, for every $0 \leq p \leq \varsigma - 1$, when considering u_p as a function of $\mathcal{E}_p$ and values in $\mathbb{E}$.

We notice that the appearance of the particular linear fractional transform $t \mapsto \frac{t}{1+k_0\epsilon t}$, i.e. an homography, in the main equation under study is motivated by the fact that the change of variable $t = 1/s$, and therefore the change of the unknown function $u(t,z,\epsilon) = X(1/t,z,\epsilon)$, transforms equation (1) into a singularly perturbed PDE combined with small shift operator $T_{k_0,\epsilon}X(s,z,\epsilon) = X(s+k_0\epsilon,z,\epsilon)$, which has been studied in the literature in the field of asymptotic analysis of functional equations. We refer to [6, 8, 9], also [21] in the framework of singularly perturbed elliptic partial differential equations, and [2, 3, 7, 11, 12, 13] as examples of advances in the study of difference equations.

The paper is organized as follows: in Section 2, the precise statement of the main problem (1) and auxiliary problem (17) are established. Sections 3 and 4 describe the construction of two families of analytic solutions of (17). In both sections, the geometry of the problem is analysed together with the definition and main properties of the Banach space where the solutions belong. The construction of the analytic solutions to the main problem in Section 5 is made regarding their different nature. Section 6 is devoted to the study of the main asymptotic results of the work (Theorems 1 and 3) in which the existence of a formal solution is attained, being the common formal Gevrey asymptotic expansion of all the analytic solutions, with respect to the perturbation parameter near the origin. The work concludes with two final sections on known facts on Fourier transform and Lambert W function.

2 Statement of the main problem and related auxiliary problems

Let $\delta_D \geq 2$ be an integer. Let $I \subseteq \mathbb{N}$ be a finite set of indices. For every $\ell \in I$, we choose non-negative integers Δ_ℓ, d_ℓ and δ_ℓ. We assume that

$$\Delta_\ell > \delta_\ell - d_\ell, \tag{4}$$

and

$$\delta_\ell > 2d_\ell, \quad \delta_D \geq d_\ell + 2, \tag{5}$$

for every $\ell \in I$. We also choose positive integers δ_0 and k_0.

Let $Q(X), R_D(X), R_0(X) \in \mathbb{C}[X]$, and $R_\ell(X) \in \mathbb{C}[X]$ for every $\ell \in I$, under the following conditions

$$\frac{Q(im)}{R_0(im)} \in S_{Q,R_0}, \quad R_0(im) \neq 0, \qquad m \in \mathbb{R}, \tag{6}$$

where

$$S_{Q,R_0} = \{z \in \mathbb{C} : \alpha_{Q,R_0} < \arg(z) < \beta_{Q,R_0}, \quad r_{Q,R_0} \leq |z| \leq R_{Q,R_0}\},$$

for some $0 < r_{Q,R_0} < R_{Q,R_0}$ and $\alpha_{Q,R_0} < \beta_{Q,R_0}$.

For all $\ell \in I$ one has that

$$\deg(R_\ell) \leq \deg(R_D). \tag{7}$$

In addition to this,

$$\frac{Q(im)}{R_D(im)} \in S_{Q,R_D}, \quad R_D(im) \neq 0, \qquad m \in \mathbb{R}, \tag{8}$$

where

$$S_{Q,R_D} = \{z \in \mathbb{C} : \alpha_{Q,R_D} < \arg(z) < \beta_{Q,R_D}, \quad r_{Q,R_D} \leq |z| \leq R_{Q,R_D}\},$$

for some $0 < r_{Q,R_D} < R_{Q,R_D}$ and $\alpha_{Q,R_D} < \beta_{Q,R_D}$. Observe from the previous assumptions that

$$\deg(R_\ell) \leq \deg(Q) \tag{9}$$

for every $\ell \in I$.

We consider the main problem under study

$$\begin{aligned} Q(\partial_z)u(t,z,\epsilon) = \epsilon^{\delta_D}(t^2\partial_t)^{\delta_D}R_D(\partial_z)u(t,z,\epsilon) + \epsilon^{\delta_0}\left((t^2\partial_t)^{\delta_0}R_0(\partial_z)u\right)\left(\frac{t}{1+k_0\epsilon t}, z, \epsilon\right) \\ + \sum_{\ell \in I} \epsilon^{\Delta_\ell} t^{\delta_\ell} \partial_t^{d_\ell} c_\ell(z,\epsilon) R_\ell(\partial_z) u\,(t,z,\epsilon) + f(t,z,\epsilon), \end{aligned} \tag{10}$$

for null initial data $u(0,z,\epsilon) \equiv 0$. In the previous equation, ϵ acts as a small complex perturbation parameter, and the coefficients $c_\ell(z,\epsilon)$ and the forcing term $f(t,z,\epsilon)$ are constructed as follows.

Let $\epsilon_0, \beta > 0$. For every $\ell \in I$, the function $c_\ell(z,\epsilon)$ is holomorphic on $H_{\beta'} \times D(0,\epsilon_0)$ for all $0 < \beta' < \beta$, where

$$H_{\beta'} = \{z \in \mathbb{C} : |\mathrm{Im}(z)| \leq \beta'\},$$

and it is constructed as the inverse Fourier transform

$$c_\ell(z,\epsilon) := \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} C_\ell(m,\epsilon) e^{izm} dm,$$

with $m \mapsto C_\ell(m,\epsilon)$ being a continuous function for $m \in \mathbb{R}$ and satisfying uniform bounds with respect to the perturbation parameter ϵ in $D(0,\epsilon_0)$. More precisely, there exists $\mathcal{C}_\ell > 0$ such that

$$\sup_{\epsilon \in D(0,\epsilon_0)} |C_\ell(m,\epsilon)| \leq \frac{\mathcal{C}_\ell}{(1+|m|)^\mu} \exp(-\beta|m|), \quad m \in \mathbb{R}, \tag{11}$$

for some $\mu > 1$. Observe from Annex 1 that the previous property coincides with

$$\sup_{\epsilon \in D(0,\epsilon_0)} \|m \mapsto C_\ell(m,\epsilon)\|_{\beta,\mu} \leq \mathcal{C}_\ell,$$

for all $\ell \in I$.

Let $\psi : \mathbb{C} \times \mathbb{R} \times D(0,\epsilon_0) \to \mathbb{C}$ be an entire function with respect to its first variable, continuous on $\mathbb{R}$ in its second variable, and holomorphic on the disc $D(0,\epsilon_0)$ with respect to its third variable. Moreover, we assume there exists $\mathcal{C}_{\psi}, \beta, \nu > 0$ such that ψ satisfies the following upper bounds:

$$|\psi(\tau, m, \epsilon)| \leq \frac{\mathcal{C}_{\psi}}{(1+|m|)^{\mu}} e^{-\beta|m|} \exp(\nu|\tau|)|\tau|, \tag{12}$$

for every $(\tau, m, \epsilon) \in \mathbb{C} \times \mathbb{R} \times D(0, \epsilon_0)$. We define the function

$$F(T, z, \epsilon) := \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{L_d} \psi(u, m, \epsilon) \exp\left(-\frac{u}{T}\right) e^{izm} \frac{du}{u} dm,$$

where $L_d = [0, \infty) e^{d\sqrt{-1}}$ can spin around the origin in order to guarantee that F is a holomorphic function on $D(0, r_F)$ for $0 < r_F < 1/\nu$, with respect to T by analytic continuation. The forcing term $f(t, z, \epsilon)$, defined by

$$f(t, z, \epsilon) = F(\epsilon t, z, \epsilon) \tag{13}$$

turns out to be holomorphic on $D(0, r_f) \times H_{\beta'} \times D(0, \epsilon_0)$, for every $0 < \beta' < \beta$, where $r_f > 0$ satisfies $\epsilon_0 r_f < r_F$.

We search for solutions of (10) in the form $u(t, z, \epsilon) = U(\epsilon t, z, \epsilon)$, for some function $U(T, z, \epsilon)$ which becomes a solution the auxiliary problem

$$\begin{aligned} Q(\partial_z)U(T, z, \epsilon) &= (T^2\partial_T)^{\delta_D} R_D(\partial_z) U(T, z, \epsilon) + \Big((T^2\partial_T)^{\delta_0} R_0(\partial_z) U\Big)\left(\frac{T}{1+k_0 T}, z, \epsilon\right) \\ &+ \sum_{\ell \in I} \epsilon^{\Delta_\ell - \delta_\ell + d_\ell} T^{\delta_\ell} \partial_T^{d_\ell} c_\ell(z, \epsilon) R_\ell(\partial_z) U(T, z, \epsilon) + F(T, z, \epsilon). \end{aligned} \tag{14}$$

In addition to this, we explore solutions of (14) (and consequently of the main problem (10)) in the form of a Laplace-like and Fourier transform, i.e.

$$U(T, z, \epsilon) := \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{\mathcal{L}} \omega(u, m, \epsilon) e^{-\frac{u}{T}} e^{izm} \frac{du}{u} dm, \tag{15}$$

where $\mathcal{L}$ is an infinite path which can be of different nature, to be described in the work. As a matter of fact, the auxiliary function $\omega(u, m, \epsilon)$ turns out to be a solution of a second auxiliary equation, on certain domains to be specified.

We display some relations provided by the action of the operators involved in (14). At first, these properties are considered to be symbolic, but they will become analytic provided that convergence is guaranteed in the sequel.

Lemma 1 *Given $U(T, z, \epsilon)$ in the form (15), it holds that*

$$T^2\partial_T U(T, z, \epsilon) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{\mathcal{L}} u\omega(u, m, \epsilon) e^{-\frac{u}{T}} e^{izm} \frac{du}{u} dm,$$

and for all positive $m \in \mathbb{N}$

$$T^m U(T, z, \epsilon) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{\mathcal{L}} \left(\frac{u}{\Gamma(m)} \int_0^u (u-s)^{m-1} \omega(s, m, \epsilon) \frac{ds}{s}\right) e^{-\frac{u}{T}} e^{izm} \frac{du}{u} dm.$$

In addition to this, one has

$$U\left(\frac{T}{1+k_0 T}, z, \epsilon\right) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{\mathcal{L}} e^{-uk_0} \omega(u, m, \epsilon) e^{-\frac{u}{T}} e^{izm} \frac{du}{u} dm.$$

The following lemma ([22],p. 3630) will help on finding an auxiliary writing of our main problem.

Lemma 2 *Let $m \in \mathbb{N}$ be a positive integer. Then it holds that*

$$T^{2m}\partial_T^m = (T^2\partial_T)^m + \sum_{1\le p\le m-1} A_{m,p}T^{m-p}(T^2\partial_T)^p,$$

for some real numbers $A_{m,p}$, $1 \le p \le m-1$.

In view of (5), we define the positive integer $d_{\ell,1}$ by

$$\delta_\ell = 2d_\ell + d_{\ell,1}, \tag{16}$$

for every $\ell \in I$.

Taking into account (16), one can apply Lemma 2 together with Lemma 1 and the properties of Fourier transform (see Section 7) in order to have that $\omega(\tau, m, \epsilon)$ solves the following second auxiliary problem

$$(17)\quad Q(im)\omega(\tau,m,\epsilon) = \tau^{\delta_D}R_D(im)\omega(\tau,m,\epsilon) + \tau^{\delta_0}e^{-\tau k_0}R_0(im)\omega(\tau,m,\epsilon)$$
$$+\frac{1}{(2\pi)^{1/2}}\sum_{\ell\in I}\int_{-\infty}^{\infty} C_\ell(m-m_1,\epsilon)R_\ell(im_1)\epsilon^{\Delta_\ell-\delta_\ell+d_\ell}\left(\frac{\tau}{\Gamma(d_{\ell,1})}\int_0^\tau (\tau-s)^{d_{\ell,1}-1}s^{d_\ell}\omega(s,m_1,\epsilon)\frac{ds}{s}\right.$$
$$\left.+\sum_{1\le p\le d_\ell-1} A_{d_\ell,p}\frac{\tau}{\Gamma(d_{\ell,1}+d_\ell-p)}\int_0^\tau (\tau-s)^{d_{\ell,1}+d_\ell-p-1}s^p\omega(s,m_1,\epsilon)\frac{ds}{s}\right)dm_1 + \psi(\tau,m,\epsilon),$$

at least from a formal point of view. The analytic functional spaces in which the solution of (17) is defined will be described subsequently in the paper. In that framework, (15) defines a holomorphic function in adequate domains, providing an actual solution of (14), and consequently of (10).

Two different families of solutions of the auxiliary problem (17) will be provided. We give detail on each type of solution and describe the situation for each case separately. The elements in the first family, studied in Section 3, are related to different branches of Lambert W function whereas the elements of the second family, analysed in Section 4, are linked to the classical Borel-Laplace summability procedure.

3 First family of analytic solutions of (17)

We depart from the main and auxiliary problems described in Section 2, together with the assumptions made on the elements in their construction.

In a first subsection, Section 3.1, we describe the geometry of the problem. It is worth mentioning a particular case, considered in Section 3.2, whose geometry serves as a model for the more complicated geometry of the general setting. For this reason, we provide a detailed proof of Lemma 5 within the particular case, which can be adapted to the general one under minor modifications.

3.1 Geometry of the problem

Under the assumptions made on the problem, equation (17) reads as follows:

$$\mathcal{D}(\tau,m)\omega(\tau,m,\epsilon) = \tau^{\delta_D}R_D(im)\omega(\tau,m,\epsilon) + \frac{1}{(2\pi)^{1/2}}\sum_{\ell\in I}\int_{-\infty}^{\infty} C_\ell(m-m_1,\epsilon)R_\ell(im_1)$$
$$\times\,\epsilon^{\Delta_\ell-\delta_\ell+d_\ell}\left(\frac{\tau}{\Gamma(d_{\ell,1})}\int_0^\tau(\tau-s)^{d_{\ell,1}-1}s^{d_\ell}\omega(s,m_1,\epsilon)\frac{ds}{s} + \sum_{1\le p\le d_\ell-1}A_{d_\ell,p}\frac{\tau}{\Gamma(d_{\ell,1}+d_\ell-p)}\right.$$
$$\left.\times\int_0^\tau(\tau-s)^{d_{\ell,1}+d_\ell-p-1}s^p\omega(s,m_1,\epsilon)\frac{ds}{s}\right)dm_1 + \psi(\tau,m,\epsilon), \tag{18}$$

where

$$\mathcal{D}(\tau,m) = Q(im) - \tau^{\delta_0}e^{-\tau k_0}R_0(im).$$

In the next result, we show that the roots of $\mathcal{D}(\tau,m)$ are related to Lambert W function.

Lemma 3 *Let $\mathcal{U}$ be some neighborhood of the origin in $\mathbb{C}$. We define*

$$H := \cup_{m\in\mathbb{R}}\{\tau\in\mathbb{C} : \mathcal{D}(\tau,m)=0\}$$

Then, provided that $r_{Q,R_0}>0$ is large enough, it holds that

$$H\cap\hat{H}=\emptyset,$$

with $\hat{H} := \mathcal{U}\cup\left(\bigcup_{k\in\mathbb{Z}}L_k\right)$, and where the set L_k consists of the complex numbers with negative real part which belong to a horizontal strip-like set, for every $k\in\mathbb{Z}$.

Proof Let $m\in\mathbb{R}$. It holds that $\mathcal{D}(\tau,m)=0$ if and only if

$$\tau^{\delta_0}e^{-\tau k_0} = \frac{Q(im)}{R_0(im)}. \tag{19}$$

Therefore, $\omega e^\omega = A(m)$, with $\omega = -\tau\frac{k_0}{\delta_0}$, $A(m) = -\frac{k_0}{\delta_0}B(m)$, and $B(m)$ stands for one δ_0 root of $Q(im)/R_0(im)$. In other words, $\omega = W(A(m))$, where W is Lambert W function. In view of (6), and the properties of Lambert W function (see Section 8), we derive that

$$|A(m)| = \left|-\frac{k_0}{\delta_0}B(m)\right| = \frac{k_0}{\delta_0}\left|\frac{Q(im)}{R_0(im)}\right|^{\frac{1}{\delta_0}} \ge \frac{k_0}{\delta_0}r_{Q,R_0}^{\frac{1}{\delta_0}} =: \tilde{r}_{Q,R_0} > 0.$$

In addition to this,

$$\frac{2\pi j+\alpha_{Q,R_0}}{\delta_0} < \arg(-A(m)) < \frac{2\pi j+\beta_{Q,R_0}}{\delta_0},$$

for $j\in\{0,1,\dots,\delta_0-1\}$. The two previous conditions describe δ_0 sectors, say $\{S_{Q,R_0,j}\}_{j=0\dots,\delta_0-1}$. Therefore,

$$\bigcup_{m\in\mathbb{R}}A(m)\subseteq\bigcup_{j=0}^{\delta_0-1}S_{Q,R_0,j}.$$

We write

$$S_{Q,R_0,j} = \{z\in\mathbb{C} : \alpha_{Q,R_0,j} < \arg(z) < \beta_{Q,R_0,j},\quad \tilde{r}_{Q,R_0}\le|z|\}$$

for each $j = 0, \ldots, \delta_0 - 1$, for certain $\alpha_{Q,R_0,j}, \beta_{Q,R_0,j}$ with

$$0 \leq \alpha_{Q,R_0,0} < \beta_{Q,R_0,0} < \alpha_{Q,R_0,1} < \beta_{Q,R_0,1} < \ldots < \alpha_{Q,R_0,\delta_0-1} < \beta_{Q,R_0,\delta_0-1} < 2\pi.$$

At this point, we describe the set $W(S_{Q,R_0,j})$ for $j = 0, \ldots, \delta_0 - 1$. For this purpose, let $j \in \{0, \ldots, \delta_0 - 1\}$ and write $z = \rho \exp(\sqrt{-1}\theta) \in S_{Q,R_0,j}$. We have $\tilde{r}_{Q,R_0} \leq \rho$ and $\alpha_{Q,R_0,j} < \theta < \beta_{Q,R_0,j}$. For the sake of simplicity, we write $\tilde{r}$, α_j and β_j for $\tilde{r}_{Q,R_0}$, $\alpha_{Q,R_0,j}$ and $\beta_{Q,R_0,j}$, respectively. Let $\omega = W(z)$ and write $\omega = \xi + \eta\sqrt{-1}$. Then, one has

$$\rho\cos(\theta) = \exp(\xi)(\xi\cos(\eta) - \eta\sin(\eta)), \qquad \rho\sin(\theta) = \exp(\xi)(\eta\cos(\eta) + \xi\sin(\eta)).$$

From the previous equalities we derive

$$\rho = \exp(\xi)\sqrt{\xi^2 + \eta^2}, \qquad \tan(\theta) = \frac{\eta\cos(\eta) + \xi\sin(\eta)}{\xi\cos(\eta) - \eta\sin(\eta)}.$$

Therefore,

$$W(S_{Q,R_\ell,j}) = \Big\{\omega = \xi + \eta\sqrt{-1} \in \mathbb{C} : \exp(\xi)\sqrt{\xi^2 + \eta^2} \geq \tilde{r}, \quad \tan(\theta) = \frac{\eta\cos(\eta) + \xi\sin(\eta)}{\xi\cos(\eta) - \eta\sin(\eta)}, \text{for } \theta \in (\alpha_j, \beta_j)\Big\}. \tag{20}$$

On the one hand, the set $\{\xi + \eta\sqrt{-1} \in \mathbb{C} : \exp(\xi)\sqrt{\xi^2 + \eta^2} \geq \tilde{r}\}$ is an infinite domain with boundary given by the curve $\{(\xi, \eta) \in \mathbb{R}^2 : \exp(\xi)\sqrt{\xi^2 + \eta^2} = \tilde{r}\}$ which intersects the vertical axis at the points $\eta = \pm\tilde{r}$, and the horizontal axis at the unique positive solution of $\exp(\xi)\xi = \tilde{r}$, which tends to infinity when $\tilde{r}$ tends to infinity. The domain contains the positive semiplane, except from a bounded set. An example of such domain for $\tilde{r} = 1$ is represented in Figure 1 (left). On the other hand, we are interested in the subset of $\{(\xi, \eta) \in \mathbb{R}^2 : \xi > 0\}$ determined by

$$\tan(\theta) = \frac{\eta\cos(\eta) + \xi\sin(\eta)}{\xi\cos(\eta) - \eta\sin(\eta)}, \quad \text{for } \theta \in (\alpha_j, \beta_j)$$

which is an infinite domain consisting of an infinite union of horizontal strip-like sets. Figure 1 (right) shows such subset for $\tan(\theta) \in (-\frac{1}{2}, \frac{1}{2})$, together with the ranges of the branches of Lambert W function. It is worth remarking that the principal branch of Lambert W function is contained in the horizontal strip $\{\omega \in \mathbb{C} : -\pi < \mathrm{Im}(\omega) < \pi\}$ and its boundary tends to the boundary of that strip, when the real part becomes larger. Let $M > 0$. The set of complex numbers contained in the set $\{\omega \in \mathbb{C} : \mathrm{Re}(\omega) \geq M\}$ which belong to the $k-$th branch of Lambert W function, for $k \in \mathbb{Z}\setminus\{0\}$, are contained in the horizontal strip $\{\omega \in \mathbb{C} : 2k\pi - \delta(M) < \mathrm{Im}(\omega) < (2k+1)\pi + \delta(M)\}$ for positive k (resp. in $\{\omega \in \mathbb{C} : (2k-1)\pi - \delta(M) < \mathrm{Im}(\omega) < 2k\pi + \delta(M)\}$ for negative k). Here, $\delta = \delta(M)$ is a positive decreasing function of M, which tends to 0 if M approaches infinity. A similar behavior can be observed when restricting the branches of Lambert W to the complex numbers of arguments in (α_j, β_j). It is worth mentioning the asymptotic behavior of Lambert W function resembling the complex logarithmic function at infinity.

Taking into account the previous remarks and the fact that ω is defined by $\omega = -\tau\frac{k_0}{\delta_0}$, we conclude the result.

□

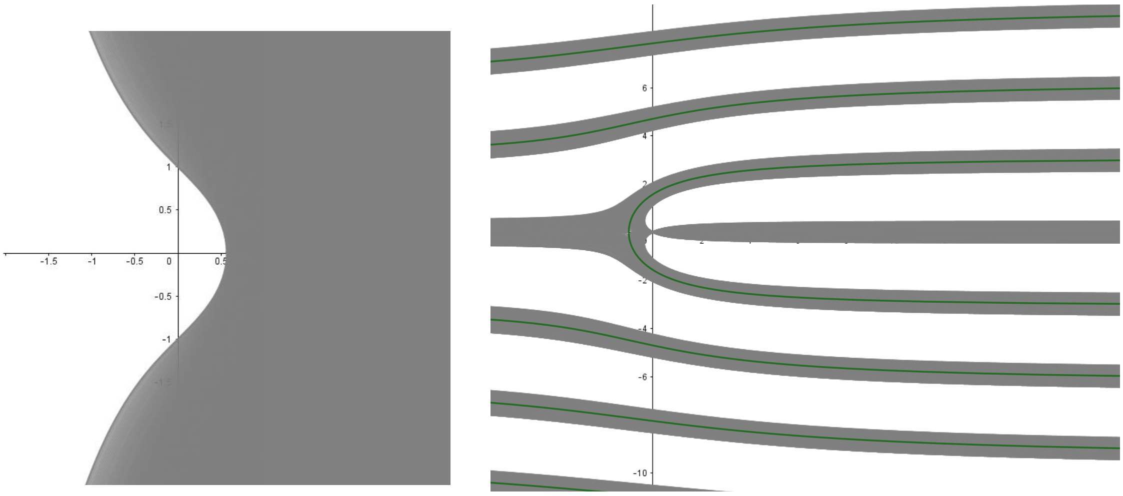

Figure 1: Domains $\exp(\xi)\sqrt{\xi^2+\eta^2} > \tilde{r} = 1$ (left) and $-\frac{1}{2} < \frac{\eta\cos(\eta)+\xi\sin(\eta)}{\xi\cos(\eta)-\eta\sin(\eta)} < \frac{1}{2}$ (right)

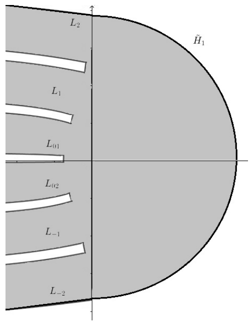

Figure 2: Example of domain $\tilde{H}$, resembling an octopus!

Definition 1 *One can slightly diminish the size of $\hat{H}$ in such a way that the distance from H to $\hat{H}$ is positive, while preserving its geometry. Let $\{L_k\}_{k\in\mathbb{Z}}$ denote the set of strip-like sets which conform $\hat{H}$. We have*

$$L_k = \{z \in \mathbb{C} : Re(z) < 0, \alpha_k(z) < Im(z) < \beta_k(z)\}, \tag{21}$$

for some real functions $\alpha_k(z) < \beta_k(z)$. We remark that several strip-like sets may arise within each branch, depending on the arguments of the elements in S_{Q,R_0}. We maintain the same notation for the strips for the sake of simplicity. We write $\tilde{H}$ for the connected component of $\hat{H}$ which contains $\mathcal{U}$. Figure 2 illustrates the geometric configuration of $\tilde{H}$.

Remark 1: Observe that the number of horizontal strip-like sets conforming $\tilde{H}$ is a positive increasing function of r_{Q,R_0}, which tends to infinity when r_{Q,R_0} becomes larger. As a matter of fact, we will assume that r_{Q,R_0} is large enough in order that the strip-like set contains a horizontal strip. Observe this is always possible due to the geometry of Lambert W function is asymptotically as that of the logarithm. Therefore, and for practical reasons, one can choose the functions $\alpha_k(z)$ and $\beta_k(z)$ in (21) to be constants, say $\alpha_k(z) \equiv \alpha < \beta \equiv \beta_k(z)$.

The details proof of Lemma 4 are postponed after Lemma 5, which states the same result in a particular case which helps to illustrate the technique of the proof of Lemma 4.

Lemma 4 *There exists* $C_1 > 0$ *such that*

$$\left| \frac{\tau^{\delta_D} R_D(im)}{Q(im) - \tau^{\delta_0} e^{-\tau k_0} R_0(im)} \right| \leq C_1,$$

for all $m \in \mathbb{R}$ *and* $\tau \in \hat{H}$.

In the next subsection, we describe the geometry in a particular case in which the strip-like sets L_k are indeed horizontal strips.

3.2 A particular case

This subsection is devoted to the particular case, under the further assumption

$$\delta_0 = 0. \tag{22}$$

We find this is a situation which helps to illustrate the geometry of the problem. Under assumption (22), we have

$$\mathcal{D}(\tau, m) = Q(im) - e^{-\tau k_0} R_0(im).$$

It holds that $\mathcal{D}(\tau, m) = 0$ if and only if

$$\tau = \tau_{m,k} = -\frac{1}{k_0} \left(\log \left| \frac{Q(im)}{R_0(im)} \right| + \left(\arg \left(\frac{Q(im)}{R_0(im)} \right) + 2\pi k \right) \sqrt{-1} \right), \quad k \in \mathbb{Z}, m \in \mathbb{R}.$$

Under condition (6), one has that the roots of $\mathcal{D}(\tau, m)$, $\tau_{m,k}$ for $k \in \mathbb{Z}$ and $m \in \mathbb{R}$, belong to a family of horizontal strips $\{H_k\}_{k \in Z}$, where

$$H_k = \{\tau \in \mathbb{C} : \mathrm{Re}(\tau) < -M, \quad \alpha + \frac{2k\pi}{k_0} < \mathrm{Im}(\tau) < \beta + \frac{2k\pi}{k_0}\},$$

for $M = \frac{1}{k_0} \log(r_{Q,R_0}) > 0$, which does not depend on $m \in \mathbb{R}$ nor $k \in \mathbb{Z}$, and $\alpha = -\frac{1}{k_0}\beta_{Q,R_0}$, $\beta = -\frac{1}{k_0}\alpha_{Q,R_0}$.

We write $\hat{H}$ for

$$\hat{H} = \{\tau \in \mathbb{C} : -M < \mathrm{Re}(\tau) < 0\} \bigcup \left(\bigcup_{k \in \mathbb{Z}} L_k \right),$$

where

$$L_k = \{\tau \in \mathbb{C} : \mathrm{Re}(\tau) < 0, \beta + \frac{2k\pi}{k_0} < \mathrm{Im}(\tau) < \alpha + \frac{2(k+1)\pi}{k_0}\}. \tag{23}$$

One may slightly reduce $\hat{H}$ in such a way that its distance to $\cup_{m \in \mathbb{R}} \{\tau : \mathcal{D}(\tau, m) = 0\}$ is positive. Figure 3 illustrates the geometric situation. Let $R > 0$.

We write $\tilde{H}$ for the connected component of $D(0, R) \bigcup \left(\bigcup_{k \in \mathbb{Z}} L_k \right)$ containing the origin of coordinates.

Remark 2: We observe that the number of horizontal strips in $\tilde{H}$ grows to infinity when $R \to \infty$.

Remark 3: The previous sets have already appeared as natural domains of solutions of problems previously studied by the authors, [14].

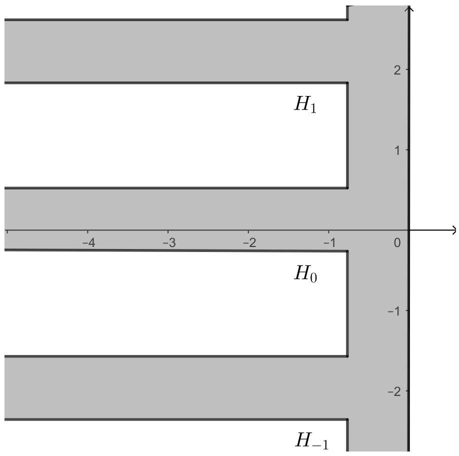

Figure 3: Example of configuration of the family $\{H_k\}_{k\in\mathbb{Z}}$

Lemma 5 *There exists $C_1 > 0$ such that*

$$\left| \frac{\tau^{\delta_D} R_D(im)}{Q(im) - e^{-\tau k_0} R_0(im)} \right| \leq C_1,$$

for all $m \in \mathbb{R}$, and $\tau \in \hat{H}$.

Proof Let $k \in \mathbb{Z}$, $\tau \in L_k$ and $m \in \mathbb{R}$. One can write $\tau = \tau_{m,k} + (a + i\theta)$ for a well chosen θ in a bounded interval and $a \in (-\infty, C \log \left|\frac{Q(im)}{R_0(im)}\right|]$ for some $C > 0$, since the real part of $\tau_{m,k}$ is $-1/k_0 \log \left|\frac{Q(im)}{R_0(im)}\right|$. This entails that

$$Q(im) - e^{-\tau k_0} R_0(im) = Q(im) - e^{-(\tau_{m,k}+a+i\theta)k_0} R_0(im) := A.$$

We observe that $\mathcal{D}(\tau_{m,k}, m) = 0$ which yields

$$A = Q(im)(1 - e^{-(a+i\theta)k_0}),$$

and therefore

$$\left| \frac{\tau^{\delta_D} R_D(im)}{Q(im) - e^{-\tau k_0} R_0(im)} \right| = \frac{|\tau_{m,k} + a + i\theta|^{\delta_D} |R_D(im)|}{|1 - e^{-(a+i\theta)k_0}||Q(im)|}.$$

By construction of L_k, we get a constant $C_{11} > 0$ with

$$|1 - e^{-(a+i\theta)k_0}| \geq C_{11} \tag{24}$$

for all $a \in (-\infty, C \log |\frac{Q(im)}{R_0(im)}|]$, all $m \in \mathbb{R}$, all $\theta \in I$ (a well chosen bounded interval). Furthermore, since e^{-ak_0} grows exponentially as a tends to $-\infty$, and since $\tau_{m,k}$ remains in a bounded domain for all $m \in \mathbb{R}$ (see assumption (6)), we deduce that the quantity

$$\frac{|\tau_{m,k} + a + i\theta|^{\delta_D}}{|1 - e^{-(a+i\theta)k_0}|}$$

remains bounded provided that $a < -M$ for some fixed $M > 0$, for all $m \in \mathbb{R}$, all $\theta \in I$. On the other hand, provided that $a \in (-M, C\log|\frac{Q(im)}{R_0(im)}|]$, the quantity

$$|\tau_{m,k} + a + i\theta|^{\delta_D}$$

remains bounded (again from (6)), for all $m \in \mathbb{R}$, all $\theta \in I$. As a result, from (24), the quotient

$$\frac{|\tau_{m,k} + a + i\theta|^{\delta_D}}{|1 - e^{-(a+i\theta)k_0}|}$$

remains bounded for all $m \in \mathbb{R}$, all $\theta \in I$ and all $a \in (-M, C\log|\frac{Q(im)}{R_0(im)}|]$.

At last, when $\tau \in D(0,R)$. According to (6) and provided that $r_{Q,R_0} > 0$ is large enough, we get a constant $\tilde{C}_{11} > 0$ with

$$|\frac{Q(im)}{R_0(im)} - e^{-\tau k_0}| > \tilde{C}_{11}$$

for all $\tau \in D(0,R)$, all $m \in \mathbb{R}$. As a result, the quotient

$$|\frac{\tau^{\delta_D} R_D(im)}{Q(im) - e^{-\tau k_0} R_0(im)}| = |\frac{R_D(im)}{R_0(im)}||\tau^{\delta_D}|\frac{1}{|\frac{Q(im)}{R_0(im)} - e^{-\tau k_0}|}$$

remains bounded, for all $m \in \mathbb{R}$, all $\tau \in D(0,R)$.

□

proof of Lemma 4:

Let $\tau \in L_k$, $m \in \mathbb{R}$. Let $\tau_{m,k}$ be a solution of (19) associated to the $k-$th branch of Lambert W function. We can write $\tau = \tau_{m,k} + a + i\theta$ for some well chosen a (on an interval detailed below) and θ on a well chosen bounded interval I. Furthermore,

$$\begin{aligned} Q(im) - \tau^{\delta_0} e^{-\tau k_0} R_0(im) \\ = Q(im) - \frac{(\tau_{m,k} + a + i\theta)^{\delta_0}}{\tau_{m,k}^{\delta_0}} \tau_{m,k}^{\delta_0} \exp(-(\tau_{m,k} + a + i\theta)k_0) R_0(im) \\ = Q(im)\left(1 - (\frac{\tau_{m,k} + a + i\theta}{\tau_{m,k}})^{\delta_0} \exp(-(a+i\theta)k_0)\right) \end{aligned}$$

Therefore,

$$\left|\frac{\tau^{\delta_D} R_D(im)}{Q(im) - \tau^{\delta_0} e^{-\tau k_0} R_0(im)}\right| = \frac{|\tau_{m,k} + a + i\theta|^{\delta_D}}{|1 - (\frac{\tau_{m,k}+a+i\theta}{\tau_{m,k}})^{\delta_D} e^{-(a+i\theta)k_0}|}\left|\frac{R_D(im)}{Q(im)}\right|$$

By construction, we know that

$$\tau_{m,k} = -\frac{\delta_0}{k_0} W_k(A(m))$$

where $A(m) \in \cup_{j=1}^{\delta_0 - 1} S_{Q,R_0,j}$ satisfies in particular

$$|A(m)| = \frac{k_0}{\delta_0}\left|\frac{Q(im)}{R_0(im)}\right|^{1/\delta_0}$$

We recall that $W_k(\tau)$ is close to the $k-$branch of the logarithm $\log(z) + 2i\pi k$ as $|z|$ tends to infinity where $\log(z)$ denotes the principal branch of the logarithm. Provided that $r_{Q,R_0} > 0$ is large enough, we deduce that $\tau_{m,k}$ is close to the quantity

$$-\frac{\delta_0}{k_0}(\log(A(m)) + 2i\pi k)$$

for all $m \in \mathbb{R}$. From now on, the proof follows similar arguments as the one of Lemma 5. However, we provide sharp bounds that will have crucial importance later on in the work.

Since e^{-ak_0} grows exponentially as a tends to $-\infty$, we deduce the next bounds for the quotient

$$\frac{|\tau_{m,k} + a + i\theta|^{\delta_D}}{|1 - (\frac{\tau_{m,k}+a+i\theta}{\tau_{m,k}})^{\delta_D} e^{-(a+i\theta)k_0}|} \leq \frac{\sum_{p+q=\delta_D} \frac{\delta_D!}{p!q!}|\tau_{m,k}|^p |a+i\theta|^q}{|1 - (\frac{\tau_{m,k}+a+i\theta}{\tau_{m,k}})^{\delta_D} e^{-(a+i\theta)k_0}|} \leq D_1(\log|Q(im)/R_0(im)|)^{\tilde{\delta}}$$

provided that $a \in (-\infty, C\log|Q(im)/R_0(im)|]$ for a well chosen constant $C, D_1 > 0$ and $\tilde{\delta} > 0$ (depending on $\delta_0, k_0, k, \delta_D$) for all $m \in \mathbb{R}$. As a result, we get bounds of the form

$$\left|\frac{\tau^{\delta_D} R_D(im)}{Q(im) - \tau^{\delta_0}e^{-\tau k_0}R_0(im)}\right| \leq D_1(\log|Q(im)/R_0(im)|)^{\tilde{\delta}} \left|\frac{R_D(im)}{Q(im)}\right|$$

for all $\tau \in L_k$, all $m \in \mathbb{R}$.

In the last part of the proof, we provide bounds for τ on $\mathcal{U}$. According to (6), provided that r_{Q,R_0} is taken large enough, we get a constant $\hat{C}_{11} > 0$ with

$$\left|\frac{Q(im)}{R_0(im)} - \tau^{\delta_0}e^{-k_0\tau}\right| > \hat{C}_{11}$$

for all $\tau \in \mathcal{U}$, all $m \in \mathbb{R}$. As a result, the quotient

$$\left|\frac{\tau^{\delta_D} R_D(im)}{Q(im) - \tau^{\delta_0}e^{-\tau k_0}R_0(im)}\right| = \left|\frac{R_D(im)}{R_0(im)}\right| |\tau|^{\delta_D} \frac{1}{|\frac{Q(im)}{R_0(im)} - \tau^{\delta_D}e^{-\tau k_0}|} \tag{25}$$

remains bounded, for all $m \in \mathbb{R}$, all $\tau \in \mathcal{U}$.

□

Taking into account (9), the proof of Lemma 4 can also be directly applied to the next result.

Lemma 6 *Let $\gamma_1 > 0$ and let $a \in \mathbb{C}$ be a complex number. For every $\ell \in I$, there exists $C_1(\gamma_1, a, \ell) > 0$ such that*

$$\left|\frac{(a+\tau)^{\gamma_1} R_\ell(im)}{Q(im) - \tau^{\delta_0}e^{-\tau k_0}R_0(im)}\right| \leq C_1(\gamma_1, a, \ell)$$

for all $m \in \mathbb{R}$, all $\tau \in \hat{H}$.

Remark 4: Observe that the constant C_1 in Lemma 4 (also in Lemma 5) depends on r_{Q,R_D} (see (8)) in such a way that $C_1 \to 0$ if $r_{Q,R_D} \to 0$.

3.3 Auxiliary Banach spaces of functions and solution of the auxiliary problem

Let $\mathcal{U}$ be some convex neighborhood of the origin in $\mathbb{C}$, for example $\mathcal{U} := D(0, R)$ for some $R > 0$. From the assumptions made on the geometry of the problem, one can restrict the domain of study to a horizontal strip, as in Figure 3.

Let L be a horizontal strip of the form

$$L = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0, \alpha < \mathrm{Im}(z) < \beta\}, \tag{26}$$

with $\alpha < \beta$. Moreover, we assume that $\mathcal{U} \cap L \neq \emptyset$.

Definition 2 *Let $\mu, \beta, \nu > 0$. The set $F^L_{(\nu,\beta,\mu)}$ consists of all continuous functions $(\tau, m) \mapsto h(\tau, m)$ defined on $\mathcal{U} \times \mathbb{R}$, holomorphic on $\mathcal{U}$ with respect to the first variable, such that*

1) For every $m \in \mathbb{R}$, the function $\tau \mapsto h(\tau, m)$ can be extended analytically to L.

2) For every $h \in F^L_{(\nu,\beta,\mu)}$ it holds that

$$\|h(\tau, m)\|_{(\nu,\beta,\mu),L} := \sup_{\tau \in (\mathcal{U} \cup L), m \in \mathbb{R}} (1 + |m|)^{\mu} \frac{1 + |\tau|^2}{|\tau|} \exp(\beta|m| - \nu|\tau|)|h(\tau, m)| < \infty.$$

The pair $(F^L_{(\nu,\beta,\mu)}, \|\cdot\|_{(\nu,\beta,\mu),L})$ is a complex Banach space.

One has the following result, whose proof is a direct consequence of the definition of the previous Banach space.

Lemma 7 *Let $\mu, \beta, \nu > 0$. Let $(\tau, m) \mapsto a(\tau, m)$ be a continuous function defined on $(\mathcal{U} \cup L) \times \mathbb{R}$, holomorphic with respect to the first variable on $\mathcal{U} \cup L$ such that*

$$\sup_{(\tau,m) \in (\mathcal{U} \cup L) \times \mathbb{R}} |a(\tau, m)|$$

is upper bounded. Then, for every $f \in F^L_{(\nu,\beta,\mu)}$ it holds that $(\tau, m) \mapsto a(\tau, m)f(\tau, m)$ belongs to $F^L_{(\nu,\beta,\mu)}$, and

$$\|a(\tau, m)f(\tau, m)\|_{(\nu,\beta,\mu),L} \leq \left(\sup_{(\tau,m) \in (\mathcal{U} \cup L) \times \mathbb{R}} |a(\tau, m)| \right) \|f(\tau, m)\|_{(\nu,\beta,\mu),L}.$$

The next result describes continuous operators regarding the previous Banach space and $(E_{(\beta,\mu)}, \|\cdot\|_{(\beta,\mu)})$ (see Annex 1).

Lemma 8 *Let $\mu > 1$ and $\beta, \nu > 0$. For every $f \in E_{(\beta,\mu)}$ and $g \in F^L_{(\nu,\beta,\mu)}$, the function $\phi(\tau, m)$ defined by*

$$\phi(\tau, m) := \int_{-\infty}^{\infty} f(m_1)g(\tau, m - m_1)dm_1$$

belongs to $F^L_{(\nu,\beta,\mu)}$. In addition to this, there exists $D_1 > 0$ such that

$$\|\phi(\tau, m)\|_{(\nu,\beta,\mu),L} \leq D_1 \|f\|_{(\beta,\mu)} \|g\|_{(\nu,\beta,\mu),L}.$$

Proof Let $f \in E_{(\beta,\mu)}$ and $g \in F^{L}_{(\nu,\beta,\mu)}$. From the definition of the Banach spaces $F^{L}_{(\nu,\beta,\mu)}$ and $E_{(\beta,\mu)}$ we get

$$
\begin{aligned}
(27)\quad & \|\phi(\tau,m)\|_{(\nu,\beta,\mu),L} \\
&= \sup_{(\tau,m)\in(\mathcal{U}\cup L)\times\mathbb{R}} (1+|m|)^{\mu}\frac{1+|\tau|^2}{|\tau|}\exp(\beta|m|-\nu|\tau|)\int_{-\infty}^{\infty}(1+|m_1|)^{\mu}\exp(\beta|m_1|)|f(m_1)| \\
&\qquad \times (1+|m-m_1|)^{\mu}\frac{1+|\tau|^2}{|\tau|}\exp(\beta|m-m_1|-\nu|\tau|)|g(\tau,m-m_1)|\mathcal{G}(\tau,m,m_1)dm_1,
\end{aligned}
$$

with

$$\mathcal{G}(\tau,m,m_1) := \frac{1}{(1+|m_1|)^{\mu}}\exp(-\beta_1|m_1|)\frac{1}{(1+|m-m_1|)^{\mu}}\frac{|\tau|}{1+|\tau|^2}\exp(-\beta|m-m_1|+\nu|\tau|).$$

This entails that

$$\|\phi(\tau,m)\|_{(\nu,\beta,\mu),L} \le \sup_{m\in\mathbb{R}}(1+|m|)^{\mu}\int_{-\infty}^{\infty}\frac{1}{(1+|m_1|)^{\mu}(1+|m-m_1|)^{\mu}}dm_1 \,\|f\|_{(\beta,\mu)}\,\|g\|_{(\nu,\beta,\mu),L}\,.$$

At this point, one can apply Lemma 2.2 in [5] or Lemma 4 in [19] to conclude the result. □

The following Proposition can be proved under minor modifications following analogous arguments as those in Proposition 5, [18], or Proposition1 [15]. We give the details of the proof for a self-contained presentation.

Proposition 1 *Let $\gamma_1 \ge 0$, $\eta_2 > -1$ be real numbers. Let $\nu_2 > 0$ be an integer. We also consider the function $a_{\gamma_1}(\tau)$, holomorphic on $\mathcal{U}\cup L$, continuous up to its boundary, such that*

$$|a_{\gamma_1}(\tau)| \le \frac{1}{(1+|\tau|)^{\gamma_1}}, \quad \tau\in\mathcal{U}\cup L.$$

We assume that $\gamma_1 > \nu_2+\eta_2+1$. Then, there exists $D_2>0$ which only depends on the values of the previous parameters, such that

$$\left\| a_{\gamma_1}(\tau)\int_0^{\tau}(\tau-s)^{\eta_2}s^{\nu_2}f(s,m)ds\right\|_{(\nu,\beta,\mu),L} \le D_2\,\|f(\tau,m)\|_{(\nu,\beta,\mu),L}$$

for every $f(\tau,m)\in F^{L}_{(\nu,\beta,\mu)}$. In the previous bounds, the integral is performed along a path totally contained in $\mathcal{U}\cup L$.

Proof Let $f(\tau,m)\in F^{L}_{(\nu,\beta,\mu)}$ From the definition of the norm, one has

$$
\begin{aligned}
(28)\quad & \\
&\left\| a_{\gamma_1}(\tau)\int_0^{\tau}(\tau-s)^{\eta_2}s^{\nu_2}f(s,m)ds\right\|_{(\nu,\beta,\mu),L} = \sup_{\tau\in\mathcal{U}\cup L, m\in\mathbb{R}}(1+|m|)^{\mu}\frac{1+|\tau|^2}{|\tau|}\exp(\beta|m|-\nu|\tau|) \\
&\qquad \times\left| a_{\gamma_1}(\tau)\int_0^{\tau}(1+|m|)^{\mu}e^{\beta|m|}\exp(-\nu|s|)\frac{1+|s|^2}{|s|}f(s,m)\mathcal{F}(\tau,s,m)ds\right|,
\end{aligned}
$$

with

$$\mathcal{F}(\tau,s,m) = \frac{1}{(1+|m|)^{\mu}}e^{-\beta|m|}\frac{\exp(\nu|s|)}{1+|s|^2}|s|(\tau-s)^{\eta_2}s^{\nu_2}.$$

The path of integration relying 0 and τ is given by the next construction. When $\tau \in \mathcal{U}$, the path is merely given by the segment $[0,\tau]$. When $\tau \in L$, take some $\tau_1 \in \mathcal{U}$ such that $\mathrm{Im}(\tau) = \mathrm{Im}(\tau_1)$, then the path is given by the union of the two segment $L_1 = [0,\tau_1]$ and $L_2 = [\tau_1,\tau]$. The integral in (28) can be parametrized by splitting the integration path as described above. We show upper bounds in the two following cases:

a) If $\tau \in \mathcal{U}$, we are reduced to give upper estimates for

$$\frac{1+|\tau|^2}{|\tau|}e^{-\nu|\tau|}\frac{1}{(1+|\tau|)^{\gamma_1}}\int_0^{|\tau|}\frac{e^{\nu h}}{1+h^2}h^{1+\nu_2}(|\tau|-h)^{\eta_2}dh$$

which is bounded by a constant since $\mathcal{U}$ is bounded.

b) In the case that $\tau \in L$, the integration along L_1 implies to give bounds for the next quantity

$$\frac{1+|\tau|^2}{|\tau|}e^{-\nu|\tau|}\frac{1}{(1+|\tau|)^{\gamma_1}}\int_0^{|\tau_1|}\frac{e^{\nu h}}{1+h^2}h^{1+\nu_2}(|\tau|-h)^{\eta_2}dh$$

for $|\tau| \geq 1$. The previous expression is upper bounded by

$$\frac{1+|\tau|^2}{|\tau|}e^{\nu(|\tau_1|-|\tau|)}\frac{1}{(1+|\tau|)^{\gamma_1}}|\tau|^{1+\nu_2+\eta_2}\int_0^{|\tau_1|}\frac{1}{1+h^2}dh,$$

which is upper bounded for any value of the parameters involved.

On the other hand, for the integration along L_2, we parametrize L_2 by $h \mapsto s(h) = -h+\sqrt{-1}\mathrm{Im}(\tau_1)$ on the segment $[|\mathrm{Re}(\tau_1)|,|\mathrm{Re}(\tau)|]$. We are reduced to give bounds for the quantity

$$\frac{1+|\tau|^2}{|\tau|}e^{-\nu|\tau|}\frac{1}{(1+|\tau|)^{\gamma_1}}\int_{|\mathrm{Re}(\tau_1)|}^{|\mathrm{Re}(\tau)|}\frac{\exp(\nu(\sqrt{h^2+(\mathrm{Im}(\tau_1))^2}))}{1+h^2+(\mathrm{Im}(\tau_1))^2} \times (\sqrt{h^2+(\mathrm{Im}(\tau_1))^2})^{\nu_2+1}\times|\tau-(-h+\sqrt{-1}\mathrm{Im}(\tau_1))|^{\eta_2}dh \tag{29}$$

for $\tau \in L$. The expression in (29) is upper estimated by

$$\frac{1+|\tau|^2}{|\tau|}\frac{1}{(1+|\tau|)^{\gamma_1}}\left(\int_0^{\infty}\frac{dh}{1+h^2}\right)|\tau|^{\nu_2+1}(2|\tau|)^{\eta_2}.$$

One concludes the proof regarding the assumption made on the parameters.

□

Lemma 9 *Let the auxiliary equation (17) be constructed as in Section 2, and let $k \in \mathbb{Z}$ such that $L_k \cap \mathcal{U} \neq \emptyset$. Here, L_k and $\mathcal{U}$ stand for the sets of $\mathbb{C}$ of Lemma 3. Let $\beta,\nu,\varpi > 0$ and $\mu > 1$. Then, there exists $\xi_\psi > 0$ and r_{Q,R_D} such that if $\mathcal{C}_\psi \leq \xi_\psi$ ($\mathcal{C}_\psi$ is the constant in (12), then for every $\epsilon \in D(0,\epsilon_0)$ the following properties of the map*

$$\mathcal{H}_\epsilon(\omega(\tau,m)) := \frac{\tau^{\delta_D}R_D(im)\omega(\tau,m,\epsilon)}{\mathcal{D}(\tau,m)} + \frac{1}{(2\pi)^{1/2}}\sum_{\ell\in I}\int_{-\infty}^{\infty}C_\ell(m-m_1,\epsilon)R_\ell(im_1)\epsilon^{\Delta_\ell-\delta_\ell+d_\ell}$$
$$\times\left(\frac{\tau}{\mathcal{D}(\tau,m)\Gamma(d_{\ell,1})}\int_0^\tau(\tau-s)^{d_{\ell,1}-1}s^{d_\ell}\omega(s,m_1,\epsilon)\frac{ds}{s} + \sum_{1\leq p\leq d_\ell-1}A_{d_\ell,p}\frac{\tau}{\mathcal{D}(\tau,m)\Gamma(d_{\ell,1}+d_\ell-p)}\right.$$
$$\left.\times\int_0^\tau(\tau-s)^{d_{\ell,1}+d_\ell-p-1}s^p\omega(s,m_1,\epsilon)\frac{ds}{s}\right)dm_1 + \frac{1}{\mathcal{D}(\tau,m)}\psi(\tau,m,\epsilon)$$

hold:

1. $\mathcal{H}_\epsilon : \overline{D}(0,\varpi) \to \overline{D}(0,\varpi)$, *where* $\overline{D}(0,\varpi)$ *stands for the closed disc of radius* ϖ *in the Banach space* $F^{L_k}_{(\nu,\beta,\mu)}$.

2. *For every* $\omega_1,\omega_2 \in \overline{D}(0,\varpi) \subseteq F^{L_k}_{(\nu,\beta,\mu)}$, *it holds that*

$$\|\mathcal{H}_\epsilon(\omega_1) - \mathcal{H}_\epsilon(\omega_2)\|_{(\nu,\beta,\mu),L_k} \leq \frac{1}{2}\|\omega_1-\omega_2\|_{(\nu,\beta,\mu),L_k}.$$

Proof Let $\epsilon \in D(0,\epsilon_0)$, $\omega \in F^{L_k}_{(\nu,\beta,\mu)}$. In view of Lemma 4 and Lemma 7 we derive that

$$\left\|\frac{\tau^{\delta_D}R_D(im)}{\mathcal{D}(\tau,m)}\omega(\tau,m)\right\|_{(\nu,\beta,\mu),L_k} \leq C_1\|\omega(\tau,m)\|_{(\nu,\beta,\mu),L_k}. \tag{30}$$

In view of Lemma 8, we observe for every $\ell \in I$

$$\begin{aligned}&\left\|\frac{\tau}{\mathcal{D}(\tau,m)}\int_{-\infty}^{\infty}C_\ell(m-m_1,\epsilon)R_\ell(im_1)\int_0^\tau(\tau-s)^{d_{\ell,1}-1}s^{d_\ell}\omega(s,m_1,\epsilon)\frac{ds}{s}\right\|_{(\nu,\beta,\mu),L_k}\\&\leq D_1\|C_\ell(m)\|_{(\beta,\mu)}\left\|\frac{\tau R_\ell(im)}{\mathcal{D}(\tau,m)}\int_0^\tau(\tau-s)^{d_{\ell,1}-1}s^{d_\ell}\omega(s,m,\epsilon)\frac{ds}{s}dm_1\right\|_{(\nu,\beta,\mu),L_k}\end{aligned} \tag{31}$$

and for every $1 \leq p \leq d_\ell - 1$,

$$\begin{aligned}&\left\|\frac{\tau}{\mathcal{D}(\tau,m)}\int_{-\infty}^{\infty}C_\ell(m-m_1,\epsilon)R_\ell(im_1)\int_0^\tau(\tau-s)^{d_{\ell,1}+d_\ell-p-1}s^{p}\omega(s,m_1,\epsilon)\frac{ds}{s}dm_1\right\|_{(\nu,\beta,\mu),L_k}\\&\leq D_1\|C_\ell(m)\|_{(\beta,\mu)}\left\|\frac{\tau R_\ell(im)}{\mathcal{D}(\tau,m)}\int_0^\tau(\tau-s)^{d_{\ell,1}+d_\ell-p-1}s^{p}\omega(s,m,\epsilon)\frac{ds}{s}\right\|_{(\nu,\beta,\mu),L_k}\end{aligned} \tag{32}$$

for some constant $D_1 > 0$.

Let $a \in \mathbb{C}$ be a complex number with $\mathrm{dist}(L_k \cup \mathcal{U}, -a) > 0$. From Lemma 6, for any given $\gamma_1 > 0$, we get a constant $C_1(\gamma_1+1,a,\ell) > 0$ with

$$\left|\frac{\tau R_\ell(im)}{\mathcal{D}(\tau,m)}\right| = \left|\frac{\tau}{(a+\tau)^{\gamma_1+1}}\right|\left|\frac{(a+\tau)^{\gamma_1+1}R_l(im)}{\mathcal{D}(\tau,m)}\right| \leq D_3\frac{C_1(\gamma_1+1,a,\ell)}{(1+|\tau|)^{\gamma_1}}$$

for every $\ell \in I$, some $D_3 > 0$, provided that $\tau \in L_k \cup \mathcal{U}$.

We apply Lemma 7 and Proposition 1 to (31) and (32) with $\gamma_1 > \max_{\ell\in I}\{\delta_\ell - d_\ell + 1\}$ to arrive at

$$\left\|\frac{\tau R_\ell(im)}{\mathcal{D}(\tau,m)}\int_0^\tau(\tau-s)^{d_{\ell,1}-1}s^{d_\ell}\omega(s,m,\epsilon)\frac{ds}{s}dm_1\right\|_{(\nu,\beta,\mu),L_k} \leq D_3C_1(\gamma_1+1,a,\ell)D_2\|\omega(\tau,m)\|_{(\nu,\beta,\mu),L_k}, \tag{33}$$

and

$$\left\|\frac{\tau R_\ell(im)}{\mathcal{D}(\tau,m)}\int_0^\tau(\tau-s)^{d_{\ell,1}+d_\ell-p-1}s^{p}\omega(s,m,\epsilon)\frac{ds}{s}\right\|_{(\nu,\beta,\mu),L_k} \leq D_3C_1(\gamma_1+1,a,\ell)D_2\|\omega(\tau,m)\|_{(\nu,\beta,\mu),L_k}. \tag{34}$$

On the other hand, in view of (12), Lemma 7 and the slight diminishing of $\tilde{H}$ we get

$$\left\| \frac{1}{\mathcal{D}(\tau,m)} \psi(\tau,m,\epsilon) \right\|_{(\nu,\beta,\mu),L_k} \leq \left(\sup_{\tau \in (\mathcal{U} \cup L_k), m \in \mathbb{R}} \frac{1}{|\mathcal{D}(\tau,m)|} \right) \mathcal{C}_{\psi} \leq C_2 \xi_{\psi}, \tag{35}$$

for some $C_2 > 0$.

In addition to this, the remark after the proof of Lemma 4, we derive that if the geometry of the problem is such that $r_{Q,R_D} > 0$ is large enough, then the constant C_1 can be chosen arbitrary small. In this situation, assume that $\xi_{\psi} > 0$ and ϵ_0 are small enough in such a way that

$$C_1 \varpi + \frac{1}{(2\pi)^{1/2}} D_1 D_2 D_3 \sum_{\ell \in I} \epsilon_0^{\Delta_\ell - \delta_\ell + d_\ell} C_1(\gamma_1 + 1, a, \ell) \| C_\ell \|_{(\beta,\mu)} \left(\frac{1}{\Gamma(d_{\ell,1})} \varpi \right. \\ \left. + \sum_{1 \leq p \leq d_\ell - 1} |A_{d_\ell,p}| \frac{1}{\Gamma(d_{\ell,1} + d_\ell - p)} \varpi \right) + C_2 \xi_{\psi} \leq \varpi. \tag{36}$$

Then, the application of (30), (31), (32), (33), (34), (35), and (4) to the definition of $\mathcal{H}_\epsilon$ yields the first part of the proof. For the second part, let $\omega_1, \omega_2 \in \overline{D}(0,\varpi) \subseteq F^{L_k}_{(\nu,\beta,\mu)}$. Then, analogous estimates as in the first part of the proof guarantee that if C_1, ϵ_0 are small enough so that

$$C_1 + \frac{D_1 D_2 D_3}{(2\pi)^{1/2}} \sum_{\ell \in I} \epsilon_0^{\Delta_\ell - \delta_\ell + d_\ell} C_1(\gamma_1 + 1, a, \ell) \| C_\ell \|_{(\beta,\mu)} \left(\frac{1}{\Gamma(d_{\ell,1})} + \sum_{1 \leq p \leq d_\ell - 1} \frac{|A_{d_\ell,p}|}{\Gamma(d_{\ell,1} + d_\ell - p)} \right) \leq \frac{1}{2}.$$

then the second statement is attained. We recall that if $r_{Q,R_D} > 0$ is large enough, then C_1 becomes closer to 0. □

Proposition 2 *Under the hypotheses of Lemma 9, for every $\varpi > 0$ and $\epsilon_0 > 0$ there exists $\xi_\psi > 0$ such that if $\mathcal{C}_\psi \leq \xi_\psi$ and r_{Q,R_D} is large enough, then the auxiliary equation (17) admits a solution $\omega_{L_k}(\tau,m,\epsilon)$ for every $\epsilon \in D(0,\epsilon_0)$ with $\omega_{L_k}(\tau,m,\epsilon) \in F^{L_k}_{(\nu,\beta,\mu)}$ and $\|\omega_{L_k}(\tau,m,\epsilon)\|_{(\nu,\beta,\mu),L_k} \leq \varpi$.*

Proof In view of Lemma 9, for every $\epsilon \in D(0,\epsilon_0)$, the map $\mathcal{H}_\epsilon$ is contractive from $\overline{D}(0,\varpi)$ into itself. The fixed point theorem in Banach spaces guarantees the existence of a fixed point for $\mathcal{H}_\epsilon$, say $\omega_{L_k}(\tau,m,\epsilon)$. The function $\omega_{L_k}(\tau,m,\epsilon)$ belongs to $F^{L_k}_{(\nu,\beta,\mu)}$ for every $\epsilon \in D(0,\epsilon_0)$, with $\|\omega_{L_k}(\tau,m,\epsilon)\|_{(\nu,\beta,\mu),L_k} \leq \varpi$. The definition of $\mathcal{H}_\epsilon$ guarantees that ω_{L_k} is a solution of (17), and also the holomorphy with respect to ϵ in $D(0,\epsilon_0)$. □

4 Second family of analytic solutions of (17)

We depart from the construction of the main and auxiliary problems described in Section 2, together with the assumptions made on the elements involved therein. The structure of this section is similar to that of Section 3.

4.1 Geometry of the problem II

We rewrite equation (17) in the form

$$
\begin{aligned}
(37)\quad \mathcal{D}_2(\tau,m)\omega(\tau,m,\epsilon) &= \tau^{\delta_0}e^{-\tau k_0}R_0(im)\omega(\tau,m,\epsilon)\\
&+\frac{1}{(2\pi)^{1/2}}\sum_{\ell\in I}\int_{-\infty}^{\infty}C_\ell(m-m_1,\epsilon)R_\ell(im_1)\epsilon^{\Delta_\ell-\delta_\ell+d_\ell}\left(\frac{\tau}{\Gamma(d_{\ell,1})}\int_0^\tau(\tau-s)^{d_{\ell,1}-1}s^{d_\ell}\omega(s,m_1,\epsilon)\frac{ds}{s}\right.\\
&\left.+\sum_{1\le p\le d_\ell-1}A_{d_\ell,p}\frac{\tau}{\Gamma(d_{\ell,1}+d_\ell-p)}\int_0^\tau(\tau-s)^{d_{\ell,1}+d_\ell-p-1}s^p\omega(s,m_1,\epsilon)\frac{ds}{s}\right)dm_1+\psi(\tau,m,\epsilon),
\end{aligned}
$$

with

$$\mathcal{D}_2(\tau,m)=Q(im)-\tau^{\delta_D}R_D(im).$$

The next result describes the geometry of the roots of $\mathcal{D}_2(\tau,m)$. The technique used for dealing with this case resembles that of [17]. We provide a detailed proof of the results for the sake of completeness in a self-contained work.

Lemma 10 *There exists $d\in\mathbb{R}$ and $\rho>0$ such that if the opening of the sector S_{Q,R_D} in (8) is small enough, then there exist positive constants M_1,M_2 such that*

$$\sup_{\tau\in(D(0,\rho)\cup S_d)}\frac{|\mathcal{D}_2(\tau,m)|}{(1+|\tau|)^{\delta_D-1}}\ge M_1^{\delta_D-1}M_2r_{Q,R_D}^{1/\delta_D}|R_D(im)|, \tag{38}$$

for every $m\in\mathbb{R}$. Here, S_d stands for an unbounded sector of bisecting direction d, and small opening, i.e. $S_d=\{\tau\in\mathbb{C}:|arg(\tau)-d|<\eta\}$, for some $\eta>0$.

Proof We factorize the polynomial $\tau\mapsto\mathcal{D}_2(\tau,m)$ in the form

$$\mathcal{D}_2(\tau,m)=-R_D(im)\prod_{\ell=0}^{\delta_D-1}(\tau-q_\ell(m)),$$

where

$$q_\ell(m)=\left(\frac{|Q(im)|}{|R_D(im)|}\right)^{\frac{1}{\delta_D}}\exp\left(\sqrt{-1}\left(\arg(\frac{Q(im)}{R_D(im)})\frac{1}{\delta_D}+\frac{2\pi\ell}{\delta_D}\right)\right),$$

for every $0\le\ell\le\delta_D-1$ and $m\in\mathbb{R}$. We choose small enough $\rho>0$ and S_d with small enough opening satisfying the following properties:

1) There exists a constant $M_1>0$ such that

$$|\tau-q_\ell(m)|\ge M_1(1+|\tau|) \tag{39}$$

for every $m\in\mathbb{R}$ and $\tau\in S_d\cup\overline{D}(0,\rho)$.

Proof The condition (8) guarantees the existence of $\rho>0$ such that for every $0\le\ell\le\delta_D-1$ one has $|q_\ell(m)|>2\rho$ for all $m\in\mathbb{R}$. We observe that the set $\cup_{m\in\mathbb{R}}\{q_\ell(m)\}_{0\le\ell\le\delta_D-1}$ is contained in the union of unbounded sectors with vertex at the origin that do not cover a punctured neighborhood of the origin, provided that the opening of S_{Q,R_D} is small enough. Condition 1) follows from here. □

2) There exists a constant $M_2 > 0$ and $0 \leq \ell_1 \leq \delta_D - 1$ such that

(40) $$|\tau - q_{\ell_1}(m)| \geq M_2 |q_{\ell_1}(m)|$$

for every $m \in \mathbb{R}$ and $\tau \in S_d \cup \overline{D}(0,\rho)$.

Proof Observe that the previous choice of S_d and $\rho > 0$ guarantees that for any fixed $0 \leq \ell_1 \leq \delta_D - 1$, the distance from the set $\{\tau/q_{\ell_1}(m) : \tau \in (S_d \cup \overline{D}(0,\rho)), m \in \mathbb{R}\}$ to the complex number 1 is positive. Therefore, statement 2) holds for some $M_2 > 0$. □

Taking into account (39), (40), and the factorization of $\mathcal{D}_2(\tau,m)$, we conclude that

$$|\mathcal{D}_2(\tau,m)| \geq |R_D(im)| M_1^{\delta_D - 1} (1+|\tau|)^{\delta_D - 1} M_2 \left(\frac{|Q(im)|}{|R_D(im)|} \right)^{1/\delta_D}$$

(41) $$\geq |R_D(im)| M_1^{\delta_D - 1} M_2 r_{Q,R_D}^{1/\delta_D} (1+|\tau|)^{\delta_D - 1}$$

for every $\tau \in S_d \cup \overline{D}(0,\rho)$ and $m \in \mathbb{R}$. The result follows directly from the previous inequality. □

Remark 5: Observe from the proof of Lemma 10 that the direction $d \in \mathbb{R}$ can be chosen such that $-\frac{\pi}{2} < d < \frac{\pi}{2}$.

Lemma 11 *Let $\rho > 0$ and $-\frac{\pi}{2} < d < \frac{\pi}{2}$ from Lemma 10. Then, there exists $\tilde{C}_1(0) > 0$ such that*

$$\left| \tau^{\delta_0} \frac{e^{-\tau k_0} R_0(im)}{\mathcal{D}_2(\tau,m)} \right| \leq \tilde{C}_1(0),$$

for all $m \in \mathbb{R}$ and $\tau \in S_d \cup \overline{D}(0,\rho)$.

Proof In view of the choice of d, if the opening of S_d is small enough, there exists $\Delta_1 > 0$ such that $\mathrm{Re}(\tau) \geq \Delta_1 |\tau|$ for every $\tau \in S_d$. Therefore, the expression

$$\sup_{\tau \in S_d} \left| \frac{\tau^{\delta_0}}{(1+|\tau|)^{\delta_D - 1}} e^{-\tau k_0} \right| = \sup_{r \geq 0} \frac{r^{\delta_0}}{(1+r)^{\delta_D - 1}} e^{-\Delta_1 r}$$

is bounded from above by a positive constant, say $\tilde{C}_{1.1}$. In addition to this, we observe that the same holds for

$$\sup_{\tau \in \overline{D}(0,\rho)} \left| \frac{\tau^{\delta_0}}{(1+|\tau|)^{\delta_D - 1}} e^{-\tau k_0} \right| \leq \tilde{C}_{1.2},$$

for some $\tilde{C}_{1.2} > 0$. Taking into account the previous statements, and Lemma 10, one derives that for all $m \in \mathbb{R}$ and $\tau \in S_d \cup \overline{D}(0,\rho)$ one has

$$\left| \tau^{\delta_0} \frac{e^{-\tau k_0} R_0(im)}{\mathcal{D}_2(\tau,m)} \right| \leq \frac{\max\{\tilde{C}_{1.1}, \tilde{C}_{1.2}\}}{M_1^{\delta_D - 1} M_2 r_{Q,R_D}^{1/\delta_D}} \sup_{m \in \mathbb{R}} \frac{|R_0(im)|}{|R_D(im)|} = \tilde{C}_1(0),$$

for every $\ell \in I$. □

Remark 6: In the work, for technical reasons, we need that both constants $C_1 > 0$ introduced in Lemma 4 (from the geometry of the problem I) and the above constant $\tilde{C}_1(0) > 0$ from Lemma 11 (from the geometry of the problem II) must be chosen small enough (at least strictly less than 1/2). We explain now how to achieved both constraints. Namely, from the proof of Lemma 4 together with (6),(8) the next two conditions are required :

A) Condition 1:

$$(\log(R_{Q,R_0}))^{\tilde{\delta}}/r_{Q,R_D}$$

is small (stemming from the bounds on L_k),

B) Condition 2:

$$R_{Q,R_0}/r_{Q,R_D}$$

is small (stemming from the bounds on $\mathcal{U}$). Furthermore, from Lemma 11 and (6),(8) , the next condition must hold:

C) Condition 3:

$$\frac{R_{Q,R_D}}{r_{Q,R_D}^{1/\delta_D}} \times \frac{1}{r_{Q,R_0}}$$

is small.

In order to have both constants C_1 and $\tilde{C}_1(0)$ small, we will need to make r_{Q,R_D} large enough (as in the problem I) but also $r_{Q,R_0} > 0$ must be chosen large in a related manner as explained below.

Indeed, we make the assumption that R_{Q,R_D} is very close to r_{Q,R_D} and that R_{Q,R_0} is very close to r_{Q,R_0} in other words the annulus S_{Q,R_D} from (8) and S_{Q,R_0} from (6) are very thin.

We assume that

$$R_{Q,R_0} = r_{Q,R_D}/10. \tag{42}$$

In other words, R_{Q,R_0} (and hence r_{Q,R_0}) is proportional to r_{Q,R_D} with a small factor of proportionality.

From (42), we see that Condition 2 above is fulfilled. For the condition 1 above, observe that

$$(\log(R_{Q,R_0}))^{\tilde{\delta}}/r_{Q,R_D} = (\log(1/10) + \log(r_{Q,R_D}))^{\tilde{\delta}}/r_{Q,R_D}$$

is small provided that r_{Q,R_D} is taken large enough. In concern with the condition 3, we see that

$$\frac{R_{Q,R_D}}{r_{Q,R_D}^{1/\delta_D}} \times \frac{1}{r_{Q,R_0}} \sim \frac{r_{Q,R_D}}{r_{Q,R_0}} \frac{1}{(r_{Q,R_D})^{1/\delta_D}} \sim (\frac{r_{Q,R_D}}{R_{Q,R_0}}) \frac{1}{(r_{Q,R_D})^{1/\delta_D}} \sim \frac{10}{(r_{Q,R_D})^{1/\delta_D}}$$

which is small whenever r_{Q,R_D} is chosen large enough (where the symbol $\sim$ means that the quantities are comparable).

4.2 Auxiliary Banach spaces of functions and solution of the auxiliary problem II

The structure and results of this section is analogous to that of Section 3.3. We omit unnecessary repetitions. Let $d \in \mathbb{R}$ with $-\frac{\pi}{2} < d < \frac{\pi}{2}$, and S_d be an infinite sector with bisecting direction d. We also choose $\rho > 0$.

Definition 3 *Let $\mu, \beta, \nu > 0$. We write $F^d_{(\nu,\beta,\mu)}$ for the set of all continuous functions $(\tau, m) \mapsto h(\tau, m)$ defined on $(S_d \cup \overline{D}(0, \rho)) \times \mathbb{R}$, holomorphic with respect to the first variable on $S_d \cup D(0, \rho)$ and such that*

$$||h(\tau, m)||_{(\nu,\beta,\mu),d} := \sup_{\tau \in (S_d \cup D(0,\rho)), m \in \mathbb{R}} (1 + |m|)^{\mu} \frac{1 + |\tau|^2}{|\tau|} \exp(\beta|m| - \nu|\tau|)|h(\tau, m)| < \infty.$$

The pair $(F^d_{(\nu,\beta,\mu)}, ||\cdot||_{(\nu,\beta,\mu),d})$ is a complex Banach space.

Lemma 12 *Let $\mu, \beta, \nu > 0$. Let $(\tau, m) \mapsto a(\tau, m)$ be a continuous function defined on $(S_d \cup \overline{D}(0,\rho)) \times \mathbb{R}$, holomorphic with respect to the first variable on $S_d \cup D(0,\rho)$ such that*

$$\sup_{(\tau,m)\in(S_d\cup\overline{D}(0,\rho))\times\mathbb{R}} |a(\tau,m)|,$$

is upper bounded. Then, for every $f \in F^d_{(\nu,\beta,\mu)}$ it holds that $(\tau,m) \mapsto a(\tau,m)f(\tau,m)$ belongs to $F^d_{(\nu,\beta,\mu)}$, and

$$\|a(\tau,m)f(\tau,m)\|_{(\nu,\beta,\mu),d} \leq \left(\sup_{(\tau,m)\in(S_d\cup\overline{D}(0,\rho))\times\mathbb{R}} |a(\tau,m)|\right) \|f(\tau,m)\|_{(\nu,\beta,\mu),d}\,.$$

The proof of the next lemma follows the same lines as that of Lemma 8, so its proof is omitted.

Lemma 13 *Let $\mu > 1$ and $\beta, \nu > 0$. For every $f \in E_{(\beta,\mu)}$ and $g \in F^d_{(\nu,\beta,\mu)}$, the function $\phi(\tau,m)$ defined by*

$$\phi(\tau,m) := \int_{-\infty}^{\infty} f(m_1)g(\tau, m-m_1)dm_1$$

belongs to $F^d_{(\nu,\beta,\mu)}$. Moreover, there exists $D_1 > 0$ such that

$$\|\phi(\tau,m)\|_{(\nu,\beta,\mu),d} \leq D_1 \|f\|_{(\beta,\mu)} \|g\|_{(\nu,\beta,\mu),d}\,.$$

Lemma 14 *Let $\gamma_1 > 0$, γ_2, γ_3 be real numbers such that*

$$\gamma_2 + 1 > 0, \quad \gamma_3 + 2 > 0, \quad \gamma_2 + \gamma_3 + 2 \geq 0, \quad \gamma_1 \geq \gamma_3 + 2. \tag{43}$$

Let $f \in F^d_{(\nu,\beta,\mu)}$ and $a_{\gamma_1}(\tau,m)$ continuous on $(\overline{S}_d \cup \overline{D}(0,\rho)) \times \mathbb{R}$, holomorphic w.r.t τ on $S_d \cup D(0,\rho)$ such that

$$|a_{\gamma_1}(\tau,m)| \leq \frac{1}{(1+|\tau|)^{\gamma_1}} \tag{44}$$

for all $\tau \in \overline{S}_d \cup \overline{D}(0,\rho)$, $m \in \mathbb{R}$.

Then, the function

$$\Phi_f(\tau,m) := a_{\gamma_1}(\tau,m)\tau \int_0^\tau (\tau-s)^{\gamma_2} s^{\gamma_3} f(s,m)ds \tag{45}$$

belongs to $F^d_{(\nu,\beta,\mu)}$. In addition to this, there exists a constant $D_4 > 0$ such that

$$\|\Phi_f(\tau,m)\|_{(\nu,\beta,\mu),d} \leq D_4 \|f(\tau,m)\|_{(\nu,\beta,\mu),d}\,. \tag{46}$$

Proof We give details to clarify the conditions declared in the statement of the result.

Let $f \in F^d_{(\nu,\beta,\mu)}$. We write $\Phi_f(\tau,m)$ in terms of the parametrization $s = \tau u$ for $0 \leq u \leq 1$:

$$\Phi_f(\tau,m) = a_{\gamma_1}(\tau,m)\tau^{\gamma_2+\gamma_3+2} \int_0^1 (1-u)^{\gamma_2} u^{\gamma_3} f(\tau u, m)du \tag{47}$$

for $\tau \in S_d \cup D(0,\rho)$ and $m \in \mathbb{R}$. As $\gamma_2 + \gamma_3 + 2 \geq 0$ and the regularity conditions on $a_{\gamma_1}(\tau,m)$ and $f(\tau,m)$ we guarantee that $\Phi_f(\tau,m)$ is holomorphic on $S_d \cup D(0,\rho)$ w.r.t τ, continuous on the closure of the previous set, and continuous with respect to $m \in \mathbb{R}$. In addition to this, for all $\tau \in D(0,\rho)$ and $m \in \mathbb{R}$ one has

$$|f(\tau,m)| \leq \|f(\tau,m)\|_{(\nu,\beta,\mu),d} \frac{|\tau|}{1+|\tau|^2} \exp(\nu|\tau| - \beta|m|)(1+|m|)^{-\mu}.$$

Therefore, taking into account the representation (47) and in view of Beta function formula (see [1], Appendix B3) one has

$$|\Phi_f(\tau,m)| \leq \frac{\Gamma(\gamma_2+1)\Gamma(\gamma_3+2)}{\Gamma(\gamma_2+\gamma_3+3)} |a_{\gamma_1}(\tau,m)| \, \|f(\tau,m)\|_{(\nu,\beta,\mu),d}$$

$$\times \sup_{u \in D(0,\rho), m \in \mathbb{R}} |u|^{\gamma_2+\gamma_3+3} \exp(\nu|u| - \beta|m|)(1+|m|)^{-\mu} \tag{48}$$

for all $\tau \in D(0,\rho)$, $m \in \mathbb{R}$.

Let $\tau \in S_d \times \mathbb{R}$. From the fact that $f \in F^d_{(\nu,\beta,\mu)}$, one derives

$$|\Phi_f(\tau,m)| \leq \|f(\tau,m)\|_{(\nu,\beta,\mu),d} \frac{|\tau|}{(1+|\tau|)^{\gamma_1}} (1+|m|)^{-\mu} \int_0^{|\tau|} (|\tau|-h)^{\gamma_2} h^{\gamma_3+1} \exp(\nu h - \beta|m|) dh$$

for all $\tau \in S_d$, all $m \in \mathbb{R}$.

Let $B(x)$ be defined by

$$B(x) = \int_0^x \exp(\nu h) h^{1+\gamma_3} (x-h)^{\gamma_2} dh \tag{49}$$

for $x \geq 0$. An analogous argument as in the proof Proposition 1 [18] yields

$$B(x) \leq K_2 x^{\gamma_3+1} e^{\nu x}, \quad x \geq 1,$$

for some $K_2 > 0$. The previous estimation, together with the assumptions in (43) yield the conclusion. □

Lemma 15 *Let the auxiliary equation (17) be constructed as described in Section 2. Let $\beta, \nu, \varpi > 0$ and $\mu > 1$ and assume there exist $\epsilon_0, \tilde{\xi}_\psi, r_{Q,R_D} > 0$ such that $\mathcal{C}_\psi \leq \tilde{\xi}_\psi$ ($\mathcal{C}_\psi$ is the constant in (12) and $r_{Q,R_D}, r_{Q,R_0} > 0$ are chosen in accordance with the conditions described in Remark 6 . Then, if one defines for every $\epsilon \in D(0,\epsilon_0)$ the operator*

$$\tilde{\mathcal{H}}_\epsilon(\omega(\tau,m)) := \tau^{\delta_0} \frac{e^{-\tau k_0} R_0(im)}{\mathcal{D}_2(\tau,m)} \omega(\tau,m,\epsilon) + \frac{1}{(2\pi)^{1/2}} \sum_{\ell \in I} \int_{-\infty}^{\infty} C_\ell(m-m_1,\epsilon) R_\ell(im_1) \epsilon^{\Delta_\ell - \delta_\ell + d_\ell}$$
$$\times \left(\frac{\tau}{\mathcal{D}_2(\tau,m)\Gamma(d_{\ell,1})} \int_0^\tau (\tau-s)^{d_{\ell,1}-1} s^{d_\ell} \omega(s,m_1,\epsilon) \frac{ds}{s} + \sum_{1 \leq p \leq d_\ell - 1} A_{d_\ell,p} \frac{\tau}{\mathcal{D}_2(\tau,m)\Gamma(d_{\ell,1}+d_\ell-p)} \right.$$
$$\left. \times \int_0^\tau (\tau-s)^{d_{\ell,1}+d_\ell-p-1} s^p \omega(s,m_1,\epsilon) \frac{ds}{s} \right) dm_1 + \frac{1}{\mathcal{D}_2(\tau,m)} \psi(\tau,m,\epsilon), \tag{50}$$

one has:

1. $\tilde{\mathcal{H}}_{\epsilon} : \overline{D}(0,\varpi) \to \overline{D}(0,\varpi)$*, where* $\overline{D}(0,\varpi)$ *stands for the closed disc of radius* ϖ *in* $F^{d}_{(\nu,\beta,\mu)}$.

2. For every $\omega_1,\omega_2 \in \overline{D}(0,\varpi) \subseteq F^{d}_{(\nu,\beta,\mu)}$*, one has*

$$\left\| \tilde{\mathcal{H}}_{\epsilon}(\omega_1) - \tilde{\mathcal{H}}_{\epsilon}(\omega_2) \right\|_{(\nu,\beta,\mu),d} \leq \frac{1}{2} \left\| \omega_1 - \omega_2 \right\|_{(\nu,\beta,\mu),d}.$$

Proof

Let $\epsilon \in D(0,\epsilon_0)$, $\omega \in F^{d}_{(\nu,\beta,\mu)}$. We first observe from Lemma 12, Lemma 13 and Lemma 14, together with the assumptions (4), (5) and (41) that for all $\ell \in I$ one has

$$\left\| \int_{-\infty}^{\infty} C_{\ell}(m-m_1,\epsilon) R_{\ell}(im_1) \epsilon^{\Delta_{\ell}-\delta_{\ell}+d_{\ell}} \frac{\tau}{\mathcal{D}_2(\tau,m)} \int_0^{\tau} (\tau-s)^{d_{\ell,1}-1} s^{d_{\ell}} \omega(s,m_1,\epsilon) \frac{ds}{s} \right\|_{(\nu,\beta,\mu),d} \leq \left\| C_{\ell}(m,\epsilon) \right\|_{(\beta,\mu)} \epsilon_0^{\Delta_{\ell}-\delta_{\ell}+d_{\ell}} \sup_{m\in\mathbb{R}} \left| \frac{R_{\ell}(im)}{R_D(im)} \right| \frac{1}{M_1^{\delta_D-1} M_2 r_{Q,R_D}^{1/\delta_D}} D_4 \varpi, \tag{51}$$

and

$$\left\| \int_{-\infty}^{\infty} C_{\ell}(m-m_1,\epsilon) R_{\ell}(im_1) \epsilon^{\Delta_{\ell}-\delta_{\ell}+d_{\ell}} \frac{\tau}{\mathcal{D}_2(\tau,m)} \int_0^{\tau} (\tau-s)^{d_{\ell,1}+d_{\ell}-p-1} s^{p} \omega(s,m_1,\epsilon) \frac{ds}{s} \right\|_{(\nu,\beta,\mu),d} \leq \left\| C_{\ell}(m,\epsilon) \right\|_{(\beta,\mu)} \epsilon_0^{\Delta_{\ell}-\delta_{\ell}+d_{\ell}} \sup_{m\in\mathbb{R}} \left| \frac{R_{\ell}(im)}{R_D(im)} \right| \frac{1}{M_1^{\delta_D-1} M_2 r_{Q,R_D}^{1/\delta_D}} D_4 \varpi. \tag{52}$$

Taking into account Lemma 11 and Lemma 12 one obtains that

$$\left\| \frac{\tau^{\delta_0} e^{-\tau k_0} R_0(im)}{\mathcal{D}_2(\tau,m)} \omega(\tau,m) \right\|_{(\nu,\beta,\mu),d} \leq \tilde{C}_1(0) \left\| \omega(\tau,m) \right\|_{(\nu,\beta,\mu),d}, \tag{53}$$

for every $\ell \in I$. Also, one gets from (12), Lemma 12 and (41) that

$$\begin{aligned} \left\| \frac{1}{\mathcal{D}_2(\tau,m)} \psi(\tau,m,\epsilon) \right\|_{(\nu,\beta,\mu),d} &\leq \left(\sup_{\tau\in(S_d\cup\overline{D}(0,\rho)),m\in\mathbb{R}} \frac{1}{|\mathcal{D}_2(\tau,m)|} \right) \tilde{C}_{\psi} \\ &\leq \frac{1}{\min_{m\in\mathbb{R}} |R_D(im)|} \frac{1}{M_1^{\delta_D-1} M_2 r_{Q,R_D}^{1/\delta_D}} \tilde{\xi}_{\psi} = \tilde{C}_2 \xi_{\psi}. \end{aligned} \tag{54}$$

We choose small enough $\epsilon_0 > 0$ and $\xi_{\psi} > 0$, and large enough $r_{Q,R_D}, r_{Q,R_0} > 0$ chosen according to Remark 6, such that for all $\ell \in I$

$$\tilde{C}_1(0)\varpi + \frac{1}{(2\pi)^{1/2}} \frac{1}{M_1^{\delta_D-1} M_2 r_{Q,R_D}^{1/\delta_D}} D_4 \sum_{\ell\in I} \mathcal{C}_{\ell} \sup_{m\in\mathbb{R}} \left| \frac{R_{\ell}(im)}{R_D(im)} \right| \epsilon_0^{\Delta_{\ell}-\delta_{\ell}+d_{\ell}} \left(\frac{1}{\Gamma(d_{\ell,1})} + \sum_{1\leq p\leq d_{\ell}-1} |A_{d_{\ell},p}| \frac{1}{\Gamma(d_{\ell,1}+d_{\ell}-p)} \right) \varpi + \tilde{C}_2 \xi_{\psi} \leq \varpi. \tag{55}$$

The first part of the proof follows from this choice and (51), (52), (53) together with (54). For the second part of the proof, let $\omega_1, \omega_2 \in \overline{D}(0,\varpi) \subseteq F^d_{(\nu,\beta,\mu)}$. Under the assumption that $\epsilon_0 > 0$, $\xi_\psi > 0$ are small enough, and $r_{Q,R_D} > 0$ is chosen according to the facts described in Remark 6 (and therefore $\tilde{C}_1(0)$ is close to zero) such that

$$\tilde{C}_1(0) + \frac{1}{(2\pi)^{1/2}} \frac{1}{M_1^{\delta_D - 1} M_2 r_{Q,R_D}^{1/\delta_D}} D_4 \sum_{\ell \in I} \mathcal{C}_\ell \sup_{m \in \mathbb{R}} \left| \frac{R_\ell(im)}{R_D(im)} \right| \epsilon_0^{\Delta_\ell - \delta_\ell + d_\ell} \times \left(\frac{1}{\Gamma(d_{\ell,1})} + \sum_{1 \le p \le d_\ell - 1} |A_{d_\ell,p}| \frac{1}{\Gamma(d_{\ell,1} + d_\ell - p)} \right) \le \frac{1}{2}. \tag{56}$$

Then, analogous estimates as before yield the second statement of the result. □

The proof of the next result is analogous to that of Proposition 2.

Proposition 3 *Under the hypotheses of Lemma 15, then the auxiliary equation (17) admits a solution $\omega_d(\tau,m,\epsilon)$ for every $\epsilon \in D(0,\epsilon_0)$ with $\omega_d(\tau,m,\epsilon) \in F^d_{(\nu,\beta,\mu)}$ and $\|\omega_d(\tau,m,\epsilon)\|_{(\nu,\beta,\mu),d} \le \varpi$.*

5 Analytic solutions of the main problem

In this section, we construct analytic solutions of the main problem (10). Regarding Section 3 and Section 4, two different situations arise in this respect, which are described in the following subsections.

5.1 First family of analytic solutions of the main problem

Let (10) be the main problem under study, whose elements are detailed in Section 2. We depart from the situation described in Section 3, and consider the auxiliary problem (17). Following the results achieved in Section 3, one can guarantee the existence of a solution of (17), say $\omega(\tau,m,\epsilon)$, which is defined in $D(0,\rho) \times \mathbb{R} \times D(0,\epsilon_0)$ for some $\rho > 0$ and $\epsilon_0 > 0$. Let $\beta, \nu, \varpi > 0$ and $\mu > 1$. As a result of Proposition 2, such solution can be extended to $\omega_L(\tau,m,\epsilon)$, defined in $(\mathcal{U} \cup L) \times \mathbb{R} \times D(0,\epsilon_0)$, where $\mathcal{U} = D(0,\rho)$ and L is a horizontal strip contained in one of the strip-like domains L_k related to Lambert W function. This extension satisfies that there exists $C_\omega > 0$ with

$$|\omega_L(\tau,m,\epsilon)| \le C_\omega \frac{1}{(1+|m|)^\mu} \frac{|\tau|}{1+|\tau|^2} \exp(-\beta|m| + \nu|\tau|), \tag{57}$$

for all $\tau \in \mathcal{U} \cup L$, $m \in \mathbb{R}$ and $\epsilon \in D(0,\epsilon_0)$. At this point, we define $\mathcal{L}_L$ as the following path, contained in $\mathcal{U} \cup L$: $\mathcal{L}_L = \mathcal{L}_{L,1} + \mathcal{L}_{L,2}$, where $\mathcal{L}_{L,1}$ stands for the segment $[0, r_L e^{\sqrt{-1}\theta_L}]$, for some $\pi/2 < \theta_L < 3\pi/2$ and $r_L < \rho$ such that $r_L e^{\sqrt{-1}\theta_L} \in L$; and $\mathcal{L}_{L,2}$ is a horizontal ray departing from $r_L e^{\sqrt{-1}\theta_L}$. More precisely, $\mathcal{L}_{L,2}$ can be parametrized by $[-s_0, +\infty) \ni s \mapsto -s + \sqrt{-1}h$, for some fixed $s_0, h \in \mathbb{R}$ with $s_0 < 0$ and $h \neq 0$.

Proposition 4 *In the previous situation, let $0 < \beta' < \beta$, and let $\mathcal{T}$ and $\mathcal{E}$ be bounded sectors in $\mathbb{C}^\star$ with $\mathcal{E} \subseteq D(0,\epsilon_0)$ and $\mathcal{T} \subseteq D(0,r_{\mathcal{T}})$, for some small enough $r_{\mathcal{T}} > 0$ and such that there exists $\Delta > 0$ with*

$$|arg(\epsilon t) - arg(u)| \le \frac{\pi}{2} - \Delta, \tag{58}$$

for every $\epsilon \in \mathcal{E}$, $t \in \mathcal{T}$ and all $u \in \mathcal{L}_L$. Then, the expression

$$u(t,z,\epsilon) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{\mathcal{L}_L} \omega_L(u,m,\epsilon) e^{-\frac{u}{\epsilon t}} e^{izm} \frac{du}{u} dm \tag{59}$$

defines a holomorphic function on $\mathcal{T} \times H_{\beta'} \times \mathcal{E}$, where the set $H_{\beta'}$ stands for the horizontal strip

$$H_{\beta'} = \{z \in \mathbb{C} : |Im(z)| < \beta'\}.$$

Proof Bearing in mind (57), and the parametrization of $\mathcal{L}_{L,1}$ $[0, r_L] \ni \rho \mapsto \rho \exp(\sqrt{-1}\theta_L)$, one has that for every $(t,z,\epsilon) \in \mathcal{T} \times H_{\beta'} \times \mathcal{E}$

$$\left| \int_{-\infty}^{\infty} \int_{\mathcal{L}_{L,1}} \omega_L(u,m,\epsilon) e^{-\frac{u}{\epsilon t}} e^{izm} \frac{du}{u} dm \right| \leq C_\omega \left(\int_{-\infty}^{\infty} \frac{1}{(1+|m|)^\mu} e^{(|\text{Im}(z)|-\beta)|m|} dm \right) \times \int_0^{r_L} \frac{1}{1+\rho^2} \exp(\nu\rho - \frac{\rho}{|\epsilon t|} \cos(\theta_L - \arg(t\epsilon))) d\rho. \tag{60}$$

If $r_{\mathcal{T}} > 0$ is small enough, then in view of (58), it holds that

$$\nu \epsilon_0 r_{\mathcal{T}} < \cos(\theta_L - \arg(t\epsilon)),$$

for every $\epsilon \in \mathcal{E}$ and $t \in \mathcal{T}$. Therefore, one has

$$\int_0^{r_L} \frac{1}{1+\rho^2} \exp(\nu\rho - \frac{\rho}{|\epsilon t|} \cos(\theta_L - \arg(t\epsilon))) d\rho \leq \int_0^{\infty} \frac{1}{1+\rho^2} \exp(\rho(\nu - \frac{1}{|\epsilon t|} \cos(\theta_L - \arg(t\epsilon))) d\rho \leq \int_0^{\infty} \frac{1}{1+\rho^2} d\rho < \infty.$$

The definition of $H_{\beta'}$ yields

$$\int_{-\infty}^{\infty} \frac{1}{(1+|m|)^\mu} e^{(|\text{Im}(z)|-\beta)|m|} dm < \infty. \tag{61}$$

On the other hand, we parametrize $\mathcal{L}_{L,2}$ by $[-s_0, \infty) \ni s \mapsto -s + \sqrt{-1}h$, for some constant $h \in \mathbb{R}$, and where $s_0 = r_L \cos(\theta_L) < 0$. We have

$$\left| \int_{-\infty}^{\infty} \int_{\mathcal{L}_{L,2}} \omega_L(u,m,\epsilon) e^{-\frac{u}{\epsilon t}} e^{izm} \frac{du}{u} dm \right| \leq \tilde{C}_\omega \left(\int_{-\infty}^{\infty} \frac{1}{(1+|m|)^\mu} e^{(|\text{Im}(z)|-\beta)|m|} dm \right) \times \int_{-s_0}^{\infty} \frac{1}{1+s^2+h^2} \exp\left(\sqrt{s^2+h^2} \left(\nu - \frac{\cos(\arctan(-h/s) - \arg(t\epsilon))}{|\epsilon t|} \right) \right) ds, \tag{62}$$

for some $\tilde{C}_\omega > 0$. Taking into account (58), we derive that if $r_{\mathcal{T}} > 0$ is small enough then

$$\frac{\cos(\arctan(-h/s) - \arg(t\epsilon))}{|\epsilon t|} - \nu \geq \Delta_0 > 0$$

for all $s \in [-s_0, \infty)$, $t \in \mathcal{T}$ and $\epsilon \in \mathcal{E} \subseteq D(0, \epsilon_0)$. The bound in (61) also holds in this situation. Moreover, the last integral in (62) can be upper bounded by

$$\int_{-s_0}^{\infty} \frac{1}{1+s^2+h^2} \exp\left(-\Delta_0 \sqrt{s^2+h^2} \right) ds < \infty.$$

□

Remark 7: Observe that in order that (58) holds, the arguments of ϵt, for all $t \in \mathcal{T}$ and $\epsilon \in \mathcal{E}$ should be close enough to direction π. This comes as a consequence of the manner that $\mathcal{L}_L$ approaches infinity following horizontal strips with negative real part.

Proposition 5 *Under the assumptions of Proposition 4, the function $u(t,z,\epsilon)$ constructed in Proposition 4, which is holomorphic on $\mathcal{T} \times H_{\beta'} \times \mathcal{E}$, is a solution of (10) in $\mathcal{T} \times H_{\beta'} \times \mathcal{E}$.*

Proof It is a direct consequence of the properties of Fourier and the Laplace-like transform and Proposition 4. □

5.2 Second family of analytic solutions of the main problem

We now depart from the assumptions made in Section 4. The results in that section guarantee that for well chosen $d \in (-\pi/2, \pi/2)$ as described in Lemma 10, if the opening of the sector S_d is small enough, a solution of (17) exists, say $\omega_d(\tau, m, \epsilon)$, defined in $(D(0,\rho) \cup S_d) \times \mathbb{R} \times D(0,\epsilon_0)$, for some $\rho > 0$. In addition to this, there exists $\tilde{C}_\omega > 0$ such that

$$|\omega_d(\tau,m,\epsilon)| \leq \tilde{C}_\omega \frac{1}{(1+|m|)^\mu} \frac{|\tau|}{1+|\tau|^2} \exp(-\beta|m| + \nu|\tau|), \tag{63}$$

for all $\tau \in D(0,\rho) \cup S_d$, $m \in \mathbb{R}$ and $\epsilon \in D(0,\epsilon_0)$.

Proposition 6 *Let $0 < \beta' < \beta$, and let $\mathcal{T}$ and $\mathcal{E}$ be bounded sectors in $\mathbb{C}^\star$ with $\mathcal{E} \subseteq D(0,\epsilon_0)$ and $\mathcal{T} \subseteq D(0, r_{\mathcal{T}})$, for some small enough $r_{\mathcal{T}} > 0$ and such that there exists $\Delta > 0$ with*

$$|arg(\epsilon t) - d| < \frac{\pi}{2} - \Delta, \tag{64}$$

for every $\epsilon \in \mathcal{E}$ and $t \in \mathcal{T}$. Then, the expression

$$u(t,z,\epsilon) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \left(\int_{\gamma_d} \omega_d(u,m,\epsilon) e^{-\frac{u}{\epsilon t}} \frac{du}{u} \right) e^{izm} dm$$

defines a holomorphic function on $\mathcal{T} \times H_{\beta'} \times \mathcal{E}$. Here, γ_d stands for the half line $[0,\infty)e^{\sqrt{-1}d}$.

Proof

We observe that the inner integral in the definition of $u(t,z,\epsilon)$ is the classical Laplace transform along direction d. The result is a direct consequence of the condition (58) and (63). As a matter of fact, one has

$$\begin{aligned} \left| \int_{-\infty}^{\infty} \left(\int_{\gamma_d} \omega_d(u,m,\epsilon) e^{-\frac{u}{\epsilon t}} \right) \frac{du}{u} e^{izm} dm \right| &\leq \tilde{C}_\omega \left(\int_{-\infty}^{\infty} \frac{1}{(1+|m|)^\mu} e^{(|\mathrm{Im}(z)|-\beta)|m|} dm \right) \\ &\times \int_0^\infty \frac{1}{1+s^2} \exp(s(\nu - \frac{\cos(d-\arg(t\epsilon))}{|\epsilon t|}))ds < \infty, \end{aligned} \tag{65}$$

for every $t \in \mathcal{T}$, $\epsilon \in \mathcal{E}$ and $z \in H_{\beta'}$, for any fixed $0 < \beta' < \beta$. □

Proposition 7 *Under the assumptions of Proposition 6, the function $u(t,z,\epsilon)$, constructed in Proposition 6, which is holomorphic on $\mathcal{T} \times H_{\beta'} \times \mathcal{E}$, is a solution of (10) in $\mathcal{T} \times H_{\beta'} \times \mathcal{E}$.*

Proof It is a direct consequence of the properties of Fourier transform, and Laplace transform, together with Proposition 6 when taking $\mathcal{L} = \gamma_d$ in Lemma 1. □

6 Asymptotic results on the analytic solutions of the main problem

This section is devoted to the study of the asymptotic behavior of the analytic solution of (10) with respect to the perturbation parameter, near the origin. In Section 5.1 and Section 5.2, we have constructed two different families of analytic solutions of (10). One may be tempted to search for asymptotic results independently, i.e. considering analytic solutions as constructed in Section 5.1, and in a parallel way treat the asymptotic solutions of the analytic ones constructed in Section 5.2. This independent approach is not feasible, due to geometric restrictions on the problem.

Indeed, regarding the condition (58) one realizes that the argument of ϵt differs from the arguments of the elements in $\mathcal{L}_L$ less than $\pi/2$. Taking into account that $\mathcal{L}_L$ is contained in the left half-plane, direction $\theta = 0$ is not attained by $\mathcal{E}\mathcal{T} = \{\epsilon t : \epsilon \in \mathcal{E}, t \in \mathcal{T}\}$, and therefore, there is some direction in the complex plane not attained by $\mathcal{E}$. The conclusion is that it is not possible to cover a punctured disc at the origin by means of sectors only satisfying condition (58).

On the other hand, the fact that any direction of integration d associated to a sector $\mathcal{E}$ satisfies that $-\pi/2 < d < \pi/2$, together with (64) implies the existence of some directions which can not be attained. Any finite set of sectors under this condition can not conform a good covering of $\mathbb{C}^\star$ (see Definition 4).

Let us consider the main problem under study (10), under the assumptions and construction of its elements described in Section 2.

In order to describe the asymptotic behavior of the analytic solutions of the main problem with respect to the perturbation parameter near the origin, we need to introduce the concept of good covering and family associated to a good covering.

Definition 4 *Let $\varsigma \geq 2$ be an integer. A set $(\mathcal{E}_p)_{0\leq p\leq\varsigma-1}$ of finite open sectors with vertex at the origin is said to conform a good covering of $\mathbb{C}^\star$ if the following properties hold:*

- *$\mathcal{E}_p \cap \mathcal{E}_{p+1} \neq \emptyset$, for all $0 \leq p \leq \varsigma - 1$. We identify the indices ς and 0.*
- *The intersection of three different sectors of $(\mathcal{E}_p)_{0\leq p\leq\varsigma-1}$ is empty.*
- *$\cup_{p=0}^{\varsigma-1}\mathcal{E}_p = \mathcal{U} \setminus \{0\}$, where $\mathcal{U}$ stands for a neighborhood of the origin in $\mathbb{C}$.*

Definition 5 *Let $\mathcal{T}$ be a bounded sector with vertex at the origin, $\mathcal{T} \subseteq D(0, r_{\mathcal{T}})$. Let $(\mathcal{E}_p)_{0\leq p\leq\varsigma-1}$ be a good covering in $\mathbb{C}^\star$. We assume that the set $\{0, \dots, \varsigma - 1\}$ is the union of two nonempty sets with empty intersection, say J_1 and J_2. For every $p_1 \in J_1$ let L_{p_1} be a strip of the form (26) which is contained in one of the strip-like domains L_k constructed in Lemma 3 and described in Definition 1; and for every $p_2 \in J_2$ let $S_{d_{p_2}}$ be an infinite sector with vertex at the origin and bisecting direction d_{p_2}, for some $d_{p_2} \in (-\pi/2, \pi/2)$ and small opening, satisfying the constraints (39) and (40). We say that the set $\{\mathcal{T}, (\mathcal{E}_p)_{0\leq p\leq\varsigma-1}, (L_{p_1})_{p_1\in J_1}, (S_{d_{p_2}})_{p_2\in J_2}\}$ is admissible if the following conditions hold:*

- *For all $p_1 \in J_1$, there exists $\Delta > 0$ with*

  $$|arg(\epsilon t) - arg(u)| \leq \frac{\pi}{2} - \Delta, \quad (66)$$

  for every $t \in \mathcal{T}$, $\epsilon \in \mathcal{E}_{p_1}$ and all u belonging to a path $\mathcal{L}_{L_{p_1}} \subseteq L_{p_1} \cup \mathcal{U}$, which might depend on t and ϵ.

- *For all* $p_2 \in J_2$, *there exists* $\Delta > 0$ *such that*

$$|arg(\epsilon t) - \xi_{p_2}| \leq \frac{\pi}{2} - \Delta, \tag{67}$$

for all $t \in \mathcal{T}$, $\epsilon \in \mathcal{E}_{p_2}$, *and some* $\xi_{p_2} \in \mathbb{R}$, *which might depend on* t *and* ϵ, *such that* $e^{\xi_{p_2}\sqrt{-1}} \in S_{d_{p_2}}$.

Theorem 1 *Let* $\{\mathcal{T}, (\mathcal{E}_p)_{0\leq p\leq\varsigma-1}, (L_{p_1})_{p_1\in J_1}, (S_{d_{p_2}})_{p_2\in J_2}\}$ *be an admissible set. For all* $p_1 \in J_1$ *the function* $u_{p_1}(t,z,\epsilon)$, *as constructed in Proposition 4 for* $L = L_{p_1}$ *and* $\mathcal{E} = \mathcal{E}_{p_1}$, *is a holomorphic function in* $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_{p_1}$, *for all* $0 < \beta' < \beta$. *For all* $p_2 \in J_2$ *the function* $u_{p_2}(t,z,\epsilon)$, *as constructed in Proposition 6 for* $d = d_{p_2}$ *and* $\mathcal{E} = \mathcal{E}_{p_2}$, *is a holomorphic function in* $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_{p_2}$, *for all* $0 < \beta' < \beta$.

Moreover, for every $p, q \in \{0, \ldots, \varsigma - 1\}$, *with* $p \neq q$ *and such that* $\mathcal{E}_p \cap \mathcal{E}_q \neq \emptyset$, *there exist* $C, D > 0$ *such that*

$$\sup_{t\in\mathcal{T}, z\in H_{\beta'}} |u_p(t,z,\epsilon) - u_q(t,z,\epsilon)| \leq C\exp\left(-\frac{D}{|\epsilon|}\right) \tag{68}$$

for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_q$.

Proof The first part of the proof has been checked in Proposition 5 and Proposition 7, respectively. We give proof for (68) regarding three different possible frameworks.

Case (A): $p, q \in J_1$.

Let $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_q$. Then, the integration path $\mathcal{L}_{L_p}$ can be split as $\mathcal{L}_{L_p,1} + \mathcal{L}_{L_p,2}$, where $\mathcal{L}_{L_p,1} = [0, r_{L_p}\exp(\sqrt{-1}\theta_{L_p})]$ and $\mathcal{L}_{L_p,2}$ is a horizontal ray contained in L_p, departing from $r_{L_p}\exp(\sqrt{-1}\theta_{L_p})$ with decreasing real part. Both paths are detailed in the proof of Proposition 4. We proceed analogously with $\mathcal{L}_{L_q} = \mathcal{L}_{L_q,1} + \mathcal{L}_{L_q,2}$. Taking into account that the function ω is holomorphic with respect to its first variable in $D(0,\rho)$, one can perform a deformation path which transforms the integral along $\mathcal{L}_{L_p,1} - \mathcal{L}_{L_q,1}$ as the integral along a regular arc $C_{p,q}$ contained in $\{\tau \in \mathbb{C} : \mathrm{Re}(\tau) < 0\} \cap D(0,\rho)$ joining the points $r_{L_p}\exp(\sqrt{-1}\theta_{L_p})$ and $r_{L_q}\exp(\sqrt{-1}\theta_{L_q})$ (see Figure 4, left).

One has

$$u_p(t,z,\epsilon) - u_q(t,z,\epsilon) = I_1 + I_2 - I_3, \tag{69}$$

where

$$I_1 = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\int_{\mathcal{L}_{L_p,2}} \omega_{L_p}(u,m,\epsilon)e^{-\frac{u}{\epsilon t}}e^{izm}\frac{du}{u}dm, \tag{70}$$

$$I_2 = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\int_{C_{p,q}} \omega(u,m,\epsilon)e^{-\frac{u}{\epsilon t}}e^{izm}\frac{du}{u}dm,$$

$$I_3 = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\int_{\mathcal{L}_{L_q,2}} \omega_{L_q}(u,m,\epsilon)e^{-\frac{u}{\epsilon t}}e^{izm}\frac{du}{u}dm,$$

for every $t \in \mathcal{T}$, $z \in H_{\beta'}$. We first give upper bounds for $|I_1|$, which can be also applied to $|I_3|$. We observe from (57) that

$$|I_1| \leq \frac{C_\omega}{(2\pi)^{1/2}}\left(\int_{-\infty}^{\infty}\frac{1}{(1+|m|)^\mu}e^{(|\mathrm{Im}(z)|-\beta)|m|}dm\right) \times \int_{-s_0}^{\infty}\frac{1}{1+s^2+h^2}\exp\left(\sqrt{s^2+h^2}\left(\nu - \frac{\cos(\arctan(-h/s)-\arg(t\epsilon))}{|\epsilon t|}\right)\right)ds. \tag{71}$$

Here, we have used the parametrization of the path $\mathcal{L}_{L_p,2}$ in the proof of Proposition 4. From (66), we know that

$$\cos(\arctan(-h/s) - \arg(t\epsilon)) > r_{\mathcal{T}}$$

for $r_{\mathcal{T}} > 0$ small enough, for all $s \in [-s_0, +\infty)$, $t \in \mathcal{T}$, $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_q$. We deduce that

$$|I_1| \leq C_{\omega,2} \int_{-s_0}^{+\infty} \exp(\sqrt{s^2+h^2}(\nu - \frac{r_{\mathcal{T}}}{|\epsilon t|}))ds \leq C_{\omega,2} \int_{-s_0}^{+\infty} \exp(\sqrt{s^2+h^2}(-\frac{r_{\mathcal{T}}}{2|\epsilon t|}))ds \tag{72}$$

provided that $|\epsilon t| < \frac{r_{\mathcal{T}}}{2\nu}$. We deduce that

$$\begin{aligned} |I_1| &\leq C_{\omega,2} \int_{-s_0}^{+\infty} \exp(\sqrt{s^2+h^2}(-\frac{1}{2|\epsilon|}))ds \\ &\leq C_{\omega,2} \left[\int_{-s_0}^{1} \exp(\sqrt{s^2+h^2}(-\frac{1}{2|\epsilon|}))ds + \int_{1}^{+\infty} \exp(\sqrt{s^2+h^2}(-\frac{1}{2|\epsilon|}))ds \right] \\ &\leq C_{\omega,2} \left[\int_{-s_0}^{1} \exp(-\frac{|h|}{2|\epsilon|})ds + \int_{1}^{+\infty} \exp(-\frac{|s|}{2|\epsilon|})ds \right] = C_{\omega,2} \left[\exp(-\frac{|h|}{2|\epsilon|})(1+s_0) + 2|\epsilon| \exp(-\frac{1}{2|\epsilon|}) \right]. \end{aligned} \tag{73}$$

This entails that

$$|I_1| \leq \hat{C}_1 \exp\left(-\frac{\hat{D}_1}{|\epsilon|}\right), \tag{74}$$

for some $\hat{C}_1, \hat{D}_1 > 0$. The same argument can be applied to arrive at

$$|I_3| \leq \hat{C}_3 \exp\left(-\frac{\hat{D}_3}{|\epsilon|}\right), \tag{75}$$

for some $\hat{C}_3, \hat{D}_3 > 0$.

Let us consider I_2. The continuity of C_{pq} yields the existence of $d_C > 0$ such that the distance $\mathrm{dist}(C_{p,q}, 0) > d_C > 0$. Let $[s_1, s_2] \ni s \mapsto \hat{h}(s)$ be a parametrization of $C_{p,q}$. By construction of the path $C_{p,q}$, we have in particular that

$$\cos(\arg(\hat{h}(s)) - \arg(\epsilon t)) > r_{\mathcal{T}}$$

for all $s \in [s_1, s_2]$, provided that $r_{\mathcal{T}}$ is small enough. In view of (57), one gets

$$\begin{aligned} |I_2| \leq{}& \frac{1}{(2\pi)^{1/2}} \left(\int_{-\infty}^{+\infty} \frac{1}{(1+|m|)^{\mu}} \exp((|\mathrm{Im}(z)| - \beta)|m|)dm \right) \\ &\times \int_{s_1}^{s_2} \frac{1}{1+|\hat{h}(s)|^2} \exp(\nu|\hat{h}(s)| - \frac{|\hat{h}(s)|}{|\epsilon t|} \cos(\arg(\hat{h}(s)) - \arg(\epsilon t)))|\hat{h}'(s)|ds \\ \leq{}& \tilde{C}_2 \max_{s\in[s_1,s_2]} |\hat{h}'(s)| \int_{s_1}^{s_2} \exp(|\hat{h}(s)|(\nu - \frac{r_{\mathcal{T}}}{|\epsilon t|}))ds \leq \tilde{C}_{2.1} \int_{s_1}^{s_2} \exp(|\hat{h}(s)|(-\frac{r_{\mathcal{T}}}{2|\epsilon t|}))ds \\ &\leq \tilde{C}_{2.1}(s_2 - s_1) \exp(-\frac{d_C}{2|\epsilon|}) \end{aligned} \tag{76}$$

provided that $|\epsilon t| < \frac{r_{\mathcal{T}}}{2\nu}$.

We conclude that

$$|I_2| \leq \hat{C}_2 \exp\left(-\frac{\hat{D}_2}{|\epsilon|}\right), \tag{77}$$

for some $\hat{C}_2, \hat{D}_2 > 0$.

The statement (68) follows from (74), (75) and (77) applied to (69).

Case (B): $p, q \in J_2$.

Let $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_q$. The integration path γ_p is written as $\gamma_{p,1} + \gamma_{p,2}$, where $\gamma_{L_p,1}$ is the segment $[0, r_{p,q}]e^{\sqrt{-1}d_p}$, for some $0 < r_{p,q} < \rho$, and $\gamma_{p,2}$ stands for the half line $[r_{p,q}, \infty)e^{\sqrt{-1}d_p}$. The path γ_q can be divided into analogous parts, namely $\gamma_q = \gamma_{q,1} + \gamma_{q,2}$. The function ω is holomorphic with respect to its first variable in $D(0,\rho)$. Therefore, the integration with respect to that variable in $\gamma_{p,1} - \gamma_{p,2}$ can be deformed as the integral in $\tilde{C}_{p,q}$, where $\tilde{C}_{p,q}$ is the arc of circle joining the points $r_{p,q}e^{\sqrt{-1}d_p}$ and $r_{p,q}e^{\sqrt{-1}d_q}$ (see Figure 4, center).

One has

$$u_p(t,z,\epsilon) - u_q(t,z,\epsilon) = \tilde{I}_1 + \tilde{I}_2 - \tilde{I}_3, \tag{78}$$

for

$$\tilde{I}_1 = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{\gamma_{p,2}} \omega_{d_p}(u,m,\epsilon) e^{-\frac{u}{\epsilon t}} e^{izm} \frac{du}{u} dm, \tag{79}$$

$$\tilde{I}_2 = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{\tilde{C}_{p,q}} \omega(u,m,\epsilon) e^{-\frac{u}{\epsilon t}} e^{izm} \frac{du}{u} dm,$$

$$\tilde{I}_3 = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{\gamma_{q,2}} \omega_{d_q}(u,m,\epsilon) e^{-\frac{u}{\epsilon t}} e^{izm} \frac{du}{u} dm,$$

for every $t \in \mathcal{T}$, $z \in H_{\beta'}$. It holds

$$|\tilde{I}_1| \leq \frac{\tilde{C}_\omega}{(2\pi)^{1/2}} \left(\int_{-\infty}^{\infty} \frac{1}{(1+|m|)^\mu} e^{(|\mathrm{Im}(z)| - \beta)|m|} dm \right) \int_{r_{p,q}}^{\infty} \frac{1}{1+s^2} \exp\left(s\left(\nu - \frac{\cos(d_p - \arg(t\epsilon))}{|\epsilon t|}\right)\right) ds.$$

The choice of an admissible set (see (67)) guarantees that

$$\cos(d_p - \arg(t\epsilon)) \geq r_{\mathcal{T}}$$

for small enough $r_{\mathcal{T}} > 0$. Hence,

$$\begin{aligned} |\tilde{I}_1| &\leq \tilde{C}_{\omega,2} \int_{r_{p,q}}^{+\infty} \exp(s(\nu - \frac{r_{\mathcal{T}}}{|\epsilon t|}))ds \\ &\leq \tilde{C}_{\omega,2} \int_{r_{p,q}}^{+\infty} \exp(-\frac{s r_{\mathcal{T}}}{2|\epsilon t|})ds \leq \tilde{C}_{\omega,2} \int_{r_{p,q}}^{+\infty} \exp(-\frac{s}{2|\epsilon|})ds = \tilde{C}_{\omega,2} 2|\epsilon| \exp(-\frac{r_{p,q}}{2|\epsilon|}) \end{aligned} \tag{80}$$

provided that $|\epsilon t| < \frac{r_{\mathcal{T}}}{2\nu}$. The same bounds apply for $\tilde{I}_3$.

Finally, we provide upper estimates for $|\tilde{I}_2|$ as follows:

$$\begin{aligned} |\tilde{I}_2| \leq \frac{\tilde{C}_\omega}{(2\pi)^{1/2}} &\left(\int_{-\infty}^{\infty} \frac{1}{(1+|m|)^\mu} e^{(|\mathrm{Im}(z)| - \beta)|m|} dm \right) \frac{r_{p,q}}{1+r_{p,q}^2} \exp(\nu r_{p,q}) \\ &\times \int_{d_p}^{d_q} \exp\left(-\frac{r_{p,q}\cos(\theta - \arg(\epsilon t))}{|\epsilon t|}\right) d\theta. \end{aligned} \tag{81}$$

Under the constraint (67), we know that

$$\cos(\theta - \arg(t\epsilon)) \geq r_{\mathcal{T}}$$

for some small $r_{\mathcal{T}} > 0$, all $\theta \in (d_p, d_q)$. We conclude that

$$|\tilde{I}_2| \leq \hat{C}_5 \exp\left(-\frac{\hat{D}_5}{|\epsilon|}\right), \tag{82}$$

for some $\hat{C}_5, \hat{D}_5 > 0$.

Case (C): $p \in J_2$ and $q \in J_1$.

This case is equivalent to that in which $p \in J_1$ and $q \in J_2$, and can be reduced to cases (A) and (B). Indeed, let $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_q$ and consider the splitting of the path γ_p as the sum of $\gamma_{p,1} + \gamma_{p,2}$, where $\gamma_{p,2}$ resembles the corresponding path of Case (B), departing from some point $r_{p,q}e^{\sqrt{-1}d_p}$ with $0 < r_{p,q} < \rho$ and direction d_p to infinity, and $\gamma_{p,1}$ is the segment $[0, r_{p,q}]e^{\sqrt{-1}d_p}$. The path $\mathcal{L}_{L_q}$ is written as the sum of $\mathcal{L}_{L_q,1} + \mathcal{L}_{L_q,2}$, where $\mathcal{L}_{L_q,2}$ stands for the horizontal ray contained in L_q, having $r_{p,q}e^{\sqrt{-1}\theta_{L_q}}$ as its endpoint, for some $\pi/2 < \theta_{L_q} < 3\pi/2$. The path $\mathcal{L}_{L_q,1}$ is the segment joining the origin and $r_{p,q}e^{\sqrt{-1}\theta_{L_q}}$. The fact that ω is holomorphic on $D(0,\rho)$, the integral along the path $\gamma_{p,1} - \mathcal{L}_{L_q,1}$ allows to deform the path as $\hat{C}_1 + \hat{C}_2$, where $\hat{C}_1$ is an arc of circle joining $r_{p,q}e^{\sqrt{-1}d_p}$ and some point $r_{p,q}e^{\sqrt{-1}\theta_p} \in \mathcal{U}$ (see Section 5.1), for some $\theta_p \in (\pi/2, 3\pi/2)$. The path $\hat{C}_2$ is a finite path contained in $\mathcal{U}$, which avoids 0, is contained in the left half-plane, and with endpoints $r_{p,q}e^{\sqrt{-1}\theta_p}$ and $r_{p,q}e^{\sqrt{-1}\theta_{L_q}}$. This path can be omitted in the splitting in the case that $\theta_p = \theta_{L_q}$. Figure 4 (right) illustrates an example of this splitting. Observe that the difference of the solutions can be written in the form

$$u_p(t,z,\epsilon) - u_q(t,z,\epsilon) = \hat{I}_1 + \hat{I}_2 - \hat{I}_3,$$

where

$$\hat{I}_1 = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{\gamma_{p,2}} \omega_{d_p}(u,m,\epsilon) e^{-\frac{u}{\epsilon t}} e^{izm} \frac{du}{u} dm, \tag{83}$$
$$\hat{I}_2 = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{\hat{C}_1+\hat{C}_2} \omega(u,m,\epsilon) e^{-\frac{u}{\epsilon t}} e^{izm} \frac{du}{u} dm,$$
$$\hat{I}_3 = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{\mathcal{L}_{L_q,2}} \omega_{L_q}(u,m,\epsilon) e^{-\frac{u}{\epsilon t}} e^{izm} \frac{du}{u} dm,$$

for every $t \in \mathcal{T}$, $z \in H_{\beta'}$. The expression $|\hat{I}_1|$ can be estimated as $|\tilde{I}_1|$ (see Case (B)). The expression $|\hat{I}_2|$ can be estimated in the same way as $|I_2|$ (see Case (A)) together with $|\tilde{I}_1|$ (see Case (B)). Finally, $|\hat{I}_3|$ is upper bounded in the same way as $|I_1|$ (see Case (A)). This entails that (68) holds.

□

The next cohomological result has been widely applied in the version of functional spaces with coefficients in a Banach space to guarantee the existence of a common asymptotic expansion related to the analytic solutions, with respect to the perturbation parameter. Here, the asymptotic behavior is preserved whereas summability results can not be attained, as it has been mentioned above. The classical Ramis-Sibuya theorem can be found in [1], p. 121 and [10], Lemma XI-2-6.

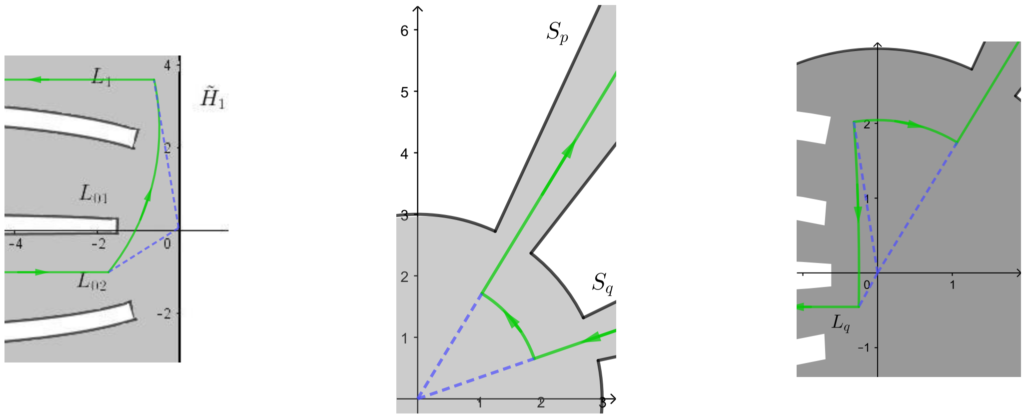

Figure 4: Example of the deformation of paths in Case (A) (left) Case (B) (center) and Case (C) (right)

Theorem 2 (RS) *Let $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ be a Banach space over $\mathbb{C}$, and let $(\mathcal{E}_p)_{0\le p\le\varsigma-1}$ be a good covering in $\mathbb{C}^\star$. For every $0 \le p \le \varsigma - 1$, let $G_p : \mathcal{E}_p \to \mathbb{E}$ be a holomorphic function, and define $\Theta_p := G_{p+1} - G_p$ defined on $Z_p = \mathcal{E}_p \cap \mathcal{E}_{p+1}$ (with the conventions $G_p := G_0$ and $\mathcal{E}_p := \mathcal{E}_0$). We assume that*

- *The function $G_p(\epsilon)$ is bounded for $\epsilon \in \mathcal{E}_p$ approaching the origin of $\mathbb{C}$ for every $0 \le p \le \varsigma-1$.*
- *For all $0 \le p \le \varsigma - 1$, the function $\Theta_p(\epsilon)$ is exponentially flat in Z_p, i.e. there exist $C_p, D_p > 0$ such that*

$$\|\Theta_p(\epsilon)\|_{\mathbb{E}} \le C_p e^{-\frac{D_p}{|\epsilon|}}.$$

Then, there exists a common formal power series $\hat{G}(\epsilon) = \sum_{n\ge 0} H_n \epsilon^n \in \mathbb{E}[[\epsilon]]$ which satisfies that $\hat{G}$ is the common Gevrey asymptotic expansion of order 1 of the function G_p, for all $0 \le p \le \varsigma-1$ on $\mathcal{E}_p$. This means that for every $0 \le p \le \varsigma - 1$ there exist $A_p, B_p > 0$ such that

$$\left\| G_p(\epsilon) - \sum_{n=0}^{N-1} H_n \epsilon^n \right\|_{\mathbb{E}} \le A_p (B_p)^N N! |\epsilon|^N,$$

for every $N \ge 1$ and $\epsilon \in \mathcal{E}_p$.

Let $\mathbb{E}$ be the banach space of holomorphic and bounded functions defined on $\mathcal{T} \times H_{\beta'}$, for some fixed $0 < \beta' < \beta$. As a consequence of (RS) Theorem and Theorem 1 we conclude the following:

Theorem 3 *Under the assumptions of Theorem 1, there exists a formal power series*

$$\hat{u}(t, z, \epsilon) = \sum_{n\ge 0} H_n(t, z) \frac{\epsilon^n}{n!} \in \mathbb{E}[[\epsilon]] \tag{84}$$

which is the common asymptotic expansion of the solution of (10) $u_p(t,z,\epsilon)$ of Gevrey order 1 in $\mathcal{E}_p$, for every $0 \leq p \leq \varsigma - 1$, when considering u_p as a function on $\mathcal{E}_p$, with values in $\mathbb{E}$.

Proof Let $\{u_p(t,z,\epsilon)\}_{0\leq p\leq\varsigma-1}$ be the family of solutions of (10) constructed in Theorem 1. For all $0 \leq p \leq \varsigma - 1$, we put $G_p(\epsilon) := (t,z) \mapsto u_p(t,z,\epsilon)$, which defines a holomorphic and bounded function defined in $\mathcal{E}_p$ with values in $\mathbb{E}$, fixed above. We observe that (68) entails that for all $0 \leq p \leq \varsigma - 1$, the difference $\Theta_p := G_{p+1} - G_p$ is exponentially flat in $Z_p := \mathcal{E}_p \cap \mathcal{E}_{p+1}$. Theorem (RS) can be applied to guarantee the existence of a formal power series $\hat{G}(\epsilon) \in \mathbb{E}[[\epsilon]]$, which is the common Gevrey asymptotic expansion of $u_p(t,z,\epsilon)$ in $\mathcal{E}_p$, for all$0 \leq p \leq \varsigma - 1$. We conclude the result by putting $\hat{u} := \hat{G}$.

□

7 Annex I: Fourier transform and related properties

In this section, we recall the definition of inverse Fourier transform, together with some algebraic properties held when applied on the elements of certain Banach spaces of functions of exponential decay at infinity, introduced in [17], and successfully applied in previous works by the authors.

Definition 6 *Let $\beta, \mu \in \mathbb{R}$. We consider the vector space $E_{(\beta,\mu)}$ of continuous functions $h : \mathbb{R} \to \mathbb{C}$ satisfying*

$$\|h(m)\|_{(\beta,\mu)} := \sup_{m\in\mathbb{R}} (1+|m|)^{\mu} \exp(\beta|m|)|h(m)| < \infty.$$

The pair $(E_{(\beta,\mu)}, \|\cdot\|_{(\beta,\mu)})$ is a Banach space.

We refer to [17] for further details on the proof of the next properties satisfied by inverse Fourier transform acting on the elements of the previous Banach space.

Definition 7 *Let $\beta > 0$ and $\mu > 1$. Given $f \in E_{(\beta,\mu)}$, the inverse Fourier transform of f is defined by*

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(m) \exp(ixm) dm, \qquad x \in \mathbb{R}.$$

The domain of definition of $\mathcal{F}^{-1}(f)$ can be extended to the set

$$H_{\beta'} = \{z \in \mathbb{C} : |Im(z)| < \beta'\},$$

for any $0 < \beta' < \beta$, providing an analytic and bounded function on $H_{\beta'}$. Moreover, the following properties hold:

- *Let φ be given by $m \mapsto \varphi(m) = imf(m)$. Then, $\varphi \in E_{(\beta,\mu-1)}$ and it holds that the function $\partial_z \mathcal{F}^{-1}(f)$ coincides with $\mathcal{F}^{-1}(\varphi)$, in H_β.*
- *Let $g \in E_{(\beta,\mu)}$ and consider the convolution product of f and g, namely*

$$\psi(m) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(m-m_1)g(m_1)dm_1, \quad m \in \mathbb{R}.$$

  Then $\psi \in E_{(\beta,\mu)}$, and it holds that $\mathcal{F}^{-1}(\psi)$ coincides with $\mathcal{F}^{-1}(f) \cdot \mathcal{F}^{-1}(g)$ in H_β, with $\cdot$ being the usual product.

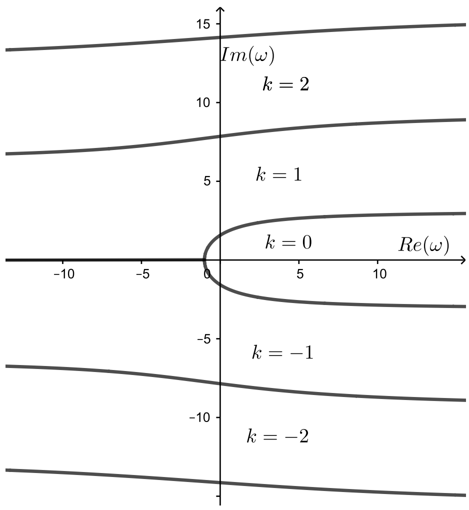

Figure 5: Branches of Lambert W function

8 Annex II: On Lambert W function

In this section, we provide some information of Lambert W function to be used in the work. We only focus on the elements used in the present study, and refer to [4] and the references therein for further details.

Lambert W function is defined as the complex function satisfying

$$W(z)e^{W(z)} = z.$$

Lambert W function turns out to be a multivalued function, partitioning the $w = W(z)$ plane into an infinite countable number of regions, corresponding to each branch of the function. Each of the branches of Lambert W function is denoted by W_k for $k \in \mathbb{Z}$. It holds that $W_0(z)$, the principal branch of Lambert W function is defined in $\mathbb{C} \setminus (-\infty, -e^{-1}]$ whereas $W_k(z)$ for $k \neq 0$, displays a branch cut along the negative real axis, and is defined in $\mathbb{C} \setminus (-\infty, 0]$. The curves defining the boundary of the different branches, as curves in $\mathbb{R}^2$ are given by

$$\{(-t\cot(t), t) \in \mathbb{R}^2 : -\pi < t < \pi\}$$

for the principal branch. The curve separating W_1 and W_{-1} is the half line $(-\infty, -1]$, and all the other branches are distinguished by the curves

$$\{(-t\cot(t), t) \in \mathbb{R}^2 : 2k\pi < \pm t < (2k+1)\pi\}, \qquad k \in \mathbb{Z} \setminus \{0\}$$

The image of each branches of Lambert W function and the curves separating the different branches are shown in Figure 5.

Each branch as described above is a holomorphic bijective map, when restricted to the domains described above.

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