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Computation of Moore–Penrose Generalized Inverses of Matrices with Meromorphic Function Entries.

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Abstract

In this paper, given a field with an involutory automorphism, we introduce the notion of Moore–Penrose field by requiring that all matrices over the field have Moore–Penrose inverse. We prove that only characteristic zero fields can be Moore–Penrose, and that the field of rational functions over a Moore-Penrose field is also Moore–Penrose. In addition, for a matrix with rational functions entries with coefficients in a field $\mathbb{K}$, we find sufficient conditions for the elements in $\mathbb{K}$ to ensure that the specialization of the Moore–Penrose inverse is the Moore–Penrose inverse of the specialization of the matrix. As a consequence, we provide a symbolic algorithm that, given a matrix whose entries are rational expression over $\mathbb{C}$ of finitely many meromorphic functions being invariant by the involutory automorphism, computes its Moore-Penrose inverve by replacing the functions by new variables, and hence reducing the problem to the case of matrices with complex rational function entries.

keywords: Generalized inverses, Moore–Penrose fields, meromorphic functions, matrices of functions.

AMS subject classification: 15A09, 15A54.

1 Introduction

Moore–Penrose generalized inverses were introduced, independently, by E.H. Moore in [16] and by R. Penrose in [18]; see e.g. Theorem 1.1.1. in [4] for the equivalence of the two results, also in [2], or in Appendix A in [4], there is a nice summary and restatement of the paper by E.H. Moore. Since then, Moore–Penrose generalized inverses have been studied and applied by many authors (see e.g. [3], [4], [8], [20], [21], [27]). Most of these applications and analyses have been mainly developed for matrices with complex numbers as entries. Nevertheless, the theory can be extended to matrices over different fields and even over rings. For this purpose, one needs to consider, jointly with the ground algebraic structure, say a field $\mathbb{K}$, an involutory automorphism that plays the role of conjugation (we recall that a homomorphism φ is involutory if $\varphi \circ \varphi = \mathrm{Id}_{\mathbb{K}}$). In this situation, the Penrose axioms (see (1)) can be directly generalized. This line of research has been followed by different authors; see e.g. the books [10] and [5] or the papers [6], [7], [15], [22], or [17] where the case of idempotent semirings is treated, and hence the case of tropical matrices is included. Also, contributions to the case of polynomial matrices can be found [11],[12],[13],[26]. In addition, the symbolic treatment of the problem of computing generalized inverses have been addressed in [19],[23],[24], [25].

Some of the papers mentioned above provide characterizations to ensure the existence of the Moore–Penrose inverse of a given matrix with entries over different types of rings or over a field. In the first part of this paper (see Section 3) we consider a slightly different problem. We study those fields $\mathbb{K}$ (with an involutory automorphism, that we will denote by φ) such that all matrices with entries in $\mathbb{K}$ have generalized inverse. We call such fields Moore–Penrose fields (see Definition 3). We show that only fields of characteristic zero can be Moore–Penrose (see Corollary 5). The main result in Section 3 is the following. We prove that if $(\mathbb{K}, \varphi)$ is a Moore–Penrose field, there exists a natural extension φ^{e} of φ from the field $\mathbb{K}$ to the field of rational functions $\mathbb{K}(\mathbf{x})$, where $\mathbf{x}$ is a tuple of variables, such that $(\mathbb{K}(\mathbf{x}), \varphi^{\mathrm{e}})$ is also Moore–Penrose (see Theorem 12).

In [24] we have shown how to reduce the computation of Drazin inverses over certain computable fields to the computation of Drazin inverses of matrices with rational functions as entries; in particular, the method is applied to matrices with meromorphic functions as entries. The key idea in [24] is to derive a polynomial providing a criterium to ensure that the evaluation of the inverse is the inverse of the evaluation. In the second part of this paper (see Section 4), we analyse the applicability of these ideas to the case of Moore–Penrose inverses. So we focus on the behaviour of the Moore–Penrose inverse under specialization. More precisely, we consider a field $(\mathbb{K}, \varphi)$ and its natural extension $(\mathbb{K}(\mathbf{x}), \varphi^{\mathrm{e}})$. In this situation, if A is a matrix with entries in $\mathbb{K}(\mathbf{x})$, whose Moore–Penrose inverse exists, and $\mathbf{a}$ is a tuple of self-adjoint elements of $\mathbb{K}$ (i.e. $\varphi(\mathbf{a}) = \mathbf{a}$), if the denominator of all the entries of A, and the denominator of all the entries of its generalized inverse, do not vanish at $\mathbf{a}$ it holds that the Moore–Penrose inverse of the specialization is the specialization of the Moore–Penrose inverse (see

Theorem 13 and Theorem 15). Note that, if $(\mathbb{K}, \varphi)$ is Moore–Penrose, the condition of the existence of the Moore–Penrose inverse of A is not required.

The last section of the paper (see Section 5) is devoted to the case of matrices with functions as entries. More precisely, we consider the field $\mathrm{Mer}(\Omega)$ of meromorphic functions over a connected open subset of $\mathbb{C}$, and the involutory automorphism $\varphi(f(z)) = \overline{f(\overline{z})}$, where $\overline{\cdot}$ is the usual conjugation in $\mathbb{C}$. In this situation, we take a finite subset $\mathcal{F} \subset \mathrm{Mer}(\Omega)$ of self-adjoint functions, and we consider matrices with entries in $(\mathbb{C}(\mathcal{F}), \varphi_{|_{\mathbb{C}(\mathcal{F})}})$. As an application of the results in the previous sections, we prove that the computation of the Moore–Penrose inverse of a matrix with entries in $\mathbb{C}(\mathcal{F})$ can be reduced to the computation of the Moore–Penrose inverse of a matrix with rational functions as entries. As a consequence, we present a symbolic algorithm to compute the Moore–Penrose inverse of matrices in $\mathbb{C}(\mathcal{F})$. We finish Section 5 with a brief empirical analysis, based on two tests. In the first test, we check the performance of Algorithm 1 when the number of function entries is fixed, but the order of the matrix increases, while, in the second test, we fix the order of the matrix and we increase the number of functions. The results in both tests are satisfactory.

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2 Preliminaries on Generalized Inverses

Let $(\mathbb{K}, \varphi)$ be a field with an involutory automorphism φ; that is, φ is a field automorphism and $\varphi \circ \varphi = \mathrm{Id}$. We observe that an involutory map is always bijective. Let $\mathcal{M}_{m\times n}(\mathbb{K})$ be the ring of the $m \times n$ matrices with entries in $\mathbb{K}$. By abuse of notation, for $A = (a_{ij}) \in \mathcal{M}_{m\times n}(\mathbb{K})$, we will write $\varphi(A)$ to denote the matrix $(\varphi(a_{ij})) \in \mathcal{M}_{m\times n}(\mathbb{K})$. Also, we denote by A^* the transpose of the matrix $\varphi(A)$. In this situation, the generalized inverse of A is introduced as a matrix $X \in \mathcal{M}_{n\times m}(\mathbb{K})$ such that

$$\text{[Penrose axioms]} \quad \begin{cases} AXA = A, \\ XAX = X \\ AX = X^*A^* \\ XA = A^*X^* \end{cases} \tag{1}$$

If X, satisfying the above conditions, exists then it is unique (see Theorem 1 in [10] pp. 112), and we call it the (Moore–Penrose) generalized inverse of A. If the generalized inverse of A exists, we denote it as $A^\dagger$.

The existence of $A^\dagger$ does depend on the field $\mathbb{K}$ and the involutory automorphism φ. A useful characterization is the following (see Theorem 3 in [10], pp. 116).

Theorem 1 *$A^\dagger$ exists if and only if* $\mathrm{rank}(AA^*) = \mathrm{rank}(A) = \mathrm{rank}(A^*A)$.

The following theorem (see Theorem 3 in [10], pp. 116) provides an expression of $A^\dagger$ when it exists.

Theorem 2 *Let $r = \operatorname{rank}(A)$, and let us assume that $A^\dagger$ exists, then*

$$A^\dagger = C^*(B^*AC^*)^{-1}B^*$$

where B is any $m \times r$ submatrix of A containing exactly r linearly independent columns of A, and C is any $r \times n$ submatrix of A containing exactly r linearly independent rows of A.

Remark 1

1. *If $\operatorname{rank}(A) = n$ then C is regular and $B = A$. Therefore, in that case,*

   $$A^\dagger = C^*(B^*AC^*)^{-1}B^* = C^*(A^*AC^*)^{-1}A^* = C^*(C^*)^{-1}(A^*A)^{-1}A^* = (A^*A)^{-1}A^*.$$

   Similarly, if $\operatorname{rank}(A) = m$ then

   $$A^\dagger = A^*(AA^*)^{-1}.$$

2. *Theorem 2 implies that, if $A^\dagger$ exists then $A^\dagger \in \mathcal{M}_{n\times m}(\mathbb{K})$.*

3 Moore–Penrose Fields

Theorem 1 provides a criterium to decide whether the generalized inverse of a particular matrix exists. In the following we look for conditions on the field $\mathbb{K}$ to ensure that all matrices over $\mathbb{K}$ have Penrore generalized inverse. For this purpose, throughout this section, $(\mathbb{K}, \varphi)$ is a field with an involutory automorphim. First, we introduce the notion of Moore–Penrose field.

Definition 3 *We say that $(\mathbb{K}, \varphi)$ is a* Moore–Penrose field *if for all $A \in \mathcal{M}_{m\times n}(\mathbb{K})$ the generalized inverse $A^\dagger$ exists.*

We know that $(\mathbb{C}, \overline{\cdot})$, where $\overline{\cdot}$ is the complex number conjugation, is a Moore–Penrose field. However, $(\mathbb{Z}_2, \mathrm{id})$ is not (see [10], Example 2, pp. 117).

The following theorem provides a characterization for the notion of Moore–Penrose field.

Theorem 4 *The following statements are equivalent*

1. *$(\mathbb{K}, \varphi)$ is a Moore–Penrose field.*
2. *If $\sum_{k=1}^{\ell} a_k\varphi(a_k) = 0$, with $a_k \in \mathbb{K}$, then $a_1 = \cdots = a_\ell = 0$.*

Proof: Taking into account Remark 8 and Corollary 6, both in [10], pp. 117, one deduces that statement (2) implies statement (1). Let us see that statement (1) implies statement (2). Let $\sum_{k=1}^{\ell} a_k\varphi(a_k) = 0$, with $a_k \in \mathbb{K}$. We consider the matrix $A = (a_1, \ldots, a_\ell) \in \mathcal{M}_{1\times\ell}(\mathbb{K})$. Since $(\mathbb{K}, \varphi)$ is Moore–Penrose, by Theorem 1, $\operatorname{rank}(A) = \operatorname{rank}(AA^*) = \operatorname{rank}\Big(\Big(\sum_{k=1}^{\ell} a_k\varphi(a_k)\Big)\Big) = 0$. Therefore $a_1 = \cdots = a_\ell = 0$. $\square$

Corollary 5 *If $(\mathbb{K}, \varphi)$ is a Moore–Penrose field, then the characteristic of $\mathbb{K}$ is zero.*

Proof: Let $\mathbb{K}$ has characteristic $p > 0$. We denote by $1_{\mathbb{K}}$ and $0_{\mathbb{K}}$ the neutral element of $\mathbb{K}$ w.r.t. the multiplication and addition, respectively. Then, since φ is a field homomorphism, then $\varphi(1_{\mathbb{K}}) = 1_{\mathbb{K}}$, Let $A = (1_{\mathbb{K}}, \ldots, 1_{\mathbb{K}}) \in \mathcal{M}_{1\times p}(\mathbb{K})$. Then, $\operatorname{rank}(A) = 1$ and $\operatorname{rank}(AA^*) = \operatorname{rank}((1_{\mathbb{K}} + \cdots + 1_{\mathbb{K}})) = \operatorname{rank}((0_{\mathbb{K}})) = 0$. $\square$

Let us see an example of a field of characteristic 0 that is not a Moore–Penrose field.

Example 6 *Let $\mathbb{K} = \mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. We consider the involutory automorphism*

$$\varphi : \mathbb{Q}(\sqrt{2}) \longrightarrow \mathbb{Q}(\sqrt{2}); a + b\sqrt{2}, a, b \in \mathbb{Q} \longmapsto a - b\sqrt{2}.$$

Let $A = (\sqrt{2} + 2, \sqrt{2}) \in \mathcal{M}_{1\times 2}(\mathbb{Q}(\sqrt{2}))$. We have that

$$\begin{array}{lll} A & \textit{and} & \operatorname{rank}(A) = 1 \\ AA^* = (0) & \textit{and} & \operatorname{rank}(AA^*) = 0 \\ A^*A = \begin{pmatrix} 2 & -2 + 2\sqrt{2} \\ -2 - 2\sqrt{2} & -2 \end{pmatrix} & \textit{and} & \operatorname{rank}(A^*A) = 1. \end{array}$$

Therefore, applying Theorem 1, we get that $A^\dagger$ does not exist, and hence $(\mathbb{Q}(\sqrt{2}), \varphi)$ is not Moore–Penrose.

We now introduce the subset of elements in $\mathbb{K}$ that are φ–invariant (or self-adjoint) and we prove that it is a subfield of $\mathbb{K}$.

Definition 7 *We denote by $\mathbb{K}_\varphi$ the set $\mathbb{K}_\varphi = \{x \in \mathbb{K} \mid \varphi(x) = x\}$.*

Lemma 8 *$\mathbb{K}_\varphi$ is a field.*

Proof: Since φ is a field homomorphism, then $\varphi(0) = 0$ and $\varphi(1) = 1$, and hence $0, 1 \in \mathbb{K}_\varphi$. Moreover, if $x, y \in \mathbb{K}_\varphi$ then $\varphi(x+y) = \varphi(x)+\varphi(y) = x+y$. So, $x+y \in \mathbb{K}_\varphi$. Similarly, if $x \in \mathbb{K}_\varphi \setminus \{0\}$, then $\varphi(x^{-1}) = \varphi(x)^{-1} = x^{-1}$. Thus, $x^{-1} \in \mathbb{K}_\varphi$. Therefore, $\mathbb{K}_\varphi$ is a subfield of $\mathbb{K}$. $\square$

In the following we analyse the extension of the automorphism φ, first, to the polynomial ring $\mathbb{K}[\mathbf{x}]$, where $\mathbf{x}$ denotes a tuple of r variables, and, second, to the field of rational functions $\mathbb{K}(\mathbf{x})$. For this purpose, polynomials in $\mathbb{K}[\mathbf{x}]$, will be represented as $\sum a_j T_j$ where the sum has finitely many summands, $a_j \in \mathbb{K}$, and T_j are terms of the variables in $\mathbf{x}$.

We consider the ring homomorphism

$$\varphi^\diamond : \mathbb{K}[\mathbf{x}] \to \mathbb{K}[\mathbf{x}]; P(\mathbf{x}) = \sum a_i T_i(\mathbf{x}) \mapsto \sum \varphi(a_i) T_i(\mathbf{x}), \tag{2}$$

and the field homomorphism

$$\varphi^{\mathrm{e}} : \mathbb{K}(\mathbf{x}) \to \mathbb{K}(\mathbf{x}); \frac{P(\mathbf{x})}{Q(\mathbf{x})} \mapsto \frac{\varphi^\diamond(P(\mathbf{x}))}{\varphi^\diamond(Q(\mathbf{x}))}. \tag{3}$$

Observe that $\varphi^\diamond$ and φ^{e} are involutory, and hence bijective. We prove several lemmas that will be used in the proof of Theorem 12.

Lemma 9 *Let* $(\mathbb{K}[\mathbf{x}])_{\varphi^\diamond} = \{P \in \mathbb{K}[\mathbf{x}] \mid \varphi^\diamond(P) = P\}$. *Then,* $\mathbb{K}_\varphi[\mathbf{x}] = (\mathbb{K}[\mathbf{x}])_{\varphi^\diamond}$.

Proof: Clearly $\mathbb{K}_\varphi[\mathbf{x}] \subset (\mathbb{K}[\mathbf{x}])_{\varphi^\diamond}$. For the other inclusion, let $P \in (\mathbb{K}[\mathbf{x}])_{\varphi^\diamond}$. Then, $\varphi^\diamond(P) - P = 0$. Therefore, $\varphi^\diamond(\sum a_j T_j) - \sum a_j T_j = \sum(\varphi(a_j) - a_j)T_j = 0$. Therefore, for all a_j, $\varphi(a_j) = a_j$. So, $P \in \mathbb{K}_\varphi[\mathbf{x}]$. $\square$

For the next lemmas, we recall that $\mathbf{x}$ is an r-tuple of variables. Furthermore, in the sequel, we will denote by $\mathbb{K}_\varphi^r$ the r–dimensional vector space $\mathbb{K}_\varphi^r := \mathbb{K}_\varphi \times \cdots \times \mathbb{K}_\varphi$ (see Definition 7 and Lemma 8).

Lemma 10 *Let* $P \in \mathbb{K}(\mathbf{x})$. *For every* $\mathbf{a} \in \mathbb{K}_\varphi^r$, *such that* $P(\mathbf{a})$ *is defined, it holds that* $\varphi^{\mathrm{e}}(P)(\mathbf{a}) = \varphi(P(\mathbf{a}))$.

Proof: We prove it first for polynomials. Say that $P = \sum b_i T_i(\mathbf{x}) \in \mathbb{K}[\mathbf{x}]$, where $T_i(\mathbf{x})$ are the terms appearing in P. Then, $\varphi^{\mathrm{e}}(P) = \sum \varphi(b_i) T_i(\mathbf{x})$ and $\varphi^{\mathrm{e}}(P)(\mathbf{a}) = \sum \varphi(b_i) T_i(\mathbf{a})$. On the other hand, $\varphi(P(\mathbf{a})) = \varphi(\sum b_i T_i(\mathbf{a})) = \sum \varphi(b_i)\varphi(T_i(\mathbf{a}))$, and since $\mathbf{a} \in \mathbb{K}_\varphi^r$, then $\varphi(T_i(\mathbf{a})) = T_i(\mathbf{a})$. So, $\varphi^{\mathrm{e}}(P)(\mathbf{a}) = \varphi(P(\mathbf{a}))$. Now, for the case of rational functions, the result follows from the definition on φ^{e}. $\square$

In the sequel, we will use the following notation. Let $A = (f_{i,j}(\mathbf{x}))$ be a matrix whose entries are functions depending on the tuple of variables $\mathbf{x}$, and let $\mathbf{a}$ be in the intersection of the domains of all the functions $f_{i,j}$. Then, we denote by $A_{|\mathbf{x}=\mathbf{a}}$ the matrix $(f_{i,j}(\mathbf{a}))$. In addition, if the entries of A are rational functions, we denote by $\mathrm{den}(A)$ the least common multiple of all denominators in A.

Lemma 11 *Let* $A \in \mathcal{M}_{m\times n}(\mathbb{K}(\mathbf{x}))$. *For every* $\mathbf{a} \in \mathbb{K}_\varphi^r$, *such that* $\mathrm{den}(A)(\mathbf{a}) \neq 0$, *it holds that*

$$(A^*)_{|\mathbf{x}=\mathbf{a}} = (A_{|\mathbf{x}=\mathbf{a}})^*.$$

Proof: It is a direct consequence of Lemma 10 $\square$

Theorem 12 *Let $(\mathbb{K}, \varphi)$ be a Moore–Penrose field. Then, $(\mathbb{K}(\mathbf{x}), \varphi^{\mathrm{e}})$ is Moore–Penrose.*

Proof: Let us assume that $(\mathbb{K}(\mathbf{x}), \varphi^{\mathrm{e}})$ is not Moore–Penrose. Then, there exists $A \in \mathcal{M}_{m\times n}(\mathbb{K}(\mathbf{x}))$ such that $A^\dagger$ does not exist. By Theorem 1, at least one of the rank equalities does not hold. Let us assume that $\mathrm{rank}(A) \neq \mathrm{rank}(A^*A)$; similarly if $\mathrm{rank}(A) \neq \mathrm{rank}(AA^*)$. Let $\mathcal{R}$ be the set containing all denominators appearing during the Gaussian triangulation process of both A and A^*A. Let B and C be the triangular matrices output by the Gaussian triangulation process when applied to A and A^*A, respectively. Let $\mathcal{S}$ be the set containing all numerators of the non-zero entries in B and C. Note that $\mathcal{R}, \mathcal{S} \subset \mathbb{K}[\mathbf{x}]$. In addition, let $T = \prod_{P\in\mathcal{R}\cup\mathcal{S}} P \in \mathbb{K}[\mathbf{x}]$. Let $\mathcal{T}$ be the hypersurface defined by T in $\mathbb{K}_\varphi^r$; r is the length of the tuple $\mathbf{x}$. Since $T \neq 0$, and since $\mathbb{K}_\varphi$ is infinity because $\mathbb{K}$ is of characteristic zero (see Corollary 5), one has that $\mathbb{K}_\varphi^r \setminus \mathcal{T} \neq \emptyset$. We take an element $\mathbf{a} \in \mathbb{K}_\varphi^r \setminus \mathcal{T}$. Then, it holds that

1. $B_{|\mathbf{x}=\mathbf{a}}$ is the triangularization of $A_{|\mathbf{x}=\mathbf{a}}$, and $\mathrm{rank}(A) = \mathrm{rank}(B) = \mathrm{rank}(B_{|\mathbf{x}=\mathbf{a}}) = \mathrm{rank}(A_{|\mathbf{x}=\mathbf{a}})$.
2. $C_{|\mathbf{x}=\mathbf{a}}$ is the triangularization of $(A^*A)_{|\mathbf{x}=\mathbf{a}}$, and $\mathrm{rank}(A^*A) = \mathrm{rank}(C) = \mathrm{rank}(C_{|\mathbf{x}=\mathbf{a}}) = \mathrm{rank}((A^*A)_{|\mathbf{x}=\mathbf{a}})$.
3. $(A^*A)_{|\mathbf{x}=\mathbf{a}}$ is well-defined, and since $\mathbf{a} \in \mathbb{K}_\varphi^r$, we have, by lemma 11, that $(A^*A)_{|\mathbf{x}=\mathbf{a}} = (A^*)_{|\mathbf{x}=\mathbf{a}} A_{|\mathbf{x}=\mathbf{a}} = (A_{|\mathbf{x}=\mathbf{a}})^* A_{|\mathbf{x}=\mathbf{a}}$.

In this situation, let $M = A_{|\mathbf{x}=\mathbf{a}} \in \mathcal{M}_{m\times n}(\mathbb{K})$. By (3) and (2), we have that $\mathrm{rank}(M^\star M) = \mathrm{rank}(A^\star A)$. By (1), $\mathrm{rank}(M) = \mathrm{rank}(A)$. Thus, using that $\mathrm{rank}(A) \neq \mathrm{rank}(A^*A)$, we get that $\mathrm{rank}(M) \neq \mathrm{rank}(M^*M)$. Therefore, by Theorem 1, $M^\dagger$ does not exists which is a contradiction with the fact that $(\mathbb{K}, \varphi)$ is Moore–Penrose. $\square$

4 Moore–Penrose Inverse under Specializations

In this section, we consider matrices with rational functions entries and we analyse the behaviour of the Moore–Penrose inverse when the variables are substituted by field elements. We will use the notation $A_{|\mathbf{x}=\mathbf{a}}$ introduced in the previous section.

Theorem 13 *Let $(\mathbb{K}, \varphi)$ be a Moore–Penrose field. We consider the Moore–Penrose field $(\mathbb{K}(\mathbf{x}), \varphi^{\mathrm{e}})$. Let $A \in \mathcal{M}_{m\times n}(\mathbb{K}(\mathbf{x}))$, with $\mathbf{x} = (x_1, \ldots, x_r)$, and let $\mathbf{a} \in \mathbb{K}_\varphi^r$ such that*

$$\mathrm{den}(A)(\mathbf{a})\mathrm{den}(A^\dagger)(\mathbf{a}) \neq 0.$$

Then

$$(A^\dagger)_{|\mathbf{X}=\mathbf{a}} = (A_{|\mathbf{X}=\mathbf{a}})^\dagger.$$

Proof: First we observe that $A_{|\mathbf{X}=\mathbf{a}}$ and $(A^\dagger)_{|\mathbf{X}=\mathbf{a}}$ are well defined. In addition, let P be any denominator in A^*. Then, $\varphi^{\mathrm{e}}(P)$ is a denominator in A. Furthermore, by Lemma 10, $0 \neq \varphi^{\mathrm{e}}(P)(\mathbf{a}) = \varphi(P(\mathbf{a}))$. So $P(\mathbf{a}) \neq 0$, and $(A^*)_{|\mathbf{X}=\mathbf{a}}$ is also well-defined. Let $M = A_{|\mathbf{X}=\mathbf{a}}$. In this situation, from the Penrose axioms (see (1)), we have that

- $M = (AA^\dagger A)_{|\mathbf{X}=\mathbf{a}} = A_{|\mathbf{X}=\mathbf{a}}(A^\dagger)_{|\mathbf{X}=\mathbf{a}}A_{|\mathbf{X}=\mathbf{a}} = M\,(A^\dagger)_{|\mathbf{X}=\mathbf{a}}M$
- $(A^\dagger)_{|\mathbf{X}=\mathbf{a}} = (A^\dagger AA^\dagger)_{|\mathbf{X}=\mathbf{a}} = (A^\dagger)_{|\mathbf{X}=\mathbf{a}}A_{|\mathbf{X}=\mathbf{a}}(A^\dagger)_{|\mathbf{X}=\mathbf{a}} = (A^\dagger)_{|\mathbf{X}=\mathbf{a}}M\,(A^\dagger)_{|\mathbf{X}=\mathbf{a}}$
- $M\,(A^\dagger)_{|\mathbf{X}=\mathbf{a}} = A_{|\mathbf{X}=\mathbf{a}}(A^\dagger)_{|\mathbf{X}=\mathbf{a}} = (AA^\dagger)_{|\mathbf{X}=\mathbf{a}} = ((A^\dagger)^*A^*)_{|\mathbf{X}=\mathbf{a}} = ((A^\dagger)^*)_{|\mathbf{X}=\mathbf{a}}(A^*)_{|\mathbf{X}=\mathbf{a}}$.

  So, by Lemma 11, we get

  $M\,(A^\dagger)_{|\mathbf{X}=\mathbf{a}} = ((A^\dagger)_{|\mathbf{X}=\mathbf{a}})^*(A_{|\mathbf{X}=\mathbf{a}})^* = ((A^\dagger)_{|\mathbf{X}=\mathbf{a}})^*M^*$.
- Reasoning as above, $(A^\dagger)_{|\mathbf{X}=\mathbf{a}}M = ((A^\dagger)_{|\mathbf{X}=\mathbf{a}})^*M^*$.

Therefore, $(A^\dagger)_{|\mathbf{X}=\mathbf{a}}$ satisfies the Penrose axioms for M. Thus, by the uniqueness of the Moore–Penrose inverse, we conclude the proof. $\square$

Example 14 *Let $(\mathbb{K},\varphi) = (\mathbb{C}, \overline{\cdot})$. We consider its extension $(\mathbb{C}(x_1,x_2,x_3), \overline{\cdot}^{\mathrm{e}})$; see (3) for the definition of the involutory automorphism $\overline{\cdot}^{\mathrm{e}}$. Note that $\mathbb{K}_\varphi = \mathbb{R}$. We consider the matrix*

$$A = \begin{pmatrix} \dfrac{x_1x_2}{x_3} & x_1x_3 & x_2x_3 \\ \dfrac{x_1x_2}{x_3} & x_1x_3 & 0 \end{pmatrix} \in \mathcal{M}_{2\times 3}(\mathbb{C}(x_1,x_2,x_3)).$$

The Moore–Penrose inverse is (see last part of this section for details on how to compute $A^\dagger$)

$$A^\dagger = \begin{pmatrix} 0 & \dfrac{x_2x_3}{x_1\,({x_3}^4 + {x_2}^2)} \\ 0 & \dfrac{{x_3}^3}{x_1\,({x_3}^4 + {x_2}^2)} \\ \dfrac{1}{x_2x_3} & -\dfrac{1}{x_2x_3} \end{pmatrix}.$$

By Theorem 13, we know that for all $a,b,c \in \mathbb{R}\setminus\{0\}$ it holds that

$$(A_{|(x_1,x_2,x_3)=(a,b,c)})^\dagger = (A^\dagger)_{|(x_1,x_2,x_3)=(a,b,c)}.$$

For instance

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{\dagger} = (A_{|_{(x_1,x_2,x_3)=(1,1,1)}})^{\dagger} = (A^{\dagger})_{|_{(x_1,x_2,x_3)=(1,1,1)}} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \\ 1 & -1 \end{pmatrix}.$$

However, the specialization property does not hold, in general, if the values are not taken in $\mathbb{K}_{\varphi}$. *For instance* (i *denotes the imaginary unit, that is* $\mathrm{i} = \sqrt{-1}$),

$$\begin{pmatrix} 0 & \frac{1}{3} - \mathrm{i}\frac{1}{3} \\ 0 & \frac{1}{3} \\ \frac{1}{2} - \mathrm{i}\frac{1}{2} & -\frac{1}{2} + \mathrm{i}\frac{1}{2} \end{pmatrix} = (A_{|_{(x_1,x_2,x_3)=(1,1+\mathrm{i},1)}})^{\dagger} \neq$$

$$(A^{\dagger})_{|_{(x_1,x_2,x_3)=(1,1+\mathrm{i},1)}} = \begin{pmatrix} 0 & \frac{3}{5} - \mathrm{i}\frac{1}{5} \\ 0 & \frac{1}{5} - \mathrm{i}\frac{2}{5} \\ \frac{1}{2} - \mathrm{i}\frac{1}{2} & -\frac{1}{2} + \mathrm{i}\frac{1}{2} \end{pmatrix}.$$

In addition, within the same example, we may consider a different field extension. We take $(\mathbb{K}, \varphi) = (\mathbb{C}(x_1), \overline{\cdot}^{\mathrm{e}})$ *and we extend it to* $(\mathbb{C}(x_1)(x_2, x_3), \varphi^{\mathrm{e}})$*, where* $\varphi^{\mathrm{e}} = (\overline{\cdot}^{\mathrm{e}})^{\mathrm{e}}$*; see* (3) *for the definition of the involutory automorphisms* $\overline{\cdot}^{\mathrm{e}}$ *and* $(\overline{\cdot}^{\mathrm{e}})^{\mathrm{e}}$*,also we recall that* $\mathbb{C}(x_1)(x_2, x_3)$ *is the field of complex rational functions in the variable* $\{x_2, x_3\}$ *with coefficients in* $\mathbb{C}(x_1)$*. Now,* $\mathbb{K}_{\varphi} = \mathbb{R}(x_1)$*; we recall that* $\mathbb{R}(x_1)$ *is the field real rational functions in the variable* x_1*. In this situation,* $A \in \mathcal{M}_{3\times 2}(\mathbb{C}(x_1)(x_2, x_3))$*, and for every* $b, c \in \mathbb{R}(x_1) \setminus \{0\}$ *we have that*

$$(A_{|_{(x_2,x_3)=(b,c)}})^{\dagger} = (A^{\dagger})_{|_{(x_2,x_3)=(b,c)}}.$$

For instance,

$$\begin{pmatrix} 1 & {x_1}^4 & {x_1}^5 \\ 1 & {x_1}^4 & 0 \end{pmatrix}^{\dagger} = (A_{|_{(x_2,x_3)=(x_1^2,x_1^3)}})^{\dagger} = (A^{\dagger})_{|_{(x_2,x_3)=(x_1^2,x_1^3)}} = \begin{pmatrix} 0 & \frac{1}{1+x_1^8} \\ 0 & \frac{{x_1}^4}{{x_1}^8+1} \\ \frac{1}{x_1^5} & -\frac{1}{x_1^5} \end{pmatrix}.$$

In the first part of this section we have been working with Moore–Penrose fields. Now, we see that in the case of fields, where the Moore–Penrose field property is not guaranteed, a similar treatment can be performed. So, we assume that we are given a field with an involutory automorphism, $(\mathbb{K}, \varphi)$, and we consider $(\mathbb{K}(\mathbf{x}), \varphi^{\mathrm{e}})$, where φ^{e} is defined as in (3). However, we do not ask $(\mathbb{K}, \varphi)$ to be Moore–Penrose, and hence Theorem 12 cannot be applied. So, in general, we cannot ensure the existence of the Moore–Penrose inverse. In this situation, Theorem 13 can be reformulated as follows.

Theorem 15 *We consider the fields $(\mathbb{K}, \varphi)$ and $(\mathbb{K}(\mathbf{x}), \varphi^{\mathrm{e}})$. Let $A \in \mathcal{M}_{m\times n}(\mathbb{K}(\mathbf{x}))$, with $\mathbf{x} = (x_1, \ldots, x_r)$, such that $A^\dagger$ exists. Let $\mathbf{a} \in \mathbb{K}^r_\varphi$ such that*

$$\mathrm{den}(A)(\mathbf{a})\mathrm{den}(A^\dagger)(\mathbf{a}) \neq 0.$$

Then $(A_{|\mathbf{x}=\mathbf{a}})^\dagger$ exists and

$$(A^\dagger)_{|\mathbf{x}=\mathbf{a}} = (A_{|\mathbf{x}=\mathbf{a}})^\dagger.$$

In the last part of this section, we discuss different possibilities for computing the Moore–Penrose inverse of matrices of rational functions. We assume that $(\mathbb{K}, \varphi)$ is a Moore–Penrose field. So, by Theorem 12, $(\mathbb{K}(\mathbf{x}), \varphi^{\mathrm{e}})$ is also Moore–Penrose. Let us assume that $\mathbb{K}$ is a computable field (i.e. a field where the basic arithmetic is computable) and that an algorithm to compute $\varphi(\mathbf{a})$, with $\mathbf{a} \in \mathbb{K}^r$, is available. Then, it follows that $\mathbb{K}(\mathbf{x})$ is also computable and the algorithm to compute φ can be naturally extended to an algorithm to compute φ^{e}. Therefore, Theorem 2, and in particular Remark 1, provide an algorithmic method to compute $A^\dagger$ in $(\mathbb{K}(\mathbf{x}), \varphi^{\mathrm{e}})$.

Now, let us consider the particular case of $\mathbb{K} = \mathbb{C}$, taking $\varphi = \overline{\cdot}$ as the usual conjugation of complex numbers. There exist different algorithms for computing $A^\dagger$ with $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ (see e.g. [3], [4]). In addition, different approaches for matrices with real polynomials and real rational functions have been presented in [11], [12], [13], [19], [25], [26]; see also [1], [9] for computing the inverse via limits. Now, let $A \in \mathcal{M}_{m\times n}(\mathbb{C}(\mathbf{x}))$. We observe that if $\lambda \in \mathbb{C}(\mathbf{x})_\varphi \setminus \{0\}$, i.e. $\overline{\lambda} = \lambda$, then $(\lambda A)^\dagger = 1/\lambda A^\dagger$. Therefore, since $\mathbb{R}(\mathbf{x}) \subset \mathbb{C}(\mathbf{x})_\varphi$, the algorithms for real polynomial matrices mentioned above extend naturally to the case of real rational function matrices. Moreover, if $A \in \mathcal{M}_{m\times n}(\mathbb{C}(\mathbf{x}))$, normalizing the complex rational functions, and multiplying the matrix by the least common multiple of all denominators, the computation of the Moore–Penrose inverse in $\mathcal{M}_{m\times n}(\mathbb{C}(\mathbf{x}))$ is reduced to the computation in $\mathcal{M}_{m\times n}(\mathbb{C}[\mathbf{x}])$, where $\mathbb{C}[\mathbf{x}]$ denotes the polynomial ring of the complex polynomials in the variables $\mathbf{x}$.

In [23], elimination theory techniques are applied to compute the Drazin inverse of matrices with rational function entries. For the case of the Moore–Penrose inverse of a matrix one may try to proceed similarly. However, a main difference appears here, namely the conjugation of the polynomials. Nevertheless, one could use the following scheme. We express the unknown matrix X in (1) as $X = Y + \mathrm{i}Z$ where Y, Z are unknown matrices with undetermined entries in $\mathbb{R}(\mathbf{x})$. Then, if we write X, Y as $X = (\alpha_{ij}), Y = (\beta_{ij})$, the equalities in (1) provide a set $\mathcal{S} = \{H_k(\alpha_{11}, \ldots, \beta_{nm}) = 0\}_{k\in I}$, where I is a finite set of indexes, and where $H_k \in \mathbb{C}(\mathbf{x})[\alpha_{11}, \ldots, \beta_{nm}]$, that is H_k is a polynomial in the variables α_{ij}, β_{ij} with coefficients in $\mathbb{C}(\mathbf{x})$; note that the total degree of each H_K, w.r.t. $\{\alpha_{11}, \ldots, \beta_{nm}\}$, is at most 2. Now, taking the real part and the imaginary parts of H_k we get a new algebraic system defined by real polynomials. Since, the solution is unique, the system is zero-dimensional, and applying Gröbner bases techniques one derives the Moore–Penrose inverse.

Alternatively, one can also design an algorithm based on interpolation techniques as follows. Let $A \in \mathcal{M}_{m\times n}(\mathbb{C}[\mathbf{x}])$. By Theorem 2 and Remark 1, one may deduce an upper bound of the degree w.r.t. x_i of the numerators and denominators of the entries in $A^\dagger$. On the other hand, by Theorem 13, we know that for every $\mathbf{a} \in \mathbb{R}^r$, such that neither the denominator of A nor the denominator of $A^\dagger$ vanishes at $\mathbf{a}$, it holds that the specialization of the Moore-Penrose inverse is the Moore-Penrose inverse of the specialization. Therefore, taking sufficiently many tuples of real numbers, and using rational function interpolation, we may derive a candidate for the Moore–Penrose inverse of A. The correctness of the computed solution can be checked by substituting it in (1).

In spite of these disquisitions, in this paper we do not investigate in this direction. We focus on the application of Theorems 13 and 15 to matrices with meromorphic functions (see Section 5) and, for this purpose, one may use any of the available algorithms to compute the Moore–Penrose inverses of rational function matrices that are involved in the process. In our case, we use the Maple implementation of the pseudoinverse function.

5 Application to Matrices of Functions

In this section we deal with matrices whose entries as functions. More precisely, let Ω be a connected open subset of $\mathbb{C}$. We consider the integral domain $\mathcal{O}(\Omega)$ of the holomorphic functions over Ω, as well as its field of fractions $\mathrm{Mer}(\Omega)$ of the meromorphic functions over Ω (see [14]). Furthermore, in $\mathrm{Mer}(\Omega)$ we introduce the involutory automorphism

$$\varphi : \mathrm{Mer}(\Omega) \to \mathrm{Mer}(\Omega); f(z) \mapsto \overline{f(\,\overline{z}\,)}. \tag{4}$$

We consider $(\mathbb{C}(z), \overline{\cdot}^e)$ (see (3)); that is, $\mathbb{C}(z)$ is the field of complex rational functions in z, and

$$\overline{\cdot}^e : \mathbb{C}(z) \to \mathbb{C}(z); R(z) := \frac{\sum_{i=0}^{k_1} a_i z^i}{\sum_{i=0}^{k_2} b_i z^i} \longmapsto \overline{R(z)}^e = \frac{\sum_{i=0}^{k_1} \overline{a_i} z^i}{\sum_{i=0}^{k_2} \overline{b_i} z^i}. \tag{5}$$

Thus, since $\varphi(z) = \overline{\overline{z}} = z$ (see (4)), we have that the restriction of φ to $\mathbb{C}(z)$ is $\overline{\cdot}^e$; that is, $\varphi_{|_{\mathbb{C}(z)}} = \overline{\cdot}^e$. In addition, we observe that the elements of the field $\mathbb{R}(z)$, of the real rational functions in the variable z, are φ-invariant. So $\mathbb{R}(z) \subset \mathrm{Mer}(\Omega)_\varphi$. Other φ-invariant meromorphic functions are, for instance, e^z, $\sin(z)$, $\cos(z)$, $\sinh(z)$, $\cosh(z)$, $\Gamma(z) \in \mathrm{Mer}(\Omega)_\varphi$.

In this situation, we take a subset

$$\mathcal{F} = \{f_1(z), \ldots, f_k(z)\} \subset \mathrm{Mer}(\Omega)_\varphi \tag{6}$$

being algebraically independent over $\mathbb{C}$; this means that there exists none non-zero polynomial $P \in \mathbb{C}[\mathbf{w}]$ such that $P(f_1, \ldots, f_k) = 0$. Let us use the notation $\mathbf{w} = (w_1, \ldots, w_k)$, and $\mathbf{f} = (f_1(z), \ldots, f_k(z))$. Then, we consider the field extension

$$\mathbb{C}(\mathcal{F}) = \left\{ \frac{P(\mathbf{f})}{Q(\mathbf{f})} \text{ with } P, Q \in \mathbb{C}[\mathbf{w}] \text{ and } Q(\mathbf{f}) \neq 0 \right\} \subset \mathrm{Mer}(\Omega).$$

Let $F \in \mathbb{C}(\mathcal{F})$, and let $P, Q \in \mathbb{C}[\mathbf{w}]$, with $Q(\mathbf{f}) \neq 0$, be such that $F = P(\mathbf{f})/Q(\mathbf{f})$. Let us write P and Q as $P = \sum a_i T_i(\mathbf{w}), Q = \sum b_i T_i(\mathbf{w})$ where T_i are terms in the variables $\mathbf{w}$ and $a_i, b_i \in \mathbb{C}$. Then,

$$\varphi(F) = \frac{\overline{\sum a_i T_i(\mathbf{f}(\overline{z}))}}{\overline{\sum b_i T_i(\mathbf{f}(\overline{z}))}} = \frac{\sum \overline{a_i} T_i(\,\overline{\mathbf{f}(\overline{z})}\,)}{\sum \overline{b_i} T_i(\,\overline{\mathbf{f}(\overline{z})}\,)} = \frac{\sum \overline{a_i} T_i(\mathbf{f}(z))}{\sum \overline{b_i} T_i(\mathbf{f}(z))} \in \mathbb{C}(\mathcal{F}).$$

Therefore the restriction of φ to $\mathbb{C}(\mathcal{F})$ is also an involutory automorphism that, abusing of the notation, we denote again as φ. Thus, we have $(\mathbb{C}(\mathcal{F}), \varphi) \subset (\mathrm{Mer}(\Omega), \varphi)$.

Lemma 16 *Let $P \in \mathbb{R}[\mathbf{w}]$. Then, $\varphi(P(\mathbf{f})) = P(\mathbf{f})$.*

Proof: Let P be expressed as $P = \sum a_i T_i(\mathbf{w})$, where $a_i \in \mathbb{R}$ and T_i are terms in the variables $\mathbf{w}$. Then

$$\varphi(P(\mathbf{f}(z))) = \overline{\sum a_i T_i(\mathbf{f}(\,\overline{z}\,))} = \sum \overline{a_i} T_i(\,\overline{\mathbf{f}(\,\overline{z}\,)}\,). = \sum \overline{a_i} T_i(\varphi(f_1(z)), \ldots, \varphi(f_k(z))).$$

Now, the result follows taking into account that $a_i \in \mathbb{R}$ and that $f_i \in \mathrm{Mer}(\Omega)_\varphi$. $\square$

Lemma 17 *Let $F \in \mathbb{C}(\mathcal{F})$ be written as*

$$F = \frac{A(\mathbf{f}(z)) + \mathrm{i}B(\mathbf{f}(z)))}{C(\mathbf{f}(z)) + \mathrm{i}D(\mathbf{f}(z))}$$

with $A, B, C, D \in \mathbb{R}[\mathbf{w}]$. Then

$$F \cdot \varphi(F) = \frac{A(\mathbf{f}(z))^2 + B(\mathbf{f}(z))^2}{C(\mathbf{f}(z))^2 + D(\mathbf{f}(z))^2}.$$

Proof: Taking into account that A, B, C, D are real polynomials and that $f_i \in \mathrm{Mer}(\Omega)_\varphi$, by Lemma 16, one has that

$$\begin{aligned}
F \cdot \varphi(F) &= \frac{A(\mathbf{f}(z)) + \mathrm{i}B(\mathbf{f}(z)))}{C(\mathbf{f}(z)) + \mathrm{i}D(\mathbf{f}(z))} \cdot \frac{\overline{A(\mathbf{f}(\,\overline{z}\,)) + \mathrm{i}B(\mathbf{f}(\,\overline{z}\,)))}}{\overline{C(\mathbf{f}(\,\overline{z}\,)) + \mathrm{i}D(\mathbf{f}(\,\overline{z}\,))}} \\
&= \frac{A(\mathbf{f}(z)) + \mathrm{i}B(\mathbf{f}(z)))}{C(\mathbf{f}(z)) + \mathrm{i}D(\mathbf{f}(z))} \cdot \frac{\overline{A(\mathbf{f}(\,\overline{z}\,))} - \mathrm{i}\overline{B(\mathbf{f}(\,\overline{z}\,)))}}{\overline{C(\mathbf{f}(\,\overline{z}\,))} - \mathrm{i}\overline{D(\mathbf{f}(\,\overline{z}\,))}} \\
&= \frac{A(\mathbf{f}(z)) + \mathrm{i}B(\mathbf{f}(z)))}{C(\mathbf{f}(z)) + \mathrm{i}D(\mathbf{f}(z))} \cdot \frac{A(\mathbf{f}(z)) - \mathrm{i}B(\mathbf{f}(z)))}{C(\mathbf{f}(z)) - \mathrm{i}D(\mathbf{f}(z))} = \frac{A(\mathbf{f}(z))^2 + B(\mathbf{f}(z))^2}{C(\mathbf{f}(z))^2 + D(\mathbf{f}(z))^2}.
\end{aligned}$$

$\square$

Theorem 18 $(\mathbb{C}(\mathcal{F}), \varphi)$ *is a Moore–Penrose field.*

Proof: Let $F_1/G_1, \ldots, F_k/G_k \in \mathbb{C}(\mathcal{F})$ such that $\sum_{j=1}^{k} F_j/G_j\varphi(F_j/G_j) = 0$. We prove that, $F_1/G_1 = \cdots = F_k/G_k = 0$. Let F_i, G_i be expressed as $F_i = P_i(\mathbf{f})$ and $G_i = Q_i(\mathbf{f})$ where $P_i = A_i + \mathrm{i}B_i, Q_i = C_i + \mathrm{i}D_i$ with $A_i, B_i, C_i, D_i \in \mathbb{R}[\mathbf{w}]$. By Lemma 17, we have that

$$0 = \sum_{j=1}^{k} \frac{F_j}{G_j}\varphi\left(\frac{F_j}{G_j}\right) = \sum_{j=1}^{k} \frac{A_j(\mathbf{f})^2 + B_j(\mathbf{f})^2}{C_j(\mathbf{f})^2 + D_j(\mathbf{f})^2}.$$

Let $H_j(\mathbf{w}) = C_j(\mathbf{w})^2 + D_j(\mathbf{w})^2$. Then,

$$0 = \sum_{j=1}^{k} \left(\prod_{i \neq j} H_i(\mathbf{f})\right) (A_j(\mathbf{f})^2 + B_j(\mathbf{f})^2).$$

Since $\mathcal{F}$ is algebraically independent, we have that

$$0 = \sum_{j=1}^{k} \left(\prod_{i \neq j} H_i(\mathbf{w})\right) (A_j(\mathbf{w})^2 + B_j(\mathbf{w})^2).$$

Let us assume that there exists j_0 such that either A_{j_0} or B_{j_0} are not zero; say that $A_{j_0} \neq 0$. By construction, since G_i is not zero, we have that $H_i(\mathbf{w})$ is not zero for all i. Let $\mathbf{a} \in \mathbb{R}^k$ such that $A_{j_0}(\mathbf{a})H_1(\mathbf{a}) \cdots H_k(\mathbf{a}) \neq 0$. Then

$$0 = \sum_{j=1}^{k} \left(\prod_{i \neq j} H_i(\mathbf{a})\right) (A_j(\mathbf{a})^2 + B_j(\mathbf{a})^2),$$

which is a contradiction since it is a sum of positive real numnbers, and one of them, namely $\left(\prod_{i \neq j_0} H_i(\mathbf{a})\right) (A_{j_0}(\mathbf{a})^2 + B_{j_0}(\mathbf{a})^2)$, is not zero. So, $A_j = B_j = 0$ for all j. Thus, $P_j = 0$, and hence $F_j = P_j(\mathbf{f}) = 0$ for all j. Now, the result follows from Theorem 4. $\square$

Theorem 19 $(\mathbb{C}(\mathcal{F})(\mathbf{x}), \varphi^{\mathrm{e}})$ *is a Moore–Penrose field.*

Proof: It follows from Theorem 18 and Theorem 12. $\square$

Lemma 20 $\mathbb{C}(\mathcal{F})_\varphi = \mathbb{R}(\mathcal{F})$.

Proof: Let $F/G \in \mathbb{R}(\mathcal{F})$. Say that $F = P(\mathbf{f})$ and $G = Q(\mathbf{f})$ with $P, Q \in \mathbb{R}[\mathbf{w}]$. Then, by Lemma 16, one has that

$$\varphi\left(\frac{F}{G}\right) = \frac{\varphi(P(\mathbf{f}))}{\varphi(Q(\mathbf{f}))} = \frac{P(\mathbf{f})}{Q(\mathbf{f})} = \frac{F}{G}.$$

So, $\mathbb{R}(\mathcal{F}) \subset \mathbb{C}(\mathcal{F})_\varphi$. Conversely, let $F/G \in \mathbb{C}(\mathcal{F})_\varphi$. Then, $\varphi(F/G) = F/G$. We express F and G as $F = P(\mathbf{f}), G = Q(\mathbf{f})$, where $P, Q \in \mathbb{C}[\mathbf{w}]$, and let $P = A + \mathrm{i}B, Q = C + \mathrm{i}D$ with $A, B, C, D \in \mathbb{R}[\mathbf{w}]$. Then, using Lemma 16, we have that

$$\frac{A(\mathbf{f}) + \mathrm{i}B(\mathbf{f})}{C(\mathbf{f}) + \mathrm{i}D(\mathbf{f})} = \frac{P(\mathbf{f})}{Q(\mathbf{f})} = \frac{F}{G} = \varphi\left(\frac{F}{G}\right) = \frac{\varphi(F)}{\varphi(G)} = \frac{A(\mathbf{f}) - \mathrm{i}B(\mathbf{f})}{C(\mathbf{f}) - \mathrm{i}D(\mathbf{f})}.$$

This implies that

$$A(\mathbf{f})D(\mathbf{f}) - B(\mathbf{f})C(\mathbf{f}) = 0.$$

So,

$$\frac{F}{G} = \frac{P(\mathbf{f})}{Q(\mathbf{f})} = \frac{A(\mathbf{f}) + \mathrm{i}B(\mathbf{f})}{C(\mathbf{f}) + \mathrm{i}D(\mathbf{f})} = \frac{(A(\mathbf{f})C(\mathbf{f}) + B(\mathbf{f})D(\mathbf{f})) + \mathrm{i}(-A(\mathbf{f})D(\mathbf{f}) + B(\mathbf{f})C(\mathbf{f}))}{C(\mathbf{f})^2 + D(\mathbf{f})^2}$$

Therefore,

$$\frac{F}{G} = \frac{(A(\mathbf{f})C(\mathbf{f}) + B(\mathbf{f})D(\mathbf{f}))}{C(\mathbf{f})^2 + D(\mathbf{f})^2} \in \mathbb{R}(\mathcal{F}).$$

□

We consider the following map, that transforms a matrix with function entries into a matrix with rational function entries.

$$\begin{array}{rlcl} \mathrm{Rat}: & \mathcal{M}_{m\times n}(\mathbb{C}(\mathcal{F})) & \rightarrow & \mathcal{M}_{m\times n}(\mathbb{C}(\mathbf{w})) \\ & (a_{i,j}(\mathbf{f})) & \mapsto & (a_{i,j}(\mathbf{w})) \end{array}$$

In the following we relate, via the map Rat, the Moore–Penrose inverse of matrices in $\mathcal{M}_{m\times n}(\mathbb{C}(\mathcal{F}))$ to the Moore–Penrose inverse of matrices in $\mathcal{M}_{m\times n}(\mathbb{C}(\mathbf{w}))$. We recall that in $\mathcal{M}_{m\times n}(\mathbb{C}(\mathcal{F}))$ we use the automorphism φ introduced in (4), and in $\mathcal{M}_{m\times n}(\mathbb{C}(\mathbf{w}))$ we take

$$\begin{array}{rrcl} \overline{\cdot}^{\,e}: & \mathbb{C}(\mathbf{w}) & \rightarrow & \mathbb{C}(\mathbf{w}) \\ & R(\mathbf{w}) := \dfrac{\sum_{I=(i_1,\ldots,i_k)} a_I w_1^{i_1}\cdots w_k^{i_k}}{\sum_{I=(i_1,\ldots,i_k)} b_I w_1^{i_1}\cdots w_k^{i_k}} & \longmapsto & \overline{R(z)}^{\,e} = \dfrac{\sum_{I=(i_1,\ldots,i_k)} \overline{a_I} w_1^{i_1}\cdots w_k^{i_k}}{\sum_{I=(i_1,\ldots,i_k)} \overline{b_I} w_1^{i_1}\cdots w_k^{i_k}}. \end{array}$$

Theorem 21 *Let $A \in \mathcal{M}_{m\times n}(\mathbb{C}(\mathcal{F}))$. Then*

$$(\mathrm{Rat}(A)^\dagger)_{|_{\mathbf{w}=\mathbf{f}}} = A^\dagger.$$

Proof: $\mathrm{Rat}(A) \in \mathcal{M}_{m\times n}(\mathbb{C}(\mathbf{w})) \subset \mathcal{M}_{m\times n}(\mathbb{C}(\mathcal{F})(\mathbf{w}))$. By Theorem 19, $(\mathbb{C}(\mathcal{F})(\mathbf{w}), \varphi^{\mathrm{e}})$ is Moore–Penrose and, by Lemma 20, $\mathbb{C}(\mathcal{F})_\varphi = \mathbb{R}(\mathcal{F})$. Now, since $\mathcal{F}$ is algebraically independent, it holds that $\mathrm{Rat}(A)(\mathbf{f})\mathrm{den}(\mathrm{Rat}(A)^\dagger)(\mathbf{f}) \neq 0$, and hence the result follows from Theorem 13. □

In the first part of this section, we have assumed that $\mathcal{F}$ (see (6)) is algebraically independent. This hypothesis has been used in two places. First for the definability of the map Rat, and secondly in the proof of Theorem 19. In the following theorem, we see how to proceed when $\mathcal{F}$ is algebraically dependent.

Theorem 22 *Let $A \in \mathcal{M}_{m\times n}(\mathbb{C}(\mathcal{F}))$, with $\mathcal{F}$ not necessarily algebraically independent. If*

$$\mathrm{den}(\mathrm{Rat}(A)^{\dagger})(\mathbf{f}) \neq 0$$

Then $A^{\dagger}$ exists and

$$(\mathrm{Rat}(A)^{\dagger})_{|_{\mathbf{w}=\mathbf{f}}} = A^{\dagger}.$$

Proof: We observe that $(\mathbb{C}(\mathbf{w}), \varphi_{|_{\mathbb{C}(\mathbf{w})}})$ is Moore–Penrose (see Theorem 12). So, $\mathrm{Rat}(A)^{\dagger}$ exists. $\mathrm{Rat}(A) \in \mathcal{M}_{m\times n}(\mathbb{C}(\mathbf{w})) \subset \mathcal{M}_{m\times n}(\mathbb{C}(\mathcal{F})(\mathbf{w}))$. By Lemma 20, $\mathbb{C}(\mathcal{F})_{\varphi} = \mathbb{R}(\mathcal{F})$. By construction, $\mathrm{den}(\mathrm{Rat}(A))(\mathbf{f}) \neq 0$. So, the result follows from Theorem 15. □

Using the previous results one can derive the following algorithm for computing the Moore–Penrose inverses of matrices with entries in $\mathbb{C}(\mathcal{F})$.

Algorithm 1 Moore–Penrose inverse computation of matrices of functions

Given $\mathcal{F}$ as in (6), not necessarily algebraically independent, and $A \in \mathcal{M}_{m\times n}(\mathbb{C}(\mathcal{F}))$, the algorithm computes the Moore–Penrose inverse $A^{\dagger}$.

1: For $i \in \{1, \ldots, k\}$ replace in A the function $f_i(z)$ by the variable w_i; that is, compute the matrix $\mathrm{Rat}(A)$. Let $B(\mathbf{w}) \in \mathcal{M}_{m\times n}(\mathbb{C}(\mathbf{w}))$ be the resulting matrix of the execution of this step.
2: Compute $B^{\dagger}$.
3: For $i \in \{1, \ldots, k\}$ replace in $B^{\dagger}$ the variable w_i by the function $f_i(z)$. Let $C(\mathbf{f}) \in \mathcal{M}_{m\times n}(\mathbb{C}(\mathcal{F}))$ be the resulting matrix of the execution of this step.
4: If $\mathrm{den}(B^{\dagger})(\mathbf{f}) = 0$ return that the method fails.
5: Return $C(\mathbf{f})$.

Remark 2

1. *In Step 1, we replace the input matrix A, that depends on the meromorphic functions in $\mathcal{F}$, by the matrix B that depends on rational functions. For this purpose, each function in $\mathcal{F}$, appearing in A, is replaced by a new variable. Thus, the entries in B are rational functions that depend on as many variables as functions we have in $\mathcal{F}$. Therefore the running time of the algorithm is affected by the number of different meromorphic functions in $\mathcal{F}$. In Subsection 5.1 we show an empirical test (see Test-2) where this phenomenon is analyzed.*

2. *For the execution of Step 2 of the algorithm see the last part of Section 4, where we have commented different possibilities.*

We illustrate the above ideas with a couple of examples; we recall that $\mathrm{i} = \sqrt{-1}$.

Example 23 *Let $\mathcal{F} = \{\Gamma(z), \cos(z), e^z, \sin(z)\}$. Note that $\mathcal{F}$ is algebraically dependent. We take $\mathbf{w} = (w_1, w_2, w_3, w_4)$ and the matrix*

$$A = \left(\frac{\mathrm{i}}{e^z + \mathrm{i}\sin(z)} \quad \frac{\Gamma(z)}{\cos(z) + \mathrm{i}\sin(z)} \right).$$

In the Step 1 of the Algorithm 1 we get the matrix

$$B = \mathrm{Rat}(A) = \left(\frac{\mathrm{i}}{w_3 + \mathrm{i}w_4}, \frac{w_1}{w_2 + \mathrm{i}w_4} \right).$$

In the Step 2 we get the Moore–Penrose inverse of B

$$B^\dagger = \begin{pmatrix} -\dfrac{(w_2(\mathrm{i}w_3 - w_4))^2 + (w_4(\mathrm{i}w_3 - w_4))^2}{{w_1}^2{w_3}^2 + {w_1}^2{w_4}^2 + {w_2}^2 + {w_4}^2} \\ \dfrac{({w_3}^2 + {w_4}^2)\, w_1 (w_2 + \mathrm{i}w_4)}{{w_1}^2{w_3}^2 + {w_1}^2{w_4}^2 + {w_2}^2 + {w_4}^2} \end{pmatrix}.$$

Since, $\mathrm{den}(B^\dagger)$ does not vanish at $\mathbf{f} = (\Gamma(z), \cos(z), e^z, \sin(z))$, the Moore–Penrose inverse of A is

$$A^\dagger = (B^\dagger)_{|\mathbf{w}=\mathbf{f}} =$$

$$\begin{pmatrix} \dfrac{\mathrm{i}e^z - \sin(z)}{\cos(z)^2\Gamma(z)^2 - \Gamma(z)^2 e^{2z} - \Gamma(z)^2 - 1} \\ \dfrac{\Gamma(z)\left(-\mathrm{i}\cos(z)^2\sin(z) + ie^{2z}\sin(z) - \cos(z)^3 + \cos(z)e^{2z} + i\sin(z) + \cos(z)\right)}{-\cos(z)^2\Gamma(z)^2 + \Gamma(z)^2 e^{2z} + \Gamma(z)^2 + 1} \end{pmatrix}.$$

Example 24 *Let $\mathcal{F} = \{\cos(z), e^z\}$. We take $\mathbf{w} = (w_1, w_2)$ and the matrix*

$$A = \begin{pmatrix} \dfrac{\mathrm{i}\,e^z}{\mathrm{i}\,e^z + \cos(z)} & \dfrac{\mathrm{i}\,e^z}{\cos(z) - \mathrm{i}\,e^z} & \dfrac{\mathrm{i}\,e^z}{\mathrm{i}\,e^z + \cos(z)} & \frac{\mathrm{i}\,e^z}{\cos(z) - \mathrm{i}\,e^z} & \dfrac{\mathrm{i}\,e^z}{\mathrm{i}\,e^z + \cos(z)} \\ \dfrac{\cos(z)e^z + \mathrm{i}\,e^z}{\cos(z) - \mathrm{i}\,e^z} & \dfrac{\cos(z)e^z + \mathrm{i}\,e^z}{\mathrm{i}\,e^z + \cos(z)} & \dfrac{\cos(z)e^z + \mathrm{i}\,e^z}{\cos(z) - \mathrm{i}\,e^z} & \dfrac{\cos(z)e^z + \mathrm{i}\,e^z}{\mathrm{i}\,e^z + \cos(z)} & \dfrac{\cos(z)e^z + \mathrm{i}\,e^z}{\cos(z) - \mathrm{i}\,e^z} \end{pmatrix}.$$

In the Step 1 of the Algorithm 1 we get the matrix

$$B = \mathrm{Rat}(A) = \begin{pmatrix} \dfrac{\mathrm{i}\,w_2}{\mathrm{i}\,w_2 + w_1} & \dfrac{\mathrm{i}\,w_2}{w_1 - \mathrm{i}\,w_2} & \dfrac{\mathrm{i}\,w_2}{\mathrm{i}\,w_2 + w_1} & \dfrac{\mathrm{i}\,w_2}{w_1 - \mathrm{i}\,w_2} & \dfrac{\mathrm{i}\,w_2}{\mathrm{i}\,w_2 + w_1} \\ \dfrac{w_1w_2 + \mathrm{i}\,w_2}{w_1 - \mathrm{i}\,w_2} & \dfrac{w_1w_2 + \mathrm{i}\,w_2}{\mathrm{i}\,w_2 + w_1} & \dfrac{w_1w_2 + \mathrm{i}\,w_2}{w_1 - \mathrm{i}\,w_2} & \dfrac{w_1w_2 + \mathrm{i}\,w_2}{\mathrm{i}\,w_2 + w_1} & \dfrac{w_1w_2 + \mathrm{i}\,w_2}{w_1 - \mathrm{i}\,w_2} \end{pmatrix}.$$

In the Step 2 we get the Moore–Penrose inverse of B

$$B^\dagger = \begin{pmatrix} \frac{-\left({w_1}^2+{w_2}^2\right)\left(\mathrm{i}\,w_2-w_1\right)}{12w_1{w_2}^2} & \frac{-\left({w_1}^2+{w_2}^2\right)\left(\mathrm{i}\,{w_1}^2+\mathrm{i}\,w_2-w_1w_2+w_1\right)}{12{w_2}^2w_1\left({w_1}^2+1\right)} \\ \frac{-\left({w_1}^2+{w_2}^2\right)\left(w_1+\mathrm{i}\,w_2\right)}{8w_1{w_2}^2} & \frac{\left({w_1}^2+{w_2}^2\right)\left(\mathrm{i}\,{w_1}^2-\mathrm{i}\,w_2+w_1w_2+w_1\right)}{8{w_2}^2w_1\left({w_1}^2+1\right)} \\ \frac{-\left({w_1}^2+{w_2}^2\right)\left(\mathrm{i}\,w_2-w_1\right)}{12w_1{w_2}^2} & \frac{-\left({w_1}^2+{w_2}^2\right)\left(\mathrm{i}\,{w_1}^2+\mathrm{i}\,w_2-w_1w_2+w_1\right)}{8{w_2}^2w_1\left({w_1}^2+1\right)} \\ \frac{-\left({w_1}^2+{w_2}^2\right)\left(w_1+\mathrm{i}\,w_2\right)}{8w_1{w_2}^2} & \frac{\left({w_1}^2+{w_2}^2\right)\left(\mathrm{i}\,{w_1}^2-\mathrm{i}\,w_2+w_1w_2+w_1\right)}{8{w_2}^2w_1\left({w_1}^2+1\right)} \\ \frac{-\left({w_1}^2+{w_2}^2\right)\left(\mathrm{i}\,w_2-w_1\right)}{12w_1{w_2}^2} & \frac{-\left({w_1}^2+{w_2}^2\right)\left(\mathrm{i}\,{w_1}^2+\mathrm{i}\,w_2-w_1w_2+w_1\right)}{12{w_2}^2w_1\left({w_1}^2+1\right)} \end{pmatrix}.$$

Since, $\mathrm{den}(B^\dagger)$ *does not vanish at* $\mathbf{f} = (\cos(z), e^z)$*, the Moore–Penrose inverse of* A *is* $A^\dagger = (B^\dagger)_{|_{\mathbf{w}=\mathbf{f}}}$.

5.1 Some empirical tests

In this subsection we execute two different empirical tests to analyse the behaviour of Algorithm 1. For this purpose, we have implemented Algorithm 1 in the computer algebra system Maple. We consider on one hand, the Maple command to compute directly the Moore–Penrose Inverse of matrices in $\mathcal{M}_{m\times n}(\mathbb{C}(\mathcal{F}))$, namely MatrixInverse($A$ method=pseudo), and on the other the implementation of Algorithm 1. In the implementation of Algorithm 1, when we execute Step 2, we apply the Maple command mentioned above.

In the first test, we check the performance when the number of functions in $\mathcal{F}$ is fixed but the order of the matrix increases. However, in second test, we fix the order of the matrix and we increase the number of functions. In both cases, the results are satisfactory.

All executions were done on a Intel(R) Core(TM) i7-6600U with 2.60GHz and 16,0 GB RAM using Maple 18.

We recall that i denotes the imaginary unit, that is $\mathrm{i} = \sqrt{-1}$.

Test-1. In this test, we fix the number of functions appearing in the input matrix to 4, that is $\#(\mathcal{F}) = 4$, and we increase the order of the matrix from 1 to 8. More precisely, we take $\mathcal{F} = \{\cos(z), \sinh(z), \cosh(z), e^z\}$, and we consider the matrix

$$A = \left(\frac{\cos(z)\sinh(z)\cosh(z)e^{zj_1} + \mathrm{i}(1+\sinh(z))}{\cos(z)\cosh(z) + \mathrm{i}(1+e^{zj_2})}\right)_{1\le j_1,j_2\le 8} \in \mathcal{M}_{8\times 8}(\mathbb{C}(\mathcal{F})).$$

Then, we run Algorithm 1 for computing $A^\dagger_{k_1k_2}$ for all the $k_1 \times k_2$ principal submatrices $A_{k_1k_2}$ of A, where $k_1, k_2 \in \{1,\ldots,8\}$ (see Table 1). Alternatively we have directly executed the Maple command for computing $A^{\dagger}_{k_1 k_2}$ (see Table 2), such that if the execution time is bigger than 30 minutes, the computation is stopped.

$A^{\dagger}_{k_1k_2}$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$	$k_2 = 4$	$k_2 = 5$	$k_2 = 6$	$k_2 = 7$	$k_2 = 8$
$k_1 = 1$	0.046	0.016	0.031	0.062	0.125	0.202	0.484	0.717
$k_1 = 2$	0.016	0.172	0.281	0.702	1.685	3.323	6.599	16.349
$k_1 = 3$	0.031	0.047	0.702	1.934	4.259	8.221	18.876	45.459
$k_1 = 4$	0.015	0.063	0.171	3.323	8.393	15.491	36.395	88.639
$k_1 = 5$	0.016	0.078	0.172	0.561	13.494	25.897	59.358	151.353
$k_1 = 6$	0.032	0.078	0.187	0.577	1.669	38.937	89.857	220.304
$k_1 = 7$	0.015	0.109	0.266	0.624	1.809	4.462	121.728	310.473
$k_1 = 8$	0.031	0.094	0.312	0.671	1.872	4.571	12.121	417.272

Table 1: Time in seconds of the execution of Algorithm 1 to get $A^{\dagger}_{k_1 k_2}$ in Test 1.

$A^{\dagger}_{k_1k_2}$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$	$k_2 = 4$	$k_2 = 5$	$k_2 = 6$	$k_2 = 7$	$k_2 = 8$
$k_1 = 1$	0.280	0.749	2.449	6.194	10.218	16.614	33.540	85.957
$k_1 = 2$	0.437	2.979	10.983	38.111	83.117	146.563	$> 30m$	$> 30m$
$k_1 = 3$	0.234	0.250	126.330	$> 30m$	$> 30m$	$> 30m$	$> 30m$	$> 30m$
$k_1 = 4$	0.577	0.733	1.326	$> 30m$	$> 30m$	$> 30m$	$> 30m$	$> 30m$
$k_1 = 5$	0.561	0.640	1.404	3.416	$> 30m$	$> 30m$	$> 30m$	$> 30m$
$k_1 = 6$	0.452	0.780	1.310	3.557	11.466	$> 30m$	$> 30m$	$> 30m$
$k_1 = 7$	0.453	0.858	1.451	3.869	13.322	41.325	$> 30m$	$> 30m$
$k_1 = 8$	0.546	0.905	1.560	4.883	14.664	47.253	114.005	$> 30m$

Table 2: Time in seconds of the direct execution of the Maple command to get $A^{\dagger}_{k_1 k_2}$ in Test 1.

We observe that all executions in Algorithm 1 took less than 7 minutes, and all but 5 cases took less than 1.5 minutes. However, the direct application of the Maple command failed (i.e. was stopped after 30 minutes) when the orders were not small.

Test-2. In this test, we fix the order of the matrix, and we increase the number of functions. More precisely,

$$\left\{ A_n := \begin{pmatrix} \dfrac{\cos(z) + k_1 \mathrm{i} \sum_{j=1}^{n} e^{z^j}}{k_2 \sin(z) + \mathrm{i}\, e^z} \end{pmatrix}_{1 \le k_1 \le 4, 1 \le k_2 \le 5} \right\}_{n \in \mathbb{N}}.$$

So, $A_n \in \mathcal{M}_{4\times 5}(\mathcal{F})$, where $\#(\mathcal{F}) = n + 2$. In this situation, we apply Algorithm 1 (see Table 3), as well as the Maple direct command (see Table 4), to compute $\{A^{\dagger}_n\}_{n\in\mathbb{N}}$ till the time of execution is bigger than 20 minutes. We observe that Algorithm 1 runs under 20 minutes till order 9, and hence with rational functions involving at most 11 variables. However, the direct application of the Maple command run till order 3.

$A_n^\dagger$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n\geq 10$
	0.858	1.669	3.962	7.722	7.286	31.059	69.858	497.580	911.483	$> 20m$

Table 3: Time in seconds of the execution of Algorithm 1 to get $A_n^\dagger$ in Test 2.

$A_n^\dagger$	$n=1$	$n=2$	$n=3$	$n\geq 4$
	110.449	237.824	512.042	$> 20m$

Table 4: Time in seconds of the direct execution of the Maple command to get $A_n^\dagger$ in Test 2.

6 Conclusions

We have introduced the notion of Moore–Penrose field and we have seen that the characteristic of a Moore–Penrose field has to be zero, and that the field of rational functions preserves the property of being Moore–Penrose (see Corollary 5 and Theorem 12). From the computational point of view we have found a criterium to guarantee that the specialization of a matrix, with rational functions, commute with its Moore–Penrose inverse (see Theorems 13 and 15). In addition, the applications of these results provide a symbolic algorithm to reduce the computation of the Moore–Penrose inverse of matrices, whose entries are rational expression of finitely many self-adjoint meromorphic functions, to the case of the Moore-Penrose inverse of matrices with complex rational functions entries. As future work one may deal with the question of speeding up the running time when the order of the matrix or the number of variables in the rational functions is not small, for instance with interpolation techniques, or to extend the current results to matrices with rational functions with coefficients on other fields.

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