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Bounds on spectrum graph coloring

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Abstract

We propose two vertex-coloring problems for graphs, endorsing the spectrum of colors with a matrix of interferences between pairs of colors. In the Threshold Spectrum Coloring problem, the number of colors is fixed and the aim is to minimize the maximum interference at a vertex (interference threshold). In the Chromatic Spectrum Coloring problem, a threshold is settled and the aim is to minimize the number of colors (among the available ones) for which respecting that threshold is possible. We prove general upper bounds for the solutions to each problem, valid for any graph and any matrix of interferences. We also show that both problems are NP-hard and perform experimental results, proposing a DSATUR-based heuristic for each problem, in order to study the gap between the theoretical upper bounds and the values obtained.

Keywords: graph coloring, threshold spectrum coloring, chromatic spectrum coloring, DSATUR, frequency assignment, WiFi channel assignment

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1 Introduction

Graph coloring is one of those problems in Discrete Mathematics appealing both mathematicians and engineers [6,8], with frequency assignment being one of its prominent applications [1]. Inspired by WiFi channel assignment, we consider an abstract undirected graph G together with a spectrum of colors $S = \{c_1, \dots, c_s\}$ (representing available channels) for which we have a symmetric matrix W of non-negative distances $W_{ij} = W(c_i, c_j)$ between each pair of colors (representing interferences between each pair of channels). Thus, a coloring c of the graph induces an *interference* at each vertex v:

$$I_v(G, W, c) = \sum_{u \in N(v)} W(c(u), c(v)).$$

In our work [4] we have applied succesfully this model to find efficient frequency assignments in Wireless Networks. In the present work, we introduce the Threshold Spectrum Coloring (TSC) problem, which considers a graph G and a spectrum of k colors endorsed with a $k \times k$ matrix W of interferences between them, with the goal of determining the minimum threshold $t \in \mathbb{R}_{\geq 0}$ such that (G, W) admits a k-coloring c in which the interference at every vertex is at most t, i.e., $I_v(G, W, c) \leq t, \ \forall v$. Such a minimum t will be called the *minimum k-chromatic threshold* of (G, W), denoted as $T_k(G, W)$.

We also introduce the counterpart Chromatic Spectrum Coloring (CSC) problem, which considers a threshold $t \in \mathbb{R}_{\geq 0}$, letting the size of the spectrum to be the number $|V(G)|$ of vertices, with the goal of determining the minimum number of colors $k \in \mathbb{N}$ such that (G, W) admits a k-coloring c in which the interference at every vertex is at most that threshold t. Such a minimum k will be called the *t-interference chromatic number* of (G, W), denoted as $\chi_t(G, W)$.

Figure 1 illustrates the TSC problem for the paw graph PG with a spectrum of $k = 3$ colors endorsed with a matrix W_{2ed} in which interferences decay exponentially with base 2, leading to $T_3(PG, W_{2ed}) = 1$. For the CSC problem, the corresponding 4×4 matrix and interference threshold $t = 1$ lead to $\chi_1(PG, W_{2ed}) = 3$.

As for related work, Araujo et al. [2] consider a weight function w on the edges of the graph G, instead of a matrix W of interferences between colors, defining the interference at a vertex to be the sum of weights of incident monochromatic edges. Many other works impose conditions to the colors of the endpoints of any edge, many of which can be framed into $L(p_1, \dots, p_k)$-*labellings* [3], where vertices at distance i must get colors at distance $\geq p_i$.

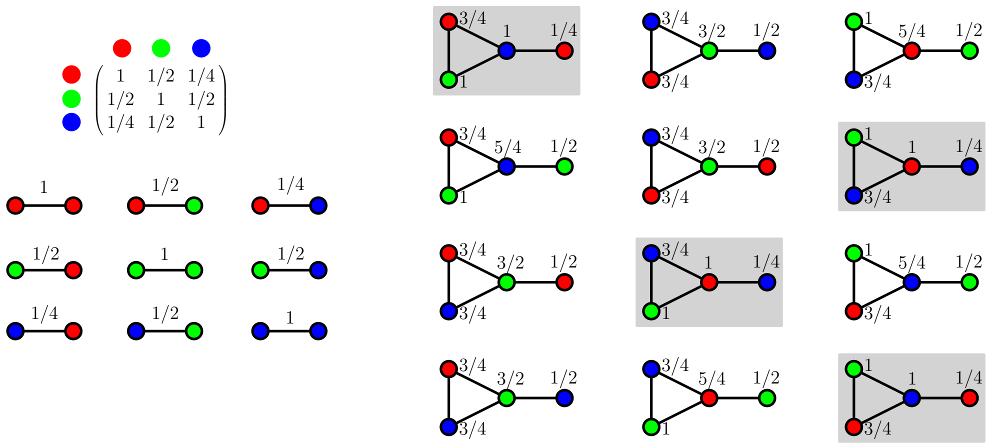

Fig. 1. Left: Matrix W_{2ed} and induced interferences on the possible colorings of edges. Right: Proper 3-colorings of PG and the interferences they induce at every vertex. Highlighted colorings achieve interference ≤ 1 at every vertex.

2 Theoretical results

Theorem 2.1 *The* Chromatic Spectrum Coloring *and the* Threshold Spectrum Coloring *problems are NP-hard.*

Proof. The NP-hard [5] standard Vertex Coloring (VC) problem is a particular case of our CSC problem, using the identity matrix I and any threshold $t < 1$. Furthermore, for any fixed $k \geq 2$ it is NP-complete to decide if there exists a k-coloring achieving a threshold $t \geq 2$, see [2] and the references therein. Hence, our TSC problem is also NP-hard. □

Given a graph G, a matrix W, and a coloring c, we denote the *potential interference* at vertex v if this was colored i as $I_v^i(G, W, c) = \sum_{u \in N(v)} W(c(u), i)$.

A k-coloring c of G will be said to be W*-balanced* if, for every vertex, the actual interference is not greater than any of the potential interferences, i.e., if for every vertex v we have $I_v(G, W, c) \leq I_v^j(G, W, c)$ for all $j \in \{1, \ldots, k\}$.

Proposition 2.2 *Given a graph G and a spectrum S of size $|S| \geq 2$ endorsed with a matrix W of interferences, there exists a W-balanced coloring of G.*

Proof. Let $I = \sum_{uv \in E} W(c(u), c(v))$ be the sum of edge-interferences for the current coloring c. Observe that any coloring achieving the minimum for this sum will be a W-balanced coloring of G. Such a balanced coloring can be found by a random greedy algorithm: Start with a random coloring of the graph $G = (V, E)$ and, if there exist a vertex $v \in V$ and a color $j \in S$ such that the potential interference $I_v^j(G,W,c)$ for that color is smaller than the current interference $I_v^{c(v)}(G,W,c)$, then recolor the vertex v with color j and repeat. At each step of the procedure, I is increased by $I_v^j(G,W,c)$ and decreased by $I_v^{c(v)}(G,W,c)$ (with c being the initial coloring before the step). Hence, the sum of edge-interferences I decreases by a positive amount $I_v^{c(v)}(G,W,c) - I_v^j(G,W,c)$ and the algorithm ends giving a W-balanced coloring. □

Lemma 2.3 *Any W-balanced k-coloring c of a graph G fulfills that, for each vertex v, $k\, I_v^{c(v)}(G,W,c) \leq \deg(v)\, ||W||_\infty$, where $||W||_\infty = \max_i \sum_j W_{ij}$ is the natural norm of the matrix W.*

Proof. Such a coloring fulfills $k\, I_v^{c(v)}(G,W,c) \leq \sum_{j=1}^k I_v^j(G,W,c)$, since c balanced implies $I_v^{c(v)}(G,W,c) \leq I_v^j(G,W,c)$ for all j. The result follows restating as $\sum_{u\in N(v)|c(u)=1} \sum_j W_{1j} + \cdots + \sum_{u\in N(v)|c(u)=k} \sum_j W_{kj}$ the right-hand side. □

Theorem 2.4 *Given a graph G and a spectrum S of size $|S|$ endorsed with a matrix W of interferences, for any fixed natural number $2 \leq k \leq |S|$, the following bound holds for the minimum k-chromatic threshold $T_k(G,W)$:*

$$T_k(G,W) \leq \frac{\Delta(G)\ ||W||_\infty}{k},$$

where $\Delta(G)$ is the maximum vertex-degree in the graph G. Furthermore, there are instances of k, G, W for which the bound is tight.

Proof. We prove the existence of a coloring c_0 of G with k colors for which the interference at every vertex does not exceed the threshold $t_0 = \Delta(G)\ ||W||_\infty/k$.

By Proposition 2.2 and $k \geq 2$, there exists a W-balanced coloring c_0 of G using k colors. This c_0 fulfills the interference condition above since

$$k\, T_k(G,W) \leq k\, I_v^{c_0(v)}(G,W,c_0) \leq \deg(v)\, ||W||_\infty,$$

holds for every v with maximum interference: The leftmost inequality follows from $T_k(G,W)$ being minimum and the rightmost inequality comes from Lemma 2.3. For the final claim, considering $k=2$ and $W=I$ our bound becomes $T_2(G,W) \leq \lceil \Delta(G)/2 \rceil$, which is tight for a cycle of odd length. □

Theorem 2.5 *Given a graph G and a spectrum S of size $|S| \geq 2$ endorsed with a matrix W of interferences, for any fixed threshold t being a multiple of $\gcd(W)$ and such that $t \geq \Delta(G)\ ||W||_\infty - \gcd(W)(|S|-1)/|S|$ the follow-*

ing bound holds for the t-interference chromatic number $\chi_t(G, W)$*:*

$$\chi_t(G, W) \le \left\lceil \frac{\Delta(G)\ ||W||_\infty + \gcd(W)}{t + \gcd(W)} \right\rceil,$$

where the gcd of non-integer numbers is defined with x *dividing* y *if* $y/x \in \mathbb{Z}$*. Furthermore, there are instances of* t, G, W *for which the bound is tight.*

Proof. The number of colors has to be at least one and the bound is trivially fulfilled for $\chi_t(G, W) = 1$, so let us focus on the case $\chi_t(G, W) > 1$. We prove that there exists a coloring c_0 of G with $k_0 = \left\lceil \frac{\Delta(G)\ ||W||_\infty + \gcd(W)}{t+\gcd(W)} \right\rceil \le |S|$ colors for which the interference at every vertex does not exceed the threshold t.

By Proposition 2.2 and $k_0 \ge 2$, there exists a W-balanced coloring c_0 of G using k_0 colors. By contradiction, suppose that for the coloring c_0 there is a vertex v in G with an interference above the threshold t, i.e., $I_v^{c_0(v)}(G, W, c_0) > t$ which, because of t and the interferences around a vertex being multiples of $\gcd(W)$, implies $I_v^{c_0(v)}(G, W, c_0) \ge t + \gcd(W)$. Lemma 2.3 leads to $\Delta(G)\,||W||_\infty \ge \deg(v)\,||W||_\infty \ge k_0\, I_v^{c_0(v)}(G, W, c_0) \ge k_0\,(t + \gcd(W))$ and our choice of k_0 implies the contradiction $\Delta(G)\,||W||_\infty \ge \Delta(G)\ ||W||_\infty + \gcd(W)$. For the final claim, considering $W = I$ and $t = 0$ our bound becomes $\chi_0(G, I) = \chi(G) \le \Delta(G) + 1$, which is tight for a cycle of odd length. □

3 Experimental results

We have generated 10 Erdős-Renyi random graphs for each combination of $n \in \{60, 70, 80\}$ and $p \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and we have taken the matrix of interferences as the W_{2ed} above. For the TSC problem we have chosen $k \in \{4, 6, 11\}$ and for the CSC problem $t \in \{np/4, np/2, 3np/4\}$. With 20 repetitions of each experiment, we have developed and tested DSATUR-inspired heuristics for TSC and CSC, together with a nonlinear optimizer. Figure 2 shows, for each setting, the average gap between our theoretical upper bounds and the best values obtained in the experiments, expressed as a percentage of the bound. The interested reader can find details in our work [7].

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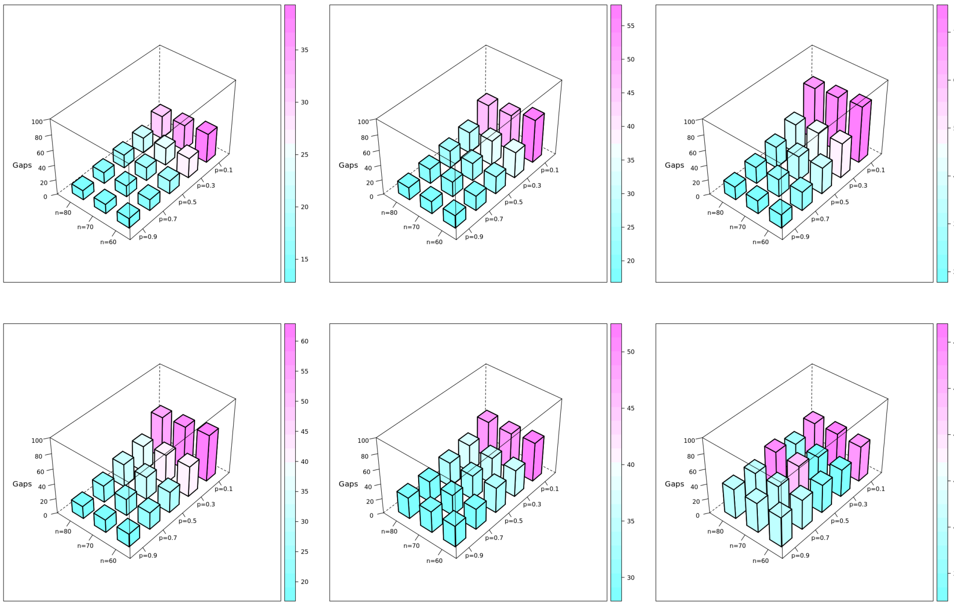

Fig. 2. Top (left-to-right): TSC for $k \in \{4, 6, 11\}$. Bottom (left-to-right): CSC for $t \in \{np/4, np/2, 3np/4\}$.

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