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Ultraquadrics associated to affine and projective automorphisms*

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Abstract

The concept of ultraquadric has been introduced by the authors as a tool to algorithmically solve the problem of simplifying the coefficients of a given rational parametrization in $\mathbb{K}(\alpha)(t_1, \dots, t_n)$ of an algebraic variety of arbitrary dimension over a field extension $\mathbb{K}(\alpha)$. In this context, previous work in the one-dimensional case has shown the importance of mastering the geometry of 1-dimensional ultraquadrics (hypercircles). In this paper we study, for the first time, the properties of some higher dimensional ultraquadrics, namely, those associated to automorphisms in the field $\mathbb{K}(\alpha)(t_1, \dots, t_n)$, defined by linear rational (with common denominator) or by polynomial (with inverse also polynomial) coordinates. We conclude, among many other observations, that ultraquadrics related to polynomial automorphisms can be characterized as varieties $\mathbb{K}-$isomorphic to linear varieties, while ultraquadrics arising from projective automorphisms are isomorphic to the Segre embedding of a blowup of the projective space along an ideal and, in some general case, linearly isomorphic to a toric variety. We conclude with some further details about the real-complex, 2-dimensional case, showing, for instance, that this family of ultraquadrics can be presented as a collection of ruled surfaces described by pairs of hypercircles.

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1 Introduction

The study and analysis of ultraquadrics was introduced in [2] as a higher dimensional generalization of the concept of hypercircle (cf. [1], [8], [9], [10], [14], [15]) and as a fundamental tool to solve the problem of the optimal algebraic reparametrization of rational varieties of arbitrary dimension (e.g. rational surfaces, see [3] and [4] for applications to some families of surfaces relevant in computer aided design).

Given a rational variety $\mathcal{V}$, presented by a rational parametrization with n parameters $t_1, \ldots, t_n$ and coefficients in a certain extension $\mathbb{K}(\alpha)$ of a ground field $\mathbb{K}$, it is natural to ask for the possibility of reparametrizing $\mathcal{V}$ over $\mathbb{K}$ (i.e. the problem of the $\mathbb{K}$-algebraic optimality for unirational varieties). The search for parametrizations with optimal coefficients has been studied by other authors from different perspectives. For instance [11] analyzes the complex-real case for surfaces given implicitly, and [12] approached the same problem for the implicit curve case. A natural context for this type of problems is the field of computer aided design, where rational surfaces, and their parametrizations, are usually required to be defined over the reals; different examples of this statement can be found, for instance, in the papers [6], [7]. The optimality of parametrization coefficients is also specifically sought in some concrete applications, such as computing quadrics intersection [5].

In our case, within this research line, we focus on the case of parametrically given varieties. For this purpose the paper [2] introduces the concept of "ultraquadrics" as varieties associated to automorphisms of the field $\mathbb{K}(\alpha)(t_1, \ldots, t_n)$, and describes its application to the reparametrization of $\mathcal{V}$ over $\mathbb{K}$, when possible. The reparametrization problem, in the case of ruled and swung surfaces, two families of surfaces of interest in CAD, has already been successfully addressed ([3] and [4]) in the context of our theory of ultraquadrics, in each case by developing some "ad hoc" methods.

Now, in the case of dimension one varieties, i.e. when ultraquadrics receive the specific name of hypercircles (cf. [8]), increasingly effective algorithms to simplify the given parametrization came by hand of a deeper understanding of the geometry of hypercircles. Likewise, we believe that the detailed study of ultraquadrics associated to specific families of automorphisms should provide a similar understanding and, therefore, will allow the design of more efficient and systematic algorithms for dealing with these varieties in the context of the search for an optimal reparametrization of a given variety.

Thus, in this paper, we study the ultraquadrics associated to some important kind of automorphisms in the field $\mathbb{K}(\alpha)(t_1, \ldots, t_n)$, such as those defined by linear rational (with common denominator) or polynomial (with inverse also polynomial) coordinates. After introducing (Section 2) the main notation and general properties of ultraquadrics, we analyze (Section 3) ultraquadrics related to polynomial automorphisms, yielding its characterization as varieties $\mathbb{K}-$isomorphic to linear varieties (cf. Theorem 3.6).

Section 4 is devoted to ultraquadrics derived from linear fractional automorphisms with a common denominator, concluding that, projectively speaking, these ultraquadrics are isomorphic to the Segre embedding of the projective space along some precise ideal (see Theorem 4.1); in particular, the affine part of such ultraquadric is always smooth and, in some general case, linearly isomorphic to a toric variety. Section 4 concludes with some further details about the real-complex, 2-dimensional case. In particular, this family of ultraquadrics is presented as a collection of ruled surfaces described by means of some hypercircles (Theorem 4.9).

2 Notation and Preliminaries

In this section we introduce the main notation used throughout the paper and we recall the basic notion and properties of ultraquadrics.

2.1 Notation

In the sequel, $\mathbb{K}$ is a field of characteristic zero, α is an algebraic element over $\mathbb{K}$, $\mathbb{L}$ is the field extension $\mathbb{K}(\alpha)$ and $\mathbb{F}$ is the algebraic closure of $\mathbb{L}$. So $\mathbb{K} \subset \mathbb{L} = \mathbb{K}(\alpha) \subset \mathbb{F}$. We assume that $[\mathbb{K} : \mathbb{L}] = r$. We use the notation

$$\bar{t} = (t_1, \dots, t_n) \text{ and } \overline{T} = (t_0 : \dots : t_n)$$

for affine –respectively, projective– coordinates.

On the other hand, we will consider the following three groups of automorphisms under composition:

1. $\mathbf{B}_{\mathbb{L}}$ is the group of all $\mathbb{L}$-birational transformations (i.e. $\mathbb{L}$-definable) of $\mathbb{F}^n$ onto $\mathbb{F}^n$.

2. $\mathbf{A}_{\mathbb{L}}$ is the group of all $\mathbb{L}$-automorphism of the affine space $\mathbb{F}^n$; that is, the subgroup of $\mathbf{B}_{\mathbb{L}}$ where the transformation and its inverse are both described through polynomial coordinates.

3. $\mathbf{PGL}_{\mathbb{L}}(n)$ is the group of all $\mathbb{L}$-automorphism of the projective space $\mathbb{P}^n(\mathbb{F})$. Elements in $\mathbf{PGL}_{\mathbb{L}}(n)$ are represented by a $(n+1) \times (n+1)$ regular matrix L

$$\mathbb{P}^n(\mathbb{F}) \to \mathbb{P}^n(\mathbb{F}); \overline{T} \mapsto L \cdot (\overline{T}^{\,t}) = [L_0(\overline{T}) : \cdots : L_n(\overline{T})] \qquad (1)$$

   where the rows L_i of L represent linear forms.

In addition, let $\mathbf{B}_{\mathbb{K}}$ be the group of all $\mathbb{K}$-birational transformations of $\mathbb{F}^n$ onto $\mathbb{F}^n$. We consider the following binary relation in $\mathbf{B}_{\mathbb{L}}$: for $\Psi_1, \Psi_2 \in \mathbf{B}_{\mathbb{L}}$, we say that $\Psi_1 \mathcal{R} \Psi_2$ iff there exists $\phi \in \mathbf{B}_{\mathbb{K}}$ such that $\Psi_1 \circ \phi = \Psi_2$. We observe that $\mathcal{R}$ is in fact an equivalence relation, and we denote by $[\Phi]$ the equivalence class of $\Phi \in \mathbf{B}_{\mathbb{L}}$.

2.2 Ultraquadrics

Let us start with the notion of hypercircle; for further details on hypercircles see [8]. Let Φ be a $\mathbb{L}$-birational map from $\mathbb{F}$ onto $\mathbb{F}$. Then, if we denote by u a generic point in $\mathbb{F}$, the transformation $\Phi(u)$ is described by a linear rational function with coefficients in $\mathbb{L}$. That is

$$\Phi(u) = \frac{au+b}{cu+d} \text{ where } ad-cb \neq 0.$$

Now, express Φ in the basis $\{1,\ldots,\alpha^{r-1}\}$ of the algebraic extension, as follows

$$\Phi(u) = \phi_0(u) + \cdots + \alpha^{r-1}\phi_{r-1}(u),$$

where $\phi_i \in \mathbb{K}(u)$. We define the hypercircle associated with Φ, and we denote it by $\mathrm{Hyper}(\Phi)$, as the rational curve of $\mathbb{F}^r$ parametrized by $(\phi_0(u),\ldots,\phi_{r-1}(u))$. Furthermore, we denote by $\mathrm{H}(\Phi)$ the parametrization $(\phi_0(u),\ldots,\phi_{r-1}(u))$ of $\mathrm{Hyper}(\Phi)$.

A similar construction can be done when the $\mathbb{L}$-birational map is taken from $\mathbb{F}^n$ onto $\mathbb{F}^n$ yielding to the notion of ultraquadrics (see [2] for further details). More precisely, let $\Psi = (\psi_1,\ldots,\psi_n) \in \mathbf{B}_{\mathbb{L}}$. Then, we express Ψ in the basis $\{1,\ldots,\alpha^{r-1}\}$ as

$$\Psi(\bar{t}\,) = \left(\sum_{j=0}^{r-1}\psi_{1,j}\alpha^j,\ldots,\sum_{j=0}^{r-1}\psi_{n,j}\alpha^j\right),$$

where $\psi_{ij} \in \mathbb{K}(\bar{t}\,)$. Then, using this notation, we consider the expansion map

$$\begin{array}{rlll} \mathrm{U}: & \mathbf{B}_{\mathbb{L}} & \rightarrow & \mathbb{K}(\bar{t}\,)^{nr} \\ & \Psi(\bar{t}\,) & \mapsto & \mathrm{U}(\Psi(\bar{t}\,)) = \\ & & & (\psi_{10}(\bar{t}\,),\ldots,\psi_{1(r-1)}(\bar{t}\,),\ldots,\psi_{n0}(\bar{t}\,),\ldots,\psi_{n(r-1)}(\bar{t}\,)) \end{array} \tag{2}$$

We define the ultraquadric associated with Ψ, and we denote it by $\mathrm{Ultra}(\Psi)$, as the rational variety of $\mathbb{F}^{nr}$ parametrized by $\mathrm{U}(\Psi(\bar{t}\,))$.

If $\Psi \in \mathbf{PGL}_{\mathbb{L}}(n)$, say $\Psi(\overline{T}\,) = [L_0(\overline{T}\,) : \ldots : L_n(\overline{T}\,)]$, we will denote as $\mathrm{Ultra}(\Psi)$ the (affine) ultraquadric generated by the associated affine mapping

$$\Psi_a(\bar{t}\,) = \left(\frac{L_1(1,t_1,\ldots,t_n)}{L_0(1,t_1,\ldots,t_n)},\ldots,\frac{L_n(1,t_1,\ldots,t_n)}{L_0(1,t_1,\ldots,t_n)}\right) \tag{3}$$

That is, $\mathrm{Ultra}(\Psi) = \mathrm{Ultra}(\Psi_a)$.

Note that $\mathrm{Ultra}(\Psi)$ is the same variety for all maps in $[\Psi]$. So, we will write either $\mathrm{Ultra}(\Psi)$ or $\mathrm{Ultra}([\Psi])$. Furthermore, we observe that

$$\mathbb{F}(\bar{t}\,) = \mathbb{F}(\Psi(\bar{t}\,)) \subset \mathbb{F}(\mathrm{U}(\Psi(\bar{t}\,)) \subset \mathbb{F}(\bar{t}\,).$$

So, $\mathbb{F}(\mathrm{U}(\Psi(\bar{t}\,))) = \mathbb{F}(\bar{t}\,)$. In addition, it also holds that $\mathbb{F}(\bar{t}\,) = \mathbb{F}(\mathrm{U}(\Psi^*(\bar{t}\,)))$ for all $\Psi^* \in [\Psi]$. Thus we have the following result.

Lemma 2.1. *Let $\Psi \in \mathbf{B}_{\mathbb{L}}$. For every $\Psi^* \in [\Psi]$, $\mathrm{U}(\Psi^*(\bar{t}\,))$ is a proper parametrization of* $\mathrm{Ultra}(\Psi^*)$.

This, in particular, implies that $\dim(\mathrm{Ultra}(\Psi)) = n$. Furthermore, the following lemma holds.

Lemma 2.2. *Let $\mathcal{P}(\bar{t}\,) = (P_{10}, \ldots, P_{1(r-1)}, \ldots, P_{n0}, \ldots, P_{n(r-1)})$ be a $\mathbb{K}$-definable proper parametrization of an ultraquadric* $\mathrm{Ultra}(\Psi)$. *Then,*

$$\mathcal{Q}(\bar{t}\,) := \left(\sum_{j=0}^{r-1} P_{1,j}\alpha^j, \ldots, \sum_{j=0}^{r-1} P_{n,j}\alpha^j \right) \in [\Psi] \subseteq \mathbf{B}_{\mathbb{L}}.$$

Proof. Observe that Ψ, $\mathrm{U}(\Psi)$ (see Lemma 2.1) and $\mathcal{P}$ are invertible; the inverse map goes, respectively, from $\mathbb{F}^n$ to $\mathbb{F}^n$ and from $\mathrm{Ultra}(\Psi)$ to $\mathbb{F}^n$. Moreover, Ψ^{-1} is $\mathbb{L}$-definable and $\mathcal{P}^{-1}$ and $\mathrm{U}(\Psi)^{-1}$ are $\mathbb{K}$-definable. Furthermore, since $\mathrm{U}(\Psi)$ and $\mathcal{P}$ are proper parametrizations over $\mathbb{K}$ of the same variety, there exist automorphisms $R, S \in \mathbb{K}(\bar{t}\,)$ such that

$$\mathcal{P}(S(\bar{t}\,)) = \mathrm{U}(\Psi)(\bar{t}\,),\ \mathrm{U}(\Psi)(R(\bar{t}\,)) = \mathcal{P}(\bar{t}\,);$$

indeed, $S = \mathcal{P}^{-1} \circ \mathrm{U}(\Psi)$ and $R = \mathrm{U}(\Psi)^{-1} \circ \mathcal{P}$. In addition, since $R, S \in \mathbb{K}(\bar{t}\,)$ one has that $\mathcal{Q}(S(\bar{t}\,)) = \Psi(\bar{t}\,)$, $\Psi(R(\bar{t}\,)) = \mathcal{Q}(\bar{t}\,)$ and $\mathcal{Q} \in [\Psi] \subseteq \mathbf{B}_{\mathbb{L}}$. $\square$

Finally, we recall the relationship from $\mathrm{U}(\Psi)$ to the conjugate parametrizations of Ψ (see [2]). Let $\alpha = \alpha_1$ and let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be the conjugates of α over $\mathbb{K}$ in $\mathbb{F}$. And let $\sigma_1 = \mathrm{Id}, \sigma_2, \ldots, \sigma_r$ be $\mathbb{K}$-automorphims of $\mathbb{F}$ such that $\sigma_i(\alpha) = \alpha_i$. Let $\Psi = (\psi_1, \ldots, \psi_n) \in \mathbf{B}_{\mathbb{L}}$ and $\mathrm{U}(\Psi)$ the parametrization of $\mathrm{Ultra}(\Psi)$ (see (2)). Let $\mathcal{V}$ be the variety of $\mathbb{F}^{nr}$ parametrized by $\Psi(\bar{t}\,) \times \Psi^{\sigma_2}(\bar{t}\,) \times \cdots \times \Psi^{\sigma_r}(\bar{t}\,)$, where Ψ^{σ_i} denotes the conjugate birational map that is obtained from Ψ by substituting α by α_i in Ψ. Then it is easy to conclude the following.

Lemma 2.3. $\mathrm{Ultra}(\Psi)$ *and the variety $\mathcal{V}$ of $\mathbb{F}^{nr}$ parametrized by $\Psi(\bar{t}\,) \times \Psi^{\sigma_2}(\bar{t}\,) \times \cdots \times \Psi^{\sigma_r}(\bar{t}\,)$, are $\mathbb{L}$-isomorphic by the linear transformation induced by the Vandermonde matrix*

$$\begin{pmatrix} \psi_1 & \ldots & \psi_n \\ \psi_1^{\sigma_2} & \ldots & \psi_n^{\sigma_2} \\ & \ldots & \\ \psi_1^{\sigma_r} & \ldots & \psi_n^{r-1} \end{pmatrix} = \begin{pmatrix} 1 & \alpha & \ldots & \alpha^{r-1} \\ 1 & \alpha_2 & \ldots & \alpha_2^{r-1} \\ & & \ldots & \\ 1 & \alpha_r & \ldots & \alpha_r^{r-1} \end{pmatrix} \begin{pmatrix} \psi_{10} & \ldots & \psi_{n0} \\ \psi_{11} & \ldots & \psi_{n1} \\ & \ldots & \\ \psi_{1(r-1)} & \ldots & \psi_{n(r-1)} \end{pmatrix}$$

where the ψ_{ij} are as in (2).

3 Ultraquadrics associated to $\mathbf{A}_{\mathbb{L}}$

In this section we analyze the properties of ultraquadrics associated to automorphisms from $\mathbf{A}_{\mathbb{L}}$. For hypercircles (i.e. one-dimensional ultraquadrics) we proved in [8] that being a hypercircle defined by a polynomial automorphism is equivalent to being defined by an automorphism defined by a linear polynomial and is also equivalent to being a $\mathbb{K}$-definable line; indeed the $\mathbb{K}$-parametrizable line $(at+b, ct+d)$ is the hypercircle $\mathrm{Hyper}((a+\alpha c)t+(b+\alpha d))$. Thus, before dealing with the central question of this section, it is natural to analyze whether every $\mathbb{K}$-definable linear variety of dimension n in $\mathbb{F}^{nr}$ –a n-plane– is an ultraquadric (for a suitable algebraic element α of degree r).

Contrary to the one-dimensional case, we conclude here that this fact, in general, is not true. Let us provide a simple example. We take $\alpha = \mathrm{i}$ (the imaginary unit), $\mathbb{K} = \mathbb{R}$, and we consider the real plane in $\mathbb{C}^4$

$$\mathcal{P}(\bar{t}\,) = (1,1,1,5)s + (1,-1,5,-1)t.$$

$\mathcal{P}(\bar{t}\,)$ is a real proper parametrization of a plane but it can not parametrize an i-ultraquadric since (notation as in Lemma 2.2) $\mathcal{Q}(\bar{t}\,) = ((1+\mathrm{i})s+(1-\mathrm{i})t, (1+5\mathrm{i})s+(5-\mathrm{i})t) \notin \mathbf{B}_{\mathbb{R}}$; note that

$$\begin{vmatrix} 1+\mathrm{i} & 1-\mathrm{i} \\ 1+5\mathrm{i} & 5-\mathrm{i} \end{vmatrix} = 0.$$

More generally, let $\mathcal{P}(\bar{t}\,) = (a_1,\ldots,a_4)s + (b_1,\ldots,b_4)t + (c_1,\ldots,c_4)$ be a real plane in $\mathbb{C}^4$; that is, $a_i, b_i, c_i \in \mathbb{R}$ and $(a_1,\ldots,a_4), (b_1,\ldots,b_4)$ linearly independent. Let $\mathcal{Q}(\bar{t}\,) = ((a_1+\mathrm{i}a_2)s+(b_1+\mathrm{i}b_2)t+(c_1+\mathrm{i}c_2), (a_3+\mathrm{i}a_4)s+(b_3+\mathrm{i}b_4)t+(c_3+\mathrm{i}c_4))$. Then, $\mathcal{Q} \in \mathbf{B}_{\mathbb{R}}$ iff

$$\begin{vmatrix} a_1+\mathrm{i}a_2 & b_1+\mathrm{i}b_2 \\ a_3+\mathrm{i}a_4 & b_3+\mathrm{i}b_4 \end{vmatrix} \neq 0.$$

But, this condition is not equivalent to the property of $(a_1,\ldots,a_4), (b_1,\ldots,b_4)$ being linearly independent. This fact motivates the following definition.

Definition 3.1. We say that a $\mathbb{K}$-definable n-plane Π in $\mathbb{F}^{nr}$ is non-degenerated (w.r.t. α) if there exists a basis $\{(a^1_{10},\ldots,a^1_{n(r-1)}),\ldots,(a^n_{10},\ldots,a^n_{1(r-1)})\} \subset \mathbb{K}^{nr}$ of Π such that $\{(\sum_{j=0}^{r-1} a^1_{\ell,j}\alpha^j)_{1\leq \ell\leq n},\ldots,(\sum_{j=0}^{r-1} a^n_{\ell,j}\alpha^j)_{1\leq \ell\leq n}\}$ is a basis of $\mathbb{F}^n$.

As a consequence of 2.2, we can rephrase the above definition as follows:

Lemma 3.2. *A $\mathbb{K}$-definable n-plane in $\mathbb{F}^{nr}$ is a ultraquadric if and only if the n-plane is non-degenerated.*

Notice that, by the same Lemma 2.2, the existence of just one basis, with the properties described in the definition of non-degenerate n-planes, implies the same property holds for all bases.

Now let us state the main characterization of ultraquadrics associated to polynomial automorphisms. First we consider the case of ultraquadrics defined by automorphisms with linear polynomials as coordinates:

Lemma 3.3. *Let $\Psi \in \mathbf{B}_{\mathbb{L}}$. The following statements are equivalent:*

1. $\mathrm{Ultra}(\Psi)$ *is a $\mathbb{K}$-definable n-plane in $\mathbb{F}^{nr}$.*

2. *There exists a linear automorphism in* $[\Psi]$.

Proof. (1) implies (2) follows by taking a linear parametrization, over $\mathbb{K}$, of the n-plane, and applying Lemma 2.1. On the other hand (2) implies (1) is a direct consequence of Lemma 2.2. $\square$

Corollary 3.4. *Let* $\Psi = [L_0 : \ldots : L_n] \in \mathbf{PGL}_{\mathbb{L}}(n)$ *such that* $L_0 \in \mathbb{K}[\overline{T}\,]$. *Then* $\mathrm{Ultra}(\Psi)$ *is an* n*-plane.*

Proof. Let $L_0(\overline{T}\,) = f_0 t_0 + f_1 t_1 + \cdots + f_n t_n$. If $f_1 = \cdots = f_n = 0$ the result follows from Lemma 3.3. Let us assume w.l.o.g. that $f_1 \neq 0$. Then, we observe that (see (3) for the notation Ψ_a) the denominator of $\Psi^*(\bar{t}\,) = \Psi_a(t_1 - f_2 t_2 - \cdots - f_n t_n - f_0, t_2, \ldots, t_n) \in [\Psi_a(\bar{t}\,)]$ is t_1. Now, $\Psi^{**} = \Psi^*(\frac{1}{t_1}, \frac{t_2}{t_1}, \ldots, \frac{t_n}{t_1}) \in [\Psi_a]$ and it is polynomial. So, the result follows from Lemma 3.3. $\square$

Corollary 3.5. *Let* $\Psi \in \mathbf{B}_{\mathbb{L}} \cap \mathbb{K}(\bar{t}\,)^n$, *then* $\mathrm{Ultra}(\Psi)$ *is an* n*-plane.*

Proof. Note that $(\bar{t}\,) \in [\Psi]$. $\square$

Unlike the hypercircle case, for general ultraquadrics it is not true that being defined by a linear polynomial automorphism is equivalent to being defined by a polynomial automorphism. Still, there is a close relationship, as stated in the following result that generalizes Lemma 3.3.

Theorem 3.6. *Let* $\Psi \in \mathbf{B}_{\mathbb{L}}$. *The following statements are equivalent*

1. $\mathrm{Ultra}(\Psi)$ *is* $\mathbb{K}$*-isomorphic to* $\mathbb{F}^n$.

2. $[\Psi] \cap \mathbf{A}_{\mathbb{L}} \neq \emptyset$.

Proof. If (1) holds, then there exists a polynomial proper parametrization $\Phi(\bar{t}\,)$, over $\mathbb{K}$, of $\mathrm{Ultra}(\Psi)$. Now, from Lemma 2.2, we know that $\Phi(\bar{t}\,)$ defines an element, say $\varphi(\bar{t}\,)$, in $\mathbf{B}_{\mathbb{L}}$. By construction, φ is polynomial. Let us see that φ^{-1} is also polynomial. By Lemma 2.3 (see also Theorem 6 in [2]) it holds that

$$V_\alpha^{-1} \circ \Phi^{-1} \circ V_\alpha = (\varphi \times \varphi^{\sigma_1} \times \cdots \times \varphi^{\sigma_{r-1}})^{-1}$$

where φ^{σ_i} denotes each of the conjugates of φ w.r.t. α, and V_α is the Vandermonde matrix in Lemma 2.3. Therefore, since Φ has polynomial inverse, the inverse of φ is polynomial, too.

Conversely, let (2) hold. Because of Lemma 2.1, we can assume w.l.o.g. that $\Psi \in \mathbf{A}_{\mathbb{L}}$. Let $\bar{\alpha} = (1, \alpha, \ldots, \alpha^{r-1})$. Now, we consider the map

$$\begin{array}{rlcl} \xi: & \mathrm{Ultra}(\Psi) \subset \mathbb{F}^{nr} & \longrightarrow & \mathbb{F}^n \\ & (\bar{x}_1, \ldots, \bar{x}_n) & \mapsto & \Psi^{-1}(\bar{\alpha} \cdot \bar{x}_1, \ldots, \bar{\alpha} \cdot \bar{x}_n), \end{array}$$

where $\bar{x}_i = (x_{i0}, \ldots, x_{i(r-1)})$. Since $\Psi \in \mathbf{A}_{\mathbb{L}}$, ξ is polynomial and defines an $\mathbb{F}$-isomorphism, being its inverse $\mathrm{U}(\Psi) : \mathbb{F}^n \to \mathrm{Ultra}(\Psi)$. Let ξ be expressed as

$$\xi(\bar{x}) = (\bar{\alpha} \cdot (\xi_{10}(\bar{x}), \ldots, \xi_{1(r-1)}(\bar{x})), \ldots, \bar{\alpha} \cdot (\xi_{n0}(\bar{x}), \ldots, \xi_{n(r-1)}(\bar{x}))),$$

where $\xi_{ij}(\bar{x}) \in \mathbb{K}[\bar{x}]$. If we prove that, for all i, for $j > 0$ and $\bar{x} \in \mathrm{Ultra}(\Psi)$, $\xi_{ij}(\bar{x}) = 0$, then we will get that ξ is in fact a $\mathbb{K}$-isomorphism from $\mathrm{Ultra}(\Psi)$ and $\mathbb{F}^n$. Indeed, since $\xi \circ \mathrm{Ultra}(\Psi) = \mathrm{Id}_{\mathbb{F}^n}$ it holds that for each i

$$\sum_{j=0}^{r-1} \xi_{ij}(\mathrm{U}(\Psi)(\bar{t}))\alpha^j = t_i.$$

Therefore, since $\xi_{ij}(\mathrm{U}(\Psi)(\bar{t})) \in \mathbb{K}[\bar{t}]$, then, for $j > 0$, $\xi_{ij}(\mathrm{U}(\Psi)(\bar{t})) = 0$. $\square$

4 Ultraquadrics associated to $\mathbf{PGL}_{\mathbb{L}}(n)$

In this section, we assume that the birational transformation $\Psi = L$ is an element of $\mathbf{PGL}_{\mathbb{L}}(n)$, and we describe the structure of $\mathrm{Ultra}(\Psi)$ as a blowup of $\mathbb{P}^n(\mathbb{F})$. We write Ψ as

$$\Psi(\overline{T}) = L \cdot \overline{T}^{\,t} = [L_0(\overline{T}) : L_1(\overline{T}) : \ldots : L_n(\overline{T})]$$

where L_i is the linear form represented by the i-th row of L. In addition, let $\sigma_1, \ldots, \sigma_r$ be as in Lemma 2.3, and let g_i be the form of degree $r-1$ that is the product of all conjugate forms $\{L_0^{\sigma_1}, \ldots, L_0^{\sigma_r}\}$ with the exception of $L_0^{\sigma_i}$. Furthermore, let $I = (g_1, \ldots, g_r)$ be the homogeneous ideal generated by $\{g_1, \ldots, g_r\}$ in $\mathbb{F}[t_0, \ldots, t_n]$.

In the following theorem we relate the ultraquadric associated to Ψ with the blowup of the projective space $\mathbb{P}^n(\mathbb{F})$ along the ideal I (see e.g. Section 7.4. in [13], for the notion of blowup along an ideal).

Theorem 4.1. *The projective closure of the ultraquadric* $\mathrm{Ultra}(\Psi)$ *is* $\mathbb{L}$*-linearly isomorphic to the Segre embedding of the blowup of* $\mathbb{P}^n(\mathbb{F})$ *along the ideal* I*.*

Proof. We consider the map

$$\begin{array}{cccc} \eta: & \mathbb{P}^n(\mathbb{F}) & \longrightarrow & \mathbb{P}^n(\mathbb{F}) \times \mathbb{P}^{r-1}(\mathbb{F}) \\ & \overline{T} & \mapsto & (\overline{T}\ ;\ (g_1(\overline{T}) : g_2(\overline{T}) : \ldots : g_r(\overline{T}))) \end{array}$$

which is a blowup of $\mathbb{P}^n(\mathbb{F})$ along I. Now, we compose this map with the Segre embedding of $\mathbb{P}^n(\mathbb{F}) \times \mathbb{P}^{r-1}(\mathbb{F})$ to get the blowup of $\mathbb{P}^n(\mathbb{F})$ as isomorphic to the subvariety $\mathcal{W}$ of $\mathbb{P}^{rn+r-1}(\mathbb{F})$ parametrized by

$$P := [t_0 g_1 : \ldots : t_0 g_r : \ldots : t_n g_1 : \ldots : t_n g_r] \tag{4}$$

On the other hand, $\mathrm{Ultra}(\Psi)$ is (linearly) $\mathbb{L}$-isomorphic to the affine variety $\mathcal{V}$ parametrized by $\Psi_a \times \Psi_a^{\sigma_2} \times \cdots \times \Psi_a^{\sigma_r}$ (see Lemma 2.3 and (3) for the notation Ψ_a). Projectively, the parametrization $\Psi_a \times \Psi_a^{\sigma_2} \times \cdots \times \Psi_a^{\sigma_r}$ can be expressed as

$$[L_0 g_1 : L_1 g_1 : \ldots : L_n g_1 : L_1^{\sigma_2} g_2 : \ldots : L_n^{\sigma_2} g_2 : \ldots : L_1^{\sigma_r} g_r : \ldots : L_n^{\sigma_r} g_r].$$

This variety is isomorphic to the subvariety of $\mathbb{P}^{nr+r-1}$ parametrized by

$$Q := [L_0 g_1 : \ldots : L_n g_1 : L_0^{\sigma_2} g_2 : \ldots : L_n^{\sigma_2} g_2 : \ldots : L_0^{\sigma_r} g_r : \ldots : L_n^{\sigma_r} g_r]$$

since $L_0^{\sigma_i} g_i = L_0 g_1$, and we are just duplicating the first coordinate of each block.

Since by definition $\Psi^{\sigma_i}(\overline{T})^t = L^{\sigma_i} \cdot \overline{T}^t$, then $(g_i \Psi^{\sigma_i})^t = L^{\sigma_i}(g_i \cdot \overline{T})^t$ where the super-index t denotes the transpose of the matrix. Therefore

$$Q = \begin{pmatrix} (g_1 \Psi^{\sigma_1})^t \\ \vdots \\ (g_r \Psi^{\sigma_r})^t \end{pmatrix} = \begin{pmatrix} L^{\sigma_1} & & \\ & \ddots & \\ & & L^{\sigma_r} \end{pmatrix} \begin{pmatrix} (g_1 \overline{T})^t \\ \vdots \\ (g_r \overline{T})^t \end{pmatrix}$$

Finally observe that the parametrization provided by the right side of the formula above is just a re-ordering of the coordinates of P. Thus, $\mathcal{W}$ is linearly isomorphic to the projective closure of $\mathrm{Ultra}(\Psi)$. □

Remark 4.2. If we rescale the matrix L so that one of the coefficients of L_0 is 1, then the coefficients of L_0 generate an intermediate extension $\mathbb{K} \subseteq \mathbb{K}(\beta) \subseteq \mathbb{K}(\alpha)$. Following the ideas developed for the case of hypercircles (see [8]), we say that an ultraquadric associated to Ψ is primitive if $\mathbb{K}(\beta) = \mathbb{K}(\alpha)$. If $\mathrm{Ultra}(\Psi)$ is not primitive, $[\mathbb{K} : \mathbb{K}(\beta)] = s < r$, then $g_i = \|L_0\|^{r/s} / L_0^{\sigma_i}$ (where $\|L_0\|$ is L_0 multiplied by all its different conjugates) and the polynomials of the parametrization of the blowup can be taken of degree s. In particular, if the denominator of Ψ has coefficients in $\mathbb{K}$, then $\mathrm{Ultra}(\Psi)$ is a n-plane (cf. Lemma 3.3).

Remark 4.3. Consider the hyperplanes defined by the conjugate linear forms $L_0 = L_0^{\sigma_1}, \ldots, L_0^{\sigma_r}$. The center of the blowup, i.e. the variety defined by the ideal I, is the union of all codimension 2 linear spaces where two different hyperplanes $L_0^{\sigma^i}$ intersect

$$\mathcal{Z} = \bigcup_{L^{\sigma_i} \neq L^{\sigma_j}} \{L_0^{\sigma_i} = L_0^{\sigma_j} = 0\}.$$

In particular, if L_0 does not have coefficients in $\mathbb{K}$, then the ultraquadric is not a n-plane.

The next corollaries follow from Theorem 4.1.

Corollary 4.4. $\mathrm{U}(\Psi)$ *is an isomorphism of* $\mathbb{P}^n(\mathbb{F}) \setminus \mathcal{Z}$ *onto its image. In particular, the affine part of* $\mathrm{Ultra}(\Psi)$ *is always smooth.*

Corollary 4.5. *Let* $r \leq n$ *and let* $L_0^{\sigma_1}, \ldots, L_0^{\sigma_r}$ *be hyperplanes in general position in* $\mathbb{P}^n(\mathbb{F})$*. Then, the ultraquadric* $\mathrm{Ultra}(\Psi)$ *is (linearly isomorphic to) a toric variety.*

Proof. Take a system of coordinates $\overline{s}$ such that $L_0^{\sigma_i} = s_i$. Then, the parametrization of the blowup (see (4)) is of the form

$$[s_0 g_1 : \ldots : s_0 g_r : \ldots : s_n g_1 : \ldots : s_n g_r]$$

where $g_i = s_1 \cdots s_{i-1}s_{i+1} \cdots s_r$. So the variety is parametrizable by monomials, and hence toric. $\square$

In some applications it is interesting to restrict to real-complex case and surfaces, see for instance [3]. Hence, we take now a closer look to the case of algebraic extensions of degree $r = 2$ and automorphisms of $\mathbb{P}^2(\mathbb{F})$. Next result describes the intersection of ultraquadrics arising in this context, with the hyperplane at infinity (cf. [8] for the hypercircle case).

Corollary 4.6. *Let $r = 2$, $\Phi = [L_0 : L_1 : L_2] \in \mathbf{PGL}_{\mathbb{L}}(2)$, let $x^2 + ax + b$ be the minimal polynomial of α over $\mathbb{K}$.*

1. *If the primitive part of L_0 is in $\mathbb{K}[s, t]$, then* $\mathrm{Ultra}(\Psi)$ *is a plane.*
2. *If the primitive part of L_0 is in $\mathbb{L}[s, t] \setminus \mathbb{K}[s, t]$, then* $\mathrm{Ultra}(\Psi)$ *is linearly isomorphic to the surface parametrized by*

$$[u^2 : uv : uw : v^2 : vw]$$

and hence a blowup of the plane at a point. In particular, it is smooth.

Moreover, let $\{L_0 = 0\}$ and $\{L_0^\sigma = 0\}$ be the lines defined, respectively, by the denominator and by its conjugate, let $p = \{L_0 = L_0^\sigma = 0\}$ be the intersection point. Then, the intersection of $\mathrm{Ultra}(\Psi)$ *with the hyperplane at infinity consists in three lines $\mathcal{L}$, $\mathcal{L}^\sigma$, E. Furthermore:*

1. $\mathrm{Ultra}(\Psi)$ *is the blowup of the plane at p.*
2. *$\mathcal{L}$ does not depend on Ψ (and hence neither does $\mathcal{L}^\sigma$), it only depends on the minimal polynomial of α. In fact $\mathcal{L} = V(\{x_0, 2x_1 - (2\alpha + a)x_2, 2x_3 - (2\alpha + a)x_4\})$.*
3. *$q = [0 : (\alpha + a/2)L_1(p) : L_1(p), (\alpha + a/2)L_2(p) : L_2(p)] \in \mathcal{L}$ is such that $\mathcal{L} \setminus \{q\}$ corresponds, by the parametrization, to $\{L_0 = 0\} \setminus \{p\}$.*
4. *$E = \langle q, q^\sigma \rangle$, the line through q and q^σ, is the exceptional divisor of the blowup.*

Proof. If $L_0 \in \mathbb{K}[s, t]$, the result follows from Corollary 3.4. In the other case, $\mathrm{Ultra}(\Psi)$ is the blowup at p by Corollary 4.5. To check the rest of the claims, we parametrize $\mathrm{Ultra}(\Psi)$ following the construction of Theorem 4.1. The parametrization of $\mathrm{Ultra}(\Psi)$ is the composition of the maps

$$r \to [L_0(r)L_0^\sigma(r) : L_1(r)L_0^\sigma(r) : L_1^\sigma(r)L_0(r) : L_2(r)L_0^\sigma(r) : L_2^\sigma(r)L_0(r)] =$$

$$= [t_0 : t_{10} : t_{11} : t_{20} : t_{21}] \to \left[t_0 : \frac{t_{10} + t_{11}}{2} : \frac{t_{10} - t_{11}}{(\alpha - \alpha^\sigma)} : \frac{t_{20} + t_{21}}{2} : \frac{t_{20} - t_{21}}{(\alpha - \alpha^\sigma)}\right]$$

If we restrict the map to $\{L_0 = 0\} \setminus \{p\}$ we have:

$$r \to [0 : L_1(r)L_0^\sigma(r) : 0 : L_2(r)L_0^\sigma(r) : 0] = [0 : L_1(r) : 0 : L_2(r) : 0] \to$$

$$\to \left[0 : \frac{L_1(r)}{2} : \frac{L_1(r)}{(\alpha-\alpha^\sigma)} : \frac{L_2(r)}{2} : \frac{L_2(r)}{(\alpha-\alpha^\sigma)}\right]$$

This is a parametrization of the line $\mathcal{L}$ and the only point that is not attained (corresponding to p) is q. The rest of the items follow easily from this observation. $\square$

Example 4.7. Consider the extension $\mathbb{R} \subseteq \mathbb{R}(\mathrm{i}) = \mathbb{C}$ and the automorphism of the plane given by $L(t_0 : t_1 : t_2) = (t_1 + \mathrm{i}t_2, t_0, t_1)$. Then $L_0 = \{t_1 + \mathrm{i}t_2 = 0\}$, $L_0^\sigma = \{t_1 - \mathrm{i}t_2 = 0\}$. The center of the blowup is the origin $(1 : 0 : 0)$. $\mathrm{Ultra}(L) = V(x_2x_3 - x_1x_4, x_3 - x_3^2 - x_4^2, x_1 - x_1x_3 - x_2x_4) \subseteq \mathbb{C}^5$. The projectivization of $\mathrm{Ultra}(L)$ intersects the hyperplane at infinity at the three lines $\mathcal{L} = V(x_0, x_1 - \mathrm{i}x_2, x_3 - \mathrm{i}x_4)$, $\mathcal{L}^\sigma = V(x_0, x_1 + \mathrm{i}x_2, x_3 + \mathrm{i}x_4)$ and $E = V(x_0, x_3, x_4)$. In this case $q = (0 : \mathrm{i} : 1 : 0 : 0)$.

In the previous corollary we have assumed that $\Psi = [L_0 : L_1 : L_2] \in \mathbf{PGL}_{\mathbb{L}}(2)$ and that $r = 2$. For the rest of this section, we keep these assumptions and we analyze how $\mathrm{Ultra}(\Phi)$ is related to hypercircles. For this purpose, and taking into account Corollary 3.4 or Corollary 4.6, we may assume w.l.o.g. that the primitive part of L_0 is not a polynomial over $\mathbb{K}$. We start with the following lemma.

Lemma 4.8. *Let $\Psi = [L_0 : L_1 : L_2] \in \mathbf{PGL}_{\mathbb{L}}(2)$ be such that the primitive part of L_0 is not over $\mathbb{K}$. There exists $\Psi^* \in [\Psi_a]$ (see (3) for the notation Ψ_a) of the form*

$$\Psi^*(s,t) = \left(a_1^* + \frac{b_1^*t + c_1^*}{s+\alpha}, a_2^* + \frac{b_2^*t + c_2^*}{s+\alpha}\right),$$

where $a_i^*, b_i^*, c_i^* \in \mathbb{L}$.

Proof. We can assume that L_0 has degree 1 in s; note that, if this is not the case, $\Psi_a(s, s+t) \in [\Psi_a]$ and this new automorphism satisfies the condition. So, let Ψ_a be expressed as

$$\Psi_a(s,t) = \left(\frac{A_1s + B_1t + C_1}{s + ft + g}, \frac{A_2s + B_2t + C_2}{s + ft + g}\right)$$

where $A_i, B_i, C_i, f, g \in \mathbb{L}$. Dividing w.r.t. s each numerator by the denominator, Ψ_a can we written as

$$\Psi_a(s,t) = \left(a_1 + \frac{b_1t + c_1}{s + ft + g}, a_2 + \frac{b_2t + c_2}{s + ft + g}\right)$$

where $a_i, b_i, c_i, f, g \in \mathbb{L}$. Now, let f and g be expressed in the $\{1, \alpha\}$ basis as

$$f = f_0 + \alpha f_1,\ g = g_0 + \alpha g_1.$$

with $f_i, g_i \in \mathbb{K}$. We distinguish two cases.

- Let $f_1 \neq 0$. Then, we consider

$$\overline{\Psi}(s,t) := \Psi_a\left(-f_0\left(\frac{1}{f_1}t - \frac{g_1}{f_1}\right) - g_0 + s, \frac{1}{f_1}t - \frac{g_1}{f_1}\right) \in [\Psi_a].$$

Thus, $\overline{\Psi}$ can be expressed as

$$\overline{\Psi}(s,t) = \left(a_1 + \frac{b_1^* t + c_1^*}{s + \alpha t}, a_2 + \frac{b_2^* t + c_2^*}{s + \alpha t}\right).$$

Finally, we get that

$$\Psi^*(s,t) := \overline{\Psi}\left(\frac{s}{t}, \frac{1}{t}\right) \in [\overline{\Psi}] = [\Psi_a].$$

Moreover, $\Psi^*(s,t)$ is of the form

$$\Psi^*(s,t) = \left(a_1 + \frac{b_1^* + c_1^* t}{s + \alpha}, a_2 + \frac{b_2^* + c_2^* t}{s + \alpha}\right).$$

- Let $f_1 = 0$, then $g_1 \neq 0$, and the common denominator is $s + f_0 t + (g_0 + \alpha g_1)$. Then

$$\Psi^*(s,t) = \Psi_a\left(sg_1 - tg_1, \frac{g_1}{f_0}t - \frac{g_0}{f_0}\right) \in [\Psi_a]$$

□

Finally, the following theorem shows that the ultraquadrics of elements in $\mathbf{PGL}_{\mathbb{L}}(2)$ are surfaces ruled by means of some hypercircles. For this purpose, we introduce the following notation. For $i \in \{1,2\}$, let $\Phi_i(u) = \phi_{i0}(u) + \alpha\phi_{i1}(u)$, with $\phi_{ij} \in \mathbb{K}(u)$, be $\mathbb{L}$-birational maps from $\mathbb{F}$ onto $\mathbb{F}$. We denote by $\mathrm{H}(\Phi_1) \odot \mathrm{H}(\Phi_2)$ the rational curve in $\mathbb{F}^4$ parametrized by

$$(\phi_{10}(u), \phi_{11}(u), \phi_{20}(u), \phi_{21}(u)).$$

In this situation, we have the following result.

Theorem 4.9. *Let* $\Psi = [L_0 : L_1 : L_2] \in \mathbf{PGL}_{\mathbb{L}}(2)$*, then* $\mathrm{Ultra}(\Psi)$ *is a ruled surface. Moreover, if the primitive part of* L_0 *is not over* $\mathbb{K}$*, there exist four* $\mathbb{L}$*-birational maps,* $\Phi_1, \ldots, \Phi_4$*, of* $\mathbb{F}$ *onto* $\mathbb{F}$*, such that* $\mathrm{Ultra}(\Psi)$ *is parametrized as*

$$(\mathrm{H}(\Phi_1)(s) \odot \mathrm{H}(\Phi_2)(s)) + t\,(\mathrm{H}(\Phi_3)(s) \odot \mathrm{H}(\Phi_4)(s)).$$

Proof. Taking into account Corollary 3.4, the theorem holds for the case where L_0 is over $\mathbb{K}$. So we assume that L_0 is in $\mathbb{L}[s,t] \setminus \mathbb{K}[s,t]$. By Lemma 4.8, we can assume w.l.o.g. that Ψ_a (see (3) for the notation Ψ_a) is expressed as

$$\Psi_a(s,t) = \left(a_1 + \frac{b_1}{s+\alpha}t + \frac{c_1}{s+\alpha}, a_2 + \frac{b_2}{s+\alpha}t + \frac{c_2}{s+\alpha}\right).$$

Let $a_i = a_{i0} + \alpha a_{i1}$, $\frac{b_i}{s+\alpha} = b_{i0}(s) + \alpha b_{i1}(s)$, $\frac{c_i}{s+\alpha} = c_{i0}(s) + \alpha c_{i1}(s)$, with $a_{ij} \in \mathbb{K}$ and $b_{ij}, c_{ij} \in \mathbb{K}(s)$. Then $\mathrm{U}(\Psi)$ can be expressed as $\mathrm{U}(\Psi)(s,t) = A(s) + tB(s)$, where

$$\begin{array}{rcl} A(s) & = & (a_{10} + c_{10}(s), a_{11} + c_{11}(s), a_{20} + c_{20}(s), a_{21} + c_{21}(s)) \\ B(s) & = & (b_{10}(s), b_{11}(s), b_{20}(s), b_{21}(s)) \end{array}$$

Finally, observe that

$$\begin{array}{rcl} A(s) & = & \mathrm{H}\left(a_1 + \frac{c_1}{s+\alpha}\right) \odot \mathrm{H}\left(a_2 + \frac{c_2}{s+\alpha}\right) \\ B(s) & = & \mathrm{H}\left(\frac{b_1}{s+\alpha}\right) \odot \mathrm{H}\left(\frac{b_2}{s+\alpha}\right) \end{array}$$

$\square$

Example 4.10. Take the automorphism given in Example 4.7, $(s,t) \to \left(\frac{1}{s+\mathrm{i}t}, \frac{s}{s+\mathrm{i}t}\right)$. Following Lemma 4.8 we get the automorphism in the same class $(s,t) \to (\frac{t}{s+\mathrm{i}}, 1 + \frac{-\mathrm{i}}{s+\mathrm{i}})$. Then $A(s) = \left(0, 0, \frac{s^2}{s^2+1}, \frac{-s}{s^2+1}\right)$ and $B(s) = \left(\frac{s}{s^2+1}, \frac{-1}{s^2+1}, 0, 0\right)$, and $\mathrm{Ultra}(\Psi)$ is parametrized as

$$(\mathrm{H}(\Phi_1)(s) \odot \mathrm{H}(\Phi_2)(s)) + t\,(\mathrm{H}(\Phi_3)(s) \odot \mathrm{H}(\Phi_4)(s)) = A(s) + tB(s).$$

In this case, neither $\mathrm{H}(\Phi_1)$ nor $\mathrm{H}(\Phi_4)$ are *true* hypercircles while $\mathrm{H}(\Phi_2) = \mathrm{H}(s/(s+\mathrm{i}))$ and $\mathrm{H}(\Phi_3) = \mathrm{H}(1/(s+\mathrm{i}))$.

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