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ULTRAQUADRICS ASSOCIATED TO AFFINE AND PROJECTIVE AUTOMORPHISMS

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ABSTRACT. In this extended abstract, we study the properties of ultraquadrics associated with automorphisms of the field $\mathbb{K}(\alpha)(t_1,\ldots,t_n)$, defined by linear rational (with common denominator) or by polynomial (with polynomial inverse) coordinates. We conclude that ultraquadrics related to polynomial automorphisms can be characterized as varieties $\mathbb{K}-$isomorphic to linear varieties, while ultraquadrics arising from projective automorphisms are isomorphic to the Segre embedding of a blowup of the projective space along an ideal and, in some general case, linearly isomorphic to a toric variety. This information helps us to compute a parametrization of some ultraquadrics.

1. INTRODUCTION

The study and analysis of ultraquadrics was introduced in [2] as a higher dimensional generalization of the concept of hypercircle (cf. [1], [4], [5], [6]) and as a fundamental computational tool to algorithmically solve the problem of the optimal algebraic reparametrization of rational varieties of arbitrary dimension (e.g. rational surfaces, see [3]).

Given a rational variety $\mathcal{V}$, presented by a rational parametrization with n parameters $t_1,\ldots,t_n$ and coefficients in a certain algebraic extension $\mathbb{K}(\alpha)$ of a ground field $\mathbb{K}$, it is natural to ask for the possibility of reparametrizing $\mathcal{V}$ over $\mathbb{K}$. For this purpose the paper [2] introduces the concept of "ultraquadrics" as varieties associated to automorphisms of the field $\mathbb{K}(\alpha)(t_1,\ldots,t_n)$, and describes its application to computing the reparametrization of $\mathcal{V}$ over $\mathbb{K}$, when possible.

In this extended abstract, we study the ultraquadrics associated to some important kind of automorphisms in the field $\mathbb{K}(\alpha)(t_1,\ldots,t_n)$, such as those defined by linear rational (with common denominator) or polynomial (with inverse also polynomial) coordinates. The provided results reinforce the computational usefulness of ultraquadrics.

A complete version of this extended abstract has been submitted to a journal.

1.1. Notation.

In the sequel, $\mathbb{K}$ is a field of characteristic zero, α is an algebraic element over $\mathbb{K}$, $\mathbb{L}$ is the field extension $\mathbb{K}(\alpha)$ and $\mathbb{F}$ is the algebraic closure of $\mathbb{L}$. So $\mathbb{K}\subset\mathbb{L}=\mathbb{K}(\alpha)\subset\mathbb{F}$. We assume that $[\mathbb{K}:\mathbb{L}]=r$. We use the notation $\overline{t}=(t_1,\ldots,t_n)$ and $\overline{T}=(t_0:\ldots:t_n)$ for affine –respectively, projective– coordinates.

On the other hand, we will consider the following three groups of automorphisms under composition:

• $\mathbf{B}_{\mathbb{L}}$ is the group of all $\mathbb{L}$-birational transformations (i.e. $\mathbb{L}$-definable) of $\mathbb{F}^n$ onto $\mathbb{F}^n$.

• $\mathbf{A}_{\mathbb{L}}$ is the group of all $\mathbb{L}$-automorphism of the affine space $\mathbb{F}^n$; that is, the subgroup of $\mathbf{B}_{\mathbb{L}}$ where the transformation and its inverse are both described through polynomial coordinates.

• $\mathbf{PGL}_{\mathbb{L}}(n)$ is the group of all $\mathbb{L}$-automorphism of the projective space $\mathbb{P}^n(\mathbb{F})$. Elements in

$\mathbf{PGL}_{\mathbb{L}}(n)$ are represented by a $(n+1)\times(n+1)$ regular matrix L

$$\mathbb{P}^n(\mathbb{F}) \to \mathbb{P}^n(\mathbb{F});\ \overline{T} \mapsto L\cdot(\overline{T}^{\,t}) = [L_0(\overline{T}):\cdots:L_n(\overline{T})] \tag{1}$$

where the rows L_i of L represent linear forms.

1.2. **Ultraquadrics.** Let $\Psi = (\psi_1,\dots,\psi_n)$ be a birational automorphism of $\mathbb{F}^n$. Then, we express Ψ in the basis $\{1,\dots,\alpha^{r-1}\}$ as

$$\Psi(\bar{t}) = \left(\sum_{j=0}^{r-1}\psi_{1,j}\alpha^j,\dots,\sum_{j=0}^{r-1}\psi_{n,j}\alpha^j\right),\quad \psi_{ij}\in\mathbb{K}(\bar{t}).$$

Then, using this notation, we consider the expansion map

$$\begin{array}{rlcl} \mathrm{U}: & \mathbf{B}_{\mathbb{L}} & \to & \mathbb{K}(\bar{t})^{nr} \\ & \Psi(\bar{t}) & \mapsto & (\psi_{10}(\bar{t}),\dots,\psi_{1(r-1)}(\bar{t}),\dots,\psi_{n0}(\bar{t}),\dots,\psi_{n(r-1)}(\bar{t})) \end{array} \tag{2}$$

We define the ultraquadric associated with Ψ, and we denote it by $\mathrm{U}ltra(\Psi)$, as the rational variety of dimension n in $\mathbb{F}^{nr}$ parametrized by $\mathrm{U}(\Psi(\bar{t}))$. Different automorphisms Ψ_1, Ψ_2 may define the same ultraquadric $\mathrm{U}ltra(\Psi_1) = \mathrm{U}ltra(\Psi_2)$. This can happen if and only if $\Psi_2 = \Psi_1\circ\Phi$ with Φ an automorphism in $\mathbf{B}_{\mathbb{L}}$ with coefficients in $\mathbb{K}$. We define $[\Psi]$ as the coset $[\Psi] = \{\Psi\circ\Phi|\Phi\in \mathbf{B}_{\mathbb{L}}$ with coefficients in $\mathbb{K}\}$.

If $\Psi \in \mathbf{PGL}_{\mathbb{L}}(n)$, say $\Psi(\overline{T}) = [L_0(\overline{T}):\dots:L_n(\overline{T})]$, we will denote as $\mathrm{U}ltra(\Psi)$ the (affine) ultraquadric generated by the associated affine mapping

$$\Psi_a(\bar{t}) = \left(\frac{L_1(1,t_1,\dots,t_n)}{L_0(1,t_1,\dots,t_n)},\dots,\frac{L_n(1,t_1,\dots,t_n)}{L_0(1,t_1,\dots,t_n)}\right) \tag{3}$$

2. Ultraquadrics associated to affine and projective automorphisms

Next statement characterizes the ultraquadrics associated with automorphisms in $\mathbf{A}_{\mathbb{L}}$.

Theorem 2.1. *Let $\Psi\in\mathbf{B}_{\mathbb{L}}$. The following statements are equivalent*

1. $\mathrm{U}ltra(\Psi)$ *is $\mathbb{K}$-isomorphic to $\mathbb{F}^n$.*
2. $[\Psi]\cap\mathbf{A}_{\mathbb{L}}\neq\emptyset$.

Moreover, $\mathrm{U}ltra(\Psi)$ is a linear variety if and only if $[\Psi]$ contains a linear automorphism.

Proof. (sketch) A $\mathbb{K}$-definable proper parametrization $\mathcal{P}(\bar{t}) = (P_{10},\dots,P_{1(r-1)},\dots,P_{n0},\dots,P_{n(r-1)})$ parametrizes $\mathrm{U}ltra(\Psi)$ if and only if $\mathcal{Q}(\bar{t}) := (\sum_{j=0}^{r-1}P_{1,j}\alpha^j,\dots,\sum_{j=0}^{r-1}P_{n,j}\alpha^j)\in[\Psi]$. Now, $\mathcal{P}^{-1}$ is the expansion map obtained from $\mathcal{Q}^{-1}$. Hence, $\mathcal{P}$ and $\mathcal{P}^{-1}$ are polynomial (resp. linear) if and only if $\mathcal{Q}$ and $\mathcal{Q}^{-1}$ are polynomial (resp. linear). □

Now, we study the case of projective automorphisms. Let $\Psi = L\in\mathbf{PGL}_{\mathbb{L}}(n)$, we describe the structure of $\mathrm{U}ltra(\Psi)$ as a blowup of $\mathbb{P}^n(\mathbb{F})$, (see [7]). Write Ψ as

$$\Psi(\overline{T}) = L\cdot\overline{T}^{\,t} = [L_0(\overline{T}):L_1(\overline{T}):\dots:L_n(\overline{T})]$$

where L_i is the linear form represented by the i-th row of L. Let $\alpha = \alpha_1,\alpha_2,\dots,\alpha_r$ be the conjugates of α in $\mathbb{F}$ and let $\sigma_1,\dots,\sigma_r$ be $\mathbb{K}$-automorphisms of $\mathbb{F}$ such that $\sigma_i(\alpha)=\alpha_i$, and let g_i be the form of degree $r-1$ that is the product of all conjugate forms $\{L_0^{\sigma_1},\dots,L_0^{\sigma_r}\}$ with the exception of $L_0^{\sigma_i}$; where L^{σ_i} is the linear form obtained from L substituting α by α_i. Furthermore, let $I=(g_1,\dots,g_r)$ be the homogeneous ideal generated by $\{g_1,\dots,g_r\}$ in $\mathbb{F}[t_0,\dots,t_n]$. Then

Theorem 2.2. *The projective closure of the ultraquadric* $\mathrm{U}ltra(\Psi)$ *is* $\mathbb{L}$*-linearly isomorphic to the Segre embedding of the blowup of* $\mathbb{P}^n(\mathbb{F})$ *along the ideal* I.

Proof. We consider the map

$$\begin{array}{rccc} \eta: & \mathbb{P}^n(\mathbb{F}) & \longrightarrow & \mathbb{P}^n(\mathbb{F}) \times \mathbb{P}^{r-1}(\mathbb{F}) \\ & \overline{T} & \mapsto & (\overline{T}\ ;\ (g_1(\overline{T}) : g_2(\overline{T}) : \ldots : g_r(\overline{T}))) \end{array}$$

which is a blowup of $\mathbb{P}^n(\mathbb{F})$ along I. Now, we compose this map with the Segre embedding of $\mathbb{P}^n(\mathbb{F}) \times \mathbb{P}^{r-1}(\mathbb{F})$ to get the blowup of $\mathbb{P}^n(\mathbb{F})$ as isomorphic to the subvariety $\mathcal{W}$ of $\mathbb{P}^{rn+r-1}(\mathbb{F})$ parametrized by $P := [t_0g_1 : \ldots : t_0g_r : \ldots : t_ng_1 : \ldots : t_ng_r]$. On the other hand, $\mathrm{U}ltra(\Psi)$ is (linearly) $\mathbb{L}$-isomorphic to the affine variety $\mathcal{V}$ parametrized by $\Psi_a \times \Psi_a^{\sigma_2} \times \cdots \times \Psi_a^{\sigma_r}$ (see [3]). Projectively, the parametrization $\Psi_a \times \Psi_a^{\sigma_2} \times \cdots \times \Psi_a^{\sigma_r}$ can be expressed as $[L_0g_1 : L_1g_1 : \ldots : L_ng_1 : L_1^{\sigma_2}g_2 : \ldots : L_n^{\sigma_2}g_2 : \ldots : L_1^{\sigma_r}g_r : \ldots : L_n^{\sigma_r}g_r]$. This variety is isomorphic to the subvariety of $\mathbb{P}^{nr+r-1}$ parametrized by

$$Q := [L_0g_1 : \ldots : L_ng_1 : L_0^{\sigma_2}g_2 : \ldots : L_n^{\sigma_2}g_2 : \ldots : L_0^{\sigma_r}g_r : \ldots : L_n^{\sigma_r}g_r]$$

since $L_0^{\sigma_i}g_i = L_0g_1$, and we are just duplicating the first coordinate of each block.

Since by definition $\Psi^{\sigma_i}(\overline{T})^t = L^{\sigma_i} \cdot \overline{T}^{\,t}$, then

$$Q = (g_i\Psi^{\sigma_i})^t = L^{\sigma_i}(g_i \cdot \overline{T})^t$$

where the super-index t denotes the transpose of the matrix. Finally observe that the parametrization provided by the right side of the formula above is just a re-ordering of the coordinates of P. Thus, $\mathcal{W}$ is linearly isomorphic to the projective closure of $\mathrm{U}ltra(\Psi)$. □

Remark 2.3. The center of the blowup, i.e. the variety defined by the ideal I, is

$$\mathcal{Z} = \bigcup_{L^{\sigma_i} \neq L^{\sigma_j}} \{L_0^{\sigma_i} = L_0^{\sigma_j} = 0\}.$$

If L_0 does not have coefficients in $\mathbb{K}$, then the ultraquadric is not a linear variety.

Corollary 2.4.

(1) $\mathrm{U}(\Psi)$ *is an isomorphism of* $\mathbb{P}^n(\mathbb{F}) \setminus \mathcal{Z}$ *onto its image. In particular, the affine part of* $\mathrm{U}ltra(\Psi)$ *is always smooth.*

(2) *Let* $r \leq n$ *and let* $L_0^{\sigma_1}, \ldots, L_0^{\sigma_r}$ *be hyperplanes in general position in* $\mathbb{P}^n(\mathbb{F})$. *Then, the ultraquadric* $\mathrm{U}ltra(\Psi)$ *is (linearly isomorphic to) a toric variety.*

In some applications it is interesting to restrict to real-complex case and surfaces, see for instance [3]. Hence, we take now a closer look to the case of algebraic extensions of degree $r = 2$ and automorphisms of $\mathbb{P}^2(\mathbb{F})$. Next result describes in this context the intersection of ultraquadrics with the hyperplane at infinity (cf. [4] for the hypercircle case).

Corollary 2.5. *Let* $r = 2$, $\Phi = [L_0 : L_1 : L_2] \in \mathbf{PGL}_{\mathbb{L}}(2)$, *let* $x^2 + ax + b$ *be the minimal polynomial of* α *over* $\mathbb{K}$.

(1) *If the primitive part of* L_0 *is in* $\mathbb{K}[s,t]$, *then* $\mathrm{U}ltra(\Psi)$ *is a plane.*

(2) *If the primitive part of* L_0 *is in* $\mathbb{L}[s,t] \setminus \mathbb{K}[s,t]$, *then* $\mathrm{U}ltra(\Psi)$ *is linearly isomorphic to a blowup of the plane at a point. In particular, it is smooth.*

Moreover, let $\{L_0 = 0\}$ *and* $\{L_0^\sigma = 0\}$ *be the lines defined, respectively, by the denominator and by its conjugate, let* $p = \{L_0 = L_0^\sigma = 0\}$ *be the intersection point. Then, the intersection of* $\mathrm{U}ltra(\Psi)$ *with the hyperplane at infinity consists in three lines* $\mathcal{L}$, $\mathcal{L}^\sigma$, E. *Furthermore:*

1. $\mathrm{U}ltra(\Psi)$ *is the blowup of the plane at* p.
2. $\mathcal{L}$ *does not depend on* Ψ *(and hence neither does* $\mathcal{L}^\sigma$*), it only depends on the minimal polynomial of* α. *In fact* $\mathcal{L} = V(\{x_0, 2x_1 - (2\alpha + a)x_2, 2x_3 - (2\alpha + a)x_4\})$.
3. $q = [0 : (\alpha + a/2)L_1(p) : L_1(p), (\alpha + a/2)L_2(p) : L_2(p)] \in \mathcal{L}$ *is such that* $\mathcal{L} \setminus \{q\}$ *corresponds, by the parametrization, to* $\{L_0 = 0\} \setminus \{p\}$.
4. $E = \langle q, q^\sigma\rangle$, *the line through* q *and* q^σ, *is the exceptional divisor of the blowup.*

Example 2.6. Consider the extension $\mathbb{R} \subseteq \mathbb{R}(\mathrm{i}) = \mathbb{C}$ and the automorphism of the plane given by $L(t_0 : t_1 : t_2) = (t_1 + \mathrm{i}t_2, t_0, t_1)$. Then $L_0 = \{t_1 + \mathrm{i}t_2 = 0\}$, $L_0^\sigma = \{t_1 - \mathrm{i}t_2 = 0\}$. The center of the blowup is the origin $(1 : 0 : 0)$. $\mathrm{U}ltra(L) = V(x_2x_3 - x_1x_4, x_3 - x_3^2 - x_4^2, x_1 - x_1x_3 - x_2x_4) \subseteq \mathbb{C}^5$. The projectivization of $\mathrm{U}ltra(L)$ intersects the hyperplane at infinity at the three lines $\mathcal{L} = V(x_0, x_1 - \mathrm{i}x_2, x_3 - \mathrm{i}x_4)$, $\mathcal{L}^\sigma = V(x_0, x_1 + \mathrm{i}x_2, x_3 + \mathrm{i}x_4)$ and $E = V(x_0, x_3, x_4)$. In this case $q = (0 : \mathrm{i} : 1 : 0 : 0)$. This information suggests to parametrize the ultraquadric by intersecting it with the pencils of hyperplanes $x_1 + ix_2 = t$, $x_3 + ix_4 = s$, yielding the parametrization $x_1 = st/(2s-1)$, $x_2 = (s-1)t/(2is - i)$, $x_3 = s^2/(2s-1)$, $x_4 = (-is^2 + is)/(2is - 1)$.

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