On the Shape of Curves that are Rational in Polar Coordinates

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Abstract

In this paper we provide a computational approach to the shape of curves which are rational in polar coordinates, i.e. which are defined by means of a parametrization $(r(t), \theta(t))$ where both $r(t), \theta(t)$ are rational functions. Our study includes theoretical aspects on the shape of these curves, and algorithmic results which eventually lead to an algorithm for plotting the "interesting parts" of the curve, i.e. the parts showing the main geometrical features of it. On the theoretical side, we prove that these curves, with the exceptions of lines and circles, cannot be algebraic (in cartesian coordinates), we characterize the existence of infinitely many self-intersections, and we connect this with certain phenomena which are not possible in the algebraic world, namely the existence of limit circles, limit points, or spiral branches. On the practical side, we provide an algorithm which has been implemented in the computer algebra system Maple to visualize this kind of curves. Our implementation makes use (and improves some aspects of) the command `polarplot` currently available in Maple for plotting curves in polar form.

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1 Introduction

Plotting and correct visualization of algebraic curves, both in the case when they are implicitly defined by means of a polynomial $f(x, y) = 0$, or by a parametrization $\varphi(t) = (x(t), y(t))$ with $x(t), y(t)$ being rational, have received a great deal of attention in the literature on scientific computation (see for example [1], [4], [6], [7], [8], [9], [11]). With this paper, we want to initiate a similar study for curves which are written in polar coordinates. Such curves may appear in Engineering and Physics, in particular in Mechanics (very specially in Celestial Mechanics), and also in Cartography. In this sense, here we address those curves which are rational when considered in polar form, i.e. curves defined by means of a parametrization $(r(t), \theta(t))$ where both $r(t), \theta(t)$ are rational functions. Arquimedes' spiral, Cote's spiral under certain conditions, Fermat's spiral or Lituus' spiral (some of these can be found in www.mathematische-basteleien.de/spiral.htm), for example, belong to this class; also, in Cartography these curves may appear when using for example conic, pseudo-conic or polyconic projections ([3], [13]).

It is quite natural to start wondering if this kind of curves can be algebraic, when considered in cartesian coordinates. However, one may prove (see Theorem 1 in Subsection 2.1) that, with the exceptions of lines and circles, this cannot happen. As a consequence they can exhibit properties that algebraic curves cannot have. Here we analyze some of them which are relevant from the point of view of plotting, in particular the existence or not of infinitely many self-intersections (see Subsection 2.2), the appearance of limit points, limit circles and spiral branches, all of them introduced in Subsection 2.3, and the relationship between both phenomena. Our results are of algorithmic nature, and so they can be used to effectively detect all these situations just from the parametrization (in polar form) defining the curve.

The possibility of detecting in advance the above phenomena is very useful to improve the plotting of these curves, which can be really "devilish". To give an idea, one may use the computer algebra system Maple, where the instruction `polarplot` is available (we have also tried other packages like Maxima, SAGE and Mathematica; however, either no similar instruction was available, or the corresponding command behaved in a very similar way to `polarplot`). This instruction allows to draw curves defined in polar coordinates either by means of an equation $r = f(\theta)$, or by a parametrization $(r(t), \theta(t))$, and has a nice performance for simple curves, but may not be enough for illustrating the behavior of a more complicated curve. As an example, one may consider the following parametrizations in polar coordinates: (1) $r = \dfrac{t^2}{t^2 - 11t + 30}, \theta = \dfrac{t^2 + 78}{t^2 + 1}$; (2) $r = t, \theta = \dfrac{t^2 + 14}{t^2 + 1}$; (3) $r = t, \theta = \dfrac{t^3 + 1}{t^2 - 3t + 2}$. If one uses `polarplot` to visualize these curves, one obtains the outputs in Fig. 1: here the curves (1), (2), (3) are displayed from left to right; in (1) and (2) we have asked Maple to plot the curve for $t \in (-\infty, \infty)$, and in the case of (3) we have chosen $t \in (-2.01, 2.01)$ (because r, θ are both non-bounded for $t \to \pm\infty$). However, in the three cases it is clear that the output is not enough to properly understand the behavior of the curve.

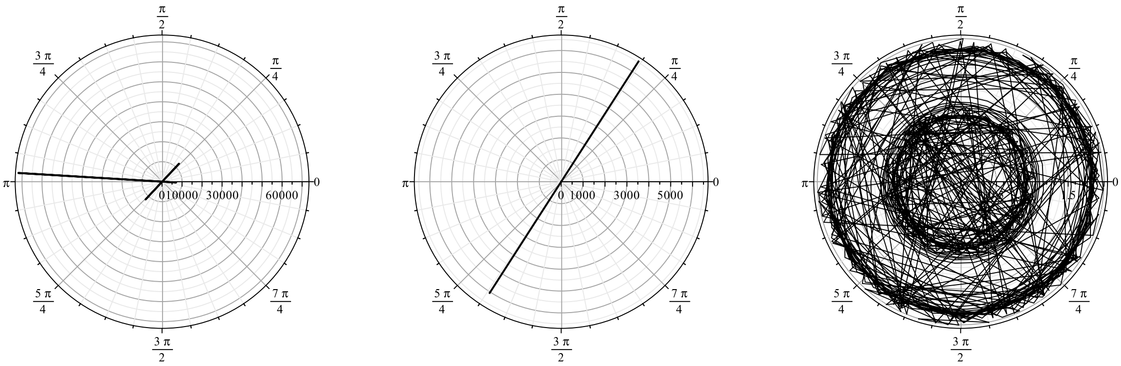


Fig. 1. Some complicated curves plotted with `polarplot`

By using our ideas, one can obtain an algorithm which provides, for a given curve of the considered kind, information and several plottings corresponding to the more interesting parts of it. We have implemented this algorithm, that we call `polares` (the Spanish word for "polar coordinates"), in Maple 15. Our algorithm analyzes the curve, detects its main features, and uses the command `polarplot`, but for plotting the curve over intervals (generated by our algorithm), each one showing certain features of the curve; together with the provided information, this helps to clarify the behavior of the curve. The outputs of our algorithm for the above curves can be checked in Subsection 3.2 (Figures 12 and 13 for (1) and (2) respectively) and Subsection 3.4 (Figure 15 for (3)).

The paper is structured as follows: in Section 2, the reader will find the theoretical results on the shape of these curves. In Section 3, we provide some details on the algorithm, together with several examples of outputs. Finally, in Section 4 we present some conclusions and suggest some future lines of research.

2 Geometrical Properties of Curves Rational in Polar Coordinates

2.1 Preliminaries and First Properties

In the present paper, given a point $P \in \mathbb{R}^2$, its polar coordinates are denoted by (r, θ), and its cartesian coordinates by (x, y). Notice that the polar coordinates of P are not unique, since (r, θ), $(r, \theta + 2k\pi)$ with $k \in \mathbb{Z}$, or $(-r, \theta + (2\ell + 1)\pi)$ with $\ell \in \mathbb{Z}$, define the same point.

We are interested in analyzing the geometry of a planar curve $\mathcal{C}$ which is *rational* when parametrized in polar coordinates. In other words, there exist four real polynomials $A(t)$, $B(t)$, $C(t)$, $D(t)$ not all of them constant, with $\gcd(A, B) = \gcd(C, D) = 1$, such that

$$\varphi(t) = (r(t), \theta(t)) = \left(\frac{A(t)}{B(t)}, \frac{C(t)}{D(t)} \right), \ t \in \mathbb{R},$$

parametrizes $\mathcal{C}$ in polar coordinates. We will refer to $\varphi(t)$ as a "polar parametrization" of $\mathcal{C}$ and say that $\mathcal{C}$ is "algebraic in polar coordinates"; certainly this does not mean that $\mathcal{C}$ is algebraic itself (i.e. that it has an implicit equation in x, y).

Let $\mathcal{C}^\star$ be the rational planar curve parametrized by $\varphi(t)$ over the (r, θ)–plane. This curve $\mathcal{C}^\star$ will be useful in this section and the next one for proving certain facts on $\mathcal{C}$. Anyway our goal is not to describe $\mathcal{C}^\star$ (which could be done, for example, by using the results in [1]), but $\mathcal{C}$.

Furthermore we assume that $\varphi(t)$, as a parametrization of $\mathcal{C}^\star$, is *proper*, i.e. that it is injective for almost all values of t (equivalently, that almost all points of $\mathcal{C}^\star$ are generated by just one t-value). In order to check this property, we consider the polynomials

$$G_1(t, s) = A(t)B(s) - A(s)B(t)$$
$$G_2(t, s) = C(t)D(s) - C(s)D(t)$$

obtained by setting $r(t) = r(s)$ and $\theta(t) = \theta(s)$, and clearing denominators. Then it is well-known that $\varphi(t)$ is proper iff $\gcd(G_1, G_2) = t - s$ (for modern references, see Theorem 4.30 in [12]); furthermore, if $\varphi(t)$ is not proper then it can always be properly reparametrized by applying the algorithm in Chapter 6 of [12]. Also, we will say that a point $P_0 \in \mathcal{C}^\star$ is *reached* by the parametrization if there exists some $t_0 \in \mathbb{C}$ such that $\varphi(t_0) = P_0$. It can be proven (see

Proposition 4.2 in [2]) that the only point of $\mathcal{C}^\star$ that may not be reached by the parametrization is

$$P_\infty = \lim_{t\to\infty}\varphi(t)$$

whenever it exists. Furthermore, if $\varphi(t)$ is proper then the real points of $\mathcal{C}^\star$ that are reached by complex, but not real, values of the parameter are, except perhaps for P_∞, isolated points of $\mathcal{C}^\star$ (see also Proposition 4.2 in [2]). Therefore, the real part of $\mathcal{C}^\star$ can be expressed as

$$\varphi(\mathbb{R}) \cup P_\infty \cup \{\text{Finitely many isolated points.}\}$$

In the sequel we will discard the isolated points of $\mathcal{C}^\star$ and we will focus on the remaining part of $\mathcal{C}^\star$. Since polar coordinates provide a natural mapping Π (not 1:1) from $\mathcal{C}^\star$ onto $\mathcal{C}$, we will say that a point of $\mathcal{C}$ is *reached* by $\varphi(t)$ if it has the form $\Pi(P)$, where $P \in \mathcal{C}^\star$ has been reached by $\varphi(t)$. In this language, notice that we are identifying $\mathcal{C} = \Pi(\tilde{\mathcal{C}}^\star)$ where $\tilde{\mathcal{C}}^\star$ is the real part of $\mathcal{C}^\star$, discarding isolated singularities.

Now if $\mathcal{C}$ is either a circle centered at the origin or a line passing through the origin, then it is algebraic in algebraic and polar form at the same time (in fact, $\mathcal{C}^\star$ is represented by an equation of the type $r - r_0 = 0$ or $\theta - \theta_0 = 0$). The next theorem proves that circles and lines are the only cases when this phenomenon happens. We acknowledge here the help of Fernando San Segundo for proving the theorem.

Theorem 1 *If $\mathcal{C}$ is an irreducible, real algebraic curve and is rational in polar coordinates, then it is either a circle centered at the origin or a line passing through the origin.*

Proof. Let $g(x,y) \in \mathbb{R}[x,y]$ be the implicit equation of $\mathcal{C}$, and let $f(r,\theta) \in \mathbb{R}[r,\theta]$ be the implicit equation of $\mathcal{C}^\star$ in polar form. Now if f depends only on r, since $\mathcal{C}$ is real and irreducible we deduce that it is a circle; similarly if f depends only on θ, for the same reason it must be a line passing through the origin. So let us assume that f depends on both r, θ. Now let $h(r,\theta) = g(r\sin(\theta), r\cos(\theta))$, in simplified form. This is an analytic function in r, θ with the property that if a point $P = (x_0, y_0) \in \mathcal{C}$ is represented by *any* pair $(r_0, \theta_0) \in \mathbb{C}^2$, then $h(r_0, \theta_0) = 0$. As a consequence, every zero of $f(r,\theta)$ is also a zero of $h(r,\theta)$. In this situation, consider the resultant $M(\theta,u,v) = \mathrm{Res}_r(f(r,\theta), h(r,u,v))$, where $h(r,u,v)$ is the function obtained when substituting formally $\sin(\theta) = u$ and $\cos(\theta) = v$ in $h(r,\theta)$. Since $\mathcal{C}^\star$ is rational then it is irreducible. Thus, since f depends on both r, θ, it has no factor only depending on r, and therefore $f(r,\theta)$ and $h(r,u,v)$ cannot have any factor in common; hence, $M(\theta,u,v)$ cannot be identically 0. Observe also that $M(\theta,u,v)$ must depend explicitly on either u or v, and on θ. Now we distinguish two situations:

(1) *$M = M(\theta, u)$ or $M = M(\theta, v)$, but M does not depend simultaneously on u, v.* Let us assume w.l.o.g. that M depends on u. Then by Lemma 4.3.1 in [14] we have that the resultant $\mathrm{Res}_r(f(r,\theta), h(r,u,v))$ specializes properly when $u = \sin(\theta)$ and $v = \cos(\theta)$, i.e.

$$M(\theta, \sin(\theta)) = \mathrm{Res}_r(f(r,\theta), h(r, \sin(\theta), \cos(\theta))).$$

Since f depends explicitly on both r, θ we can find an open interval $I \subset \mathbb{R}$ such that for every $\theta_0 \in I$, the equation $f(r, \theta_0)$ and therefore also $h(r, \theta_0) = 0$, have at least one real solution. Hence, by well-known properties of resultants, for all $\theta_0 \in I$ it holds that $\theta = \theta_0$ is a zero of $M(\theta, \sin(\theta))$, i.e. $M(\theta, \sin(\theta))$ vanishes over I. Hence, by the Identity Theorem (see page 81 in [10]), and taking into account that M is analytic, it holds that $M(\theta, \sin(\theta))$ is identically 0. Since M is a polynomial in θ and $\sin(\theta)$, this implies that θ and $\sin(\theta)$ are algebraically dependent. However, this cannot happen because $\sin(\theta)$ is a trascendental function.

(2) *$M = M(\theta, u, v)$ (i.e. M explicitly depends on u, v, at the same time).* In this case, reasoning as in case (1) we end up with an algebraic relationship $M(\theta, \sin(\theta), \cos(\theta)) = 0$. Combining this relationship with the well-known formula $\sin^2(\theta) + \cos^2(\theta) = 1$ (again substituting formally $\sin(\theta) = u$ and $\cos(\theta) = v$ and using resultants, for instance), we can also find an algebraic between $\theta, \cos(\theta)$ or $\theta, \sin(\theta)$, which again cannot happen because $\sin(\theta)$ and $\cos(\theta)$ are trascendental functions.

So, we deduce that under the considered hypotheses, f cannot simultaneously depend on r, θ, and therefore the statement follows. □

If the curve $\mathcal{C}$ in the above theorem is reducible, it suffices to reason for each component. So, we deduce the following corollary.

Corollary 2 *The only algebraic curves which are rational in polar coordinates, are circles centered at the origin, and lines passing through the origin.*

The above theorem proves that in general the curves we are dealing with are not algebraic. Hence, it is expectable that we encounter phenomena which are different from those arising in the algebraic world. Some of these phenomena are analyzed in the next subsections.

Hereafter we exclude the cases when either $r(t)$ or $\theta(t)$ are constant; notice that these cases correspond to circles centered at the origin and lines passing through the origin respectively.

Since the polar coordinates of a point are not unique, we have that a point generated by $t \in \mathbb{R}$ corresponds to a self-intersection of $\mathcal{C}$ if it exists $s \in \mathbb{R}$, $t \neq s$, and $k \in \mathbb{Z}$, such that (t, s, k) is solution of some of the following two systems:

$$(\star)_1 = \begin{cases} r(t) = r(s) \\ \theta(t) = \theta(s) + 2k\pi \end{cases} \quad (\star)_2 = \begin{cases} r(t) = -r(s) \\ \theta(t) = \theta(s) + (2k+1)\pi \end{cases}$$

Moreover, when $P_\infty = (r_\infty, \theta_\infty)$ exists, we will also have to consider the self-intersections involving P_∞, i.e. the solutions of

$$(\star)_3 = \begin{cases} r_\infty = r(s) \\ \theta_\infty = \theta(s) + 2k\pi \end{cases} \quad (\star)_4 = \begin{cases} r_\infty = -r(s) \\ \theta_\infty = \theta(s) + (2k+1)\pi \end{cases}$$

So, in order to study the self-intersections of $\mathcal{C}$ we have to study the above systems. Additionally, whenever the equation $r(t) = 0$ has more than one solution, or one solution and $r_\infty = 0$, the origin is also a self-intersection.

Now in the algebraic case, Bezout's theorem forces every algebraic curve to have finitely many self-intersections. However, in our case this does not necessarily hold: the number of self-intersections can be either finite, or infinite. Hence, in this subsection we will address the problem of detecting whether we have infinitely many self-intersections, or not.

Notice that since $r(t)$ is rational, the number of t-values reaching a certain point $P \in \mathcal{C}$ must be necessarily finite; so, the existence of infinitely many intersections implies not that a point is crossed by infinitely many real branches of $\mathcal{C}$, but that infinitely many points are crossed by more than one branch of $\mathcal{C}$. In order to detect whether we have finitely many self-intersections or not, it suffices to analyze $(\star)_1$ and $(\star)_2$; indeed, if P_∞ exists, since $r(t)$ is rational it is clear that the first equations $r_\infty = r(t)$ and $r_\infty = -r(t)$ of $(\star)_3$ and $(\star)_4$, respectively, have just finitely many solutions, and therefore that we have just finitely many self-intersections involving P_∞. So, in the sequel we focus on $(\star)_1$ and $(\star)_2$.

We start with $(\star)_1$. For this purpose, we write

$$r(t) - r(s) = \frac{A(t)B(s) - A(s)B(t)}{B(t)B(s)},$$

$$\theta(t) - \theta(s) - 2k\pi = \frac{C(t)D(s) - C(s)D(t) - 2k\pi D(t)D(s)}{D(t)D(s)}.$$

We denote the numerator of $r(t)-r(s)$ by $\alpha(t,s)$, and the numerator of $\theta(t) - \theta(s) - 2k\pi$ by $\beta(t,s,k)$.

Lemma 3 *There do not exist $a(t,s) \neq 1$ and $k_0 \in \mathbb{Z}$ (in fact, $k_0 \in \mathbb{R}$) such that $a(t,s)$ simultaneously divides $\gcd(\alpha(t,s), \beta(t,s,k_0))$ and $B(t) \cdot B(s) \cdot D(t) \cdot D(s)$.*

Proof. Assume that we have an irreducible polynomial $a(t,s)$ and $k_0 \in \mathbb{Z}$ such that $a(t,s)$ divides both $\gcd(\alpha(t,s), \beta(t,s,k_0))$ and $B(t) \cdot B(s) \cdot D(t) \cdot D(s)$. We need to show that $a(t,s) = 1$. Suppose that $a(t,s)$ divides $B(t)$ (resp. $B(s)$). Since it also divides $\gcd(\alpha(t,s), \beta(t,s,k_0))$, it divides $\alpha(t,s)$, and therefore it also divides $A(t)$ (resp. $A(s)$). On the other hand, $\gcd(A(t), B(t)) = 1$ by hypothesis. That shows that $a(t,s) = 1$. If $a(t,s)$ divides $D(t)$ (resp. $D(s)$) we argue similarly with $\beta(t,s,k)$. □

Then we are ready to proceed with the following theorem, that will have important consequences on the study of $(\star)_1$.

Theorem 4 *For any $k \neq 0$, $k \in \mathbb{Z}$, (in fact, $k \in \mathbb{R}$) the system $(\star)_1$ has finitely many solutions.*

Proof. In order to prove the statement, we need to show that for $k \in \mathbb{Z}$, $k \neq 0$, $\gcd(\alpha(t,s), \beta(t,s,k)) = 1$. For this purpose, assume by contradiction that there exists some $k \in \mathbb{Z}$, $k \neq 0$, such that $H(t,s) = \gcd(\alpha(t,s), \beta(t,s,k)) \neq 1$. Then $H(t,s)$ defines an algebraic curve $\mathcal{H}$ over $\mathbb{C}^2$; furthermore, since by hypothesis $k \neq 0$, $H(t,s)$ cannot be $t - s$. So, there are infinitely many points $(t_0, s_0) \in \mathcal{H}$ with $t_0 \neq s_0$. Now by Lemma 3 only finitely many of them fulfill $B(t) \cdot B(s) \cdot D(t) \cdot D(s) = 0$. So, for almost all points $(t_0, s_0) \in \mathcal{H}$ it holds that $r(t)$, $r(s)$, $\theta(t)$ and $\theta(s)$ are well-defined, and $r(t_0) = r(s_0)$, $\theta(t_0) = \theta(s_0) + 2k\pi$. Since for a fixed t_0 there can only be finitely many values of s such that $r(t_0) = r(s)$ (because $r(t)$ is rational) we deduce then that there are infinitely many points $(r_0, \theta_0) \in \mathcal{C}^\star$ such that $(r_0, \theta_0 + 2k\pi) \in \mathcal{C}^\star$ too. In other words, the curves defined over the (r,θ)-plane by $f(r,\theta)$ and $f(r, \theta + 2k\pi)$ have infinitely many points in common, and since f is irreducible and both have the same degree, then they must define $\mathcal{C}^\star$. Now let $(r_0, \theta_0) \in \mathcal{C}^\star$; then $(r_0, \theta_0 + 2k\pi)$ is a zero of $f(r, \theta + 2k\pi)$, and since $f(r, \theta + 2k\pi)$ also defines $\mathcal{C}^\star$ then it is also a point of $\mathcal{C}^\star$. Following the same reasoning, we conclude that $(r_0, \theta_0 + 4k\pi)$ also belongs to $\mathcal{C}^\star$, and in fact that $(r_0, \theta_0 + 2nk\pi) \in \mathcal{C}^\star$ for all $n \in \mathbb{N}$. Since

$k \neq 0$ we have that these are all different points of $\mathcal{C}^\star$. Hence, we have that $\mathcal{C}^\star$ intersects the line $r = r_0$ at infinitely many points. But this is impossible because $\mathcal{C}^\star$ is algebraic. $\square$

We can obtain similar results for $(\star)_2$. For this purpose, we denote the numerator of $r(t) + r(s)$ by $\mu(t,s)$, and the numerator of $\theta(t) - \theta(s) - (2k+1)\pi$ by $\nu(t,s,k)$. Then the following lemma, analogous to Lemma 3, holds.

Lemma 5 *There does not exist $b(t,s) \neq 1$ and $k_0 \in \mathbb{Z}$ (in fact, $k_0 \in \mathbb{R}$) such that $b(t,s)$ simultaneously divides $\gcd(\mu(t,s), \nu(t,s,k_0))$ and $B(t) \cdot B(s) \cdot D(t) \cdot D(s)$.*

The following theorem, very similar to Theorem 4, holds.

Theorem 6 *For any $k \in \mathbb{Z}$ (in fact, $k_0 \in \mathbb{R}$), it holds that $\gcd(\mu(t,s), \nu(t,s,k)) = 1$.*

Proof. Arguing by contradiction as in the proof of Theorem 4, we conclude that $f(r,\theta)$ and $f(-r, \theta + 2k\pi)$ define the same curve (namely, $\mathcal{C}^\star$). So, let $(r_0, \theta_0) \in \mathcal{C}^\star$ where $r - r_0$ does not divide $f(r,\theta)$. Then $(-r_0, \theta_0 + (2k+1)\pi) \in \mathcal{C}^\star$, and for the same reason $(r_0, \theta_0 + (4k+2)\pi) \in \mathcal{C}^\star$ too. Proceeding this way we get that all the points of the form $(r_0, 2n(k+1)\pi)$, with $n \in \mathbb{N}$, belong to $\mathcal{C}^\star$. For any value $k \in \mathbb{Z}$ all these points are different; so, we get that the intersection of $\mathcal{C}^\star$ with the line $r = r_0$ consists of infinitely many different points. But this cannot happen because $\mathcal{C}^\star$ is algebraic. $\square$

Hence, by Theorem 4 and Theorem 6, we obtain the following result on the existence of infinitely many self-intersections of $\mathcal{C}$.

Corollary 7 *$\mathcal{C}$ has infinitely many self-intersections if and only if $(\star)_1$ or $(\star)_2$ have solutions for infinitely many values $k \in \mathbb{Z}$.*

Based on Corollary 7, we have the following result which provides a sufficient condition for $\mathcal{C}$ to have finitely many self-intersections.

Theorem 8 *If $\theta(t)$ is bounded, then there are at most finitely many self-intersections.*

Proof. If $\theta(t)$ is bounded, the number of integer values of k satisfying the second equation of $(\star)_1$ or $(\star)_2$ for some (t,s) is necessarily finite. Then the result follows from Corollary 7. $\square$

The converse of Theorem 8 is not necessarily true (see Example 2). So, we still need a characterization of the existence of infinitely many self-intersections.

Lemma 9 *The polynomial $\mathrm{Res}_s(\alpha(t,s), \beta(t,s,k))$ cannot be constant. More precisely, it has positive degree in k.*

Proof. By Theorem 4, $\mathrm{Res}_s(\alpha(t,s),\beta(t,s,k))$ cannot be identically 0. Moreover, by writing explicitly the system $\mathcal{S}\equiv\{\alpha(t,s)=0,\beta(t,s,k)=0\}$,

$$\begin{cases} A(t)B(s)-A(s)B(t)=0 \\ C(t)D(s)-C(s)D(t)-2k\pi D(t)D(s)=0 \end{cases}$$

we observe that the points $(t',t',0)$, $t'\in\mathbb{R}$, are solutions of $\mathcal{S}$. So, for any $t'\in\mathbb{R}$, $\mathrm{Res}_s(\alpha(t,s),\beta(t,s,k))$ vanishes at $(t',0)$ and therefore it cannot be constant. For the same reason, $\mathrm{Res}_s(\alpha(t,s),\beta(t,s,k))$ cannot be an univariate polynomial in t. In fact, k is a divisor of $\mathrm{Res}_s(\alpha(t,s),\beta(t,s,k))$. □

Lemma 10 *The polynomial* $\mathrm{Res}_s(\mu(t,s),\nu(t,s,k))$ *cannot be constant. Moreover, if* $(\star_2)$ *has solutions for infinitely many values of* $k\in\mathbb{Z}$*, then* $\mathrm{Res}_s(\mu(t,s),\nu(t,s,k))$ *has positive degree in* k*.*

Proof. By Theorem 6, $\mathrm{Res}_s(\mu(t,s),\nu(t,s,k))$ cannot be identically 0. Moreover, by writing explicitly the system $\mathcal{S}'\equiv\{\mu(t,s)=0,\nu(t,s,k)=0\}$,

$$\begin{cases} A(t)B(s)+A(s)B(t)=0 \\ C(t)D(s)-C(s)D(t)-(2k+1)\pi D(t)D(s)=0 \end{cases}$$

we observe that if $A(t')=0$ with $D(t')\neq 0$ then there exists $k'\in\mathbb{R}$ such that the point $(t',t',-1/2)$ is a solution of $\mathcal{S}$. If $D(t')=0$, then the points (t',t',k) are solutions of $\mathcal{S}$ for any k. As a consequence, the polynomial $\mathrm{Res}_s(\mu(t,s),\nu(t,s,k))$ cannot be a constant.

Now assume that $(\star)_2$ has infinitely many solutions. That implies that there are infinitely (t',s',k') solutions of $\mathcal{S}$ with $D(t')\neq 0$, $D(s')\neq 0$, $B(t')\neq 0$ and $B(s')\neq 0$. Due to Theorem 6 and the linearity of k in $\nu(t,s,k)$,the polynomial $\mathrm{Res}_s(\mu(t,s),\nu(t,s,k))$ has positive degree in both t and k. □

Next we denote by $\xi_1(t,k)$ (resp. $\xi_2(t,k)$) the result of taking out from the square-free part of $\mathrm{Res}_s(\alpha(t,s),\beta(t,s,k))$ (resp. $\mathrm{Res}_s(\mu(t,s),\nu(t,s,k))$) the univariate factors in t.

Theorem 11 $\mathcal{C}$ *has infinitely many self-intersections if and only if either* $\xi_1(t,k)=0$ *and/or* $\xi_2(t,k)=0$ *are algebraic curves (in the* $\{t,k\}$*-plane) non-bounded in* k*.*

Proof. If $\mathcal{C}$ has infinitely many self-intersections, by Corollary 7 the systems $(\star)_1$ or/and $(\star)_2$ have solutions for infinitely many values $k\in\mathbb{Z}$. Due to the linearity of k, that implies that they are also solutions for infinitely many values of t and s. Consequently the systems $\mathcal{S}\equiv\{\alpha(t,s)=0,\beta(t,s,k)=0\}$

and/or $\mathcal{S}' \equiv \{\mu(t,s)=0, \nu(t,s,k)=0\}$) have infinitely many solutions too. Since resultants are combinations of the polynomials which define the systems $\mathcal{S}$ and $\mathcal{S}'$ and they are non-zero by Lemma 9 and Lemma 10, $\xi_1(t,k)=0$ and/or $\xi_2(t,k)=0$ are non-bounded in k.

Conversely, assume that $\xi_1(t,k)$ is non-bounded in k (we would argue in a similar way with $\xi_2(t,k)=0$). Since it has by definition no univariate factors depending on t, there are at most finitely many points (t',k') with $\xi_1(t',k')=0$ where the leading coefficient of $\alpha(t,s)$ with respect to s, $D(t)$ and $B(t)$ vanish. By the specialization property of resultants, any other point of $\xi_1(t,k)=0$ corresponds to a solution of $(\star)_1$. Since $\xi_1(t,k)=0$ is non-bounded in k, observe that it contains infinitely many points with $k \in \mathbb{Z}$ and so with $t \neq s$. Therefore $\mathcal{C}$ has infinitely many self-intersections. □

The above results are illustrated in the following examples.

Example 1. Let $\mathcal{C}$ be parametrized in polar coordinates by $\varphi(t)=(t,t)$. The plotting of this curve for $t \in [-5\pi, 5\pi]$ is shown in Figure 2. One may see that $\theta(t)=t$ is not bounded; so, there might be infinitely many self-intersections. In this case we get $\xi_1(t,k)=2k\pi$ and $\xi_2(t,k)=2t-2k\pi-\pi$. Finally, it is easy to see that $\xi_1(t,k)=0$ is bounded in k, but $\xi_2(t,k)=0$ is not. So, from Theorem 11 we conclude that $\mathcal{C}$ has infinitely many self-intersections. In fact, one may see that all these self-intersections lay on the y-axis.

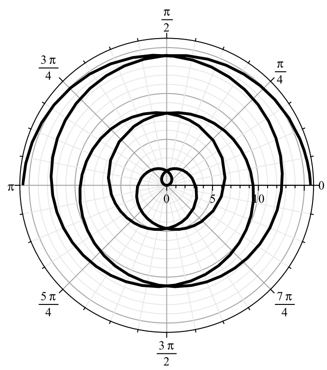


Fig. 2. $\varphi(t)=(t,t)$: $\theta(t)$ non-bounded, infinitely many self-intersections

Example 2. Let $\mathcal{C}$ be parametrized in polar coordinates by $\varphi(t)=\left(t, \dfrac{t^4}{t^2+1}\right)$. Again $\theta(t)$ is not bounded. However, the plotting of this curve for $t \in \left[-\dfrac{3}{2}, \dfrac{3}{2}\right]$, shown in Figure 3, suggests that the curve has not infinitely many self-intersections (in fact, it has no self-intersections at all). Let us check this by using Theorem 11. We get $\xi_1(t,k)=2k\pi$ and $\xi_2(t,k)=(2k+1)\pi$. Both curves are clearly bounded in k; therefore, we conclude that there are at most finitely many self-intersections.

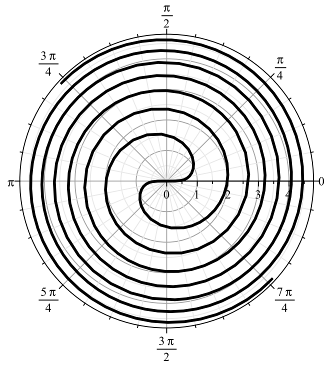


Fig. 3. $\varphi(t) = \left(t, \dfrac{t^4}{t^2+1}\right)$: $\theta(t)$ non-bounded, finitely many self-intersections

Example 3. Let $\mathcal{C}$ be parametrized by $\varphi(t) = \left(\dfrac{t}{t^2+1}, \dfrac{t^2}{t^2+1}\right)$. Since $\theta(t) = \dfrac{t^2}{t^2+1}$ is bounded, from Theorem 8 it follows that there are at most finitely many self-intersections. Furthermore, $\xi_1(t,k) = 2k\pi(2k\pi t^2 + 2k\pi - t^2 + 1)$ and $\xi_2(t,k) = \pi(2k+1)(2k\pi t^2 + 2k\pi - t^2 + \pi t^2 + 1 + \pi)$, both bounded in k. So, by Theorem 11 we derive the same conclusion. The plotting of the curve for $t \in (-\infty, \infty)$ is shown in Figure 4.

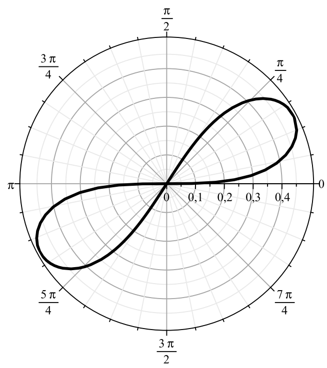


Fig. 4. $\varphi(t) = \left(\dfrac{t}{t^2+1}, \dfrac{t^2}{t^2+1}\right)$: $\theta(t)$ bounded.

2.3 Limit Circles, Limit Points and Spiral Branches

Figure 5 shows the plotting of the curves defined by $\varphi_1(t) = \left(\dfrac{t^2}{t^2+1}, \dfrac{t^3}{t^2+1}\right)$ for $t \in (0, 6\pi)$ (left), and $\varphi_2(t) = (t, 1/t)$ for $t \in (0, \pi/4)$ (right).

In the first case, one sees that the curve winds infinitely around the circle

$r = 1$, coming closer and closer to it. In the second case (which is in fact a degeneration of the first one), the curve winds infinitely around the origin of coordinates, somehow “converging” to it. Finally, in the Example 2 of Subsection 2.2, Figure 3) shows a curve that winds infinitely around the origin, but getting further and further from it. Notice that these three situations cannot arise in the algebraic world, because Bezout’s Theorem forces every algebraic curve to have finitely many intersections with every line which is not a component of it. We will refer to the first situation by saying that here the curve exhibits a limit circle ($r = 1$ in this example).

In the second case, we will say that the origin is a limit point; finally, we will refer to the third situation by saying that the curve presents a spiral branch. Along this subsection, we address these phenomena from a theoretical point of view and relate them to the appearance of infinitely many self-intersections.

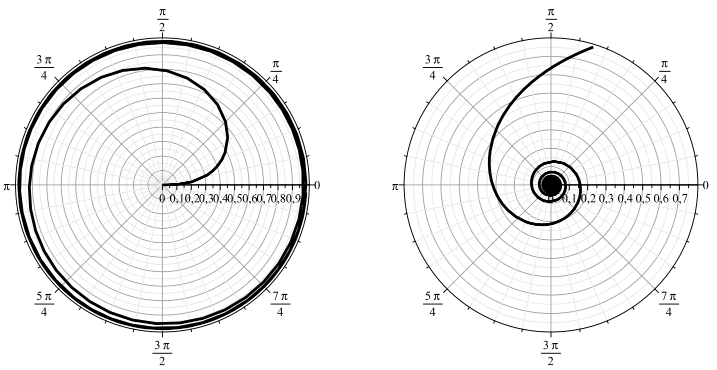


Fig. 5. Example of Limit Circle (left) and Limit Point (right)

Definition 12 *Let $t_0 \in \mathbb{R} \cup \{\pm\infty\}$. We say that $\mathcal{C}$ exhibits, for $t = t_0$:*

(1) A limit circle *if $lim_{t\to t_0} r(t) = r_0 \in \mathbb{R}$, $r_0 \neq 0$, and $lim_{t\to t_0}\theta(t) = \pm\infty$.*
(2) A limit point *at the origin if $lim_{t\to t_0} r(t) = 0$, and $lim_{t\to t_0}\theta(t) = \pm\infty$.*
(3) A spiral branch*t if $lim_{t\to t_0} r(t) = \pm\infty$, and $lim_{t\to t_0}\theta(t) = \pm\infty$*

In each case, we will say that t_0 generates the limit circle, the limit point or the spiral branch.

Notice that limit points are degenerated cases of limit circles, namely when $r_0 = 0$. One might wonder if limit circles can be centered at a point different from the origin, or if there can be limit points other than the origin. The answer, negative in both cases, is given by the following theorem.

Theorem 13 *A curve rational in polar coordinates cannot have a limit circle centered at a point different from the origin, or a limit point at a point different from the origin.*

Proof. If $\mathcal{C}$ had a limit circle centered at $P \neq (0, 0)$, or a limit point $Q \neq (0, 0)$, we would have infinitely many local maxima and minima of $r(t)$, namely the t-values generating the contact points of tangents to $\mathcal{C}$ passing through the origin (see Figure 6; local maxima and minima are shown as thick points). However, this cannot happen because $r(t)$ is a rational function, and therefore the number of t-values fulfilling $r'(t) = 0$ is finite. □

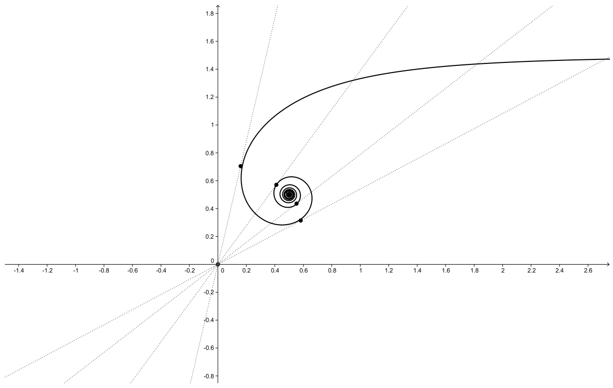


Fig. 6. A limit point different from the origin

Remark 1 *The same argument of Theorem 13 proves that a curve rational in polar coordinates cannot have any other "attractor" different from the origin, or a circle centered at the origin.*

Note that if $\mathcal{C}$ has infinitely many self-intersections, by Theorem 8 the function $\theta(t)$ is not bounded and so there must be limit points, limit circles or spiral branches. More concretely, we obtain the following result.

Proposition 14 *Let I be a subset of $\mathbb{R}$ (not necessarily an interval) with infinitely many t-values generating self-intersections of $\mathcal{C}$. Then I contains some t-value giving rise to either a limit point, or a limit circle, or a spiral branch of $\mathcal{C}$.*

Proof. By the second equation of $(\star)_1$ or $(\star)_2$ it follows that $\theta(t)$ is not bounded in I. That implies that there is $t_0 \in I$ with $\lim_{t \to t_0} \theta(t) = \pm\infty$. Thus t_0 must generate a limit point, a limit circle or a spiral branch. □

The converse of Proposition 14 is not true; for instance, in Example 2 we have a curve with a spiral branch for $t_0 = \infty$ without self-intersections.

Now we want to characterize the situation when a limit point, a limit circle or a spiral branch have infinitely many close self-intersections. This is done in the following definition.

Definition 15 *We say that a limit circle, a limit point or a spiral branch generated by $t_0 \in \mathbb{R} \cup \pm\infty$ has* infinitely many close self-intersections*, if there* *exists some real interval I verifying:*

(1) $t_0 \in I$*;*
(2) I *does not contain any other* t*-value generating a limit circle, limit point or spiral;*
(3) infinitely many $t \in I$ *generate self-intersections of* $\mathcal{C}$*.*

Definition 15 is illustrated in Figure 7; the thin lines correspond to the branch generated by the interval I appearing in Definition 15.

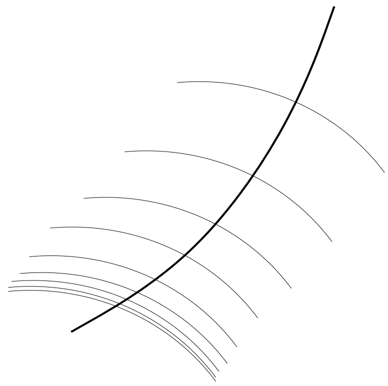


Fig. 7. Infinitely many close self-intersections

Theorem 16 *Let* $t_0 \in \mathbb{R} \cup \pm\infty$ *generating a limit circle, a limit point or a spiral branch. Then*

(1) If $t_0 \in \mathbb{R}$*, then* t_0 *has infinitely many close self-intersections if and only if* $t = t_0$ *is an asymptote of some of the curves* $\xi_1(t,k) = 0$ *or* $\xi_2(t,k) = 0$*.*
(2) If $t_0 = \pm\infty$*, then it has infinitely many close self-intersections if and only if some of the curves* $\xi_1(t,k) = 0$ *or* $\xi_2(t,k) = 0$ *exhibits an infinite branch as* $t \to \pm\infty$ *which is not an asymptote (i.e. there exists a sequence of real points* (t_n, k_n) *of the curve with* $t_n \to \pm\infty$ *and* $k_n \to \pm\infty$*).*

Proof. We prove (1); the proof of (2) is similar. Assume that there exists an interval $I \subset \mathbb{R}$ satisfying Definition 15. Then the set of points of either $\xi_1(t,k) = 0$ or $\xi_2(t,k) = 0$ with $t \in I$ must be non-bounded in k. This can only happen if there exists $t_a \in I$ such that $t = t_a$ is an asymptote of either $\xi_1(t,k) = 0$ or $\xi_2(t,k) = 0$. However, if $t = t_a$ is an asymptote then for any interval I_a containing t_a, and not containing any other t-value where $\theta(t)$ is infinite, we can find a non-bounded portion of either $\xi_1(t,k) = 0$ or $\xi_2(t,k) = 0$, therefore giving rise to infinitely many self-intersections of $\mathcal{C}$, with t-values in I_a. So, from Proposition 14 we have that $t_0 = t_a$. Using this same argument we can prove the converse statement, i.e. if $t = t_0$ is an asymptote then any interval containing it has the desired properties. □

The detection of asymptotes of an algebraic curve is addressed for example in

[15].

Corollary 17 *If $t = t_0$ has infinitely many close self-intersections, then any interval containing t_0 generates infinitely many self-intersections of $\mathcal{C}$.*

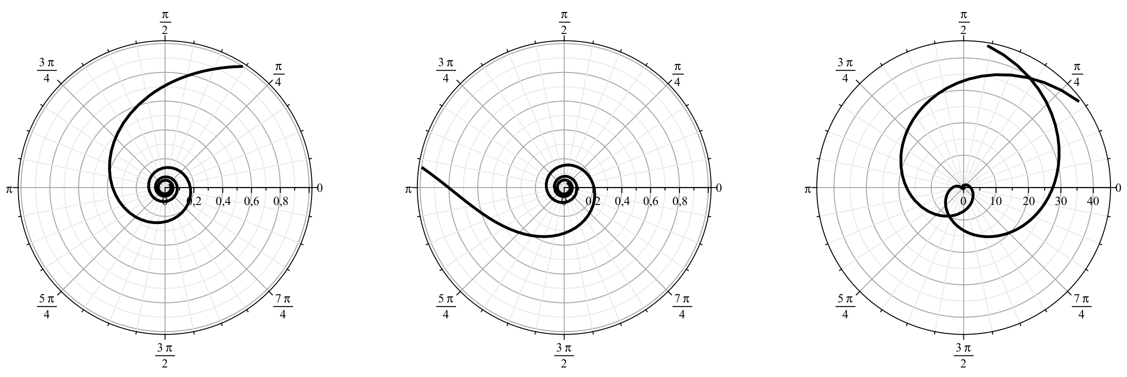
The above results are illustrated in the following examples.

Example 1 (cont.): Recall that $\xi_1(t,k) = 2k\pi$ and $\xi_2(t,k) = 2t - 2k\pi - \pi$. The second one has an infinite branch as $t \to \pm\infty$; so, from Theorem 16 we deduce that every t-interval of the form $(-\infty, a)$ or (b, ∞) contains infinitely many t-values generating self-intersections of $\mathcal{C}$.

Example 4. Consider the curve defined by $\varphi(t) = \left(\frac{1}{t^2}, \frac{t^3+t-1}{t}\right)$. This curve has a spiral branch for $t = 0$ and a limit point for $t \to \pm\infty$. In order to have a more precise idea of its behavior, one can examine Figure 8 where the curve is plotted for different values of t. A direct computation yields $\xi_1(t,k) = k(k\pi t + 1)$ and

$$\xi_2(t,k) = 4\,t^6 - 4\,\pi\,(2\,k+1)\,t^4 - 4\,t^3 + \pi^2\,(2\,k+1)^2\,t^2 + 2\,\pi\,(2\,k+1)\,t + 2.$$

The curve $\xi_2(t,k) = 0$ is empty over the reals. However a factor of $\xi_1(t,k) = 0$ corresponds to a hyperbola whose asymptotes are $t = 0$ and $k = 0$. Hence, from statement (1) of Theorem 16 we deduce that the spiral branch generated by $t_0 = 0$ has infinitely many self-intersections.



$t \in (1,5)$ $\quad$ $t \in (-5,-1)$ $\quad$ $t \in (-5,-0.15) \cup (0.15,5)$

Fig. 8. $\varphi(t) = \left(\frac{1}{t^2}, \frac{t^3+t-1}{t}\right)$

It is classical (see for example [5]) that asymptotes in polar form correspond to values $t = t_0 \in \mathbb{R} \cup \{\pm\infty\}$ such that:

- $\lim_{t \to t_0} r(t) = \pm\infty$.
- $\lim_{t \to t_0} \theta(t) = \alpha \in \mathbb{R}$.
- $\lim_{t \to t_0} r(t) \cdot (\theta(t) - \alpha) \in \mathbb{R}$.

If the above conditions hold, then the asymptote is the line parallel to the line with slope α, at distance $\delta = \lim_{t \to t_0} r(t) \cdot (\theta(t) - \alpha)$ of the origin.

Example 5. Consider the curve $\varphi_3(t) = \left(t, \dfrac{t^2}{t^2+1}\right)$. Then we have that $\lim_{t \to \pm\infty} r(t) = \pm\infty$, $\lim_{t \to \pm\infty} \theta(t) = 1$ and $\lim_{t \to \pm\infty} r(t) \cdot (\theta(t) - 1) = 0$. So, the line passing through the origin and with slope 1, i.e. $y = x$, is an asymptote of the curve for $t \to \pm\infty$ (see Figure 9).

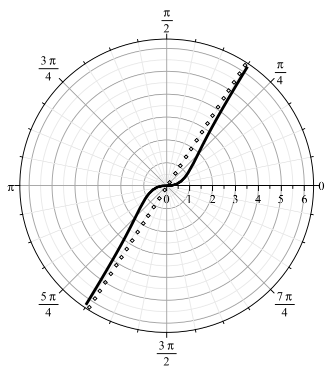


Fig. 9. An asymptote

3 Visualizing Curves that are Rational in Polar Coordinates

The goal of this section is to introduce an algorithm, `polares`, for visualizing the “interesting” part of $\mathcal{C}$, providing as well relevant information about the curve. We have implemented it in Maple 15 and tested it over several examples, some of which are presented in this section.

Next we describe the algorithm in the following different cases:

(1) $r(t)$ and $\theta(t)$ are both bounded
(2) $\theta(t)$ is bounded and $r(t)$ is not.
(3) $r(t)$ is bounded and $\theta(t)$ is not.

(4) $r(t)$ and $\theta(t)$ are both unbounded.

Obviously, the first step of the algorithm is to detect the case we are in; then the algorithm proceeds accordingly.

3.1 Algorithm `polares` *when* $r(t)$ *and* $\theta(t)$ *bounded*

`Input:` A proper polar parametrization $\varphi(t) = (r(t), \theta(t))$.
`Output:`

(1) Information about the existence of P_∞.
(2) Information about the self-intersections.
(3) Plot of $\varphi(t)$ for t in $\mathbb{R}$ using the Maple command `polarplot`.

Examples:

- $\varphi_1(t) = (\frac{t}{1+t^2}, \frac{t^2}{1+t^2})$

>polares([$\frac{t}{1+t^2}, \frac{t^2}{1+t^2}$]);
```
r and theta both bounded
Real point at the infinity such that (r, theta)=[0, 1] and the point
is [0, 0]
The point at infinity (0,0) is reached 1 times in R, so self-intersection
at the origin
```

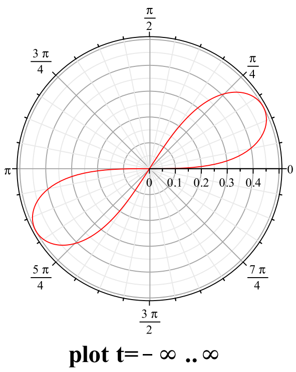


Fig. 10. $\varphi_1(t)$

- $\varphi_2(t) = (\frac{t}{1+t^2}, \frac{t^2+14}{1+t^2})$

>polares([$\frac{t}{1+t^2}, \frac{t^2+14}{1+t^2}$]);
```
r and theta both bounded
Real point at the infinity such that (r, theta)=[0, 1] and the point
is [0, 0]
```

```
The point at infinity (0,0) is reached 1 times in R, so self-intersection
at the origin
System (1) gives self-intersections for k in [[ -2,2]], k<>0
System (2) gives self-intersections for k in [[ -2,1]]
```

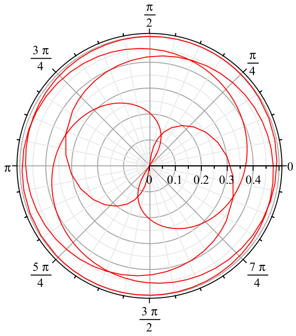


Fig. 11. $\varphi_2(t)$

3.2 Algorithm `polares` *when $\theta(t)$ is bounded and $r(t)$ not*

`Input:` A proper polar parametrization $\varphi(t) = (r(t), \theta(t))$.
`Output:`

(1) Information about the existence of P_∞.
(2) Information about the self-intersections.
(3) Information about the existence of asymptotes.
(4) Plot of $\varphi(t)$ using the Maple command `polarplot`:
   (a) If there are not values of t generating asymptotes, then we plot the curve for $t \in (-\infty, \infty)$;
   (b) Otherwise, let T_1 be the set of values of t generating asymptotes, let T_2 be the set of real values of t such that $r(t) = 0$ and, if $|\theta(t)| < 2\pi$ for $t \in \mathbb{R}$, let T_3 be the real values of t generating the maximum and minimum of $\theta(t)$.
   Let $P := T_1 \cup T_2 \cup T_3 = \{t_1, \ldots, t_m\}$ with $t_1 < \ldots < t_m$.
   Now,
      (i) If $\{t_1, t_m\} \neq \{\infty, -\infty\}$, then $P_\infty \in \mathbb{R}^2$ and its plot may be of interest. So we add $t_0 := -\infty$ and $t_{m+1} := \infty$ to the set P.
      For every $t_i \in P$, $1 < i < m$, we plot the curve $\varphi(t)$ for $t \in (\frac{t_i+t_{i-1}}{2}, \frac{t_i+t_{i+1}}{2})$. For t in $(\frac{t_i+t_{i-1}}{2}, t_i)$, the color of the plot will be red and for $(t_i, \frac{t_i+t_{i+1}}{2})$ the color will be blue.
      As for t_1 and t_m, we plot $\varphi(t)$ in the ranges $(2t_1 - \frac{t_1+t_2}{2}, \frac{t_1+t_2}{2})$ and $(\frac{t_m-t_{m-1}}{2}, 2t_m - \frac{t_m+t_{m+1}}{2})$.
      As for t_0 and t_{m+1}, we plot $\varphi(t)$ in the ranges $(-\infty, 2t_1 - \frac{t_1+t_2}{2})$

and $(2t_m - \frac{t_m+t_{m+1}}{2}, \infty)$.

(ii) If $\{t_1, t_m\} = \{\infty, -\infty\}$, then for every $t_i \in P$, $2 < i < m-1$, we plot the curve $\varphi(t)$ in the range $(\frac{t_i+t_{i-1}}{2}, \frac{t_i+t_{i+1}}{2})$. For t in $(\frac{t_i+t_{i-1}}{2}, t_i)$, the color of the plot will be red and for $(t_i, \frac{t_i+t_{i+1}}{2})$ the color will be blue.
As for t_2 and t_{m-1}, we plot $\varphi(t)$ for the ranges $(t_2 - 10, \frac{t_2+t_3}{2})$ and $(\frac{t_{m-1}-t_{m-2}}{2}, t_{m-1} + 10)$.
Finally, as for $\{\infty, -\infty\}$, we plot $\varphi(t)$ for the ranges $(t_2 - 20, t_2 - 10)$ and $(t_{m-1} + 10, t_{m-1} + 20)$.

Let us point out that we border the values of t generating the asymptotes for plotting.

Examples:

- $\varphi_3(t) = (\frac{t^2}{t^2-11\,t+30}, \frac{t^2+78}{1+t^2})$

```
>polares([t^2/(t^2 - 11 t + 30), (t^2 + 78)/(t^2 + 1)]);
r unbounded and theta bounded
Real point at the infinity such that (r, theta)=[1, 1] and the point
is [cos(1), sin(1)]
Point at infinity is not reached with k=0
Point at infinity is not reached with k<>0
System (1) gives self-intersections for k in [[ -2,2]], k<>0
The values of t generating asymptotes are {5.,6.}
Values of t considered in the plot {0.,5.,6.,∞,−∞}
```

- $\varphi_4(t) = (t, \frac{t^2+14}{1+t^2})$

```
>polares([t, (t^2 + 14)/(t^2 + 1)]);
r unbounded and theta bounded
There is no point at infinity
Values of t generating asymptotes [∞,−∞]
Values of t considered in the plot {0.,∞,−∞}
```

3.3 Algorithm `polares` *when $r(t)$ is bounded and $\theta(t)$ not*

`Input:` A proper polar parametrization $\varphi(t) = (r(t), \theta(t))$.
`Output:`

(1) Information about the existence of P_∞.
(2) Information about the existence of limit circles.
(3) Information about the existence of limit points.
(4) Information about the self-intersections.

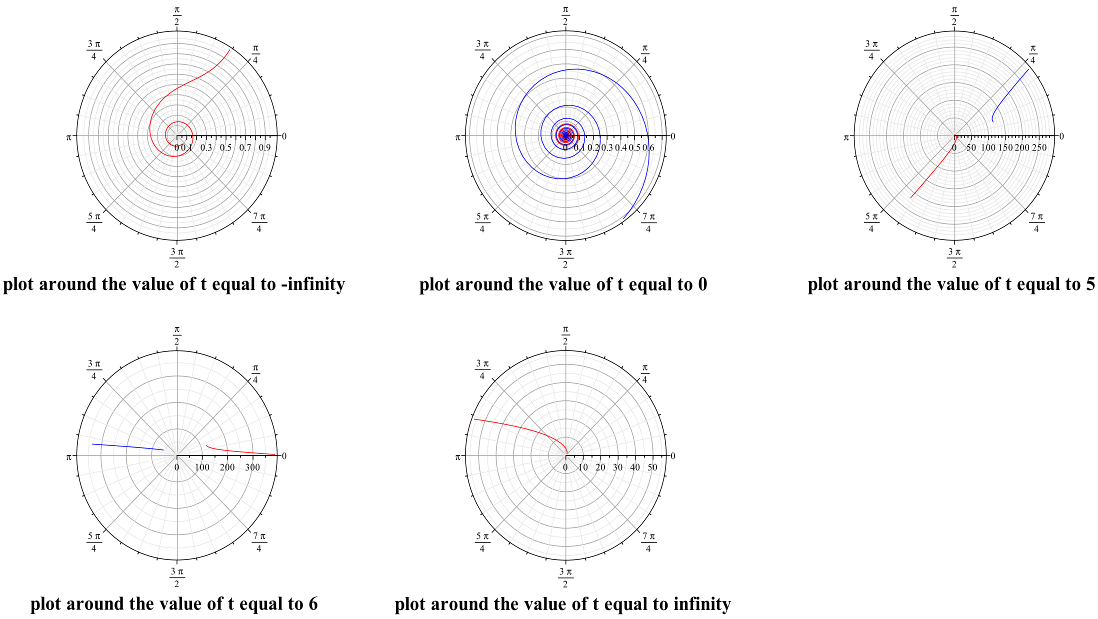


Fig. 12. $\varphi_3(t)$

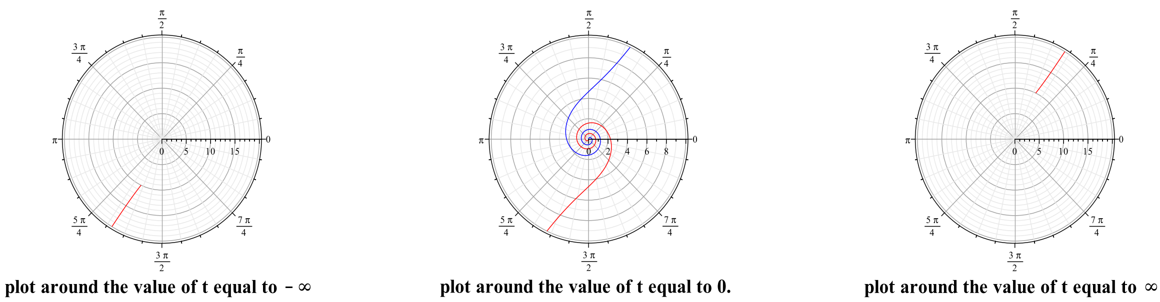


Fig. 13. $\varphi_4(t)$

(5) Plot of $\varphi(t)$ for t in $\mathbb{R}$ using the Maple command `polarplot`.

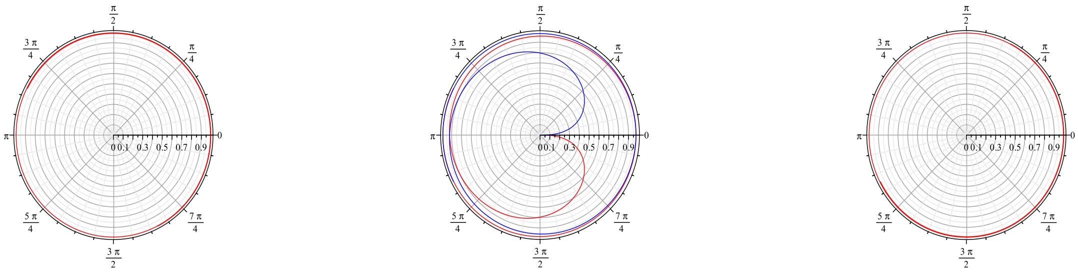
Let T_1 be the set of values of t generating limit circles, let T_2 be the set of values of t generating limit points, let T_3 be the set of real values of t such that $r(t) = 0$ and let T_4 be the real values of t generating the maximum of $|r(t)|$.

Let $P := T_1 \cup T_2 \cup T_3 \cup T_4 = \{t_1, \ldots, t_m\}$ with $t_1 < \ldots < t_m$. Then we proceed similar to the case (4) of Section 3.2. Now we border the values of t generating limit circles for plotting.

Example:

- $\varphi_5(t) = (t^2/(t^2+1), t^3/(t^2+1))$

```
>polares([t^2/(t^2+1), t^3/(t^2+1)]);
r bounded and theta unbounded
There is no point at infinity
Values of t generating limit circles [-∞, ∞]
There are infinitely many self-intersections
t=infinity has infinitely many close self-intersections
There are no limit points
Values of t considered in the plot {0., ∞, -∞}
```



plot around the value of t equal to - ∞ | plot around the value of t equal to 0. | plot around the value of t equal to ∞

Fig. 14. $\varphi_5(t)$

3.4 *Algorithm* `polares` *when both* $r(t)$ *and* $\theta(t)$ *are unbounded*

`Input:` A proper polar parametrization $\varphi(t) = (r(t), \theta(t))$.
`Output:`

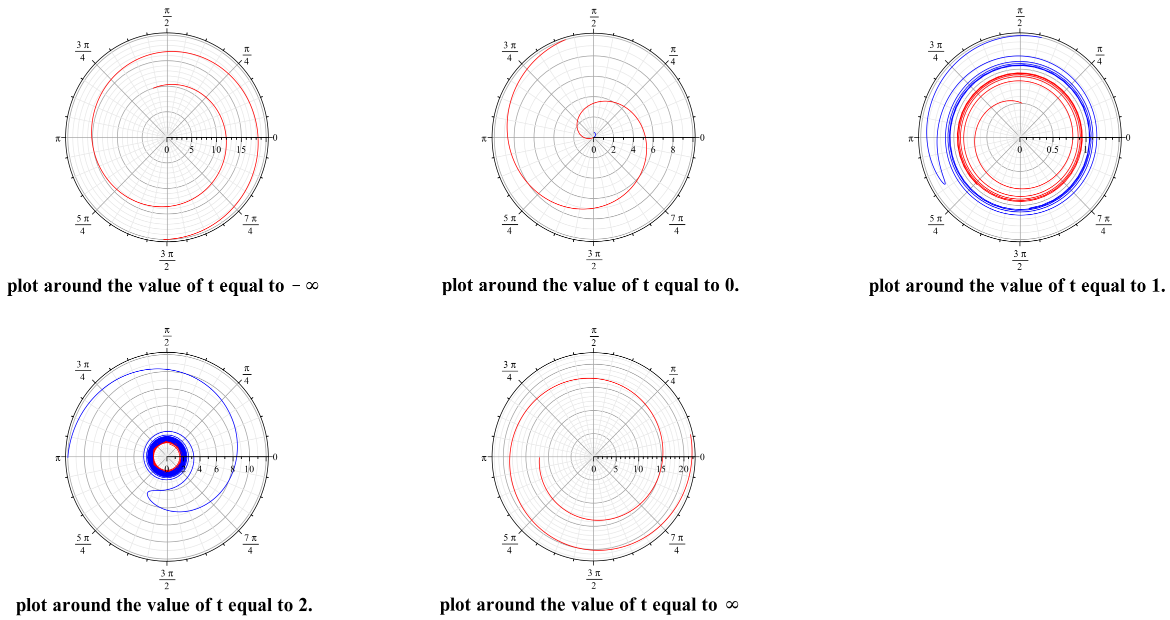
(1) Information about the existence of P_∞.
(2) Information about the existence of limit circles.
(3) Information about the existence of limit points.
(4) Information about the existence of spiral branches.
(5) Information about the existence of asymptotes.
(6) Information about the self-intersections.
(7) Plot of $\varphi(t)$ for t in $\mathbb{R}$ using the Maple command `polarplot`.

Let T_1 be the set of values of t generating limit circles, let T_2 be the set of values of t generating limit points, let T_3 be the set of values of t generating spiral branches, let T_4 be the set of values of t generating asymptotes and let T_5 be real the values of t such that $r(t) = 0$
Let $T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5 = \{t_1, \ldots, t_m\}$ with $t_1 < \ldots < t_m$.
Then we proceed similar to the case (4) of Section 3.2. Here we border the values of t generating asymptotes, limit circles and spiral branches for plotting.

Example:

- $\varphi_6(t) = (t, (t^3 + 1)/(t^2 - 3\,t + 2))$

```
>polares([t, (t^3 + 1)/(t^2 - 3 t + 2)]);
r and theta both unbounded
There is no point at infinity
Values of t generating limit circles [1., 2.]
There are no limit points
Values of t generating spiral branches [-∞, ∞]
There are not values of t generating asymptotes
There are infinitely many self-intersections
t=1 has infinitely many close self-intersections
t=2 has infinitely many close self-intersections
t=infinity has infinitely many close self-intersections
Values of t considered in the plot {-∞, 0., 1., 2., ∞}
```



plot around the value of t equal to $-\infty$

plot around the value of t equal to 0.

plot around the value of t equal to 1.

plot around the value of t equal to 2.

plot around the value of t equal to ∞

Fig. 15. $\varphi_6(t)$

4 Conclusions/Further Work

In this paper we have presented several theoretical results and an algorithm for properly plotting curves parametrized by rational functions in polar form. Our results allow to algorithmically identify phenomena which are typical of these curves, like the existence of infinitely many self-intersections, spiral branches, limit points or limit circles. Furthermore, the algorithm has been implemented in Maple 15, and provides good results. Natural extensions of the study here are space curves which are rational in spherical or cylindrical coordinates, curves which are algebraic, although non-necessarily rational, in polar coordinates (i.e. fulfilling $h(r, \theta) = 0$, with h algebraic), and similar phenomena for the case of surfaces. It would be also interesting to analyze the curves defined by (implicit) expressions of the type $f(r, \sin(\theta), \cos(\theta)) = 0$, where f is algebraic, since this class contains, and in fact extends, the class of algebraic curves; also, it includes the important subclass of curves defined by equations $r^n = g(\theta)$, with $g(\theta)$ a rational function, which often appear in Geometry and Physics. Some of these questions will be explored in the future.

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