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Departamento de Economía

1301

HOMOTHETICITY, DUALITY AND EFFICIENCY MEASURES

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This versión December, 2013

Abstract

As a result of the duality of cost and distance functions the efficiency of cost-minimizing behaviour can be compared to shadow-prizing behaviour, and conversely. In this framework we outline the form that dual efficiency measures, Muro (1982), Muro and Vera (1983), adopt for homothetic and linearly homogeneous technologies. To illustrate the subject we provide a numerical example for a technology described by a translog cost function.

Keywords: Homothetic technologies, Linearly homogeneous technologies, Efficiency measures, Scale measures, Duality theory, Polar forms.
JEL classification: D24

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[1] This paper is a version of an old 1983 paper presented at IV Encuentro hispano-francés de economistas teóricos, Bilbao (1983). All remaining errors are our own.

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1. Introduction.

Farrell (1957) focussed our attention on the notion of productive efficiency when introduced a measure of efficiency in the input space as well as the way to decompose it in its technical and allocative components. Fare and Lovell (1978) showed the difficulties inherent in such measure, namely, its uniqueness depends on the assumption of linear homogeneous technologies and its radial character (constant input mix). Forsund and Hjalmarsson (1979) introduced additional measures in the input space to identify the incidence of the scale behaviour of the technology on the productive efficiency.

However, the development of duality theory and the growing use of cost functions as support of empirical analysis of production suggest the convenience of defining efficiency measures in terms of the cost function. Muro (1982)[2], Muro and Vera (1983), drawing on previous results of Shephard (1970) and Hanoch (1970, 1978), introduced such measures.

The objective of this paper is to specialize our previous measures for homothetic and linearly homogeneous technologies. We organize the paper as follows. In section 2 we review expressions for distance and cost functions in the case of homothetic and linearly homogeneous

[2] Reproduced in Alcamentos as Muro (2011). Alcamentos 1101.

technologies. Section 3 contains dual efficiency measures. We present a numerical example in section 4. Section 5 concludes.

2. Distance and cost functions for homothetic and linearly homogeneous technologies.

Most of the efficiency studies are couched in terms of the production function but, as we are interested in the dual aspects of efficiency, this is not the more useful approach.

As Shephard (1970) has proved the production function is not a direct dual of the cost function although they are uniquely determined from each other. The basic duality is that of the distance and cost functions. We can consider the cost function as a distance function defined on the level sets of the normalized price space and the distance function as a cost function defined on the level set of the input space.

Moreover, another reason to use the distance function instead of the production function in efficiency analysis is that the former provides a natural way to define efficiency measures. The Farrell output measure of efficiency, for instance, has been shown to be the inverse of the distance function.

Since our main purpose is to characterize efficiency measures for specific technologies, an obvious first step is to elucidate what restrictions are imposed on the functions used to define the efficiency measures by the distinct production technologies. As some of these restrictions have been previously established and some are easily derived, they are presented without proof.

For the distance function, the following lemmas can be proved.

Lemma 1. If the production technology is (positively) linearly homogeneous, then

(i) $D(1/\lambda y, \underline{x}) = 1/\lambda\ D(1/y, \underline{x})$

(ii) $D(1/\lambda y, \lambda\underline{x}) = D(1/y, \underline{x})$

(iii) $D(1/y, \underline{x}) = 1/y\ D(1, \underline{x}) = 1/y\ F(\underline{x})$

Lemma 2. If the production technology is homothetic, then

(iv) $D(1/y, \underline{x}) = [g(y)]^{-1}\ D(1, \underline{x}) = [g(y)]^{-1}\ F(\underline{x})$

(v) The elasticity of scale is independent of the input mix and can be expressed as

$$\varepsilon(y) = g(y)\ [y.g'(y)]^{-1}$$

Similarly, the following lemmas can be proved for the cost function.

Lemma 3. If the production technology is (positively) linearly homogeneous, then

(vi) $C(\lambda y, \underline{w}) = \lambda\ C(y, \underline{w})$

(vii) $C(y, \underline{w}) = y\ C(1, \underline{w})$

Lemma 4. If the production technology is homothetic, then

(viii) $C(y, \underline{w}) = h(y)\, C(1, \underline{w})$

(ix) The elasticity of cost with respect to output is independent of factor prices and can be expressed by

$$\kappa(y) = \kappa'(y)\, y\, [h(y)]^{-1}$$

3. Dual measures of efficiency.

Due to the duality between cost and distance functions, it appears natural to investigate the efficiency in the dual space.

Muro (1982), Muro and Vera (1983) have introduced measures of efficiency in the normalized price space and have shown the duality between these measures and the Farrell and Forsund-Hjalmarsson measures defined in the quantity space. Both set of measures are presented in Table 1.

[Insert Table 1]

The main appeal of this dual approach is that efficiency can be measured either from observation of output and input quantities or from observation of output and normalized input prices.

The specialization of these measures for homothetic and linearly homogeneous technologies is, using the lemmas in Section 2 and the regularity conditions of both the distance and cost functions, quite straightforward. So, we shall concern ourselves with the discussion of the dual measures.

If the production technology is homothetic, the dual Farrell measures of efficiency can be written as

$$E1(y, \underline{p})= \max \{ \delta \mid C(y/ \delta, \underline{p})\geq 1 \}= \max \{ \delta \mid C(1, \underline{p})\geq [h(y/ \delta)]^{-1} \} \quad [1]$$

$$E2(y, \underline{p})= \max \{ v \mid C(y, \underline{p}/v)\geq 1 \}= \max \{ v \mid C(1, \underline{p})\geq v [h(y)^{-1}]\} \quad [2]$$

The above measures respectively provide, were the production unit to use the best practice technology, and index of the maximum increase in output that could be achieved, given the observed normalized prices, and an index of the maximum increase in normalized prices that could be obtained, given the observed level of output.[3]

With regards to the measures of scale efficiency, the assumption of a homothetic production technology allows us to write

$$S1(y, \underline{p}) = \max \{ \lambda \mu \mid C(y/ \lambda , \underline{p}/\mu)\geq 1, \kappa(y/ \lambda , \underline{p}/\mu)= 1 \} =$$
$$= \max \{ \lambda \mu \mid C(1, \underline{p})\geq \mu [h(y/ \lambda)]^{-1}, \kappa(y/ \lambda)= 1 \} \quad [3]$$

From Lemma 4 we have that $\kappa(y/ \lambda)= 1$ implies

[3] Note that in absence of complementary information nothing can be said about the causes of observed inefficiency. However, if knowledge of observed input vectors is available, the allocative and technical components of inefficiency can be evaluated by means of the Shephard's Lemma. See, in this sense, Diewert and Knops (1982).

$$[h(y/\lambda)]^{-1} = \lambda\, [h'(y/\lambda)\, y]^{-1}$$

Which substituted into (3) gives

$$S1(y, \underline{p}) = \max \{ \lambda\mu \mid C(1, \underline{p}) \geq \lambda\mu\, [h'(y/\lambda)\, y]^{-1} \} \quad [4]$$

In a similar way it is possible to obtain

$$S2(y^*, \underline{p}) = \max \{ \lambda\mu \mid C(1, \underline{p}) \geq \lambda\mu\, [h'(y^*/\lambda)\, y^*]^{-1} \} \quad [5]$$

And

$$S3(y, \underline{p}^*) = \max \{ \lambda\mu \mid C(1, \underline{p}^*) \geq \lambda\mu\, [h'(y/\lambda)\, y]^{-1} \} \quad [6]$$

Where $y^* = \max \{ y \mid C(1, \underline{p}) \geq 1 \}$

And

$$\underline{p}^* = \underline{p}[C^*]^{-1} = \underline{p}\, [\min \{ \underline{p}\, \underline{x} \mid D(1/y, \underline{x}) \geq 1 \}]^{-1}$$

These measures of efficiency provide a gross measure of scale efficiency – $S1(y, \underline{p})$ - that does not discriminate between the effects of scale and economic efficiency, and two net measures of scale efficiency – $S2(y^*, \underline{p})$ and $S3(y, \underline{p}^*)$ - that, respectively, remove the aspects of inefficiency related to output, as measured by E1(y, p), and to normalized input prices, as measured by E2(y, p).

The specialization of the dual measures of efficiency for linearly homogeneous technologies can be easily derived by the same procedure. It turns out, as expected, that all five measures of efficiency are equivalent. They reduce to an expression of the form

$$E\ (y, \underline{p}) = \max \{ \theta \mid C(1\ , \underline{p}\ ) \geq \theta\ y^{-1} \} \qquad [7]$$

4. A numerical example.

We now specify a hypothetic translog cost function to illustrate the measures developed in section 3 for non-homothetic, homothetic and linearly homogeneous technologies.

The translog function can be considered either as a “true” cost function or as a second order Taylor’s-series approximation to an arbitrary twice differentiable cost function satisfying the appropriate regularity conditions. We choose to assume it to be a “true” cost function.

As the cost function is linearly homogeneous in w, we can define a highly general homothetic translog unit cost function as

$$\ln C(y,p) = \sum_i \alpha_i \ln p_i + \frac{1}{2}\sum_i \sum_j \beta_{ij} \ln p_i \ln p_j + \beta_0 \ln y + \frac{1}{2}\beta_{00}(\ln y)^2 + \sum_i \gamma_i \ln y \ln p_i .$$

With the parametric restrictions

$$\sum_i \alpha_i = 1, \quad \sum_i \gamma_i = 0, \quad \sum_i \beta_{ij} = \sum_j \beta_{ij} = \sum_i \sum_j \beta_{ij} = 0,$$

in order to ensure linear homogeneity in input prices.

As discussed before, the unit cost function factorizes for homothetic technologies into

$$\text{Ln } C(y, \underline{p}) = \ln h(y) + \ln C(1, \underline{p}).$$

Where C(1, p) is the unit cost function for a unitary level of production and h(y) describes the scale behaviour of the technology. The scale function implicitly assumed by a homothetic translog function is

$$h(y) = y^{\beta_0 + \frac{1}{2}\beta_{00} \ln y}$$

Which, obviously reduces to h(y)= y for a linearly homogeneous technology.

Let us assume the following set of parameters

α = {0,3   0,6   0,1}

γ = {0,1 -0,2   0,1}

$$\beta = \begin{pmatrix} 0,3 & -0,1 & -0,2 \\ -0,1 & 0,1 & 0,0 \\ -0,2 & 0,0 & 0,2 \end{pmatrix}$$

$\beta_0 = 0,5$

$\beta_{00} = 0,4$

for the non-homothetic case, and the obvious additional restrictions for the homothetic case, γ = {0,0 0,0 0,0}, and the linearly homogeneous case γ = {0,0 0,0 0,0}, $\beta_0 = 1$, $\beta_{00} = 0$; so that the relevant difference is the scale behaviour of the technology. The observed production plan lny= 1,2; $\ln p_1$= -1,1; $\ln p_2$= -1,3; $\ln p_3$= -1,3; gives way to the following set of efficiency measures

	E1	E2	S1	S2	S3
Linear homogeneous technology	0,9665	0,9665	0,9665	0,9665	0,9665
Homothetic technology	0,7184	0,7075	0,7071	0,9842	0,9994
Non-homothetic technology	0,7382	0,7247	0,7246	0,5439	1,0000

5. Conclusions.

The dual approach to efficiency has some attractive features. First, the dual measures defined can be applied, if we assume the unit cost function to be a “true” cost function, without loss of information on the production technology. These measures left the technical efficiency as a residual factor but it is known that measures defined on the input space left the allocative efficiency as a residual factor. This is just another consequence of duality theory.

Moreover, the efficiency measures introduced here are also radial measures but, in the dual space contrarily to the input space, this has an obvious economic meaning: as all production units face the same vector of market prices, all observed normalized prices should be on the same ray.

As a result of the duality of cost and distance functions the efficiency of cost-minimizing behaviour can be compared to shadow-prizing behaviour, and conversely. So efficiency of private and public production units can be compared.

A further aspect to be noted is that our measures distinguish the different scale and allocative aspects of efficiency which should be useful if the measures are to be given an empirical content.

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Table 1. Dual efficiency measures, Muro (1982), Muro and Vera (1983).

Input space	**Price space**
$E1(y, \underline{x}) = \max \{ \theta \mid D(1/y\,\theta, \underline{x}) \geq 1 \}$	$E1(y, \underline{p}) = \max \{ \delta \mid C(y/\delta, \underline{p}) \geq 1 \}$
$E2(y, \underline{x}) = \max \{ \lambda \mid D(1/y, \underline{x}/\lambda) \geq 1 \}$	$E2(y, \underline{p}) = \max \{ v \mid C(y, \underline{p}/v) \geq 1 \}$
$S1(y, \underline{x}) = \max \{ \lambda\mu \mid D(1/\lambda y, \underline{x}/\mu) \geq 1, \varepsilon(\lambda y, \underline{x}/\mu) = 1 \}$	$S1(y, \underline{p}) = \max \{ \lambda\mu \mid C(y/\lambda, \underline{p}/\mu) \geq 1, \kappa(y/\lambda, \underline{p}/\mu) = 1 \}$
$S2(y^*, \underline{x}) = \max \{ \lambda\mu \mid D(1/y^*, \underline{x}/\mu) \geq 1, \varepsilon(\lambda y^*, \underline{x}/\mu) = 1 \}$	$S2(y^*, \underline{p}) = \max \{ \lambda\mu \mid C(y^*, \underline{p}/\mu) \geq 1, \kappa(y^*, \underline{p}/\mu) = 1 \}$
$S3(y, \underline{x}^*) = \max \{ \lambda\mu \mid D(1/\lambda y, \underline{x}^*) \geq 1, \varepsilon(\lambda y, \underline{x}/\mu) = 1 \}$	$S3(y, \underline{p}^*) = \max \{ \lambda\mu \mid C(y/\lambda, \underline{p}^*) \geq 1, \kappa(y/\lambda, \underline{p}/\mu) = 1 \}$

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