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ORIGINAL PAPER

A note about rational surfaces as unions of affine planes

Jorge Caravantes[1] · **J. Rafael Sendra**[1] · **David Sevilla**[2] · **Carlos Villarino**[1]

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Abstract
We prove that any smooth rational projective surface over the field of complex numbers has an open covering consisting of 3 subsets isomorphic to affine planes.

Keywords Smooth rational projective surface · Covering of surfaces · Surface blowup

Mathematics Subject Classification Primary 14E05 · 14M20; Secondary 14A10

Since all smooth rational curves are isomorphic to $\mathbb{P}^1$, they can be seen as the union of two affine lines. In dimension two, as a consequence of the structure Theorem 1.3 below, all rational surfaces admit a covering of open subsets isomorphic to the affine plane. However, up to the authors' knowledge, no general results are known on the minimal number of open subsets of such a covering, while some advances are known by computer algebrists in terms of surjectivity of parametrizations [1, 5, 6, 8]. In this short note we prove that all projective smooth rational surfaces behave like the projective plane in this aspect.

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✉ Jorge Caravantes
jorge.caravantes@uah.es

J. Rafael Sendra
rafael.sendra@uah.es

David Sevilla
sevillad@unex.es

Carlos Villarino
carlos.villarino@uah.es

1 Universidad de Alcalá, Dpto. de Física y Matemáticas, 28871 Alcalá de Henares, Madrid, Spain

2 Centro U. de Mérida, Universidad de Extremadura, Av. Santa Teresa de Jornet 38, 06800 Mérida, Badajoz, Spain

1 Main result

Theorem 1.1 *Let X be a projective smooth rational surface over the complex field. Then, there are three open subsets $U_0, U_1, U_2 \subset X$ such that:*

(1) $U_0 \cup U_1 \cup U_2 = X$.
(2) *For all $i = 0, 1, 2$, U_i is isomorphic to the affine plane.*

Remark 1.2 Note that the bound of three subsets in the covering is sharp. If the projective surface $X \subset \mathbb{P}^n$ is the union of two affine planes U_0 and U_1, then $Z = X - U_0$ is closed in X, so projective, and it is contained in $U_1 \simeq \mathbb{A}^2$, so it must be finite. Since Z is finite, there is a hyperplane $H \subset \mathbb{P}^n - Z$. Then the section $H \cap X$ is a projective curve contained in $X - Z = U_0 \simeq \mathbb{A}^2$. Since $\mathbb{A}^2$ does not contain projective varieties of positive dimension, this is a contradiction.

To prove Theorem 1.1, we will use the following well-known result:

Theorem 1.3 (see e.g. [2, Theorem V.10]) *Every non-singular rational surface can be obtained by repeatedly blowing up either $\mathbb{P}^2$ or the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ (the Hirzebruch surface Σ_n), for $n \neq 1$.*

By Theorem 1.3, there exists a chain of morphisms $\pi = \pi_1 \circ \cdots \circ \pi_r : X \to M$ such that M is either $\mathbb{P}^2$ or a Hirzebruch surface and $\pi_i : X_i \to X_{i-1}$ is the blowup of a smooth surface at a single point. Let E be the exceptional divisor of π and E_i the exceptional divisor of π_i. Then, $\pi(E) \subset M$ is a finite set of closed points and $\pi_i(E_i)$ is one closed point. Moreover, $E_i \simeq \mathbb{P}^1$ and E is a finite union of smooth rational curves (in fact, E_1 and the proper transforms of all the E_2, ..., E_r). We begin by proving Theorem 1.1 for $X = M$ with care for the centers of the blowups:

Lemma 1.4 *In the above conditions, there exist three open subsets U_0^0, U_1^0, U_2^0 such that:*

(1) $U_0^0 \cup U_1^0 \cup U_2^0 = M$.
(2) *For all $i = 0, 1, 2$, U_i is isomorphic to the affine plane.*
(3) $\pi(E) \subset U_0^0 \cap U_1^0 \cap U_2^0$.

Proof The case $M = \mathbb{P}^2$ is well-known. Since $\pi(E)$ is finite and we work over an infinite field, one can choose three different projective lines L_1, L_2 and L_3 in $\mathbb{P}^2$ such that $\pi(E) \cap (L_1 \cup L_2 \cup L_3) = \emptyset = L_1 \cap L_2 \cap L_3$.

If M is a Hirzebruch surface, then it is the projective bundle of a rank two vector bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-m)$ over $\mathbb{P}^1$. This means that there is a surjective morphism $p : M \to \mathbb{P}^1$ such that, for any point $P \in \mathbb{P}^1$, $p^{-1}(\mathbb{P}^1 - \{P\}) \simeq (\mathbb{P}^1 - \{P\}) \times \mathbb{P}^1$. Then, since we work over an infinite field, one can choose a closed point $P_0 \in \mathbb{P}^1 - p(\pi(E))$ with its isomorphism $q_0 : p^{-1}(\mathbb{P}^1 - \{P_0\}) \to \mathbb{A}^1 \times \mathbb{P}^1$. Then, we choose a line $L_0 = \mathbb{A}^1 \times \{Q_0\}$, such that $q_0(\pi(E)) \cap L_0$ is empty. With this choice, $U_0^0 = q_0^{-1}(\mathbb{A}^1 \times \mathbb{P}^1 - L_0)$ is isomorphic to $\mathbb{A}^2$ and contains $\pi(E)$.

Then, $M - U_0^0$ is the union of two rational curves $C_1 := p^{-1}(P_0)$ and $C_2 := \overline{q_0^{-1}(L_0)}$. Choosing $P_1 \in \mathbb{P}^1 - (p(\pi(E)) \cup \{P_0\})$ (again, the complement of a finite set), together with the isomorphism $q_1 : p^{-1}(\mathbb{P}^1 - \{P_1\}) \to \mathbb{A}^1 \times \mathbb{P}^1$, we have that $p^{-1}(\mathbb{P}^1 - \{P_1\})$ contains C_1 and C_2 with the exception of the point $R_1 := C_2 \cap p^{-1}(P_1)$ (the intersection of a section $\mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{P}^1$ with a fiber). We now choose a line $L_1 = \mathbb{A}^1 \times \{Q_1\}$ such that:

- $Q_1 \in \mathbb{P}^1$ is not in the second projection of $q_1(\pi(E)) \in \mathbb{A}^1 \times P^1$; and

- $L_1 \neq q_1(C_2)$ (i.e. we are asking a constant section not to coincide with a given one, which is an open condition for Q_1), so the intersection of the two curves is finite.

Then, $U_1^0 = q_1^{-1}(\mathbb{A}^1 \times \mathbb{P}^1 - L_1)$ is isomorphic to $\mathbb{A}^2$ and contains $\pi(E)$.

Now, $M - (U_0^0 \cup U_1^0) = (C_1 \cup C_2) - U_1^0$ is the finite set $A := \{R_1\} \cup q_1^{-1}(L_1 \cap q_1(C_2))$. Finally, we have again the complement of a finite set to choose $P_2 \in \mathbb{P}^1 - (p(\pi(E)) \cup p(A) \cup \{P_0, P_1\})$ with the isomorphism $q_2 : p^{-1}(\mathbb{P}^1 - \{P_2\}) \to \mathbb{A}^1 \times \mathbb{P}^1$, so we have $A \subset p^{-1}(\mathbb{P}^1 - \{P_2\})$. We now choose $L_2 = \mathbb{A}^1 \times \{Q_2\}$ such that $Q_2 \in \mathbb{P}^1$ is not in the second projection of the finite set $q_2(A \cup \pi(E)) \subset \mathbb{A}^1 \times \mathbb{P}^1$, and we define $U_2^0 = q_2^{-1}(\mathbb{A}^1 \times \mathbb{P}^1 - L_2) \simeq \mathbb{A}^2$. Then $\pi(E) \subset U_2^0$ and, since $A \subset U_2^0$, we have that $U_0^0 \cup U_1^0 \cup U_2^0 = M$. □

Remark 1.5 Let $\mathrm{Bl}_P(\mathbb{A}^2)$ be the blowup of the affine plane at a point P. Consider a line l passing through P and define U_l as the complement in $\mathrm{Bl}_P(\mathbb{A}^2)$ of the proper transform of l. Just by changing coordinates, one has that all U_l are isomorphic to each other. Since the case for l being the vertical axis is well known to be isomorphic to $\mathbb{A}^2$, by restricting the defining projection π :$\mathrm{Bl}_P(\mathbb{A}^2) \to \mathbb{A}^2$, we have morphisms $\pi_l : U_l \simeq \mathbb{A}^2 \mapsto \mathbb{A}^2$. Moreover:

(1) for $l_1 \neq l_2$, $U_{l_1} \cup U_{l_2} = \mathrm{Bl}_P(\mathbb{A}^2)$.
(2) if $E_{\mathbb{A}^2}$ is the exceptional divisor of $\mathrm{Bl}_P(\mathbb{A}^2)$, for any line l passing through P, $E_{\mathbb{A}^2} - U_l$ consists in one point, given by the isomorphism between $E_{\mathbb{A}^2}$ and the $\mathbb{P}^1$ of all lines through P.
(3) the restriction $\pi_l|_{U_l - E_{\mathbb{A}^2}}$ is an isomorphism between $U_l - E_{\mathbb{A}^2}$ and $\mathbb{A}^2 - l$.

Lemma 1.6 *Let X be a smooth rational surface such that there exist three open subsets $U_0, U_1, U_2 \subset X$ with*

(1) $U_0 \cup U_1 \cup U_2 = X$.
(2) *For all $i = 0, 1, 2$, U_i is isomorphic to the affine plane.*

Consider a finite set $A_1 \subset U_0 \cap U_1 \cap U_2$. Let $P \in (U_0 \cap U_1 \cap U_2) - A_1$ be a point and consider $\pi : Y \to X$ to be the blowup of X at P. Consider also a finite set A_2 in the exceptional divisor $E = \pi^{-1}(P) \subset Y$. Then, there are three open subsets $U_0', U_1', U_2' \subset Y$ such that

(1) $U_0' \cup U_1' \cup U_2' = Y$.
(2) *For all $i = 0, 1, 2$, U_i' is isomorphic to the affine plane.*
(3) *Both A_2 and the proper transform of A_1 are contained in $U_0' \cap U_1' \cap U_2'$*

Remark 1.7 In the conditions of Lemma 1.6, note that for any $i, j = 0, 1, 2$, $i \neq j$, $X - (U_i \cup U_j)$ is a Zariski closed subset of a projective surface which is contained in $U_k \simeq \mathbb{A}^2$, with $i \neq k \neq j$. Since it is a projective scheme in an affine space, it must be finite.

Proof Taking into account Remark 1.5, consider a line $l_0 \subset U_0 \simeq \mathbb{A}^2$ through P such that

- The intersection of l_0 with the finite set $X - (U_1 \cup U_2)$ (see Remark 1.7) is empty.
- $A_1 \cap l_0 = \emptyset$.
- The intersection point of the proper transform of l_0 with the exceptional divisor is not in A_2.

Then, we define U_0' to be the open subset U_{l_0} of the blowup of U_0 at P. U_0 is isomorphic to the affine plane, as said in Remark 1.5, and $Y - U_0'$ consists in the proper transform of $l_0 \cup (X - U_0)$. Therefore, it is one-dimensional.

Now, we choose a line $l_1 \subset U_1 \simeq \mathbb{A}^2$ such that the following open conditions are satisfied:

(1) The intersection of l_1 with the finite set $X - (U_0 \cup U_2)$ (see Remark 1.7) is empty.

(2) the intersection multiplicity of l_1 and l_0 at P is 1 (note that l_1 is smooth at P, so we are asking that l_1 is not the tangent line at P to l_0, when we see them in U_1).
(3) l_1 does not contain any point in $l_0 \cap (X - U_2)$ (note that l_0 is irreducible and $P \subset l_0 \cap U_2$, so such intersection is finite).
(4) $A_1 \cap l_1 = \emptyset$.
(5) The intersection point of the proper transform of l_1 with the exceptional divisor is not in A_2.

Since we work over an infinite field, these conditions define a nonempty Zariski open subset to choose l_1 from. Now, we define U_1' to be $U_{l_1} \simeq \mathbb{A}^2$. The whole exceptional divisor is in $U_0' \cup U_1'$. Then, $Y - (U_0' \cup U_1')$ is the proper transform of the finite sets $B_1 = [l_0 \cup (X - U_0)] \cap l_1$ and $B_2 = X - (U_0 \cup U_1)$.

Note that $P \notin B_1 \cup B_2 \subset U_2$, so we choose a last line $l_2 \subset U_2 \simeq \mathbb{A}^2$ such that

- the intersection of l_2 with the finite set $X - (U_0 \cup U_1)$ (see Remark 1.7) is empty,
- $(A_1 \cup B_1 \cup B_2) \cap l_2 = \emptyset$, and
- the intersection point of the proper transform of l_2 with the exceptional divisor is not in A_2.

Defining $U_2' = U_{l_2}$, one concludes the proof. □

Remark 1.8 It is likely that a generalisation of Lemma 1.6 to higher dimension is possible. However, it is not yet known if all rational varieties of dimension greater than 2 are covered by open subsets isomorphic to open subsets of $\mathbb{A}^n$ (see [7] for the original question). These varieties are known as plain [3] or uniformly rational [4] and it is possible that the main result can be extended to higher dimension for this type of varieties.

Proof of Theorem 1.1 By Lemma 1.4, we have that $M = X_0 = U_0^0 \cup U_1^0 \cup U_2^0$ with $U_i^0 \simeq \mathbb{A}^2$ and $\pi(E) \subset U_0^0 \cap U_1^0 \cap U_2^0$. Now we apply Lemma 1.6 to $\pi_i : X_i \to X_{i-1}$, choosing

$$A_1 = \left[\pi_i \circ \pi_{i+1}(E_{i+1}) \cup \cdots \cup \pi_i \circ \cdots \circ \pi_r(E_r)\right] - \{P_i\}$$

(i.e. the points to be the center of future blowups outside $\{P_i\}$) and

$$A_2 = \left[\pi_{i+1}(E_{i+1}) \cup \cdots \cup \pi_{i+1} \circ \cdots \circ \pi_r(E_r)\right] \cap E_i$$

(i.e. the points to be center of future blowups in E_i). Note that any curve contracted by $\pi_{i+1} \circ \cdots \circ \pi_i$ is contracted to a point in $\pi_i^{-1}(A_1) \cup A_2$. We then get U_0^i, U_1^i, U_2^i from $U_0^{i-1}, U_1^{i-1}, U_2^{i-1}$ all isomorphic to $\mathbb{A}^2$ and covering X_i, with all centers of future blowups in the intersection of the three open subsets. Then U_0^r, U_1^r and U_2^r are the three open subsets in the statement. □

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