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Determining the asymptotic family of an implicit curve ☆

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A B S T R A C T

In this paper we deal with the following problem: given an algebraic plane curve $\mathcal{C}$, implicitly defined, we determine its "asymptotic family", that is, the set of algebraic curves that have the same asymptotic behavior as $\mathcal{C}$.

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1. Introduction

The notion of an infinity branch reflects the status of a curve at the points with sufficiently large coordinates. In fact, an infinity branch is associated to a projective place centered at an infinity point, and it can be parametrized by means of Puiseux series.

An infinity branch B is a very important tool for analyzing the behavior at infinity of an implicit real algebraic plane curve. Hence, an infinity branch is applicable by itself to real world problems which are of special interest in the field of computer aided geometric design (CAGD). For instance, determining the infinity branches of a curve is an important step in sketching its graph as well as in studying its topology and, in general, in a great variety of applications in computer aided design (CAD), science and engineering, and in particular some particular problems which can be described by partial differential equations (see e.g. Arnold (1989), Arnold (1990), Bazant and Crowdy (2005), Caflisch and Papanicolau (1993), Chorin and Marsden (2000), Eggers and Fontelos (2015), Gao and Chen (2012), González-Vega and Necula (2002), Greuel et al. (2007), Hong (1996), Landau and Lifshitz (1976) and Zeng (2007)).

If a branch B can be defined by some explicit equation of the form $y = f(x)$ (or $x = g(y)$), where f (or g) is a continuous function on an infinite interval, it is easy to decide whether $\mathcal{C}$ has an asymptote at B by analyzing the existence of the limits of certain functions when x tends to ∞ (or y tends to ∞) and if these limits can be computed, we may obtain the equation

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of the asymptote of $\mathcal{C}$ at B. We recall that in analytic geometry, an asymptote of a curve is a line such that the distance between the curve and the line approaches zero as they tend to infinity. In some contexts, such as algebraic geometry, an asymptote is defined as a line which is tangent to a curve at infinity.

If the equation of the branch B cannot be converted into an explicit form, both the decision and the computation of the asymptote of $\mathcal{C}$ at B require some other tools. More precisely, an algebraic curve may have more general curves than lines describing the status of a branch at the points with sufficiently large coordinates. In this sense, we say that a curve $\widetilde{\mathcal{C}}$ is a *generalized asymptote* (or *g-asymptote*) of another curve $\mathcal{C}$ if the distance between $\widetilde{\mathcal{C}}$ and $\mathcal{C}$ tends to zero as they tend to infinity, and $\mathcal{C}$ cannot be approached by another curve of lower degree. Some important notions as the concepts of infinity branches, approaching curves and perfect curves, asymptotic behavior, etc., are introduced in previous papers (see Blasco and Pérez-Díaz (2014a), Blasco and Pérez-Díaz (2014b), Blasco and Pérez-Díaz (2015), Blasco and Pérez-Díaz (2020), Campo et al. (2022a) and Campo et al. (2022b)) and, in particular, some methods for computing the g-asymptotes for curves are presented.

In this paper, we intend to go further and given an algebraic plane curve $\mathcal{C}$, implicitly defined, we deal with the problem of determining the set of algebraic curves that have the same asymptotic behavior as $\mathcal{C}$. We refer to this set as the "asymptotic family" of the curve $\mathcal{C}$.

The case of a curve with a unique regular infinity branch is addressed in Blasco and Pérez-Díaz (2014a) (see Theorem 3). Now we focus on curves with two or more regular infinity branches. More precisely, in this paper, we prove that the asymptotic behavior of an implicit "regular curve" (an implicit curve whose infinity points are all regular) is completely determined by its homogeneous forms of degree d and $d-1$, and viceversa. That is, from f_d and f_{d-1} we can compute an asymptote for each infinity branch of the curve. Conversely, given a set of regular asymptotes (that is, regular perfect curves), we can derive the set of all the curves that are approached at its different infinity branches by those asymptotes. In other words, we can determine the family of curves whose asymptotic behavior is defined by that set of asymptotes.

We have intended this paper to be self-contained. For this reason, we have included Section 2, where we review the theory of infinity branches and introduce the notions of convergent branches (that is, branches that get closer as they tend to infinity) and approaching curves (see Blasco and Pérez-Díaz (2014b)). In Section 3, we show how the asymptotic behavior of a curve can be described and in particular, we study which terms of the implicit equation determine that asymptotic behavior. These results will allow us to construct families of curves having the same asymptotic behavior as a given one. We generalize in this way Theorem 3 and Corollary 2 for the case of arbitrary non perfect curves. Finally, in Section 4, we present the conclusions and future work. The proofs of the main results (Theorems 4 and 5) as well as some previous technical lemmas appear in Section 5.

2. Notation and previous results

In this section, we introduce the notions of *infinity branch*, *convergent branches* and *approaching curves*, and we present some properties that allow us to compare the behavior of two implicit algebraic plane curves at infinity. For further details on these concepts and results, we refer to Blasco and Pérez-Díaz (2014b).

We consider an irreducible algebraic affine plane curve $\mathcal{C}$ over $\mathbb{C}$ defined by the irreducible polynomial $f(x, y) \in \mathbb{R}[x, y]$. Let $\mathcal{C}^*$ be its corresponding projective curve, defined by the homogeneous polynomial

$$F(x, y, z) = f_d(x, y) + z f_{d-1}(x, y) + z^2 f_{d-2}(x, y) + \cdots + z^d f_0 \in \mathbb{R}[x, y, z],$$

where $d := \deg(\mathcal{C})$. We assume that $(0 : 1 : 0)$ is not an infinity point of $\mathcal{C}^*$ (otherwise, we may consider a linear change of coordinates).

In order to get the infinity branches of $\mathbb{C}$, we work in the $YZ-$chart, where the infinity points become affine. In this chart the curve is defined by the polynomial $g(y, z) = F(1 : y : z)$. The infinity branches are constructed from the Puiseux solutions of $g(y, z) = 0$ around $z = 0$. There exist exactly $\deg_y(g)$ solutions given by different Puiseux series that can be grouped into conjugacy classes. More precisely, if $\mathbb{C} \ll z \gg$ denotes the field of *formal Puiseux series*, and

$$\varphi(z) = m + a_1 z^{N_1/N} + a_2 z^{N_2/N} + a_3 z^{N_3/N} + \cdots \in \mathbb{C} \ll z \gg, \quad a_i \neq 0, \ \forall i \in \mathbb{N},$$

where $N \in \mathbb{N}$, $N_i \in \mathbb{N}$, $i = 1, \ldots$, and $0 < N_1 < N_2 < \cdots$, is a Puiseux series such that $g(\varphi(z), z) = 0$, and $\nu(\varphi) = N$ (i.e., N is the *ramification index* of φ), the series

$$\varphi_j(z) = m + a_1 c_j^{N_1} z^{N_1/N} + a_2 c_j^{N_2} z^{N_2/N} + a_3 c_j^{N_3} z^{N_3/N} + \cdots$$

where $c_j^N = 1$, $j = 1, \ldots, N$, are called the *conjugates* of φ. We easily observe that $(1 : m : 0)$, $m \in \mathbb{C}$ is an infinity point. The set of all the conjugates of φ is called the *conjugacy class* of φ and it contains $\nu(\varphi)$ different series.

Since $g(\varphi(z), z) = 0$ in some neighborhood of $z = 0$ where $\varphi(z)$ converges, there exists $M \in \mathbb{R}^+$ such that $F(1 : \varphi(t) : t) = g(\varphi(t), t) = 0$ for $t \in \mathbb{C}$ and $|t| < M$, which implies $F(t^{-1} : t^{-1}\varphi(t) : 1) = f(t^{-1}, t^{-1}\varphi(t)) = 0$, for $t \in \mathbb{C}$ and $0 < |t| < M$. We set $t^{-1} = z$, and we obtain that

$$f(z, r(z)) = 0, \quad z \in \mathbb{C} \text{ and } |z| > M^{-1}, \qquad \text{where}$$

$$r(z) = z\varphi(z^{-1}) = mz + a_1 z^{1-N_1/N} + a_2 z^{1-N_2/N} + a_3 z^{1-N_3/N} + \cdots, \quad a_i \neq 0, \forall i \in \mathbb{N}$$

$N, N_i \in \mathbb{N},\ i = 1, \ldots,$ and $0 < N_1 < N_2 < \cdots$.

Reasoning similarly with the N different series in the conjugacy class, $\varphi_1, \ldots, \varphi_N$, we get

$$r_i(z) = z\varphi_i(z^{-1}) = mz + a_1 c_i^{N_1} z^{1-N_1/N} + a_2 c_i^{N_2} z^{1-N_2/N} + a_3 c_i^{N_3} z^{1-N_3/N} + \cdots$$

where $c_1, \ldots, c_N$ are the N complex roots of $x^N = 1$. Under these conditions, we introduce the following definition.

Definition 1. An *infinity branch of an affine plane curve* $\mathcal{C}$ associated to the infinity point $P = (1 : m : 0)$, $m \in \mathbb{C}$, is a set $B = \bigcup_{j=1}^{N} L_j$, where $L_j = \{(z, r_j(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\}$, $M \in \mathbb{R}^+$, and

$$r_j(z) = z\varphi_j(z^{-1}) = mz + a_1 c_j^{N_1} z^{1-N_1/N} + a_2 c_j^{N_2} z^{1-N_2/N} + a_3 c_j^{N_3} z^{1-N_3/N} + \cdots$$

where $N, N_i \in \mathbb{N},\ i = 1, \ldots, 0 < N_1 < N_2 < \cdots$, and $c_j^N = 1,\ j = 1, \ldots, N$. The subsets $L_1, \ldots, L_N$ are called the *leaves* of the infinity branch B.

Remark 1. An infinity branch is uniquely determined from one leaf, up to conjugation. That is, if $B = \bigcup_{i=1}^{N} L_i$, where $L_i = \{(z, r_i(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M_i\}$, and

$$r_i(z) = z\varphi_i(z^{-1}) = mz + a_1 z^{1-N_1/N} + a_2 z^{1-N_2/N} + a_3 z^{1-N_3/N} + \cdots$$

then $r_j = r_i,\ j = 1, \ldots, N$, up to conjugation; i.e.

$$r_j(z) = z\varphi_j(z^{-1}) = mz + a_1 c_j^{N_1} z^{1-N_1/N} + a_2 c_j^{N_2} z^{1-N_2/N} + a_3 c_j^{N_3} z^{1-N_3/N} + \cdots$$

where $N, N_i \in \mathbb{N}$, and $c_j^N = 1,\ j = 1, \ldots, N$.

By abuse of notation, we say that $B = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\}$ (where $M := \max\{M_1, \ldots, M_N\}$). Moreover, we say that N is *the ramification index* of the branch B and we write $\nu(B) = N$. Note that B has $\nu(B)$ leaves.

Remark 2. There exists a one-to-one relation between infinity places and infinity branches. In addition, each infinity branch is associated to a unique infinity point given by the center of the corresponding infinity place. More precisely, as we stated above, there exists $M \in \mathbb{R}^+$ such that $F(1 : \varphi(t) : t) = g(\varphi(t), t) = 0$ for $|t| < M$, where

$$\varphi(z) = m + a_1 z^{N_1/N} + a_2 z^{N_2/N} + a_3 z^{N_3/N} + \cdots \in \mathbb{C} \ll z \gg .$$

Thus, for $t = 0$ we get the infinity point $P = (1 : \varphi(0) : 0) = (1 : m : 0) \in \mathcal{C}^*$.

Reciprocally, given an infinity point $P = (1 : m : 0)$, there must be, at least, one Puiseux solution φ such that $\varphi(0) = m$; this solution provides an infinity branch associated to P. Hence, we conclude that every algebraic plane curve has, at least, one infinity branch.

Remark 3. The procedure introduced above allows us to obtain the infinity branches of a curve $\mathcal{C}$, under the assumption that $(0 : 1 : 0)$ is not an infinity point of $\mathcal{C}^*$. However, a curve may have infinity branches, associated to the infinity point $(0 : 1 : 0)$, which cannot be constructed in this way. We call them Type II infinity branches and they have the form $\{(r(z), z) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\}$. A Type II infinity branch may be obtained by interchanging the variables x and y. See Blasco and Pérez-Díaz (2014b) for further details.

In the following, we introduce the notions of convergent branches and approaching curves. Intuitively speaking, two infinity branches converge if they get closer as they tend to infinity. This concept will allow us to analyze whether two curves approach each other.

Definition 2. Two infinity branches, B and $\overline{B}$, are convergent if there exist two leaves $L = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\} \subset B$ and $\overline{L} = \{(z, \overline{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > \overline{M}\} \subset \overline{B}$ such that $\lim_{z \to \infty}(\overline{r}(z) - r(z)) = 0$. In this case, we say that the leaves L and $\overline{L}$ converge.

The following theorem provides a characterization for the convergence of two infinity branches.

Theorem 1. *The following statements hold:*

1. *Two leaves $L = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$ and $\overline{L} = \{(z, \overline{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > \overline{M}\}$ are convergent if and only if the terms with non negative exponent in the series $r(z)$ and $\overline{r}(z)$ are the same.*
2. *Two infinity branches B and $\overline{B}$ are convergent if and only if for each leaf $L \subset B$ there exists a leaf $\overline{L} \subset \overline{B}$ convergent with L, and reciprocally.*
3. *Two convergent infinity branches must be associated to the same infinity point.*

For the following definition, we recall that given an algebraic plane curve $\mathcal{C}$ over $\mathbb{C}$ and a point $p \in \mathbb{C}^2$, *the distance from p to $\mathcal{C}$* is defined as $d(p, \mathcal{C}) = \min\{d(p,q) : q \in \mathcal{C}\}$. Observe that this minimum exists because $\mathcal{C}$ is a closed set (see Blasco and Pérez-Díaz (2014b)).

Definition 3. Let $\mathcal{C}$ be an algebraic plane curve with an infinity branch B. We say that a curve $\overline{\mathcal{C}}$ *approaches* $\mathcal{C}$ at its infinity branch B if there exists one leaf $L = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\} \subset B$ such that $\lim_{z\to\infty} d((z, r(z)), \overline{\mathcal{C}}) = 0$.

Theorem 2. *Let $\mathcal{C}$ be a plane algebraic curve with an infinity branch B. A plane algebraic curve $\overline{\mathcal{C}}$ approaches $\mathcal{C}$ at B if and only if $\overline{\mathcal{C}}$ has an infinity branch, $\overline{B}$, such that B and $\overline{B}$ are convergent.*

Corollary 1. *Let $\mathcal{C}$ be an algebraic plane curve with an infinity branch B. Let $\overline{\mathcal{C}}_1$ and $\overline{\mathcal{C}}_2$ be two different curves that approach $\mathcal{C}$ at B. Then $\overline{\mathcal{C}}_1$ and $\overline{\mathcal{C}}_2$ approach each other.*

Obviously, "approaching" is a symmetric concept, that is, $\mathcal{C}_1$ approaches $\mathcal{C}_2$ (at some infinity branch) if and only if $\mathcal{C}_2$ approaches $\mathcal{C}_1$. When this happens we say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *approaching curves* or that they *approach each other*.

Definition 4. We say that two curves have the same asymptotic behavior if each of them approaches the other one at all its infinity branches (see Blasco and Pérez-Díaz (2014b)).

Now, suppose that a curve $\mathcal{C}$ is approached, at one of its infinity branches, by a second curve $\overline{\mathcal{C}}$ such that $\deg(\overline{\mathcal{C}}) < \deg(\mathcal{C})$. Then one may say that $\mathcal{C}$ *degenerates*, since it behaves at infinity as a curve of smaller degree. For instance, a hyperbola is a curve of degree 2 that has two real asymptotes, which implies that the hyperbola degenerates, at infinity, in two lines. Similarly, one can check that every ellipse has two asymptotes, although they are complex lines in this case. However, the asymptotic behavior of a parabola is different, since it cannot be approached at infinity by any line. This motivates the following definition:

Definition 5. An algebraic curve of degree d is a *perfect curve* if it cannot be approached by any curve of degree less than d.

A curve that is not perfect can be approached by other curves of smaller degree. If these curves are perfect, we call them *g-asymptotes*. More precisely, we have the following definition.

Definition 6. Let $\mathcal{C}$ be a curve with an infinity branch B. A *g-asymptote* (generalized asymptote) of $\mathcal{C}$ at B is a perfect curve that approaches $\mathcal{C}$ at B.

The notion of *g-asymptote* is similar to the classical concept of asymptote. The difference is that a g-asymptote is not necessarily a line, but rather a perfect curve. Actually, it is a generalization, since every line is a perfect curve (this fact follows from Definition 5). Throughout this paper we refer to *g-asymptote* simply as *asymptote*.

Remark 4. The degree of an asymptote is less or equal than the degree of the curve it approaches. In fact, an asymptote of a curve $\mathcal{C}$ at a branch B has minimal degree among all the curves that approach $\mathcal{C}$ at B.

In Blasco and Pérez-Díaz (2014a), we show that every infinity branch of a given algebraic plane curve implicitly defined has, at least, one asymptote and we show how to compute it. For this purpose, we rewrite the equation defining a branch B (see Definition 1) as

$$r(z) = mz + a_1 z^{1-n_1/n} + \cdots + a_k z^{1-n_k/n} + a_{k+1} z^{1-N_{k+1}/N} + \cdots \tag{2.1}$$

where $0 < N_1 < \cdots < N_k \le N < N_{k+1} < \cdots$ and $\gcd(N, N_1, \ldots, N_k) = b$, $N = n\,b$, $N_j = n_j\,b$, $j = 1, \ldots, k$. That is, we have simplified the non negative exponents such that $\gcd(n, n_1, \ldots, n_k) = 1$. Note that $0 < n_1 < n_2 < \cdots$, $n_k \le n$, and $N < N_{k+1}$, i.e. the terms $a_j z^{1-N_j/N}$ with $j \ge k+1$ are those which have negative exponent. We denote these terms as

$$A(z) := \sum_{\ell=k+1}^{\infty} a_\ell z^{-q_\ell}, \quad q_\ell = 1 - N_\ell/N \in \mathbb{Q}^+, \ \ \ell \geq k+1.$$

Under these conditions, we introduce the definition of degree of a branch B:

Definition 7. Let $B = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, \ |z| > M\}$ ($r(z)$ is defined in (2.1)) be an infinity branch associated to an infinity point $P = (1 : m : 0), m \in \mathbb{C}$. We say that n is the degree of B, and we denote it by $\deg(B)$.

Taking into account Theorems 1 and 2, we have that any curve $\overline{\mathcal{C}}$ approaching $\mathcal{C}$ at B should have an infinity branch $\overline{B} = \{(z, \overline{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, \ |z| > \overline{M}\}$ such that the terms with non negative exponent in $r(z)$ and $\overline{r}(z)$ are the same. In the simplest case, if $A = 0$ (i.e. there are no terms with negative exponent; see equation (2.1)), we obtain

$$\tilde{r}(z) = mz + a_1 z^{1-n_1/n} + a_2 z^{1-n_2/n} + \cdots + a_k z^{1-n_k/n}, \tag{2.2}$$

where $a_1, a_2, \ldots \in \mathbb{C} \setminus \{0\}, m \in \mathbb{C}, \ n, n_1, n_2 \ldots \in \mathbb{N}, \ \gcd(n, n_1, \ldots, n_k) = 1$, and $0 < n_1 < n_2 < \cdots$. Note that $\tilde{r}$ has the same terms with non negative exponent as r, and $\tilde{r}$ does not have terms with negative exponent.

Let $\widetilde{\mathcal{C}}$ be the plane curve containing the branch $\widetilde{B} = \{(z, \tilde{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, \ |z| > \widetilde{M}\}$ (note that $\widetilde{\mathcal{C}}$ is unique since two different algebraic curves have finitely many common points). Observe that

$$\widetilde{\mathcal{Q}}(t) = (t^n, mt^n + a_1 t^{n-n_1} + \cdots + a_k t^{n-n_k}) \in \mathbb{C}[t]^2,$$

where $n, n_1, \ldots, n_k \in \mathbb{N}, \ \gcd(n, n_1, \ldots, n_k) = 1$, and $0 < n_1 < \cdots < n_k$, is a polynomial parametrization of $\widetilde{\mathcal{C}}$. In Blasco and Pérez-Díaz (2014a), we prove that $\widetilde{\mathcal{Q}}(t)$ is proper and that $\widetilde{\mathcal{C}}$ is an asymptote of $\mathcal{C}$ at B (see Lemma 3 and Theorem 2). Next, we illustrate the above notions and results with an example.

Example 1. Let $\mathcal{C}$ and $\mathcal{D}$ be two plane curves implicitly defined by the irreducible polynomials

$$f(x, y) = 2y^3x - y^4 + 2y^2x - y^3 - 2x^3 + x^2y + 3 \in \mathbb{R}[x, y], \text{ and}$$
$$\overline{f}(x, y) = y^3x + y^4 + y^2x + y^3 - x^3 - x^2y + 2 \in \mathbb{R}[x, y].$$

We first note that both curves have $P = (1 : 0 : 0)$ as a common infinity point. We will see that $\mathcal{C}$ approaches $\mathcal{D}$ (and reciprocally) at the infinity branch associated to P (see Fig. 1).

First, we compute the infinity branch of $\mathcal{C}$ associated to P. For this purpose, we consider the corresponding projective curve $\mathcal{C}^*$ which is defined by the homogeneous polynomial

$$F(x, y, z) = 2y^3x - y^4 + 2y^2xz - y^3z - 2x^3z + x^2yz + 3z^4 \in \mathbb{R}[x, y, z].$$

Now, we consider the curve defined by the polynomial $g(y, z) = F(1 : y : z)$ and we note that $g(p) = 0$, where $p = (0, 0)$. We compute the series expansion for the solutions of $g(y, z) = 0$. For this purpose, we use for instance the algcurves package included in the computer algebra system Maple. We get that:

$$\varphi(t) = t^{1/3} - t/3 + t^{5/3}/9 - 2t^{7/3}/81 - t^{10/3}/2 + \cdots \in \mathbb{C} \ll t \gg.$$

That is, $g(\varphi(t), t) = 0$. Since $\nu(\varphi) = 3$, we have the following three conjugate Puiseux series in the conjugacy class of φ. Thus, the infinity branch of $\mathcal{C}$ associated to P is given by $B = L_1 \cup L_2 \cup L_3$, where $L_i = \{(z, r_i(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, \ |z| > M\}$,

$$r_i(z) = z\varphi_i(z^{-1}) = c_i^2 z^{2/3} - 1/3 + 1/9c_i^2 z^{-2/3} - 2/81c_i^4 z^{-4/3} - 1/2c_i^7 z^{-7/3} + \cdots, \ c_i = 1, 2, 3,$$

c_i are the complex roots of $x^3 = 1$.

On the other side, reasoning similarly with the curve $\mathcal{D}$, we get that the infinity branch of $\mathcal{D}$ associated to P is given by $\overline{B} = \overline{L}_1 \cup \overline{L}_2 \cup \overline{L}_3$, where $\overline{L}_i = \{(z, \overline{r}_i(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, \ |z| > M\}$,

$$\overline{r}_i(z) = c_i^2 z^{2/3} - 1/3 + 1/9c_i^2 z^{-2/3} - 2/81c_i^4 z^{-4/3} - 2/3c_i^7 z^{-7/3} + \cdots, \ c_i = 1, 2, 3,$$

c_i are the complex roots of $x^3 = 1$.

We get that both branches are convergent since the terms with non-negative exponents in both series, r_i y $\overline{r}_i$, are the same.

Now, we compute the asymptotes of $\mathcal{C}$. Note that since $f_4(x, y) = -y^4 + 2y^3x$, the infinity points are $P_1 = (1 : 2 : 0)$ y $P_2 = (1 : 0 : 0)$.

For $P_1 = (1 : 2 : 0)$, we have only one infinity branch, B_1 associated to P_1, where

$$r_1(z) = 2z + \frac{3z^{-3}}{8} - \frac{9z^{-4}}{64} + \frac{27z^{-5}}{512} - \frac{81z^{-6}}{4096} + \cdots$$

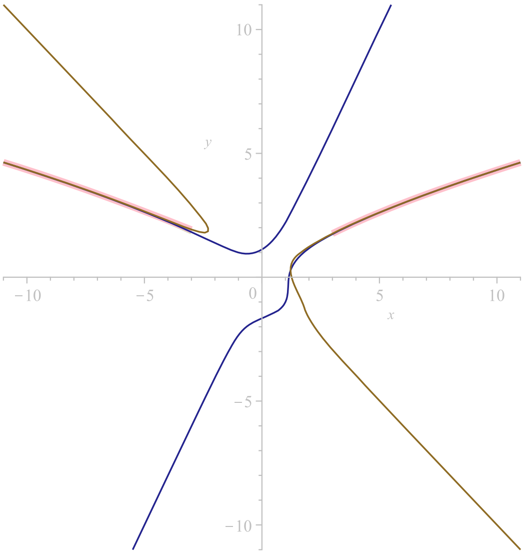

Fig. 1. Curve $\mathcal{C}$ approaching $\mathcal{D}$ at the infinity branch associated to P.

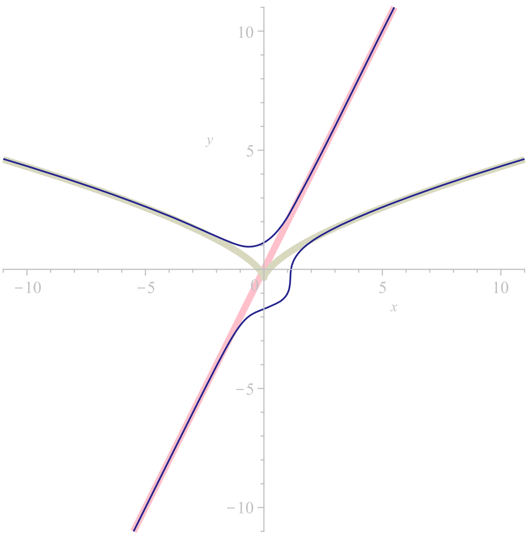

Fig. 2. Asymptotes $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$ of the curve $\mathcal{C}$.

(see the first part of this example). Then, $\widetilde{r}_1(z) = 2z$, which implies that $\widetilde{\mathcal{P}}_1(t) = (t, 2t)$ parametrically defines the asymptote $\widetilde{\mathcal{C}}_1$.

For $P_2 = (1:0:0)$, we have only one infinity branch, B_2 associated to P_2, where

$$r_2(z) = z^{2/3} - \frac{1}{3} + \frac{z^{-2/3}}{9} - \frac{2z^{-4/3}}{81} + \cdots .$$

Then, $\widetilde{r}_2(z) = z^{2/3} - 1/3$, which implies that $\widetilde{\mathcal{P}}_2(t) = (t^3, t^2 - 1/3)$ parametrically defines the asymptote $\widetilde{\mathcal{C}}_2$.

In Fig. 2, we plot the given curve $\mathcal{C}$ and the asymptotes $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$.

We complete this summary by introducing the notion of a regular perfect curve and some results concerning the asymptotic behavior of these curves.

Definition 8. A *regular perfect curve* is a curve having a unique infinity point, which is regular.

Theorem 3. *Let $\mathcal{C}$ be a regular perfect curve defined by an irreducible polynomial $f \in \mathbb{R}[x, y]$ of degree d. The asymptotic behavior of $\mathcal{C}$ is completely determined by f_d and f_{d-1}.*

Corollary 2. *Two regular perfect curves approach each other if and only if their terms of degree d and $d-1$ are the same.*

Therefore, given a regular perfect curve $\mathcal{C}$, we can obtain the whole family of curves with the same asymptotic behavior as $\mathcal{C}$, by just changing the terms of degree less than $d-1$ in the implicit equation. The set of curves obtained in this way is called the "proximity class" of $\mathcal{C}$ and has the structure of a vector space.

Proposition 1. *Let $\mathcal{C}$ be a regular perfect curve of degree d. The proximity class of $\mathcal{C}$ is isomorphic to $\mathbb{R}^{\frac{(d-1)d}{2}}$.*

In order to illustrate the above notions, let us consider a plane curve $\mathcal{C}$ of degree $d=3$ defined by the polynomial

$$f(x,y) = y^3 + x^3 + 3xy^2 + 3x^2y + 5x^2 + 3y - 1 \in \mathbb{R}[x,y].$$

Note that the homogeneous form of maximum degree is given by

$$f_3(x,y) = y^3 + x^3 + 3xy^2 + 3x^2y = (x+y)^3.$$

Thus $\mathcal{C}$ has only one infinity point, $P = (1:-1:0)$, and one may easily check that P is regular. Thus, $\mathcal{C}$ is a regular perfect curve.
The curves within the proximity class of the given curve $\mathcal{C}$ are implicitly defined by the polynomials

$$\begin{aligned}&f_3(x,y) + f_2(x,y) + a_1x + a_2y + a_3 =\\ &y^3 + x^3 + 3xy^2 + 3x^2y + 5x^2 + a_1x + a_2y + a_3, \quad a_i \in \mathbb{R},\ i = 1,2,3.\end{aligned}$$

We observe that any curve that belongs to this proximity class can be associated uniquely to the vector $(a_1, a_2, a_3) \in \mathbb{R}^3$. Further details on this subject can be found in Blasco and Pérez-Díaz (2014a).

3. Asymptotic behavior of an implicit curve

In this section we show how the asymptotic behavior of a curve can be described and we study which terms of the implicit equation determine that asymptotic behavior. These results will allow us to construct families of curves having the same asymptotic behavior as a given one. We generalize in this way Theorem 3 and Corollary 2, above, for the case of arbitrary non perfect curves.

The proofs of the main results (Theorems 4 and 5) as well as some previous technical lemmas (Lemma 1) appear in Section 5.

Let $\mathcal{C}$ be a curve with k infinity branches $B_1, \dots, B_k$, and suppose that we compute an asymptote A_i for each $B_i, i = 1 \dots, k$. Let $\mathcal{A} := \{A_1, \dots, A_k\}$ be the set of asymptotes obtained in this way. The following two conditions are satisfied:

a) Every infinity branch of $\mathcal{C}$ is approached by some asymptote of $\mathcal{A}$.
b) Every asymptote of $\mathcal{A}$ approaches some infinity branch of $\mathcal{C}$.

From Definition 4 and Corollary 1, any other curve having the same asymptotic behavior as $\mathcal{C}$ must also satisfy these two conditions. For this reason, we say that the set $\mathcal{A}$ completely describes the asymptotic behavior of $\mathcal{C}$.

Note that $\mathcal{A}$ might contain two or more asymptotes that are convergent among themselves. These asymptotes give us redundant information, so we just need one of them for describing the asymptotic behavior of the curve. This motivates the following definition.

Definition 9. An asymptotic system of an algebraic curve is a minimal set of asymptotes that completely describes its asymptotic behavior.

Note that we may consider different asymptotic systems for the same curve. However, all of them will have the same cardinality. Furthermore, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are two different asymptotic systems of a curve $\mathcal{C}$, then, for each $A_i \in \mathcal{A}_1$ there exists $A_j \in \mathcal{A}_2$ such that A_i and A_j are convergent, and viceversa. That is, there is a bijective correspondence between the elements of $\mathcal{A}_1$ and the elements of $\mathcal{A}_2$.

Remark 5. From the above considerations, given two algebraic curves $\mathcal{C}$ and $\mathcal{D}$, the following statements are equivalent:

a) $\mathcal{C}$ and $\mathcal{D}$ have the same asymptotic behavior.
b) There exists an asymptotic system valid for both curves.
c) Any asymptotic system valid for one of the curves is also valid for the other.

In Blasco and Pérez-Díaz (2014a), we provide an algorithm that allows us to obtain an asymptote for each infinity branch of a given curve, getting in this way an asymptotic system of the curve. Now we are interested in the inverse problem: we want to determine the set of curves whose asymptotic behavior is described by a given asymptotic system.

Definition 10. We define the asymptotic family of a curve $\mathcal{C}$ as the set of all curves having the same degree and the same asymptotic behavior as $\mathcal{C}$ (including $\mathcal{C}$ itself).

Note that having the same asymptotic behavior is an equivalence relation and the asymptotic families are the corresponding equivalence classes. From Remark 5, any asymptotic system defines an asymptotic family of curves. In this paper we develop a method for obtaining the asymptotic family associated to a given asymptotic system. However, we need the infinity points involved to be regular, which motivates the following definition.

Definition 11. We say that an infinity branch is regular if it is associated to a regular infinity point. In addition, we say that a curve is asymptotically regular if all its infinity branches are regular.

Remark 6. Note that two infinity branches of an asymptotically regular curve will always diverge, in fact, they will be associated to different infinity points (see Theorem 1). Otherwise, the curve would have two infinity places centered at the same infinity point and this point would be singular. Thus, an asymptotic system of an asymptotically regular curve will contain exactly one asymptote for each infinity branch.

In the following lemma, given $\mathcal{C}$ an asymptotically regular curve with k infinity branches, we compute the homogeneous form $f_d(x, y)$ of the maximum degree of $\mathcal{C}$.

Lemma 1. *Let $\mathcal{C}$ be an asymptotically regular curve with k infinity branches $B_1, B_2, \ldots, B_k$. For each $i = 1, \ldots, k$, the branch B_i is associated to the infinity point $p_i = (1 : m_i : 0)$ and it has ramification index N_i. Then, the homogeneous form of maximum degree of $\mathcal{C}$ is*

$$f_d(x, y) = \prod_{i=1}^{k} (y - m_i x)^{N_i}.$$

As a consequence, the degree of the curve is $d = N_1 + \cdots + N_k$.

From the previous lemma, we easily get the following results.

Corollary 3. *The following statements hold:*

1. *Let $\mathcal{C}$ and $\mathcal{D}$ be two curves with the same asymptotic behavior and let $\mathcal{C}$ be asymptotically regular. Then, $\mathcal{D}$ is asymptotically regular if and only if it has the same degree as $\mathcal{C}$.*
2. *The asymptotic family of an asymptotically regular curve $\mathcal{C}$ is composed of all the asymptotically regular curves with the same asymptotic behavior as $\mathcal{C}$.*
3. *Let $\mathcal{C}$ be an asymptotically regular curve and let B be an infinity branch of $\mathcal{C}$. Then, $\nu(B) = \deg(B)$.*

We want to find out which terms of the implicit equation determine the asymptotic behavior of a curve. Theorem 3 asserts that the asymptotic behavior is determined by the terms of degree d and $d-1$, in the case of a regular perfect curve. Note that a regular perfect curve is just an asymptotically regular curve with a single infinity branch. Theorem 4, below, generalizes this result for the case of an asymptotically regular curve with an arbitrary number of infinity branches.

Theorem 4. *Let $\mathcal{C}$ be an asymptotically regular curve defined by an irreducible polynomial $f \in \mathbb{R}[x, y]$ of degree d. The asymptotic behavior of $\mathcal{C}$ is completely determined by f_d and f_{d-1}.*

We have shown that the asymptotic behavior of an asymptotically regular curve can be determined from its homogeneous forms of degree d and $d-1$ (where $d = \deg(f)$). At this point, an interesting question arises: is the opposite also true? That is, can we determine the homogeneous forms f_d and f_{d-1} from the relevant terms of the Puiseux series corresponding to the different infinity branches? The answer to this question is provided by the following theorem.

Theorem 5. *Two asymptotically regular curves have the same asymptotic behavior if and only if their terms of degree d and $d-1$ are the same.*

The following remark explains how Theorem 5 can be used to obtain the asymptotic family of an arbitrary asymptotically regular curve.

Remark 7.

a) From Theorem 5, given an asymptotically regular curve $\mathcal{C}$, implicitly defined, one can get the whole family of asymptotically regular curves with the same asymptotic behavior as $\mathcal{C}$, by just changing the terms of degree less than $d-1$ in the implicit equation. We will call these terms "irrelevant terms".
b) Hence, given a set of regular perfect curves associated to different infinity points, we can obtain the asymptotic family defined by such set, that is, the family of curves which have exactly one infinity branch convergent with each of those curves (asymptotes). We do not need to compute the corresponding Puiseux expressions nor solve system (5.10). If the asymptotes are given implicitly, we can multiply their implicit polynomials and we obtain a (reducible) curve of the family. Hence we get f_d and f_{d-1}, which gives us the whole family (see Examples 2 and 3).
c) The idea just presented can be used to facilitate the computation of asymptotes. Given an implicitly defined curve we can, first of all, remove all irrelevant terms and then apply the algorithm. In fact, by doing this it is possible that the new implicit factor will be able to extract some asymptotes directly, or reduce the calculation to several curves of smaller degree.

From the proof of Theorem 5 (see Section 5), and Remark 7 (statement 1), one gets the following corollary.

Corollary 4. *Any curve having the same asymptotic behavior as $\mathcal{C}$ is asymptotically regular if and only if it has the same degree as $\mathcal{C}$. As a consequence, the family obtained by changing the irrelevant terms of $\mathcal{C}$ is precisely its asymptotic family.*

The following result is a generalization of Proposition 1 for the case of asymptotically regular curves.

Proposition 2. *The asymptotic family of a curve of degree d constitutes a vector space isomorphic to $\mathbb{R}^k$, where $k = (d-1)d/2$.*

Proof. The result follows from Remark 7, by taking into account that the number of terms of degree less than $d-1$ in a generic polynomial of degree d is $\frac{(d-1)d}{2}$, and that any curve of the asymptotic family can be associated uniquely to a vector in $\mathbb{R}^{\frac{(d-1)d}{2}}$. □

Example 2. Let $\mathcal{C}$ be the algebraic plane curve defined by the implicit polynomial

$$f(x, y) = -x^3 - 4x^2y - 5xy^2 - 2y^3 + y^2.$$

One has that

$$f_3(x, y) = -(x+2y)(y+x)^2, \quad f_2(x, y) = y^2$$

and the infinity points $(1 : -1 : 0)$, $(-2 : 1 : 0)$ are regular. Therefore, $\mathcal{C}$ is an asymptotically regular curve and the whole family of asymptotically regular curves with the same asymptotic behavior as $\mathcal{C}$ is defined by the implicit polynomial

$$f_3(x, y) + f_2(x, y) + a_1x + a_2y + a_3, \ a_i \in \mathbb{R}.$$

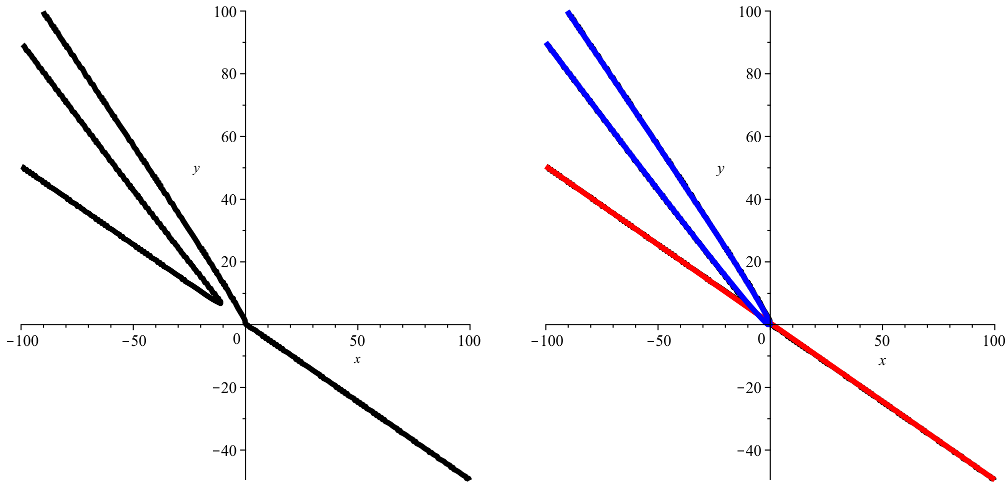

Fig. 3. Curve $\mathcal{C}$ (left) and asymptotes (right).

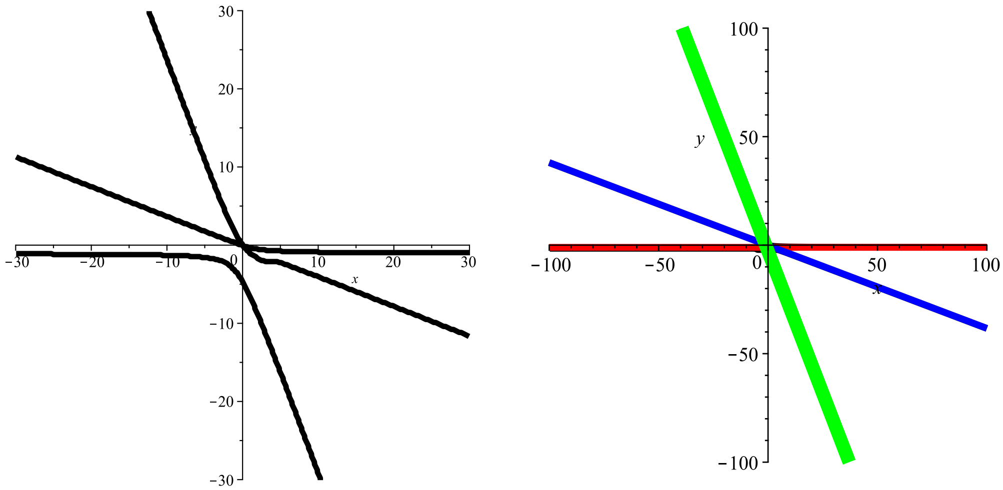

Fig. 4. Curve $\mathcal{C}$ (left) and asymptotes (right).

By applying Blasco and Pérez-Díaz (2014a), one gets that the asymptotes to this family are two plane curves defined by the proper parametrizations

$$\mathcal{Q}_1(t) = (t, -t/2 + 1/2), \qquad \mathcal{Q}_2(t) = (-t^2, t^2 - t).$$

In Fig. 3, we plot the curve $\mathcal{C}$ (left), and the asymptotes $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$ with the curve (right).

Example 3. Let $\mathcal{C}$ be the algebraic plane curve defined by the implicit polynomial

$$f(x, y) = -x^4y - 6x^3y^2 - 13x^2y^3 - 12xy^4 - 3y^5 - x^4 - 8x^3y - 24x^2y^2 - 32xy^3 - 14y^4 + y^3.$$

One has that

$$f_5(x, y) = -y(x^2 + 3xy + y^2)(x^2 + 3xy + 3y^2),$$
$$f_4(x, y) = -x^4 - 8x^3y - 24x^2y^2 - 32xy^3 - 14y^4$$

and the infinity points are regular. Therefore, $\mathcal{C}$ is an asymptotically regular curve and the whole family of asymptotically regular curves with the same asymptotic behavior as $\mathcal{C}$ is defined by the implicit polynomial

$$f_5(x, y) + f_4(x, y) + a_1x^3 + a_2yx^2 + a_3y^2x + a_4y^3 + a_5x^2 + a_6y^2 + a_7xy + a_8x + a_9y + a_{10}, \ a_i \in \mathbb{R}.$$

We apply the algorithm Blasco and Pérez-Díaz (2014a) to the curve defined by the irreducible polynomial $f_5(x, y) + f_4(x, y)$, and one gets that the asymptotes to the family of curves are the plane curves defined by the proper parametrizations

$$\mathcal{Q}_1(t) = (t, -1),$$
$$\mathcal{Q}_2(t) = (t, t(\sqrt{5}/2 - 3/2) + 3\sqrt{5}/5 - 3/2), \quad \mathcal{Q}_3(t) = (t, t(-\sqrt{5}/2 - 3/2) - 3\sqrt{5}/5 - 3/2),$$
$$\mathcal{Q}_4(t) = (t, ((tI - I)\sqrt{3})/6 - t/2 - 1/3), \quad \mathcal{Q}_5(t) = (t, ((-tI + I)\sqrt{3})/6 - t/2 - 1/3).$$

In Fig. 4, we plot the curve $\mathcal{C}$ (left), and the asymptotes $\widetilde{\mathcal{C}}_1$, $\widetilde{\mathcal{C}}_2$ and $\widetilde{\mathcal{C}}_3$ with the curve (right). Note that the asymptotes $\widetilde{\mathcal{C}}_4$ and $\widetilde{\mathcal{C}}_5$ are complex lines.

We observe that the implicit equations of the polynomials defining the asymptotes are given by

$$g_1(x, y) = y + 1,$$
$$g_2(x, y) = 10y\sqrt{5} - 3\sqrt{5} + 20x + 30y + 15, \quad g_3(x, y) = 10y\sqrt{5} - 3\sqrt{5} - 20x - 30y - 15$$
$$g_4(x, y) = 6Iy\sqrt{3} + 5I\sqrt{3} + 12x + 18y + 3, \quad g_5(x, y) = 6Iy\sqrt{3} + 5I\sqrt{3} - 12x - 18y - 3.$$

If we consider the polynomial of degree 5 defined by $\prod_{i=1}^{5} g_i(x, y)$, we obtain that the homogeneous forms of degree 5 and 4 are $f_5(x, y)$ and $f_4(x, y)$.

4. Conclusions and future work

In this paper, given an algebraic plane curve $\mathcal{C}$, implicitly defined, we determine its "asymptotic family", that is, the set of algebraic curves that have the same asymptotic behavior as $\mathcal{C}$.

More precisely, given an asymptotically regular curve $\mathcal{C}$, implicitly defined, one can get the whole family of asymptotically regular curves with the same asymptotic behavior as $\mathcal{C}$, by just changing the terms of degree less than $d-1$ in the implicit equation. We call these terms "irrelevant terms".

Hence, given a set of regular perfect curves associated to different infinity points, we can obtain the asymptotic family defined by such set, that is, the family of curves that have exactly one infinity branch convergent with each of those curves (asymptotes). We do not need to compute the corresponding Puiseux expressions. If the asymptotes are given implicitly, we can multiply their implicit polynomials and we obtain a (reducible) curve of the family. Hence we get f_d and f_{d-1}, which gives us the whole family.

The idea just presented can be used to facilitate the computation of asymptotes. Given an implicitly defined curve we can, first of all, remove all irrelevant terms and then apply the algorithm. In fact, by doing this it is possible that from the new implicit factor we will be able to extract some asymptotes directly, or reduce the calculation to several curves of smaller degree.

As a future work, we intend to deal with the input curves parametrically defined and additionally, we will try to deal with the case that the infinity points are not regular. In this sense, and in particular, we are wondering what is the form of maximum degree, f_d.

5. Proofs of the main theorems

This section is devoted to proving the main results of this paper, Theorems 4 and 5 in Section 3. For this purpose, we first prove the following lemma.

Lemma 1. *Let $\mathcal{C}$ be an asymptotically regular curve with k infinity branches $B_1, B_2, \ldots, B_k$. For each $i = 1, \ldots, k$, the branch B_i is associated to the infinity point $p_i = (1 : m_i : 0)$ and has ramification index N_i. Then, the homogeneous form of maximum degree of $\mathcal{C}$ is*

$$f_d(x, y) = \prod_{i=1}^{k} (y - m_i x)^{N_i}.$$

As a consequence, the degree of the curve is $d = N_1 + \cdots + N_k$.

Proof. Let $\mathcal{C}$ be defined by the polynomial

$$f(x, y) = f_d(x, y) + f_{d-1}(x, y) + \cdots + f_1(x, y) + f_0.$$

Then,

$$F(x, y, z) = f_d(x, y) + f_{d-1}(x, y)z + \cdots + f_1(x, y)z^{d-1} + f_0 z^d$$

and

$$g(y, z) := F(1, y, z) = f_d(1, y) + f_{d-1}(1, y)z + \cdots + f_1(1, y)z^{d-1} + f_0 z^d.$$

From Lemma 2 in Blasco and Pérez-Díaz (2014a), the term $(y - m_i x)^{N_i}$ divides $f_d(x, y)$ for each $i = 1, \ldots, k$. In addition, there cannot be any other factor $(y - mx)^N$ with $m \neq m_i$ since this would imply the existence of a new infinity point $(1 : m : 0)$ and, hence, a new infinity branch different of $B_1, B_2, \ldots, B_k$. Thus, we have that

$$f_d(x, y) = \prod_{i=1}^{k} (y - m_i x)^{\vartheta_i}, \tag{5.1}$$

with $\vartheta_i \geq N_i$ for $i = 1, \ldots, k$. In the following we prove that $\vartheta_j > N_j$ implies that $p_j = (1 : m_j : 0)$ is a singular infinity point, which contradicts the assumption that the curve is asymptotically regular.

For this purpose, let us first remark on one essential question. We are assuming that $\vartheta_j > N_j \geq 1$. Then, using Equality (5.1), we have that

$$\frac{\partial F}{\partial x}(1, m_j, 0) = \frac{\partial f_d}{\partial x}(1, m_j) = 0 \tag{5.2}$$

$$\frac{\partial F}{\partial y}(1, m_j, 0) = \frac{\partial f_d}{\partial y}(1, m_j) = 0 \tag{5.3}$$

and

$$\frac{\partial F}{\partial z}(1, m_j, 0) = f_{d-1}(1, m_j), \tag{5.4}$$

so the infinity point $p_j = (1 : m_j : 0)$ is singular if and only if $f_{d-1}(1, m_j) = 0$.

Now, let B_j be defined by the following Puiseux series:

$$r_j(z) = m_j z + \alpha_1 z^{1-1/N_j} + \alpha_2 z^{1-2/N_j} + \cdots + \alpha_{N_j} + A_j(z)$$

where $\alpha_i \in \mathbb{C}, i = 1, \ldots, N_j$ and $A_j(z)$ is an infinite series whose terms are powers of z with negative exponents. As we explain in Section 2, these terms do not affect the asymptotic behavior of the infinity branch.

We recall that the infinity branches are obtained from the Puiseux solutions of $g(y, z) = 0$ around $z = 0$ (see Section 2). More precisely, there is a solution given by

$$\phi_j(z) = m_j + \alpha_1 z^{1/N_j} + \alpha_2 z^{2/N_j} + \cdots + \alpha_{N_j} z + z A_j(1/z),$$

and r_j is obtained from ϕ_j as $r_j(z) = z\phi_j(1/z)$. Hence, there must exist $M_j > 0$ such that $g(\phi_j(z), z) = 0$ for $|z| < M_j$.

Finally, let us take a look to the different terms of $g(y, z)$. From (5.1), we have that

$$f_d(1, y) = \prod_{i=1}^{k} (y - m_i)^{\vartheta_i}.$$

We define the remaining terms in a generic way; in particular,

$$f_{d-1}(1, y) = b_0 + b_1 y + \cdots + b_{d-1} y^{d-1}$$

and

$$f_{d-2}(1, y) = c_0 + c_1 y + \cdots + c_{d-2} y^{d-2}$$

where $b_0, \ldots, b_{d-1}, c_0, \ldots, c_{d-2} \in \mathbb{R}$.

Summarizing, we have that:

$$g(y, z) = \prod_{i=1}^{k} (y - m_i)^{\vartheta_i} + (b_0 + b_1 y + \cdots + b_{d-1} y^{d-1}) z + (c_0 + c_1 y + \cdots + c_{d-2} y^{d-2}) z^2 + \cdots$$

and there exist $M_j > 0$ such that $g(\phi_j(z), z) = 0$ if $|z| < M_j$. By substituting $y = \phi_j(z)$ in g, we get

$$\begin{aligned} g(\phi_j(z), z) &= \prod_{i=1}^{k} (m_j - m_i + \alpha_1 z^{1/N_j} + \alpha_2 z^{2/N_j} + \cdots)^{\vartheta_i} + \\ &+ (b_0 + b_1 (m_j + \alpha_1 z^{1/N_j} + \alpha_2 z^{2/N_j} + \cdots) + \cdots) z + \\ &+ (c_0 + c_1 (m_j + \alpha_1 z^{1/N_j} + \alpha_2 z^{2/N_j} + \cdots) + \cdots) z^2 + \cdots \end{aligned}$$

which can be expressed in the form

$$g(\phi_j(z), z) = C_0 z + C_1 z^{1+1/N_j} + C_2 z^{1+2/N_j} + \cdots C_{N_j} z^2 + \cdots$$

Since $g(\phi_j(z), z) = 0$ in a continuous set, the coefficients $C_0, C_1, C_2, \ldots$ must be null. The result follows by noting that

$$C_0 = b_0 + b_1 m_j + \cdots + b_{d-1} m_j^{d-1} = f_{d-1}(1, m_j). \quad \square$$

Theorem 4. *Let $\mathcal{C}$ be an asymptotically regular curve defined by an irreducible polynomial $f \in \mathbb{R}[x, y]$ of degree d. The asymptotic behavior of $\mathcal{C}$ is completely determined by f_d and f_{d-1}.*

Proof. As in the proof of Lemma 1, we start by considering a curve with two infinity branches; afterwards, we will show that one may reason similarly if the curve has three or more infinity branches. Thus, let $\mathcal{C}$ be an asymptotically regular curve with two infinity branches B_1 and B_2:

- B_1 has ramification index N_1 and is centered at an infinity point $p_1 = (1 : m_1 : 0)$.
- B_2 has ramification index N_2 and is centered at an infinity point $p_2 = (1 : m_2 : 0)$.

Since the curve is asymptotically regular, p_1 and p_2 are regular points. Then, necessarily, $m_1 \neq m_2$; otherwise, we have two places centered at the same point and the point is singular (see the construction of infinity branches in Section 2).

In the following we assume that $p_1 = (1:0:0)$ (we can get this point by applying a simple change of variables), so we have that $m_1 = 0$ and $m_2 \neq 0$. Then B_1 and B_2 are defined by two Puiseux series of the form:

$$r_1(z) = \alpha_1 z^{1-1/N_1} + \alpha_2 z^{1-2/N_1} + \cdots + \alpha_{N_1} + A_1(z) \tag{5.5}$$
$$r_2(z) = m_2 z + \beta_1 z^{1-1/N_2} + \beta_2 z^{1-2/N_2} + \cdots + \beta_{N_2} + A_2(z)$$

where $\alpha_i, \beta_j \in \mathbb{C}, i = 1, \ldots, N_1, j = 1, \ldots, N_2$, and $A_1(z)$ and $A_2(z)$ are infinite series whose terms are powers of z with negative exponents. As we explain in Section 2, these terms do not affect the asymptotic behavior of the infinity branches.

We recall that r_1 and r_2 are built from the following series that allow us to trace the infinity branches on the YZ-chart:

$$\phi_1(z) = \alpha_1 z^{1/N_1} + \alpha_2 z^{2/N_1} + \cdots + \alpha_{N_1} z + zA_1(1/z)$$
$$\phi_2(z) = m_2 + \beta_1 z^{1/N_2} + \beta_2 z^{2/N_2} + \cdots + \beta_{N_2} z + zA_2(1/z)$$

More precisely, we have that $r_i(z) = z\phi_i(1/z)$. In the following, we show how the coefficients of ϕ_1 and ϕ_2 may be computed from the polynomial f which implicitly defines the curve. Let:

$$f(x, y) = f_d(x, y) + f_{d-1}(x, y) + \cdots + f_1(x, y) + f_0$$
$$F(x, y, z) = f_d(x, y) + f_{d-1}(x, y)z + \cdots + f_1(x, y)z^{d-1} + f_0 z^d$$

In order to study the different infinity points of the curve, we will consider the $YZ - chart$, where they become affine points. For this purpose we define the polynomial

$$g(y, z) := F(1, y, z) = f_d(1, y) + f_{d-1}(1, y)z + \cdots + f_1(1, y)z^{d-1} + f_0 z^d.$$

As one can see, for constructing this polynomial we need the different homogeneous forms of f to be evaluated in $x = 1$. From Lemma 1, we know that

$$f_d(1, y) = y^{N_1}(y - m_2)^{N_2}$$

where $N_1 + N_2 = d$. The remaining homogeneous forms are defined in a generic way. In particular,

$$f_{d-1}(1, y) = b_0 + b_1 y + \cdots + b_{d-1} y^{d-1}$$

and

$$f_{d-2}(1, y) = c_0 + c_1 y + \cdots + c_{d-2} y^{d-2}$$

where $b_0, \ldots, b_{d-1}, c_0, \ldots, c_{d-2} \in \mathbb{R}$.

At this point, let us remark that the infinity point $p_1 = (1:0:0)$ is regular if and only if $b_0 \neq 0$. Indeed, from Equalities (5.2), (5.3) and (5.4),

$$\frac{\partial F}{\partial x}(p_1) = 0 \quad , \quad \frac{\partial F}{\partial y}(p_1) = 0 \quad , \quad \frac{\partial F}{\partial z}(p_1) = f_{d-1}(1, 0) = b_0. \tag{5.6}$$

We will take this fact into account throughout this proof.

Once we have defined the different homogeneous forms, we have that:

$$g(y, z) = y^{N_1}(y - m_2)^{N_2} + (b_0 + b_1 y + \cdots + b_{d-1} y^{d-1})z + (c_0 + c_1 y + \cdots + c_{d-2} y^{d-2})z^2 + \cdots$$

From Section 2, we know that there exists two positive numbers M_1 and M_2 such that $g(\phi_i(z), z) = 0$ for $|z| < M_i, i = 1, 2$. By substituting $y = \phi_1(z)$ in g, we get

$$g(\phi_1(z), z) = (\alpha_1 z^{1/N_1} + \alpha_2 z^{2/N_1} + \cdots)^{N_1}(-m_2 + \alpha_1 z^{1/N_1} + \alpha_2 z^{2/N_1} + \cdots)^{N_2}$$
$$+ (b_0 + b_1(\alpha_1 z^{1/N_1} + \alpha_2 z^{2/N_1} + \cdots) + \cdots)z + (c_0 + c_1(\alpha_1 z^{1/N_1} + \alpha_2 z^{2/N_1} + \cdots) + \cdots)z^2 + \cdots$$

which can be expressed in the form

$$g(\phi_1(z), z) = C_0 z + C_1 z^{1+1/N_1} + C_2 z^{1+2/N_1} + \cdots C_{N_1} z^2 + \cdots$$

Since $g(\phi_1(z), z) = 0$ in a continuous set, the coefficients $C_0, C_1, C_2, \ldots$ must be null:

- The coefficient associated to z is

$$C_0 := \underline{\alpha_1}^{N_1}(-m_2)^{N_2} + \underline{b_0}. \tag{5.7}$$

Since $C_0 = 0$ we can compute α_1 from b_0. Furthermore, from (5.6) we know that $b_0 \neq 0$; then $\alpha_1, m_2 \neq 0$.
At this point, let us remark that, indeed, $\vartheta_1 = N_1$ (see (5.1)). Otherwise $C_0 = b_0 = 0$, which implies that the infinity point p_1 is singular (see (5.6)). Obviously, this same reasoning may be used to show that $\vartheta_2 = N_2$.
- The coefficient associated to z^{1+1/N_1} is:

$$C_1 := N_1\alpha_1^{N_1-1}\underline{\alpha_2}(-m_2)^{N_2} + \alpha_1^{N_1}N_2(-m_2)^{N_2-1}\alpha_1 + \underline{b_1}\alpha_1$$

Note that two new elements arise in this equation: α_2 and b_1. Since the rest of parameters are known and $C_1 = 0$, we can compute α_2 from b_1 (note that $N_1, \alpha_1, m_2 \neq 0$).
- The coefficient associated to z^{1+2/N_1} is:

$$C_2 := \left(N_1\alpha_1^{N_1-1}\underline{\alpha_3} + \binom{N_1}{2}\alpha_1^{N_1-2}\alpha_2^2\right)(-m_2)^{N_2} + N_1\alpha_1^{N_1-1}\alpha_2N_2(-m_2)^{N_2-1}\alpha_1$$
$$+\alpha_1^{N_1}\left(N_2(-m_2)^{N_2-1}\alpha_2 + \binom{N_2}{2}(-m_2)^{N_2-2}\alpha_1^2\right) + b_1\alpha_2 + \underline{b_2}\alpha_1^2.$$

Again, all the elements in this equation are known but α_3 and b_2. Since $C_2 = 0$, we can compute α_3 from b_2.

Reasoning in this way, we can compute $\alpha_1, \alpha_2, \ldots, \alpha_{N_1}$ from the parameters $b_0, b_1, \ldots, b_{N_1-1}$, which belong to the homogeneous form f_{d-1}. In order to compute α_{N_1+1}, we should have to consider the coefficient C_{N_1}, associated to z^2, but this coefficient involves the parameter c_0, which belongs to the homogeneous form f_{d-2}. In general, for computing the remaining terms of ϕ_1 we need to use parameters of $f_{d-2}, f_{d-3}, \ldots$. However, we are not interested in these terms since they are associated to negative powers of z in r_1 and they do not affect the asymptotic behavior of the branch. In fact, an asymptote can be obtained from $\alpha_1, \alpha_2, \ldots, \alpha_{N_1}$.

In the following, we will compute the coefficients of the series ϕ_2, which defines the second infinity branch, but first we make the following observations.

- We are assuming the infinity points of the curve to be regular; otherwise the above development could not be carried out. Indeed, if p_1 is singular we have that $b_0 = 0$ (see (5.6)); then, from (5.7), we deduce that $\alpha_1 = 0$ or $m_2 = 0$. As a consequence, the term of C_2 which contains α_2 vanishes and we cannot compute the coefficients of the series ϕ_2 as above.
- We have just shown how to compute $\alpha_1, \alpha_2, \ldots, \alpha_{N_1}$ from $b_0, b_1, \ldots, b_{N_1-1}$. Note that we could also do the opposite, that is, to compute $b_0, b_1, \ldots, b_{N_1-1}$ from $\alpha_1, \alpha_2, \ldots, \alpha_{N_1}$. Indeed, by considering C_0 (see (5.7)) we can get b_0 from α_1, by considering C_1 we can get b_1 from α_2 and so on. Thus, the implicit equation of the curve can be partially determined by one of its asymptotes. Theorem 5, below, goes further into this question.

Now we carry on computing the coefficients of ϕ_2. We have that

$$g(\phi_2(z), z) = (m_2 + \beta_1 z^{1/N_2} + \beta_2 z^{2/N_2} + \cdots)^{N_1}(\beta_1 z^{1/N_2} + \beta_2 z^{2/N_2} + \cdots)^{N_2}$$
$$+(b_0 + b_1(m_2 + \beta_1 z^{1/N_2} + \beta_2 z^{2/N_2} + \cdots) + b_2(m_2 + \beta_1 z^{1/N_2} + \beta_2 z^{2/N_2} + \cdots)^2 + \cdots)z$$
$$+(c_0 + c_1(m_2 + \beta_1 z^{1/N_2} + \beta_2 z^{2/N_2} + \cdots) + \cdots)z^2 + \cdots$$

which can be expressed as:

$$g(\phi_2(z), z) = D_0 z + D_1 z^{1+1/N_2} + D_2 z^{1+2/N_2} + \cdots D_{N_2} z^2 + \cdots$$

Now we recall that $g(\phi_2(z), z) = 0$ for every $|z| < M_2$ and we reason as above:

- The coefficient associated to z is

$$D_0 := m_2^{N_1}\underline{\beta_1}^{N_2} + (b_0 + b_1 m_2 + b_2 m_2^2 + \cdots + b_{d-1}m_2^{d-1}). \tag{5.8}$$

Since $D_0 = 0$ we may obtain β_1 from $b_0, b_1, b_2, \ldots, b_{d-1}$. Note that $b_0 + b_1 m_2 + b_2 m_2^2 + \cdots + b_{d-1}m_2^{d-1} = f_{d-1}(1, m_2) \neq 0$ since the point $p_2 = (1 : m_2 : 0)$ is regular (see (5.6)). Thus, we deduce that $\beta_1 \neq 0$.
- The coefficient associated to z^{1+1/N_2} is:

$$D_1 := N_1 m_2^{N_1-1}\beta_1\beta_1^{N_2} + m_2^{N_1}N_2\beta_1^{N_2-1}\underline{\beta_2}$$
$$+\left(b_1\beta_1 + b_2 2m_2\beta_1 + b_3\binom{3}{2}m_2^2\beta_1 + \cdots + b_{d-1}\binom{d-1}{d-2}m_2^{d-2}\beta_1\right)$$

All the coefficients in this expression are known but β_2. Since $D_1 = 0$ and $m_2, N_2, \beta_1 \neq 0$, we can compute β_2.

- The coefficient associated to z^{1+2/N_2} is:

$$D_2 := m_2^{N_1}\left(N_2\beta_1^{N_2-1}\underline{\beta_3} + \binom{N_2}{2}\beta_1^{N_2-2}\beta_2^2\right) + N_1 m_2^{N_1-1}\beta_1 N_2\beta_1^{N_2-1}\beta_2$$
$$+\left(N_1 m_2^{N_1-1}\beta_2 + \binom{N_1}{2} m_2^{N_1-2}\beta_1^2\right)\beta_1^{N_2} + b_1\beta_2 + b_2(\beta_1^2 + 2m_2\beta_2) + \cdots$$
$$+b_{d-1}\left((d-1)m_2^{d-2}\beta_3 + \binom{d-1}{2} m_2^{d-3}\beta_1^2\right).$$

Again, all the elements in this expression are known but β_3, so we can compute β_2 by taking into account that $D_2 = 0$.

We can compute in this way $\beta_1, \beta_2, \ldots, \beta_{N_2}$ from the parameters $b_0, b_1, \ldots, b_{d-1}$, which belong to the homogeneous form f_{d-1}. However, in order to compute the next term of the series ϕ_2 (that is, β_{N_2+1}), we should have to consider the coefficient D_{N_2}, associated to z^2, which involves the parameter c_0 of the homogeneous form f_{d-2}. In general, for computing the remaining terms of ϕ_2 we need to use parameters of $f_{d-2}, f_{d-3}, \ldots$, but we are not interested in these terms since they are associated to negative powers of z in r_2 and they do not affect the asymptotic behavior of the branch. In fact, an asymptote can be obtained from $\beta_1, \beta_2, \ldots, \beta_{N_2}$.

Thus, we have proved that the asymptotic behavior of the curve is determined by the homogeneous forms f_d and f_{d-1}. Finally, note that the above reasoning may be easily generalized for the case of a curve with three or more regular infinity branches. Let those branches be $B_1, B_2, \ldots, B_k$, with ramification indexes $N_1, N_2, \ldots, N_k$, and let them be centered at the infinity points $p_1 = (1:0:0), p_2 = (1:m_2:0), \ldots, p_k = (1:m_k:0)$. Now we have that

$$f_d(1, y) = y^{N_1}(y - m_2)^{N_2}\cdots(y - m_k)^{N_k}$$

and we can proceed similarly as in the case of two branches. $\square$

Theorem 5. *Two asymptotically regular curves have the same asymptotic behavior if and only if their terms of degree d and $d-1$ are the same.*

Proof. From Theorem 4 we know that two curves whose terms of degree d and $d-1$ are the same have the same asymptotic behavior. Now we prove that two curves with the same asymptotic behavior have the same terms of degree d and $d-1$.

As in the proof of Theorem 4, we start by considering an asymptotically regular curve $\mathcal{C}$ with two infinity branches, B_1 and B_2, defined by two Puiseux series of the form:

$$r_1(z) = \alpha_1 z^{1-1/N_1} + \alpha_2 z^{1-2/N_1} + \cdots + \alpha_{N_1} + A_1(z) \qquad (5.9)$$
$$r_2(z) = m_2 z + \beta_1 z^{1-1/N_2} + \beta_2 z^{1-2/N_2} + \cdots + \beta_{N_2} + A_2(z)$$

where $\alpha_i, \beta_j \in \mathbb{C}, i = 1, \ldots, N_1, j = 1, \ldots, N_2$. $A_1(z)$ and $A_2(z)$ are infinite series whose terms are powers of z with negative exponents. We show above, that these branches are associated to the infinity points $p_1 = (1:0:0)$ and $p_2 = (1:m_2:0)$ $(m_2 \neq 0)$ and that the homogeneous form of maximum degree of $\mathcal{C}$ is $f_d(1, y) = y^{N_1}(y - m_2)^{N_2}$, where $N_1 + N_2 = d$.

Now, let $\widetilde{\mathcal{C}}$ be another asymptotically regular curve having the same asymptotic behavior as $\mathcal{C}$. This means that $\widetilde{\mathcal{C}}$ has an infinity branch convergent to each infinity branch of $\mathcal{C}$ and viceversa. Consequently, both curves must have the same infinity points and the homogeneous form of maximum degree of $\widetilde{\mathcal{C}}$ must be

$$\tilde{f}_d(x, y) = y^{\vartheta_1}(y - m_2 x)^{\vartheta_2}.$$

Now, let us return to the proof of Lemma 1. There we showed that $\vartheta_1 = N_1$ and $\vartheta_2 = N_2$ must hold for $\widetilde{\mathcal{C}}$ to be asymptotically regular. Thus, evaluating in $x = 1$, we have

$$\tilde{f}_d(1, y) = y^{N_1}(y - m_2)^{N_2}.$$

We define the rest of homogeneous forms in a generic way. In particular,

$$\tilde{f}_{d-1}(1, y) = b_0 + b_1 y + \cdots + b_{d-1}y^{d-1}$$

and

$$\tilde{f}_{d-2}(1, y) = c_0 + c_1 y + \cdots + c_{d-2}y^{d-2}$$

where $b_0, \ldots, b_{d-1}, c_0, \ldots, c_{d-2} \in \mathbb{R}$.

We keep on reviewing the proof of Theorem 4. Note that, from (5.7) and the subsequent equations, one can directly compute the coefficients $b_0, b_1, \ldots, b_{N_1-1}$ from $\alpha_1, \alpha_2, \ldots, \alpha_{N_1}$. That is, some coefficients of f_{d-1} are determined by the relevant terms of the infinity branch B_1. However, if we want to compute the rest of coefficients, $b_{N_1}, b_{N_1+1}, \ldots, b_{N_1+N_2-1}$, we need to solve a linear system that involves $\beta_1, \beta_2, \ldots, \beta_{N_2}$. From (5.8), $D_0 = 0$ can be expressed as

$$b_0 + b_1 m_2 + b_2 m_2^2 + \cdots + b_{d-1} m_2^{d-1} = -m_2^{N_1} \beta_1{}^{N_2}$$

Analogously, the equalities $D_i = 0, i = 0, 1, \ldots, N_2 - 1$ are also linear equations in the variables $b_0, b_1, \ldots, b_{d-1}$. Since $b_0, \ldots, b_{N_1-1}$ are known, we have a linear system with N_2 equations and N_2 variables: $b_{N_1}, \ldots, b_{N_1+N_2-1}$ (recall that $N_1 + N_2 = d$).

Now we have to prove that this linear system has a solution and that the solution is unique. We apply the change of variables $y = Y + m_2 x$ and we obtain a new curve (for the sake of simplicity, we will denote it also by $\widetilde{\mathcal{C}}$). Thus, we have

$$\widetilde{g}(Y, z) = (Y + m_2)^{N_1} Y^{N_2} + (b_0 + b_1(Y + m_2) + \cdots + b_{d-1}(Y + m_2)^{d-1})z$$
$$+ (c_0 + c_1(Y + m_2) + \cdots + c_{d-2}(Y + m_2)^{d-2})z^2 + \cdots$$

which can also be expressed as follows:

$$\widetilde{g}(Y, z) = (Y + m_2)^{N_1} Y^{N_2} + (\widetilde{b}_0 + \widetilde{b}_1 Y + \cdots + \widetilde{b}_{d-1} Y^{d-1})z + (\widetilde{c}_0 + \widetilde{c}_1 Y + \cdots + \widetilde{c}_{d-2} Y^{d-2})z^2 + \cdots$$

We observe that the new curve has an infinity branch defined by the Puiseux series

$$\widetilde{\phi}_2(z) = \phi_2(z) - m_2 = \beta_1 z^{1/N_2} + \beta_2 z^{2/N_2} + \cdots + \beta_{N_2} z + z A_2(1/z),$$

which is associated to the infinity point $\widetilde{p}_2 = (1 : 0 : 0)$.

Reasoning as above, we can directly obtain the first N_2 coefficients of $\widetilde{f}_{d-1}$, that is, $\widetilde{b}_0, \widetilde{b}_1, \ldots, \widetilde{b}_{N_2-1}$, by taking into account that $\widetilde{g}(\widetilde{\phi}_2(z), z) = 0$ for every value of z in a continuous set. Let us show that these values determine in a unique way the N_2 coefficients $b_{N_1}, \ldots, b_{N_1+N_2-1}$. Indeed, we recall that $b_0, b_1, \ldots, b_{d-1}$ are the coefficients of $f_{d-1}(1, y) := h_{d-1}(y)$ and $\widetilde{b}_0, \widetilde{b}_1, \ldots, \widetilde{b}_{d-1}$ are the coefficients of $\widetilde{f}_{d-1}(1, y) := \widetilde{h}_{d-1}(y)$. In addition, we have that $\widetilde{h}_{d-1}(y) = h_{d-1}(y + m_2)$. From the Taylor expansion of $\widetilde{h}_{d-1}(y)$, it follows that:

$$\widetilde{b}_k = \frac{1}{k!} \frac{d^k \widetilde{h}_{d-1}}{dy^k}(0) = \frac{1}{k!} \frac{d^k h_{d-1}}{dy^k}(m_2)$$

for $k = 0, 1, 2, \ldots, d-1$. In particular, this is true for $k = 0, 1, 2, \ldots, N_2 - 1$, which leads us to a linear system of the form:

$$\begin{cases} b_0 + m_2 b_1 + m_2^2 b_2 & + \cdots + & m_2^{d-1} b_{d-1} = \widetilde{b}_0 \\ \qquad b_1 + 2 m_2 b_2 & + \cdots + & (d-1) m_2^{d-2} b_{d-1} = \widetilde{b}_1 \\ \qquad\qquad b_2 & + \cdots + & (d-1)(d-2)/2 m_2^{d-3} b_{d-1} = \widetilde{b}_2 \\ & \vdots & \end{cases}$$

This system has N_2 equations which can also be expressed as follows (recall that the coefficients $b_0, b_1, \ldots, b_{N_1-1}$ are known):

$$\begin{cases} m_2^{d-1} b_{d-1} & + m_2^{d-2} b_{d-2} & + \cdots = B_0 \\ (d-1) m_2^{d-2} b_{d-1} & + (d-2) m_2^{d-3} b_{d-2} & + \cdots = B_1 \\ (d-1)(d-2) m_2^{d-3} b_{d-1} & + (d-2)(d-3) m_2^{d-4} b_{d-2} & + \cdots = 2B_2 \\ \vdots & \vdots & \end{cases} \tag{5.10}$$

where

$$\begin{cases} B_0 = \widetilde{b}_0 - b_0 - m_2 b_1 - m_2^2 b_2 & \cdots & -m_2^{N_1-1} b_{N_1-1} \\ B_1 = \widetilde{b}_1 - b_1 - 2 m_2 b_2 & \cdots & -(N_1 - 1) m_2^{N_1-2} b_{N_1-1} \\ B_2 = \widetilde{b}_2 - b_2 & \cdots & -(N_1 - 1)(N_1 - 2)/2 m_2^{N_1-3} b_{N_1-1} \\ & \vdots & \end{cases}$$

Finally, we need to prove that the system (5.10) has a solution and that this solution is unique, which holds if and only if the following determinant is not null:

$$\begin{vmatrix} r^{d-1} & r^{d-2} & \cdots & r^{d-N_2} \\ (d-1)r^{d-2} & (d-2)r^{d-3} & \cdots & (d-N_2)r^{d-N_2-1} \\ (d-1)(d-2)r^{d-3} & (d-2)(d-3)r^{d-4} & \cdots & (d-N_2)(d-N_2-1)r^{d-N_2-2} \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{i=1}^{N_2-1}(d-i)r^{d-N_2} & \prod_{i=2}^{N_2}(d-i)r^{d-N_2-1} & \cdots & \prod_{i=N_2}^{2N_2-2}(d-i)r^{d-2N_2+1} \end{vmatrix}.$$

Indeed: note that this determinant is the Wronskian of the functions $x_1(r) = r^{d-1}, x_2(r) = r^{d-2}, \dots, x_{N_2}(r) = r^{d-N_2}$, which are well-known to be linearly independent (note that $d - N_2 \geq 0$); thus, the Wronskian is not null and the result is proved.

Note that this proof can be directly generalized for the case of a curve with three or more infinity branches. Let those branches be $B_1, B_2, \dots, B_k$ and let them be defined by the series $\phi_1, \phi_2, \dots, \phi_k$, with center at the infinity points $p_1 = (1 : 0 : 0), p_2 = (1 : m_2 : 0), \dots, p_k = (1 : m_k : 0)$. Then, for $i = 2, \dots, k$, we have to apply the change $y = Y + m_i$ and we obtain a linear system by canceling the coefficients of $g(\phi_i(z), z)$. Each of these systems has a unique solution, as happens with (5.10). □

CRediT authorship contribution statement

Authors equally contributed to this work.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

Arnold, V.I., 1989. Mathematical Methods of Classical Mechanics, second edition. Springer.
Arnold, V.I., 1990. Singularities of Caustics and Wave Fronts. Kluwer.
Bazant, M.Z., Crowdy, D., 2005. In: Yip, S. (Ed.), Handbook of Materials Modeling. Springer.
Blasco, A., Pérez-Díaz, S., 2014a. Asymptotes and perfect curves. Comput. Aided Geom. Des. 31 (2), 81–96.
Blasco, A., Pérez-Díaz, S., 2014b. Asymptotic behavior of an implicit algebraic plane curve. Comput. Aided Geom. Des. 31 (7–8), 345–357.
Blasco, A., Pérez-Díaz, S., 2015. Asymptotes of space curves. J. Comput. Appl. Math. 278, 231–247.
Blasco, A., Pérez-Díaz, S., 2020. A new approach for computing the asymptotes of a parametric curve. J. Comput. Appl. Math. 364, 1–18. 112350.
Caflisch, R.C., Papanicolau, G. (Eds.), 1993. Singularities in Fluids, Plasmas and Optics. Kluwer.
Campo, E., Fernández de Sevilla, M.A., Pérez-Díaz, S., 2022a. Computing branches and asymptotes of curves defined by a not rational parametrization. Submitted for publication.
Campo, E., Fernández de Sevilla, M.A., Pérez-Díaz, S., 2022b. A simple formula for the computation of branches of curves defined parametrically and some applications. Comput. Aided Geom. Des. 94, 102084.
Chorin, A., Marsden, J.E., 2000. A Mathematical Introduction to Fluid Mechanics. Springer.
Eggers, J., Fontelos, M.A., 2015. Singularities: Formation, Structure, and Propagation. Cambridge University Press, Cambridge.
Gao, B., Chen, Y., 2012. Finding the topology of implicitly defined two algebraic plane curves. J. Syst. Sci. Complex. 25 (2), 362–374.
González-Vega, L., Necula, I., 2002. Efficient topology determination of implicitly defined algebraic plane curves. Comput. Aided Geom. Des. 19 (9), 719–743.
Greuel, G.M., Lossen, C., Shustin, E., 2007. Introduction to Singularities and Deformations. Springer.
Hong, H., 1996. An effective method for analyzing the topology of plane real algebraic curves. Math. Comput. Simul. 42, 572–582.
Landau, L.D., Lifshitz, E.M., 1976. Mechanics. Pergamon, Oxford.
Zeng, G., 2007. Computing the asymptotes for a real plane algebraic curve. J. Algebra 316, 680–705.