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A simple formula for the computation of branches and asymptotes of curves and some applications ☆

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ABSTRACT

In this paper, we obtain a simple formula based on the computation of some derivatives for determining the branches and the asymptotes of curves that are defined by a parametrization. For this purpose, we use some previous results and notions presented in Blasco and Pérez-Díaz (2014a,b, 2015, 2020). From these results, we show how the generalized asymptotes of the input curve can be easily computed and we present some applications related to the ramification index and degree of the asymptote, the infinity form and the multiplicity of the infinity points. Furthermore, we show how to construct all the families of parametric curves having some given asymptotes. We develop this method for the plane case but it can be trivially adapted for dealing with rational curves in n-dimensional space. In addition, the formulaes presented can be similarly obtained for curves defined by a parametrization not necessarily rational.

1. Introduction

In almost any engineering sub-discipline, sometimes one encounters phenomena that have asymptotic behavior, but people don't think of studying or using those phenomena as *using asymptotes*. It comes up (among other places) whenever you want to understand the long-term behavior of a system. In more detail, suppose you have a model of a system that involves a time parameter t (among other parameters). In fact, to simplify things let's say you fix the value of all but two parameters: t and x, where x measures something. You are often interested in possible behavior of x as $t \to \infty$. If you find that there's a horizontal asymptote at some value of x, say $x = x_0$, then you know that is the long term equilibrium of that parameter, at least given your choices for all the other parameters. Asymptotes show up everywhere as for instance terminal velocity, population models, equilibrium pricing in markets, control systems/feedback loops, etc., and recognize them—they give simplified view of the function at extreme points.

The asymptotes of an infinity branch, B, of a real plane algebraic curve, $\mathcal{C}$, reflect the status of B at the points with sufficiently large coordinates (see for instance Maxwell (1962)). For example, if you look at the value of $f(x) = 1/x$ when x is very large, it is clear that $f(x)$ gets closer to 0, it *approaches* 0, without ever reaching it. That means that $f(x)$ asymptotically approaches 0 as x approaches infinity: its asymptote is $y = 0$. That is called a horizontal asymptote, because the function approaches a horizontal line. On the other side, we say that $f(x) = 1/x$ approaches infinity as x approaches 0 from the

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positive direction. We also say that $f(x) = 1/x$ approaches negative infinity as x approaches 0 from the negative direction. That is called a vertical asymptote ($x = 0$), because the function approaches a vertical line. But an asymptote do not always have a constant value. For instance, $f(x) = x + 1/x$ asymptotically approaches $f(x) = x$ as x goes to infinity. That is called an oblique asymptote ($y = x$). Note that in this case, these limits can be computed, and thus we may obtain the equation of the asymptote of $\mathcal{C}$ at B. However, if this branch B is implicitly defined and its equation cannot be converted into an explicit form, both the decision and the computation of the asymptote of $\mathcal{C}$ at B require some other tools. Even worse, a curve may have more general curves than lines describing the status of a branch at the points with sufficiently large coordinates.

Intuitively speaking, we say that a curve $\widetilde{\mathcal{C}}$ is a *generalized asymptote* (or *g-asymptote*) of another curve $\mathcal{C}$ if the distance between $\widetilde{\mathcal{C}}$ and $\mathcal{C}$ tends to zero as they tend to infinity, and $\mathcal{C}$ can not be approached by a new curve of lower degree (see Blasco and Pérez-Díaz, 2014a,b, 2015, 2020). This motivates our interest in efficiently computing these *generalized asymptotes* for a wider variety of varieties such as the curves defined in the n-dimensional space or those that are not algebraic curves. In a more general way, our purpose is to determine all the infinity branch using the input parametrization and providing some easy formulaes based on the computation of derivatives. In fact, all the infinity branch can be computed by means of the presented formula and some important results can be obtained from here as for instance, how to read from the input parametrization the ramification index and degree of the g-asymptote and how to compute the infinity form and the multiplicity and the character of the infinity points.

This problem, and in particular the study of the g-asymptotes (or generalized asymptotes) as well as some notions and results as for instance the concept of perfect curves or the asymptotic behavior is dealt in previous papers (see Blasco and Pérez-Díaz (2014a,b, 2015, 2019, 2020). In particular some methods for computing the g-asymptotes are presented for algebraic curves parametrically and implicitly defined. For this purpose, the notions of infinity branches, approaching curves and perfect curves are introduced. The new goal we solve in this paper consists in providing a simple formula based on derivatives that allow to compute easily the branches (and in particular, the g-asymptotes) of curves defined parametrically in n-dimensional space. In addition, some consequences are presented. More precisely, we show how to read from the input parametrization the ramification index and degree of the g-asymptote and how to compute the infinity form and the multiplicity and character of the infinity points. Additionally, from these results we show how to construct all the families of parametric curves having some given asymptotes. Determining the branches and, in particular, the asymptotes of a curve is an important step, for instance, in sketching its graph, but also it provides some important properties as for instance in singularity theory, where one may find in a great variety of applications in computer aided design, science and engineering, and in particular as described by partial differential equations (see e.g. Arnold (1989, 1990), Bazant and Crowdy (2005), Caflisch and Papanicolau (1993), Chorin and Marsden (2000), Eggers and Fontelos (2015), Greuel et al. (2007), Landau and Lifshitz (1976)).

We have intended the paper to be self-contained. For this reason, we have included Section 2, where we review the theory of infinity branches and introduce the notions of convergent branches (that is, branches that get closer as they tend to infinity) and approaching curves (see Blasco and Pérez-Díaz (2014a)), and Section 3, where we lay down fundamental concepts like *perfect curve* (a curve of degree d that cannot be approached by any curve of degree less than d) and *g-asymptote* (a perfect curve that approaches another curve at an infinity branch). In addition, we present the methods that allow to compute the infinity branches of a given curve implicitly and parametrically defined, and a g-asymptote for each of them (see Subsections 3.1, 3.2 and 3.2.1).

The main results of the paper are presented in Section 4. Here, we develop a method that allows to easily compute all the generalized asymptotes of a curve defined by a parametrization by only determining some simple derivatives of functions constructed from the given parametrization. The results presented are concerned with plane curves but, as we remark in the paper, they can be adapted for dealing with curves in n-dimensional space. From this formula, some important results can be obtained (see Subsection 4.1). In particular, we show how to read from the input parametrization the ramification index and degree of the g-asymptote and how to compute the infinity form and the multiplicity and the character of the infinity points. As a consequence, we also comment how this formula could provide an answer the question on determining the families of parametric curves that have some input given g-asymptotes. Finally, we finish with some conclusions, and future work (see Section 5).

2. Notation and previous results

In the following, we introduce some basic notions as the concept of *infinity branch*, *convergent branches* and *approaching curves*, and we present some properties which allow us to compare the behavior of two implicit algebraic plane curves at infinity. For more details on these concepts and results, we refer to Blasco and Pérez-Díaz (2014b) (see Sections 3 and 4).

We consider an irreducible algebraic affine plane curve $\mathcal{C}$ over $\mathbb{C}$ defined by the irreducible polynomial $f(x, y) \in \mathbb{R}[x, y]$. Let $\mathcal{C}^*$ be its corresponding projective curve, defined by the homogeneous polynomial

$$F(x, y, z) = f_d(x, y) + zf_{d-1}(x, y) + z^2 f_{d-2}(x, y) + \cdots + z^d f_0 \in \mathbb{R}[x, y, z],$$

where $d := \deg(\mathcal{C})$. We assume that $(0 : 1 : 0)$ is not an infinity point of $\mathcal{C}^*$ (otherwise, we may consider a linear change of coordinates).

In order to get the infinity branches of $\mathbb{C}$, we consider the curve defined by the polynomial $g(y, z) = F(1 : y : z)$ and we compute the series expansion for the solutions of $g(y, z) = 0$ around $z = 0$. There exist exactly $\deg_y(g)$ solutions given by different Puiseux series that can be grouped into conjugacy classes (see e.g. Chapter 2 in Sendra et al. (2007) and Example 1). More precisely, if

$$\varphi(z) = m + a_1 z^{N_1/N} + a_2 z^{N_2/N} + a_3 z^{N_3/N} + \cdots \in \mathbb{C}\langle\langle z\rangle\rangle, \quad a_i \neq 0,\ \forall i \in \mathbb{N},$$

where $N \in \mathbb{N}$, $N_i \in \mathbb{N}$, $i \in \mathbb{N}$, and $0 < N_1 < N_2 < \cdots$, is a Puiseux series such that $g(\varphi(z), z) = 0$, and $\nu(\varphi) = N$ (i.e., N is the *ramification index* of φ), the series

$$\varphi_j(z) = m + a_1 c_j^{N_1} z^{N_1/N} + a_2 c_j^{N_2} z^{N_2/N} + a_3 c_j^{N_3} z^{N_3/N} + \cdots$$

where c_j, $j = 1, \ldots, N$ are the complex roots of $x^N = 1$ (that is, $|c_j^N| = 1$, $j \in \{1, \ldots, N\}$), are called the *conjugates* of φ. The set of all the conjugates of φ is called the *conjugacy class* of φ and it contains $\nu(\varphi)$ different series.

Since $g(\varphi(z), z) = 0$ in some neighborhood of $z = 0$ where $\varphi(z)$ converges, there exists $M \in \mathbb{R}^+$ such that $F(1 : \varphi(t) : t) = g(\varphi(t), t) = 0$ for $t \in \mathbb{C}$ and $|t| < M$, which implies that $F(t^{-1} : t^{-1}\varphi(t) : 1) = f(t^{-1}, t^{-1}\varphi(t)) = 0$, for $t \in \mathbb{C}$ and $0 < |t| < M$. We set $t^{-1} = z$, and we obtain that $f(z, r(z)) = 0$ for $z \in \mathbb{C}$ and $|z| > M^{-1}$ where

$$r(z) = z\varphi(z^{-1}) = mz + a_1 z^{1-N_1/N} + a_2 z^{1-N_2/N} + \cdots + a_k z^{1-N_k/N} + a_{k+1} z^{1-N_{k+1}/N} + \cdots,$$

$a_i \in \mathbb{C} \setminus \{0\}$, $\forall i \in \mathbb{N}$, $N, N_i \in \mathbb{N}$, $i \in \mathbb{N}$, and $0 < N_1 < N_2 < \cdots$. In addition, let $N_k \le N \le N_{k+1}$, i.e. the terms $a_j z^{1-N_j/N}$ with $j \ge k+1$ have negative exponent.

Reasoning similarly with the N different series in the conjugacy class, $\varphi_1, \ldots, \varphi_N$, we get

$$r_i(z) = z\varphi_i(z^{-1}) = mz + a_1 c_i^{N_1} z^{1-N_1/N} + a_2 c_i^{N_2} z^{1-N_2/N} + a_3 c_i^{N_3} z^{1-N_3/N} + \cdots$$

where $c_1, \ldots, c_N$ are the N complex roots of $x^N = 1$. Under these conditions, we introduce the following definition.

Definition 1. An *infinity branch of an affine plane curve* $\mathcal{C}$ associated to the infinity point $P = (1 : m : 0)$, $m \in \mathbb{C}$, is a set $B = \bigcup_{j=1}^{N} L_j$, where $L_j = \{(z, r_j(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\}$, $M \in \mathbb{R}^+$, and

$$r_j(z) = z\varphi_j(z^{-1}) = mz + a_1 c_j^{N_1} z^{1-N_1/N} + a_2 c_j^{N_2} z^{1-N_2/N} + a_3 c_j^{N_3} z^{1-N_3/N} + \cdots \tag{2.1}$$

where $N, N_i \in \mathbb{N}$, $i \in \mathbb{N}$, $0 < N_1 < N_2 < \cdots$, and $|c_j^N| = 1$, $j \in \{1, \ldots, N\}$. The subsets $L_1, \ldots, L_N$ are called the *leaves* of the infinity branch B.

Remark 1. An infinity branch is uniquely determined from one leaf, up to conjugation. That is, if $B = \bigcup_{i=1}^{N} L_i$, where $L_i = \{(z, r_i(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M_i\}$, and

$$r_i(z) = z\varphi_i(z^{-1}) = mz + a_1 z^{1-N_1/N} + a_2 z^{1-N_2/N} + a_3 z^{1-N_3/N} + \cdots$$

then $r_j = r_i$, $j \in \{1, \ldots, N\}$, up to conjugation; i.e.

$$r_j(z) = z\varphi_j(z^{-1}) = mz + a_1 c_j^{N_1} z^{1-N_1/N} + a_2 c_j^{N_2} z^{1-N_2/N} + a_3 c_j^{N_3} z^{1-N_3/N} + \cdots$$

where $N, N_i \in \mathbb{N}$, and $|c_j^N| = 1$, $j \in \{1, \ldots, N\}$.

By abuse of notation, we say that $B = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\}$ (where $M := \max\{M_1, \ldots, M_N\}$). Moreover, we say that N is *the ramification index* of the branch B and we write $\nu(B) = N$. Note that B has $\nu(B)$ leaves.

The proceeding introduced above allows us to obtain the infinity branches of a curve $\mathcal{C}$, under the assumption that $(0 : 1 : 0) \notin \mathcal{C}^*$. However, a curve may have infinity branches, associated to the infinity point $(0 : 1 : 0)$, which can not be constructed in this way. We call them Type II infinity branches and they have the form $\{(r(z), z) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\}$. A Type II infinity branch may be obtained by interchanging the variables x and y. See Blasco and Pérez-Díaz (2014b) (Definition 3.3) for further details.

In the following, we introduce the notions of convergent branches and approaching curves. Intuitively speaking, two infinity branches converge if they get closer as they tend to infinity. This concept will allow us to analyze whether two curves approach each other.

Definition 2. Two infinity branches, B and $\overline{B}$, are convergent if there exist two leaves $L = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\} \subset B$ and $\overline{L} = \{(z, \overline{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > \overline{M}\} \subset \overline{B}$ such that $\lim_{z\to\infty}(\overline{r}(z) - r(z)) = 0$. In this case, we say that the leaves L and $\overline{L}$ converge.

The following theorem provides a characterization for the convergence of two infinity branches (see Lemma 4.2 and Remark 4.3 in Blasco and Pérez-Díaz (2014b)).

Theorem 1. *The following statements hold:*

1. *Two leaves $L = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\}$ and $\overline{L} = \{(z, \overline{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > \overline{M}\}$ are convergent if and only if the terms with non negative exponent in the series $r(z)$ and $\overline{r}(z)$ are the same.*
2. *Two infinity branches B and $\overline{B}$ are convergent if and only if for each leaf $L \subset B$ there exists a leaf $\overline{L} \subset \overline{B}$ convergent with L, and reciprocally.*
3. *Two convergent infinity branches must be associated to the same infinity point.*

This paper is concerned with the study of the asymptotes of a curve. The classical concept of asymptote stands for a line that *approaches* a given curve when it tends to the infinity. In the following we generalize this idea by claiming that two curves *approach each other* if they, respectively, have two infinity branches that converge (see Definition 3 and Theorem 2 below).

Definition 3. Let $\mathcal{C}$ be an algebraic plane curve with an infinity branch B. We say that a curve $\overline{\mathcal{C}}$ *approaches* $\mathcal{C}$ at its infinity branch B if there exists one leaf $L = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\} \subset B$ such that $\lim_{z\to\infty} d((z, r(z)), \overline{\mathcal{C}}) = 0$.

The following theorem is proved in Blasco and Pérez-Díaz (2014b) (see Theorem 4.11).

Theorem 2. *Let $\mathcal{C}$ be a plane algebraic curve with an infinity branch B. A plane algebraic curve $\overline{\mathcal{C}}$ approaches $\mathcal{C}$ at B if and only if $\overline{\mathcal{C}}$ has an infinity branch, $\overline{B}$, such that B and $\overline{B}$ are convergent.*

Obviously, “approaching” is a symmetric concept, that is, $\mathcal{C}_1$ approaches $\mathcal{C}_2$ if and only if $\mathcal{C}_2$ approaches $\mathcal{C}_1$. When it happens we say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *approaching curves* or that they *approach each other*. In the next section we use this concept to generalize the classical notion of asymptote of a curve.

In the following example, we illustrate all the notions included in this section.

Example 1. Let $\mathcal{C}$ be the plane curve implicitly defined by the irreducible polynomial

$$f(x, y) = xy^3 - 2y^4 - x^3 - x^2y + 2xy^2 + y^3 + 2 \in \mathbb{R}[x, y].$$

The corresponding projective curve $\mathcal{C}^*$ is defined by

$$F(x : y : z) = xy^3 - 2y^4 - x^3z - x^2yz + 2xy^2z + y^3z + 2z^4 \in \mathbb{R}[x, y, z].$$

Note that $P_1 = (1 : 1/2 : 0)$ and $P_2 = (1 : 0 : 0)$ are infinity points of $\mathcal{C}^*$. Let us compute the infinity branches associated to P_1 and P_2. For this purpose, we consider the curve implicitly defined by the polynomial $g(y, z) = F(1 : y : z)$, and we observe that $g(p_j) = 0$, where $p_1 = (1/2, 0)$, $p_2 = (0, 0)$.

We compute the series expansion for the solutions of $g(y, z) = 0$. For this purpose, we use for instance the algcurves package included in the computer algebra system Maple. We get that:

$$\varphi_1(t) = 1/2 - 4116t^3 - 98t^2 - 7/2t + \cdots, \quad \text{and}$$

$\varphi_2(t) = 13475554t^{11/3}/729 + 1213997t^{10/3}/243 + 1372t^3 + 31117t^{8/3}/81 + 8930t^{7/3}/81 + 98t^2/3 + 91t^{5/3}/9 + 10t^{4/3}/3 + 4t/3 + t^{2/3} + t^{1/3} + \cdots.$

That is, $g(\varphi_j(t), t) = 0$, $j = 1, 2$ (see e.g. Section 2.5 in Sendra et al. (2007)). Note that $\nu(\varphi_1) = 1$, which implies that we only have one Puiseux series in the conjugacy class of φ_1 and that φ_1 is associated to the infinity point P_1. However, $\nu(\varphi_2) = 3$ and then, we have three conjugate Puiseux series in the conjugacy class of φ_2, namely $\varphi_{2,j}(t)$, $j = 1, 2, 3$. Observe that φ_2 is associated to the infinity point P_2.

Thus, we obtain two infinity branches (marked above the curve)

$$B_1 = L_1 = \{(z, r_1(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\}, \qquad \text{where} \quad r_1(z) = z\varphi_1(z^{-1})$$

and $B_2 = L_{2,1} \cup L_{2,2} \cup L_{2,3}$, where $L_{2,i} = \{(z, r_{2,i}(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\}$ and $r_{2,i}(z) = z\varphi_{2,i}(z^{-1})$, $i = 1, 2, 3$. In Fig. 1, we plot the curve $\mathcal{C}$ and some points of the infinity branches B_1 and B_2 associated to the infinity points P_1, P_2, respectively.

Now, let us consider the plane curve $\overline{\mathcal{C}}$ implicitly defined by the polynomial

$$\overline{f}(x, y) = xy^3 - 2y^4 - 2x^3 + 2x^2y + xy^2 + 10y^3 + 1 \in \mathbb{R}[x, y].$$

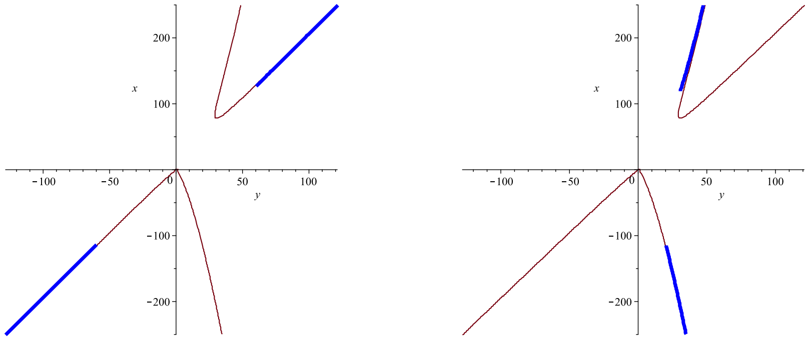

Fig. 1. Infinity branches B_1 (left), and B_2 (right).

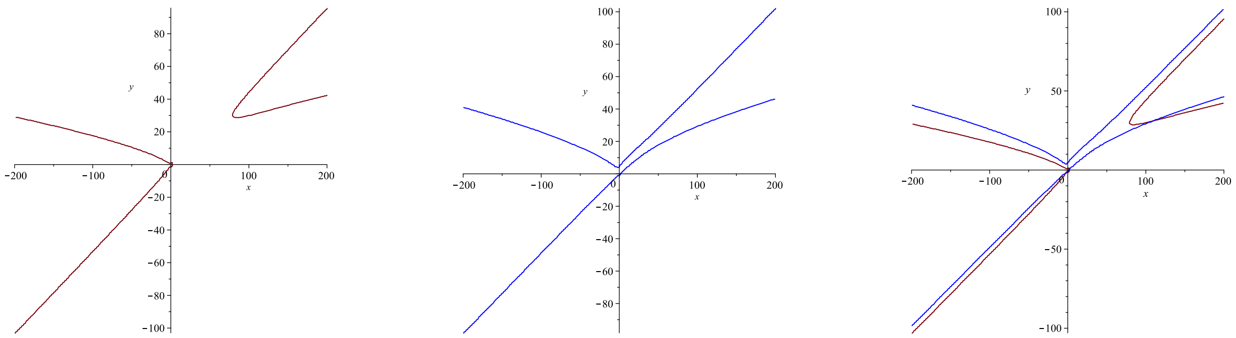

Fig. 2. $\mathcal{C}$ (left), $\overline{\mathcal{C}}$ (center), and both approaching curves (right).

Let us prove that $\mathcal{C}$ and $\overline{\mathcal{C}}$ approach each other (see Fig. 2) at the infinity branch associated to the infinity points P_1 and P_2 (note that both curves have these points as infinity points).

Reasoning as above for the curve $\mathcal{C}$, we get that the infinity branch of $\overline{\mathcal{C}}$ associated to P_1 is

$$\overline{B}_1 = \overline{L}_1 = \{(z, \overline{r}_1(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\}, \qquad \text{where} \quad \overline{r}_1(z) = z\overline{\varphi}_1(z^{-1})$$

and

$$\overline{\varphi}_1(t) = 1/2 + 1240t^3 + 60t^2 + 2t + \cdots .$$

Furthermore, the infinity branch of $\overline{\mathcal{C}}$ associated to P_2 is and $\overline{B}_2 = \overline{L}_{2,1} \cup \overline{L}_{2,2} \cup \overline{L}_{2,3}$, where $\overline{L}_{2,i} = \{(z, \overline{r}_{2,i}(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\}$ and $\overline{r}_{2,i}(z) = z\overline{\varphi}_{2,i}(z^{-1})$, $i = 1, 2, 3$, where

$\overline{\varphi}_2(t) = -5890391582 \cdot 2^{2/3}t^{11/3}/4782969 - 1237130869 \cdot 2^{1/3}t^{10/3}/1594323 - 1240t^3/3 - 3993827 \cdot 2^{2/3}t^{8/3}/39366 - 297529 \cdot 2^{1/3}t^{7/3}/6561 - 20t^2 - 1535 \cdot 2^{2/3}t^{5/3}/486 - 83 \cdot 2^{1/3}t^{4/3}/81 + t + 2^{2/3}t^{2/3})/3 + 2^{1/3}t^{(}1/3) + \cdots.$

Using Theorem 1, we conclude that both branches converge.

3. Asymptotes of an algebraic curve

Given an algebraic plane curve $\mathcal{C}$ and an infinity branch B, in Section 2, we have described how $\mathcal{C}$ can be approached at B by a second curve $\overline{\mathcal{C}}$. Now, suppose that $\deg(\overline{\mathcal{C}}) < \deg(\mathcal{C})$. Then one may say that $\mathcal{C}$ *degenerates*, since it behaves at infinity as a curve of smaller degree. For instance, a hyperbola is a curve of degree 2 that has two real asymptotes, which implies that the hyperbola degenerates, at infinity, to two lines. Similarly, one can check that every ellipse has two asymptotes, although they are complex lines in this case. However, the asymptotic behavior of a parabola is different, since it cannot be approached at infinity by any line. This motivates the following definition:

Definition 4. An algebraic curve of degree d is a *perfect curve* if it cannot be approached by any curve of degree less than d.

More properties on perfect curves can be found in Blasco and Pérez-Díaz (2014a). In particular, one has that if a given curve has an only branch and its degree is equal to the input curve, then it is perfect. For instance, a curve $\mathcal{C}$ defined by a proper parametrization of the form $(t^n, a_nt^n + a_{n-1}t^{n-1} + \cdots + a_0)$ is always perfect since it has an only branch B given by

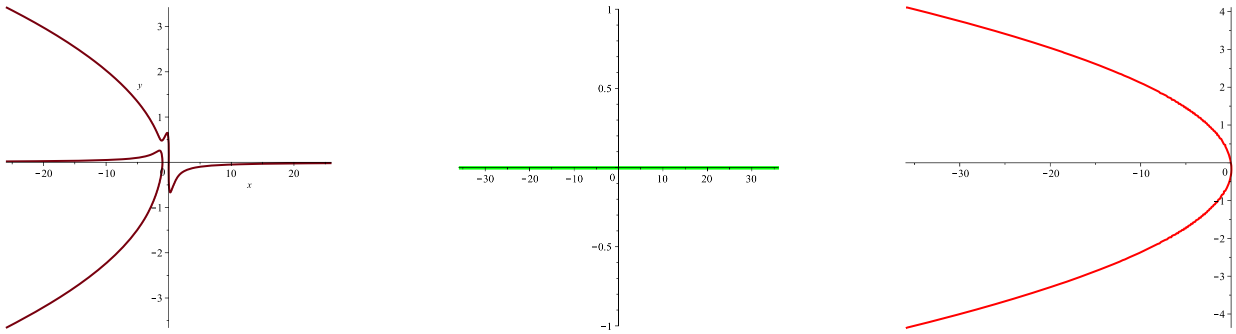

Fig. 3. $\mathcal{C}$ (left) and asymptotes (center and right).

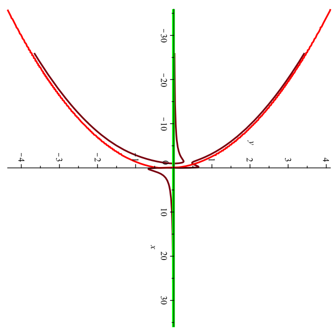

Fig. 4. Curve $\mathcal{C}$ and asymptotes.

$(z, r(z)) = (z, a_n z + a_{n-1} z^{(n-1)/n} + \cdots + a_0)$ and $\deg(\mathcal{C}) = \deg(B) = n$ (note that $\deg(B) = n$ since the given parametrization is proper).

A curve that is not perfect can be approached by other curves of smaller degree. If these curves are perfect, we call them *g-asymptotes*. More precisely, we have the following definition.

Definition 5. Let $\mathcal{C}$ be a curve with an infinity branch B. A *g-asymptote* (generalized asymptote) of $\mathcal{C}$ at B is a perfect curve that approaches $\mathcal{C}$ at B.

The notion of *g-asymptote* is similar to the classical concept of asymptote. The difference is that a g-asymptote is not necessarily a line, but a perfect curve. Actually, it is a generalization, since every line is a perfect curve (this fact follows from Definition 4). Throughout the paper we refer sometimes to *g-asymptote* simply as *asymptote*.

Remark 2. The degree of an g-asymptote is less than or equal to the degree of the curve it approaches. In fact, a g-asymptote of a curve $\mathcal{C}$ at a branch B has minimal degree among all the curves that approach $\mathcal{C}$ at B.

In Fig. 3, we plot a given curve $\mathcal{C}$ defined implicitly by the polynomial

$$f(x, y) = 4x^2y^3 - 4xy^4 + y^5 + 2x^3y - x^2y^2 + 2x^2y + 2xy^2 + x^2 + x$$

and the two g-asymptotes defined by the polynomials

$$\overline{f}_1(x, y) = 0, \quad \overline{f}_2(x, y) = 1/2x + y^2 + 1/4y + 1/64.$$

In Fig. 4, we plot the curve and the asymptotes together.

In Subsection 3.1, we show that every infinity branch of a given algebraic plane curve implicitly defined has, at least, one asymptote and we show how to compute it. For this purpose, we rewrite Equation (2.1) defining a branch B (see Definition 1) as

$$r(z) = mz + a_1 z^{1-n_1/n} + \cdots + a_k z^{1-n_k/n} + a_{k+1} z^{1-N_{k+1}/N} + \cdots \tag{3.1}$$

where $N = n \cdot b$, $N_j = n_j \cdot b$, $j \in \{1, \ldots, k\}$, and $b = \gcd(N, N_1, \ldots, N_k)$ (see Equation (2.1)). That is, we have simplified the non negative exponents such that $\gcd(n, n_1, \ldots, n_k) = 1$. Note that $0 < n_1 < n_2 < \cdots$, and $n_k \le n$, and $N < N_{k+1}$, i.e. the terms $a_j z^{-N_j/N+1}$ with $j \geq k+1$ are those which have negative exponent. We denote these terms as $A(z) := \sum_{\ell=k+1}^{\infty} a_\ell z^{-q_\ell}$, where $q_\ell = 1 - N_\ell/N \in \mathbb{Q}^+$, $\ell \geq k+1$.

Under these conditions, we introduce the definition of degree of a branch B:

Definition 6. Let $B = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M\}$ ($r(z)$ is defined in (3.1)) be an infinity branch associated to an infinity point $P = (1 : m : 0), m \in \mathbb{C}$. We say that n is the degree of B, and we denote it by $\deg(B)$.

3.1. Construction of asymptotes for curves implicitly defined

Taking into account Theorems 1 and 2, we have that any curve $\overline{\mathcal{C}}$ approaching $\mathcal{C}$ at B should have an infinity branch $\overline{B} = \{(z, \overline{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > \overline{M}\}$ such that the terms with non negative exponent in $r(z)$ and $\overline{r}(z)$ are the same. In the simplest case, if $A = 0$ (i.e. there are no terms with negative exponent; see Equation (3.1)), we obtain

$$\tilde{r}(z) = mz + a_1 z^{1-n_1/n} + a_2 z^{1-n_2/n} + \cdots + a_k z^{1-n_k/n}, \tag{3.2}$$

where $a_1, a_2, \ldots \in \mathbb{C} \setminus \{0\}$, $m \in \mathbb{C}$, $n, n_1, n_2 \ldots \in \mathbb{N}$, $\gcd(n, n_1, \ldots, n_k) = 1$, and $0 < n_1 < n_2 < \cdots$. Note that $\tilde{r}$ has the same terms with non negative exponent as r, and $\tilde{r}$ does not have terms with negative exponent.

Let $\widetilde{\mathcal{C}}$ be the plane curve containing the branch $\widetilde{B} = \{(z, \tilde{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > \widetilde{M}\}$ (note that $\widetilde{\mathcal{C}}$ is unique since two different algebraic curves have finitely many common points). Observe that

$$\widetilde{\mathcal{Q}}(t) = (t^n, mt^n + a_1 t^{n-n_1} + \cdots + a_k t^{n-n_k}) \in \mathbb{C}[t]^2,$$

where $n, n_1, \ldots, n_k \in \mathbb{N}$, $\gcd(n, n_1, \ldots, n_k) = 1$, and $0 < n_1 < \cdots < n_k$, is a polynomial parametrization of $\widetilde{\mathcal{C}}$, and it is proper (see Lemma 3 in Blasco and Pérez-Díaz (2014a)). In Theorem 2 in Blasco and Pérez-Díaz (2014a), we prove that $\widetilde{\mathcal{C}}$ is a g-asymptote of $\mathcal{C}$ at B.

From these results, we obtain the method presented in Blasco and Pérez-Díaz (2014a) and Blasco and Pérez-Díaz (2015), that computes g-asymptote that is independent of the leaf chosen to define the infinity branch. We assume that we have prepared the input curve $\mathcal{C}$, by means of a suitable linear change of coordinates, such that $(0 : 1 : 0)$ is not an infinity point of $\mathcal{C}$. We recall that throughout the paper we refer sometimes to *g-asymptote* simply as *asymptote*.

In the following, we illustrate the method with an example.

Example 2. Let $\mathcal{C}$ be the curve of degree $d = 4$ defined by the irreducible polynomial

$f(x, y) = 3645 - 11178x - 891y + 2250yx + 207y^2 + 11997x^2 - 357y^2x - 1859yx^2 + 156y^2x^2 + 496yx^3 + 21xy^3 - 25y^3 - 5039x^3 + 576x^4 + y^4 \in \mathbb{R}[x, y]$.

First, we have that $f_4(x, y) = (9x + y)(4x + y)^3$. Hence, the infinity points are $P_1 = (1 : -9 : 0)$ and $P_2 = (1 : -4 : 0)$.

We start by analyzing the point P_1: there is one infinity branch associated to P_1, $B_1 = \{(z, r_1(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_1\}$, where

$r_1(z) = -9z - 1402/z^4 + 233/z^3 - 43/z^2 + 10/z + 8 + \cdots,$

(we compute r_1 using the algcurves package included in the computer algebra system Maple; in particular we use the command puiseux).

We compute $\tilde{r}_1(z)$, and we have that $\tilde{r}_1(z) = 8 - 9z$. The parametrization of the asymptote $\widetilde{\mathcal{C}}_1$ is given by

$$\widetilde{\mathcal{Q}}_1(t) = (t,\ 8 - 9t).$$

Now, we focus on the point P_2: there is one infinity branch associated to P_2, $B_2 = \{(z, r_{2j}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_2\}$, where

$r_2(z) = -4z + 17/3 - 4/3z^{1/3} + 7z^{2/3} - 10/3z^{-1} + 598/243z^{-2/3} - 479/81z^{-1/3} + \cdots,$

We compute $\tilde{r}_2(z)$, and we have that $\tilde{r}_2(z) = -4z + 17/3 - 4/3z^{1/3} + 7z^{2/3}$. The parametrization of the asymptote $\widetilde{\mathcal{C}}_2$ is given by

$$\widetilde{\mathcal{Q}}_2(t) = (t^3,\ -4t^3 + 17/3 - 4/3t + 7t^2).$$

One may compute the polynomial defining implicitly $\widetilde{\mathcal{C}}_1, \widetilde{\mathcal{C}}_2$ (apply for instance the results in Sendra et al. (2007); see Chapter 4). We have,

$$\tilde{f}_1(x, y) = -9x + 8 - y$$
$$\tilde{f}_2(x, y) = 4913 - 6184x - 2601y + 13581x^2 + 2916yx + 459y^2 - 1728x^3 - 1296yx^2 - 324y^2x - 27y^3.$$

In Figs. 5 and 6, we plot the curve $\mathcal{C}$, and the asymptotes $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$ and the three curves together.

3.2. Construction of asymptotes for rational curves parametrically defined

Throughout this paper so far, we have dealt with algebraic plane curves implicitly defined. In this subsection, we present a method to compute infinity branches and g-asymptotes of a plane curve from their parametric (rational) representation (without implicitizing). This method is included in Blasco and Pérez-Díaz (2015) (see Section 5) and it involves the computation of Puiseux series and infinity branches. In Subsection 3.2.1, we develop a new method presented in Blasco and Pérez-Díaz (2020) that allows to easily compute the generalized asymptotes (*g-asymptotes*) by only determining some simple limits of rational functions constructed from the given parametrization. We recall that throughout the paper we refer sometimes to *g-asymptote* simply as *asymptote*.

Let $\mathcal{C}$ be a plane curve defined by the rational parametrization

$$\mathcal{P}(s) = (p_1(s), p_2(s)) \in \mathbb{R}(s)^2, \quad p_i(s) = p_{i1}(s)/p_{i2}(s), \quad \gcd(p_{i1}, p_{i2}) = 1, \; i = 1, 2.$$

If $\mathcal{C}^*$ represents the projective curve associated to $\mathcal{C}$, we have that a parametrization of $\mathcal{C}^*$ is given by $\mathcal{P}^*(s) = (p_1(s) : p_2(s) : 1)$ or, equivalently,

$$\mathcal{P}^*(s) = \left(1 : \frac{p_2(s)}{p_1(s)} : \frac{1}{p_1(s)}\right).$$

We assume that we have prepared the input curve $\mathcal{C}$, by means of a suitable linear change of coordinates (if necessary) such that $(0:1:0)$ is not a point at infinity of $\mathcal{C}^*$.

In order to compute the g-asymptotes of $\mathcal{C}$, first we need to determine the infinity branches of $\mathcal{C}$. That is, the sets

$$B = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, \, |z| > M\}, \text{ where } r(z) = z\varphi(z^{-1}).$$

For this purpose, taking into account Definition 1, we have that $f(z, r(z)) = F(1 : \varphi(z^{-1}) : z^{-1}) = F(1 : \varphi(t) : t) = 0$ around $t = 0$, where $t = z^{-1}$ and F is the polynomial defining implicitly $\mathcal{C}^*$. Observe that in this section, we are given the parametrization $\mathcal{P}^*$ of $\mathcal{C}^*$ and then, $F(\mathcal{P}^*(s)) = F(1 : p_2(s)/p_1(s) : 1/p_1(s)) = 0$. Thus, intuitively speaking, in order to compute the infinity branches of $\mathcal{C}$, and in particular the series φ, one needs to rewrite the parametrization $\mathcal{P}^*(s)$ in the form $(1 : \varphi(t) : t)$ around $t = 0$. For this purpose, the idea is to look for a value of the parameter s (in the parametrization $\mathcal{P}^*(s)$), say $\ell(t) \in \mathbb{C}\langle\langle t\rangle\rangle$, such that $\mathcal{P}^*(\ell(t)) = (1 : \varphi(t) : t)$ around $t = 0$.

Hence, from the above reasoning, we deduce that first, we have to consider the equation $1/p_1(s) = t$ (or equivalently, $p_{12}(s) - tp_{11}(s) = 0$), and we solve it in the variable s around $t = 0$. From Puiseux's Theorem, there exist solutions $\ell_1(t), \ell_2(t), \ldots, \ell_k(t) \in \mathbb{C}\langle\langle t\rangle\rangle$ such that, $p_{12}(\ell_i(t)) - tp_{11}(\ell_i(t)) = 0$, $i \in \{1, \ldots, k\}$, in a neighborhood of $t = 0$.

Thus, for each $i \in \{1, \ldots, k\}$, there exists $M_i \in \mathbb{R}^+$ such that the points $(1 : \varphi_i(t) : t)$ or equivalently, the points $(t^{-1} : t^{-1}\varphi_i(t) : 1)$, where $\varphi_i(t) = \frac{p_2(\ell_i(t))}{p_1(\ell_i(t))}$, are in $\mathcal{C}^*$ for $|t| < M_i$ (note that $\mathcal{P}^*(\ell(t)) \in \mathcal{C}^*$ since $\mathcal{P}^*$ is a parametrization of $\mathcal{C}^*$). Observe that $\varphi_i(t)$ is a Puiseux series, since $p_2(\ell_i(t))$ and $p_1(\ell_i(t))$ can be written as Puiseux series and $\mathbb{C}\langle\langle t\rangle\rangle$ is a field.

Finally, we set $z = t^{-1}$. Then, we have that the points $(z, r_i(z))$, where $r_i(z) = z\varphi_i(z^{-1})$, are in $\mathcal{C}$ for $|z| > M_i^{-1}$. Hence, the infinity branches of $\mathcal{C}$ are the sets $B_i = \{(z, r_i(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, \, |z| > M_i^{-1}\}, \quad i \in \{1, \ldots, k\}$.

Note that the series $\ell_i(t)$ satisfies that $p_1(\ell_i(t))t = 1$, for $i \in \{1, \ldots, k\}$. Then, we have that

$$\varphi_i(t) = \frac{p_2(\ell_i(t))}{p_1(\ell_i(t))} = p_2(\ell_i(t))t, \quad r_i(z) = z\varphi_i(z^{-1}) = p_2(\ell_i(z^{-1})).$$

Once we have the infinity branches, we can compute a g-asymptote for each of them by simply removing the terms with negative exponent from r_i.

Additionally we note, that some of the solutions $\ell_1(t), \ell_2(t), \ldots, \ell_k(t) \in \mathbb{C}\langle\langle t\rangle\rangle$ might belong to the same conjugacy class. Thus, we only consider one solution for each of these classes. The output asymptote $\widetilde{\mathcal{C}}$ is independent of the solutions $\ell_1(t), \ell_2(t), \ldots, \ell_k(t) \in \mathbb{C}\langle\langle t\rangle\rangle$ chosen in step 1, and of the leaf chosen to define the branch B.

In the following example, we consider a parametric plane curve with two real infinity branches. We obtain these branches and compute a g-asymptote for each of them.

Example 3. Let $\mathcal{C}$ be the plane curve introduced in Example 2 defined by the parametrization

$$\mathcal{P}(s) = \left(\frac{s^4 - s^3 + 1}{(s-1)s^3}, \frac{s^4 - 7s - 4 + s^2}{(s-1)s^3}\right) \in \mathbb{R}(s)^2.$$

We compute the asymptotes of $\mathcal{C}$. For this purpose, we determine the solutions of the equation $p_{12}(s) - tp_{11}(s) = 0$ around $t = 0$. For this purpose, we may use, for instance, the command puiseux included in the package algcurves of the computer algebra system Maple. There are two solutions that are given by the Puiseux series

$$\ell_1(t) = 1 + 327t^5 - 54t^4 + 10t^3 - 2t^2 + t + \cdots$$

$$\ell_2(t) = -10/3t^3 + 38768/19683t^{8/3} + 8546/6561t^{7/3} +$$

$$2/3t^2 + 100/243t^{5/3} + 8/81t^{4/3} - 1/3t + 1/3t^{2/3} + t^{1/3} + \cdots.$$

Now, we compute

$$r_1(z) = p_2(\ell_1(z^{-1})) = -9z + 8 + 10z^{-1} - 43z^{-2} + 233z^{-3} + \cdots$$

$$r_2(z) = p_2(\ell_2(z^{-1})) = -4z + 17/3 - 4/3z^{1/3} + 7z^{2/3} - 10/3z^{-1} + 598/243z^{-2/3} - 479/81z^{-1/3} + \cdots$$

(we may use, for instance, the command series included in the computer algebra system Maple). The curve has four infinity branches given by $B_i = \{(z, r_i(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\}$ for some $M \in \mathbb{R}^+$ (note that B_2 has three leaves).

We obtain $\tilde{r}_i(z)$ by removing the terms with negative exponent in $r_i(z)$ for $i = 1, 2$. We get

$$\tilde{r}_1(z) = -9z + 8 \quad \text{and} \quad \tilde{r}_2(z) = -4z + 17/3 - 4/3z^{1/3} + 7z^{2/3}.$$

The input curve $\mathcal{C}$ has two real asymptotes $\widetilde{\mathcal{C}}_i$ at B_i for $i = 1, 2$ that can be polynomially parametrized by (see Fig. 5):

$$\widetilde{\mathcal{Q}}_1(t) = (t,\ -9t + 8), \quad \widetilde{\mathcal{Q}}_2(t) = (t^3,\ -4t^3 + 17/3 - 4/3t + 7t^2).$$

Compare the output with the output obtained in Example 2.

3.2.1. New method for the parametric case

In this subsection, we present an improvement of the method described above, which avoids the computation of infinity branches and Puiseux series (see Blasco and Pérez-Díaz (2020)). We develop this method for the plane case but it can be adapted for dealing with rational curves in n-dimensional space. For this purpose, one may apply the same reasoning as the used in Blasco and Pérez-Díaz (2015) (see also Remark 6).

In the following we consider a rational plane curve $\mathcal{C}$ defined by the rational parametrization

$$\mathcal{P}(s) = (p_1(s), p_2(s)) \in \mathbb{R}(s)^2, \quad p_i(s) = p_{i1}(s)/p_{i2}(s), \quad \gcd(p_{i1}, p_{i2}) = 1,\ i = 1, 2.$$

We assume that $\deg(p_{i1}) \le \deg(p_{i2}) = d_i$, $i = 1, 2$ (otherwise, we apply a linear change of variables); thus, we have that $\lim_{s\to\infty} p_i(s) \ne \infty$, $i = 1, 2$ and the infinity branches of $\mathcal{C}$ will be traced when s moves around the different roots of the denominators $p_{12}(s)$ and $p_{22}(s)$. In fact, each of these roots yields an infinity branch. The following theorem (see Theorem 3 in Blasco and Pérez-Díaz (2020)) shows how to obtain a g-asymptote for each of these branches, by just computing some simple limits of rational functions constructed from $\mathcal{P}(s)$.

Theorem 3. *Let $\mathcal{C}$ be a curve defined by a parametrization*

$$\mathcal{P}(s) = (p_1(s), p_2(s)) \in \mathbb{R}(s)^2, \quad p_i(s) = p_{i1}(s)/p_{i2}(s), \quad \gcd(p_{i1}, p_{i2}) = 1,\ i = 1, 2,$$

where $\deg(p_{i1}) \le \deg(p_{i2}) = d_i$, $i = 1, 2$. Let $\tau \in \mathbb{C}$ be such that we write $p_{i2}(t) = (t - \tau)^{n_i}\overline{p}_{i2}(t)$ where $\overline{p}_{i2}(\tau) \ne 0$, $i = 1, 2$, and $n_1 \ge 1$, and let B be the corresponding infinity branch. A g-asymptote of B is defined by the parametrization

$$\widetilde{\mathcal{Q}}(t) = (t^{n_1},\ a_{n_2}t^{n_2} + a_{n_2-1}t^{n_2-1} + \ldots + a_0),$$

where

$$\begin{aligned}
a_{n_2} &= \lim_{t\to\tau} \frac{p_2(t)}{p_1(t)^{n_2/n_1}} \\
a_{n_2-1} &= \lim_{t\to\tau} p_1(t)^{1/n_1} f_1(t), && f_1(t) := \frac{p_2(t)}{p_1(t)^{n_2/n_1}} - a_{n_2} \\
a_{n_2-2} &= \lim_{t\to\tau} p_1(t)^{1/n_1} f_2(t), && f_2(t) := p_1(t)^{1/n_1} f_1(t) - a_{n_2-1} \\
&\ \vdots && \quad \vdots \\
a_{n_2-i} &= \lim_{t\to\tau} p_1(t)^{1/n_1} f_i(t), && f_i(t) := p_1(t)^{1/n_1} f_{i-1}(t) - a_{n_2-(i-1)},\ i \in \{2, \ldots, n_2\}.
\end{aligned}$$

Remark 3. From the above construction, each root τ of $p_{12}(t)$ yields an infinity branch and, hence, an infinity point P^*. Note that the parametrization $\mathcal{P}(t)$ can be expressed as $\mathcal{P}(t) = \left(\frac{q_{11}(t)}{q(t)}, \frac{q_{12}(t)}{q(t)}\right)$, where $q(t) = \mathrm{lcm}(p_{12}(t), p_{22}(t))$ and $q_{1i}(t) = p_i(t)q(t)$. Now, the corresponding projective curve is parametrized by $\mathcal{P}^*(t) = (q_{11}(t), q_{12}(t), q(t))$ and the infinity point associated to τ is $P^* = (q_{11}(\tau) : q_{12}(\tau) : 0)$.

In the following corollary, we analyze the special case of the vertical and horizontal g-asymptotes, i.e. lines of the form $x - a$ or $y - b$, where $a, b \in \mathbb{C}$ (observe that these asymptotes correspond to branches associated to the infinity points $(0 : 1 : 0)$ and $(1 : 0 : 0)$, respectively). More precisely, we prove that these asymptotes are obtained from the non–common roots of the denominators of the given parametrization. Note that in the practical design of engineering and modeling applications, the rational curves are usually presented by numerical coefficients and $\mathcal{P}(s)$ mostly satisfies that $\gcd(p_{12}, p_{22}) = 1$. A proof of this corollary can be found in Corollaries 1, 2 and 3 in Blasco and Pérez-Díaz (2020).

Corollary 1. *Let $\mathcal{C}$ be a curve defined by a parametrization*

$$\mathcal{P}(s) = (p_1(s), p_2(s)) \in \mathbb{R}(s)^2, \quad p_i(s) = p_{i1}(s)/p_{i2}(s), \quad \gcd(p_{i1}, p_{i2}) = 1,\ i = 1, 2,$$

where $\deg(p_{i1}) \leq \deg(p_{i2}),\ i = 1, 2.$

1. *Let $\tau \in \mathbb{C}$ be such that $p_{12}(t) = (t-\tau)^{n_1}\overline{p}_{12}(t)$ where $p_{22}(\tau)\overline{p}_{12}(\tau) \neq 0$, and $n_1 \geq 1$. It holds that a g-asymptote of $\mathcal{C}$ corresponding to the infinity point $(1:0:0)$ is the horizontal line $y - p_2(\tau) = 0$, defined by the parametrization $\widetilde{\mathcal{Q}}(t) = (t,\ p_2(\tau))$.*
2. *Let $\tau \in \mathbb{C}$ be such that $p_{22}(t) = (t-\tau)^{n_2}\overline{p}_{22}(t)$ where $p_{12}(\tau)\overline{p}_{22}(\tau) \neq 0$, and $n_2 \geq 1$. It holds that a g-asymptote of $\mathcal{C}$ corresponding to the infinity point $(0:1:0)$ is the vertical line $x - p_1(\tau) = 0$, defined by the parametrization $\widetilde{\mathcal{Q}}(t) = (p_1(\tau),\ t)$.*

Remark 4. The previous theorem outputs the parametrization $\widetilde{\mathcal{Q}}(t) = (t^{n_1},\ a_{n_2}t^{n_2} + a_{n_2-1}t^{n_2-1} + \ldots + a_0)$, and $n_1 \geq n_2$ (otherwise $(0:1:0)$ is an infinity point of the input curve). Note that the degree of the defined curve is not necessary n_1 since $\mathcal{Q}$ could be improper which is equivalent to $\gcd(n_1, n_2, \ldots, n_2 - j) \neq 0$ for every $j = 0, \ldots, n_2 - 1$ such that $a_{n_2-j} \neq 0$. Let us assume that $\gcd(n_1, n_2, \ldots, n_2 - j) = \beta$ for every $j = 0, \ldots, n_2 - 1$ such that $a_{n_2-j} \neq 0$. Then, let $n = n_1/\beta$ and

$$\mathcal{M}(t) = \mathcal{P}(t^{1/\beta}) = (t^n, a_{n_2}t^{n_2/\beta} + a_{n_2-1}t^{(n_2-1)/\beta} + \ldots + a_0) \in \mathbb{K}[t]^2$$

is a proper reparametrization of $\mathcal{Q}$. Then we get that the theorem outputs an asymptote since the output curve is perfect (it has an only branch and the degree of the curve which is n is equal to the degree of the branch).

By applying the above results, we can easily obtain all the g-asymptotes of any rational plane curve, as the following example shows.

Example 4. Let $\mathcal{C}$ be the plane curve introduced in Examples 2 and 3 defined by the parametrization

$$\mathcal{P}(s) = \left(\frac{s^4 - s^3 + 1}{(s-1)s^3}, \frac{s^4 - 7s - 4 + s^2}{(s-1)s^3}\right) \in \mathbb{R}(s)^2.$$

We compute the asymptotes of $\mathcal{C}$ using the new method just presented. For this purpose, we first observe that $p_{12}(s)$ has the roots $\tau_1 = 1$, $\tau_2 = 0$, with multiplicities $n_1 = 1$, and $n_2 = 3$. The multiplicities of these roots in $p_{22}(s)$ are the same and $p_{22}(s)$ does not have additional roots.
For $\tau_1 = 1$, we compute

$$a_1 = \lim_{t\to 1} \frac{p_2(t)}{p_1(t)} = -9, \qquad a_0 = \lim_{t\to 1} p_1(t) f_1(t) = 8, \quad f_1(t) := \frac{p_2(t)}{p_1(t)} - a_1.$$

Then, we obtain the asymptote $\widetilde{\mathcal{C}}_1$, defined by the proper parametrization

$$\widetilde{\mathcal{Q}}_1(t) = (t,\ -9t + 8).$$

For $\tau_2 = 0$, we compute

$$\begin{array}{ll} a_3 = \lim_{t\to 0} \frac{p_2(t)}{p_1(t)} = -4 & \\ a_2 = \lim_{t\to 0} p_1(t)^{1/3} f_1(t) = 7, & f_1(t) := \frac{p_2(t)}{p_1(t)} - a_3 \\ a_1 = \lim_{t\to 0} p_1(t)^{1/3} f_2(t) = -4/3, & f_2(t) := p_1(t)^{1/3} f_1(t) - a_2 \\ a_0 = \lim_{t\to 0} p_1(t)^{1/3} f_3(t) = 17/3, & f_3(t) := p_1(t)^{1/3} f_2(t) - a_1. \end{array}$$

Then, we obtain the asymptote $\widetilde{\mathcal{C}}_2$, defined by the proper parametrization

$$\widetilde{\mathcal{Q}}_2(t) = (t^3,\ -4t^3 + 17/3 - 4/3t + 7t^2).$$

See Figs. 5 and 6 and compare the output with the output obtained in Example 3.

The above method allows us to easily obtain all the generalized asymptotes of a rational curve. However, we should compute the roots of the denominators of the parametrization, which may entail certain difficulties if algebraic numbers are involved. This problem is solved using the notion of *conjugate points* (see Definition 12 in Pérez-Díaz (2007)), which help us to overcome this problem. The idea is collect the points whose coordinates depend algebraically on all the conjugate roots of a same irreducible polynomial (for more details see Pérez-Díaz (2007) and Subsection 4.2 in Blasco and Pérez-Díaz (2020)).

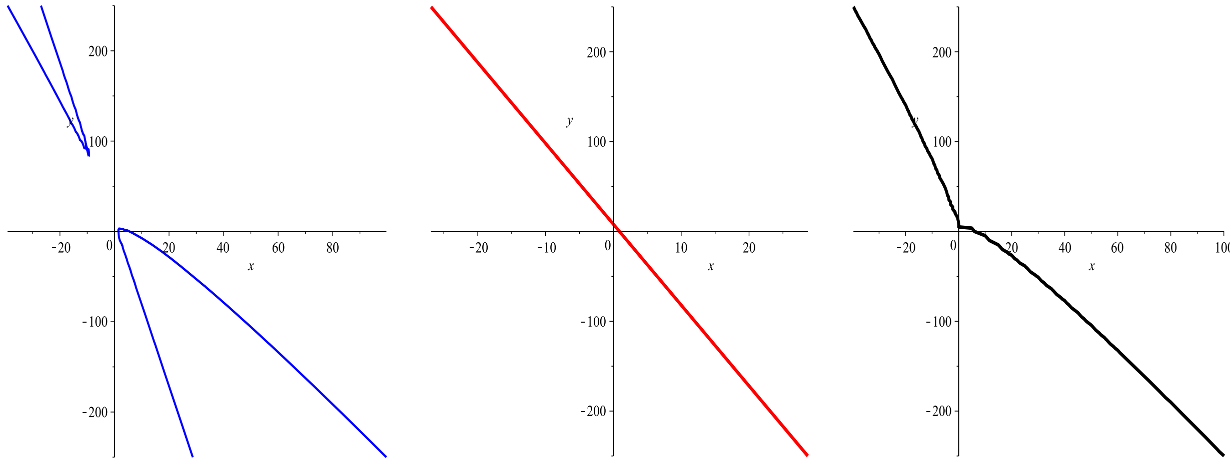

Fig. 5. Curve $\mathcal{C}$ (left) and asymptotes (center and right).

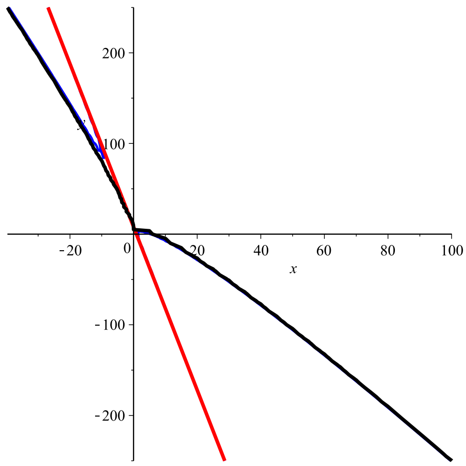

Fig. 6. Curve $\mathcal{C}$ and asymptotes.

4. Computation of branches and asymptotes using derivatives and some applications

In this section, we present the main results of the paper. More precisely, we first develop a method that allows to easily compute all the g-asymptotes and the branches of a curve defined by a parametrization by only determining some simple derivatives of functions constructed from the input parametrization. From the formula obtained, some important results can be deduced. In particular, we show how to read from the input parametrization the ramification index and degree of the g-asymptote and how to compute the infinity form and the multiplicity and the character of the infinity points. As a consequence, we also comment how this formula could provide an answer the question on determining the families of parametric curves that have some input given g-asymptotes. These consequences will be presented in Subsection 4.1.

In the following, we consider $\mathcal{C}$ be a curve defined by a rational parametrization

$$\mathcal{P}(s) = (p_1(s), p_2(s)) \in \mathbb{R}(s)^2, \quad p_i(s) = p_{i1}(s)/p(s), \quad \gcd(p_{i1}, p_{i2}, p) = 1,\ i = 1, 2.$$

We assume that all the roots of the denominators can be written as $\tau \in \mathbb{C}$ such that $p(t) = (t-\tau)^n \overline{p}(t)$ where $\overline{p}(\tau)p_1(\tau) \neq 0$, $i = 1, 2$, and $n \geq 1$. Otherwise, we consider a change of coordinates to afterwards undoing it.

Theorem 4. *Let $\mathcal{C}$ be a curve defined by a parametrization*

$$\mathcal{P}(s) = (q_1(s), q_2(s)) \in \mathbb{R}(s)^2, \quad q_i(s) = p_i(s)/p(s), \quad \gcd(p_1, p_2, p) = 1,\ i = 1, 2.$$

Let $\tau \in \mathbb{C}$ be such that we write $p(t) = (t-\tau)^n \overline{p}(t)$ where $\overline{p}(\tau)p_1(\tau) \neq 0$, $i = 1, 2$, and $n \geq 1$. Let B be the corresponding infinity branch. A g-asymptote of B is defined by the parametrization

$$\widetilde{\mathcal{Q}}(t) = (t^n,\, a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0),$$

where

$$a_n = \frac{p_2(\tau)}{p_1(\tau)},$$

and for $0 \le i \le n-1$

$$a_i = \frac{1}{(n-i)!} \cdot \frac{\partial^{n-1-i}}{\partial t^{n-1-i}} \left(\frac{\partial}{\partial t} \left(\frac{p_2(t)}{p_1(t)} \right) \cdot \left(\frac{p_1(t)}{\overline{p}(t)} \right)^{(n-i)/n} \right)(\tau)$$

Proof. First, in order to improve the notation we denote

$$\ell(t) = \frac{p_2(t)}{p_1(t)}, \quad r(t) = \left(\frac{p_1(t)}{\overline{p}(t)} \right)^{1/n}.$$

Thus, we have to prove that

$$a_n = \ell(\tau), \quad a_i = \frac{1}{(n-i)!} \cdot \frac{\partial^{n-1-i}}{\partial t^{n-1-i}} \left(\frac{\partial \ell}{\partial t}(t) \cdot r(t)^{(n-i)} \right)(\tau),\ 0 \le i \le n-1.$$

We assume w.l.o.g. that $\tau = 0$. Otherwise, one reasons similarly. First, we easily get that

$$a_n = \lim_{t \to 0} \frac{q_2(t)}{q_1(t)} = \lim_{t \to 0} \frac{p_2(t)}{p_1(t)} = \ell(0).$$

In order to prove the other equality, we observe that for $i = 1, \ldots, n$ we may write

$$a_{n-i} = \lim_{t \to 0} \frac{r\, \mu_i(t)}{t^i},$$

where

$$\mu_i(t) = \Big((\ell(t) - a_n) r^{i-1} - a_{n-1} r^{i-2} t - a_{n-2} r^{i-3} t^2 - \ldots - a_{n-i+2} r t^{i-2} - a_{n-i+1} t^{i-1} \Big).$$

We easily may check that

$$\frac{\partial^j \mu_i}{\partial t^j}(0) = 0,\ j = 0, \ldots, i-1, \quad i = 1, \ldots, n,$$

and

$$\frac{\partial \mu_1}{\partial t}(t) = \frac{\partial \ell}{\partial t}(t).$$

In addition, for $i = 2, \ldots, n$

$$\frac{\partial^i \mu_i}{\partial t^i}(t) = \left(\binom{i}{i-1} \frac{\partial^{i-1} \lambda_{i-2}}{\partial t^{i-1}}(t) \frac{\partial r}{\partial t}(t) + r(t) \frac{\partial^i \lambda_{i-2}}{\partial t^i}(t) \right)$$

where,

$$\lambda_0 = \ell(t) - a_n, \quad \lambda_{i-2} = \lambda_{i-3}\, r - a_{n-i+2} t^{i-2}, \quad i = 3, \ldots, n.$$

Indeed: let $i \in \{2, \ldots, n\}$. First, we observe that

$$\frac{\partial^i \mu_i}{\partial t^i}(t) = \frac{\partial^i\, (r\, \lambda_{i-2})}{\partial t^i}(t)$$

and since $\frac{\partial^j \lambda_{i-2}}{\partial t^j}(0) = 0,\ j = 0, \ldots, i-2$, we have that

$$\frac{\partial^i \mu_i}{\partial t^i}(0) = \left(\binom{i}{i-1} \frac{\partial^{i-1} \lambda_{i-2}}{\partial t^{i-1}}(0) \frac{\partial r}{\partial t}(0) + r(0) \frac{\partial^i \lambda_{i-2}}{\partial t^i}(0) \right).$$

Now, taking into account the previous equalities and that $r(t)$ is defined in $t = 0$, by L'Hôpital rule we get that

$$a_{n-1} = \frac{r(0)\, \frac{\partial \mu_1}{\partial t}(0)}{1!} = \left(\frac{\partial \ell}{\partial t}(t) \cdot r(t) \right)(0)$$

and for $i \in \{2,\dots,n\}$,

$$a_{n-i} = \frac{r(0)\,\frac{\partial^i \mu_i}{\partial t^i}(0)}{i!} = \frac{r(0)\left(\binom{i}{i-1}\frac{\partial^{i-1}\lambda_{i-2}}{\partial t^{i-1}}(0)\frac{\partial r}{\partial t}(0) + r(0)\frac{\partial^i \lambda_{i-2}}{\partial t^i}(0)\right)}{i!}.$$

Thus, finally we only need to prove that for $i \in \{2,\dots,n\}$

$$r(0)\left(\binom{i}{i-1}\frac{\partial^{i-1}\lambda_{i-2}}{\partial t^{i-1}}(0)\frac{\partial r}{\partial t}(0) + r(0)\frac{\partial^i \lambda_{i-2}}{\partial t^i}(0)\right) = \frac{\partial^{i-1}}{\partial t^{i-1}}\left(\frac{\partial \ell}{\partial t}(t)\cdot r(t)^i\right)(0).$$

For this purpose, we consider the factor

$$\left(\alpha_2\frac{\partial^{i-1}\lambda_{i-2}}{\partial t^{i-1}}(0) + \alpha_1\frac{\partial^i \lambda_{i-2}}{\partial t^i}(0)\right)$$

where $\alpha_1 = r(0)$ and $\alpha_2 = \binom{i}{i-1}\frac{\partial r}{\partial t}(0)$. Since $\lambda_{i-2} = \lambda_{i-3}r - a_{n-i+2}t^{i-2}$ and taking into account that $\frac{\partial^j \lambda_{i-2}}{\partial t^j}(0) = 0,\ j = 0,\dots,i-2$, we have that,

$$\left(\alpha_2\frac{\partial^{i-1}\lambda_{i-2}}{\partial t^{i-1}}(0) + \alpha_1\frac{\partial^i \lambda_{i-2}}{\partial t^i}(0)\right) = \left(\alpha_3\frac{\partial^{i-2}\lambda_{i-3}}{\partial t^{i-2}}(0) + \alpha_2\frac{\partial^{i-1}\lambda_{i-3}}{\partial t^{i-1}}(0) + \alpha_1\frac{\partial^i \lambda_{i-3}}{\partial t^i}(0)\right)$$

where $\alpha_1 = r^2(0)$, $\alpha_2 = \binom{i}{i-1}\frac{\partial r^2}{\partial t}(0)$ and $\alpha_3 = \binom{i}{i-2}\frac{\partial^2 r^2}{\partial t^2}(0)$. Since $\lambda_{i-3} = \lambda_{i-4}r - a_{n-i+3}t^{i-3}$ and taking into account that $\frac{\partial^j \lambda_{i-2}}{\partial t^j}(0) = 0,\ j = 0,\dots,i-2$, we have that,

$$\left(\alpha_3\frac{\partial^{i-2}\lambda_{i-3}}{\partial t^{i-2}}(0) + \alpha_2\frac{\partial^{i-1}\lambda_{i-3}}{\partial t^{i-1}}(0) + \alpha_1\frac{\partial^i \lambda_{i-3}}{\partial t^i}(0)\right) =$$
$$\left(\alpha_4\frac{\partial^{i-3}\lambda_{i-4}}{\partial t^{i-3}}(0) + \alpha_3\frac{\partial^{i-2}\lambda_{i-4}}{\partial t^{i-2}}(0) + \alpha_2\frac{\partial^{i-1}\lambda_{i-4}}{\partial t^{i-1}}(0) + \alpha_1\frac{\partial^i \lambda_{i-4}}{\partial t^i}(0)\right)$$

where $\alpha_1 = r^3(0)$, $\alpha_2 = \binom{i}{i-1}\frac{\partial r^3}{\partial t}(0)$, $\alpha_3 = \binom{i}{i-2}\frac{\partial^2 r^3}{\partial t^2}(0)$ and $\alpha_4 = \binom{i}{i-3}\frac{\partial^3 r^3}{\partial t^3}(0)$. Therefore, applying this process i times, we get that

$$\left(\binom{i}{i-1}\frac{\partial r}{\partial t}(0)\frac{\partial^{i-1}\lambda_{i-2}}{\partial t^{i-1}}(0) + r(0)\frac{\partial^i \lambda_{i-2}}{\partial t^i}(0)\right) =$$
$$\binom{i}{i}r^{i-1}(0)\frac{\partial^i \lambda_0}{\partial t^i}(0) + \binom{i}{i-1}\frac{\partial r^{i-1}}{\partial t}(0)\frac{\partial^{i-1}\lambda_0}{\partial t^{i-1}}(0) + \binom{i}{i-2}\frac{\partial^2 r^{i-1}}{\partial t^2}(0)\frac{\partial^{i-2}\lambda_0}{\partial t^{i-2}}(0) + \cdots + \binom{i}{i-1}\frac{\partial^i r^{i-1}}{\partial t^i}(0)\frac{\partial \lambda_0}{\partial t}(0) =$$
$$\binom{i}{i}r^{i-1}(0)\frac{\partial^i \ell}{\partial t^i}(0) + \binom{i}{i-1}\frac{\partial r^{i-1}}{\partial t}(0)\frac{\partial^{i-1}\ell}{\partial t^{i-1}}(0) + \binom{i}{i-2}\frac{\partial^2 r^{i-1}}{\partial t^2}(0)\frac{\partial^{i-2}\ell}{\partial t^{i-2}}(0) + \cdots + \binom{i}{i-1}\frac{\partial^i r^{i-1}}{\partial t^i}(0)\frac{\partial \ell}{\partial t}(0).$$

Therefore,

$$r(0)\left(\binom{i}{i-1}\frac{\partial^{i-1}\lambda_{i-2}}{\partial t^{i-1}}(0)\frac{\partial r}{\partial t}(0) + r(0)\frac{\partial^i \lambda_{i-2}}{\partial t^i}(0)\right) =$$
$$r(0)\left(\binom{i}{i}r^{i-1}(0)\frac{\partial^i \ell}{\partial t^i}(0) + \binom{i}{i-1}\frac{\partial r^{i-1}}{\partial t}(0)\frac{\partial^{i-1}\ell}{\partial t^{i-1}}(0) + \binom{i}{i-2}\frac{\partial^2 r^{i-1}}{\partial t^2}(0)\frac{\partial^{i-2}\ell}{\partial t^{i-2}}(0) + \cdots + \binom{i}{i-1}\frac{\partial^i r^{i-1}}{\partial t^i}(0)\frac{\partial \ell}{\partial t}(0)\right) =$$
$$\binom{i-1}{i-1}r^i(0)\frac{\partial^i \ell}{\partial t^i}(0) + \binom{i-1}{i-2}\frac{\partial r^i}{\partial t}(0)\frac{\partial^{i-1}\ell}{\partial t^{i-1}}(0) + \binom{i-1}{i-3}\frac{\partial^2 r^i}{\partial t^2}(0)\frac{\partial^{i-2}\ell}{\partial t^{i-2}}(0) + \cdots + \binom{i-1}{0}\frac{\partial^{i-1} r^i}{\partial t^i}(0)\frac{\partial \ell}{\partial t}(0) =$$
$$\frac{\partial^{i-1}}{\partial t^{i-1}}\left(\frac{\partial \ell}{\partial t}(t)\cdot r(t)^i\right)(0). \quad \square$$

Remark 5. Observe that as an important consequence of Theorem 4, using the formulae presented we can construct all the families of parametric curves having a given asymptote. See also Subsection 4.1 and Example 8

Corollary 2. *Let $\mathcal{C}$ be a curve defined by a parametrization*

$$\mathcal{P}(s) = (q_1(s), q_2(s)) \in \mathbb{R}(s)^2, \quad q_i(s) = p_i(s)/p(s), \quad \gcd(p_1, p_2, p) = 1,\ i = 1, 2.$$

Let $\tau \in \mathbb{C}$ be such that $p(t) = (t-\tau)^n \overline{p}(t)$ where $\overline{p}(\tau)p_1(\tau) \neq 0$, $i = 1, 2$, and $n \geq 1$. The corresponding infinity branch B is given as

$$(t^n,\ a_n t^n + a_{n-1}t^{n-1} + \ldots + a_0 + a_{-1}t^{-1} + a_{-2}t^{-2} + \ldots),$$

or equivalently

$$(t,\ a_n t + a_{n-1}t^{(n-1)/n} + \ldots + a_0 + a_{-1}t^{-1/n} + a_{-2}t^{-2/n} + \ldots),$$

where

$$a_n = \frac{p_2(\tau)}{p_1(\tau)},$$

and for $i \leq n-1$

$$a_i = \frac{1}{(n-i)!} \cdot \frac{\partial^{n-1-i}}{\partial t^{n-1-i}} \left(\frac{\partial}{\partial t}\left(\frac{p_2(t)}{p_1(t)}\right) \cdot \left(\frac{p_1(t)}{\overline{p}(t)}\right)^{(n-i)/n} \right)(\tau)$$

Remark 6. Note that we have developed this method for the plane case but it can be trivially adapted for dealing with rational curves in n-dimensional space. For instance, if $n = 3$, we have a curve $\mathcal{P}(s) = (p_1(s), p_2(s), p_3(s))$ with $p_i(s) = p_{i1}(s)/p_{i2}(s)$ and $\gcd(p_{i1}, p_{i2}) = 1,\ i = 1, 2, 3$, and the asymptotes have the form

$$\widetilde{\mathcal{Q}} = (t^{n_1},\ a_{n_2}t^{n_2} + a_{n_2-1}t^{n_2-1} + \ldots + a_0,\ b_{m_2}t^{m_2} + b_{m_2-1}t^{m_2-1} + \ldots + b_0).$$

These asymptotes can be computed by successively applying the previous results to each component (for more details on this reasoning see Blasco and Pérez-Díaz (2015)).

Example 5. Let $\mathcal{C}$ be the plane curve introduced in Examples 2, 3 and 4 defined by the parametrization

$$\mathcal{P}(s) = \left(\frac{s^4 - s^3 + 1}{(s-1)s^3}, \frac{s^4 - 7s - 4 + s^2}{(s-1)s^3}\right) \in \mathbb{R}(s)^2.$$

We compute the asymptotes of $\mathcal{C}$ using the new method just presented. For this purpose, we first observe that $p_{12}(s)$ has the roots $\tau_1 = 1$, $\tau_2 = 0$, with multiplicities $n_1 = 1$, and $n_2 = 3$. The multiplicities of these roots in $p_{22}(s)$ are the same and $p_{22}(s)$ does not have additional roots.
For $\tau_1 = 1$, we consider

$$\ell(s) := \frac{s^4 - 7s - 4 + s^2}{s^4 - s^3 + 1}$$

and

$$r(s) := \frac{s^4 - s^3 + 1}{s^3},$$

and we compute

$$a_1 = \ell(1) = -9, \qquad a_0 = \frac{\partial \ell}{\partial t}(1)r(1) = 8.$$

Then, we obtain the asymptote $\widetilde{\mathcal{C}}_1$, defined by the proper parametrization

$$\widetilde{\mathcal{Q}}_1(t) = (t,\ -9t + 8).$$

For $\tau_2 = 0$, we consider

$$r(s) := \frac{(s^4 - s^3 + 1)^{1/3}}{(s-1)^{1/3}},$$

and we compute

$$\begin{aligned}
a_3 &= \ell(0) = -4 \\
a_2 &= \tfrac{\partial \ell}{\partial t}(0) r(0) = 7, \\
a_1 &= \tfrac{1}{2!} \cdot \frac{\partial}{\partial t}\left(\frac{\partial \ell(t)}{\partial t} \cdot r(t)^2\right)(0) = -4/3, \\
a_0 &= \tfrac{1}{3!} \cdot \frac{\partial^2}{\partial t^2}\left(\frac{\partial \ell(t)}{\partial t} \cdot r(t)^3\right)(0) = 17/3.
\end{aligned}$$

Then, we obtain the asymptote $\widetilde{\mathcal{C}}_2$, defined by the proper parametrization

$$\widetilde{\mathcal{Q}}_2(t) = (t^3,\ -4t^3 + 17/3 - 4/3t + 7t^2).$$

See Figs. 5 and 6 and compare the output with the output obtained in Example 3.

Let us see that, additionally, one may compute the branches associated to each infinity point. In particular, we compute the branch for $\tau_2 = 0$ that provides the infinity point $(1 : -4 : 0)$. For this purpose, we apply Corollary 2, and we get that The corresponding infinity branch is given as

$$(t^n,\ a_3 t^3 + a_2 t^2 + a_1 t + a_0 + a_{-1} t^{-1} + a_{-2} t^{-2} + \ldots),$$

where a_3, a_2, a_1, a_0 where computed above and for $i \le -1$

$$a_i = \frac{1}{(3-i)!} \cdot \frac{\partial^{2-i}}{\partial t^{2-i}}\left(\frac{\partial}{\partial t}\left(\frac{p_2(t)}{p_1(t)}\right) \cdot \left(\frac{p_1(t)}{\overline{p}(t)}\right)^{(3-i)/3}\right)(0).$$

We get

$$\begin{aligned}
a_{-1} &= \tfrac{1}{4!} \cdot \frac{\partial^3}{\partial t^3}\left(\frac{\partial \ell(t)}{\partial t} \cdot r(t)^4\right)(0) = -479/81 \\
a_{-2} &= \tfrac{1}{5!} \cdot \frac{\partial^4}{\partial t^4}\left(\frac{\partial \ell(t)}{\partial t} \cdot r(t)^5\right)(0)) = 598/243, \\
a_{-3} &= \tfrac{1}{6!} \cdot \frac{\partial^5}{\partial t^5}\left(\frac{\partial \ell(t)}{\partial t} \cdot r(t)^6\right)(0) = -10/3, \\
a_{-4} &= \tfrac{1}{7!} \cdot \frac{\partial^6}{\partial t^6}\left(\frac{\partial \ell(t)}{\partial t} \cdot r(t)^7\right)(0) = 29402/6561 \\
&\ \vdots
\end{aligned}$$

Example 6. Let $\mathcal{C}$ be the plane curve defined by the parametrization (see Fig. 7)

$$\mathcal{P}(s) = \left(\frac{s^4 - s^3 + 5s^2 + 2s + 1}{s^4(s-1)(s-2)},\ \frac{1/27(-108 - 216s + 27s^4 - 367s^3 - 2440s^2)}{s^4(s-1)(s-2)}\right) \in \mathbb{R}(s)^2.$$

We compute the asymptotes of $\mathcal{C}$ using the new method just presented. For this purpose, we first observe that $p_3(s)$ has the roots $\tau_1 = 1$, $\tau_2 = 2$, $\tau_3 = 0$, with multiplicities $n_1 = n_2 = 1$, and $n_3 = 4$. They provide the infinity points $(1 : -388/27 : 0)$ (for τ_1, τ_2) and $(1 : -4 : 0)$ (for τ_3).
For $\tau_1 = 1$, we consider

$$\ell(s) := \frac{1/27(-108 - 216s + 27s^4 - 367s^3 - 2440s^2)}{s^4 - s^3 + 5s^2 + 2s + 1}$$

and

$$r(s) := \frac{1/27(-108 - 216s + 27s^4 - 367s^3 - 2440s^2)}{s^4(s-2)},$$

and we compute

$$a_1 = \ell(1) = -388/27, \qquad a_0 = \frac{\partial \ell}{\partial t}(1) r(1) = -405460/729.$$

Then, we obtain the asymptote $\widetilde{\mathcal{C}}_1$, defined by the proper parametrization (see Figs. 8 and 9)

$$\widetilde{\mathcal{Q}}_1(t) = (t,\ -388/27t + 1045/27).$$

For $\tau_2 = 2$, we consider

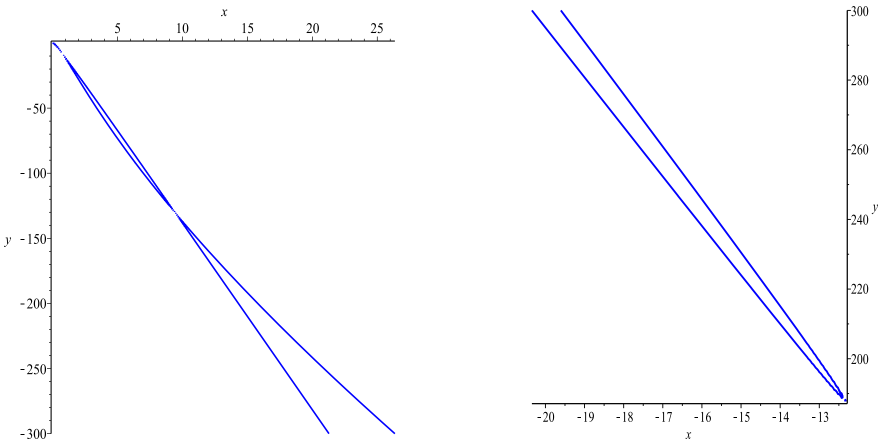

Fig. 7. Curve $\mathcal{C}$.

$$r(s) := \frac{1/27(-108 - 216s + 27s^4 - 367s^3 - 2440s^2)}{s^4(s-1)},$$

and we compute

$$a_1 = \ell(2) = -388/27, \qquad a_0 = \frac{\partial \ell}{\partial t}(2)r(2) = -67415/729.$$

Then, we obtain the asymptote $\widetilde{\mathcal{C}}_2$, defined by the proper parametrization (see Figs. 8 and 9)

$$\widetilde{\mathcal{Q}}_2(t) = (t,\ -388/27t + 695/108).$$

Here, we observe that the infinity point $(1 : -388/27 : 0)$ has two asymptotes $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$. For $\tau_3 = 0$, we consider

$$r(s) := \frac{(1/27(-108 - 216s + 27s^4 - 367s^3 - 2440s^2))^{1/4}}{(s-1)^{1/4}(s-2)^{1/4}},$$

and we compute

$$\begin{aligned}
&a_4 = \ell(0) = -4\\
&a_3 = \tfrac{\partial \ell}{\partial t}(0)r(0) = 0,\\
&a_2 = \tfrac{1}{2!} \cdot \frac{\partial}{\partial t}\left(\frac{\partial \ell(t)}{\partial t} \cdot r(t)^2\right)(0) = 950/27\sqrt{2},\\
&a_1 = \tfrac{1}{3!} \cdot \frac{\partial^2}{\partial t^2}\left(\frac{\partial \ell(t)}{\partial t} \cdot r(t)^3\right)(0) = 0\\
&a_0 = \tfrac{1}{4!} \cdot \frac{\partial^3}{\partial t^3}\left(\frac{\partial \ell(t)}{\partial t} \cdot r(t)^4\right)(0) = 6535/144.
\end{aligned}$$

Then, we obtain the asymptote $\widetilde{\mathcal{C}}_3$, defined by the parametrization $(t^4,\ -4t^4 + 950/27\sqrt{2}t^2 + 3080665/2916)$ that is not proper. A reparametrization of it provides the proper parametrization (see Figs. 8 and 9).

$$\widetilde{\mathcal{Q}}_3(t) = (t^2,\ -4t^2 + 950/27\sqrt{2}t + 6535/144).$$

4.1. Some applications: ramification index, degree of the asymptote, infinity form and multiplicity of an infinity point

Let us consider the infinity point $p = (1 : \alpha : 0)$ and the parametrization

$$\mathcal{P}(t) = (p_1(t) : p_2(t) : p_3(t)) = (p_1(t) : p_2(t) : \overline{p}_3(t)\overline{p}(t))$$

such that $s_1, \ldots, s_u$ are the (different) roots of $\overline{p}_3(t)$ and $\overline{p}(s_j) \neq 0$ and $p_2(s_j)/p_1(s_j) = \alpha$ for $j = 1, \ldots, u$. For $j = 1, \ldots, u$, we denote by N_j the multiplicity of s_j as the root of the polynomial $\overline{p}_3(t)$.

We assume that we have prepared the input curve $\mathcal{C}$, by means of a suitable linear change of coordinates (if necessary) such that $(0 : 1 : 0)$ is not a point at infinity.

Under these conditions, we have the following results that show the relation between ramification index, degree of the asymptote, infinity form and multiplicity of the infinity points.

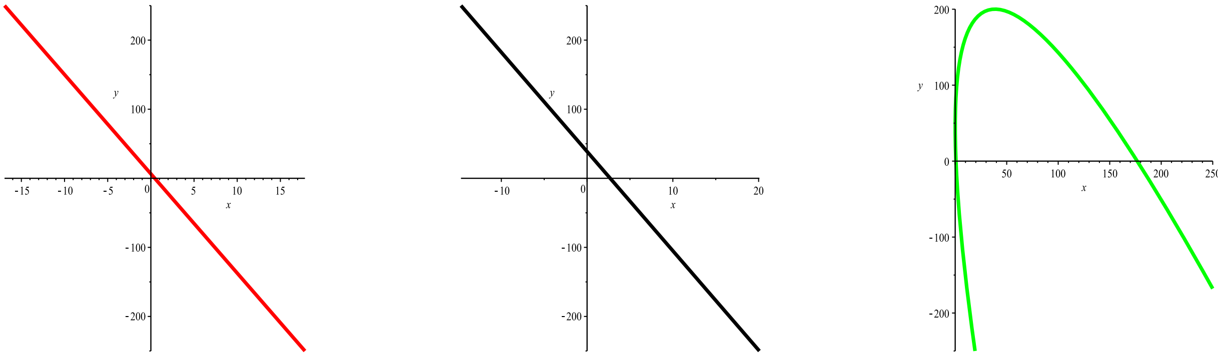

Fig. 8. Asymptotes of the curve $\mathcal{C}$.

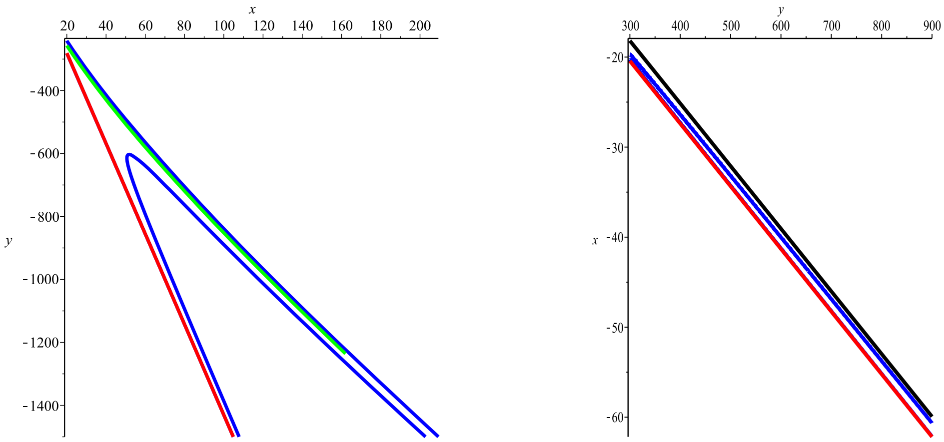

Fig. 9. Curve $\mathcal{C}$ and asymptotes.

Theorem 5. *For each $j = 1, \dots, u$, we have one branch associated to $p = (1 : \alpha : 0)$ with ramification index N_j (number of leaves) given as*

$$B_j = \{(z, r_j(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M_j\},$$

where

$$r_j(z) = \alpha z + a_{k_{1j}} z^{k_{1j}/N_j} + \cdots + a_{0j} + a_{-1j} z^{-1/N_j} + a_{-2j} z^{-2/N_j} + \cdots$$

and $0 \le k_{1j} < N_j$ is the first natural number such that $a_{k_{1j}} \neq 0$.
In addition, for each $j = 1, \dots, u$, the asymptote is defined by the rational parametrization

$$\mathcal{Q}_j(t) = (t^{N_j}, \alpha t^{N_j} + a_{k_{1j}} t^{k_j} + \cdots + a_{0j}).$$

The degree of the asymptote is $n_j = N_j/\beta_j$, where $\beta_j := \gcd(N_j, k_{1j}, \dots, k_{1j} - i)$ for every $i = 0, \dots, k_{1j} - 1$ such that $a_{ij} \neq 0$.

Proof. This theorem is obtained from Theorem 4 and Corollary 2. We note that for each $j = 1, \dots, u$, the degree of the asymptote is not necessary N_j since $\mathcal{Q}_j(t)$ could be improper which is equivalent to $\gcd(N_j, k_{1j}, \dots, k_{1j} - i) \neq 0$ for every $i = 0, \dots, k_{1j} - 1$ such that $a_{ij} \neq 0$.

Thus, under these conditions, for each $j = 1, \dots, u$, let $\beta_j := \gcd(N_j, k_{1j}, \dots, k_{1j} - i)$ and $n_j = N_j/\beta_j$ for $i = 0, \dots, k_{1j} - 1$ such that $a_{ij} \neq 0$. Thus, we consider

$$\mathcal{M}_j(t) = (t^{n_j}, \alpha t^{n_j} + a_{k_{1j}} t^{k_{1j}/\beta_j} + \cdots + a_0)$$

that is a proper reparametrization of $\mathcal{Q}_j(t)$. Then, we get that the degree of the asymptote is n_j. □

Theorem 6. *It holds that $(x - \alpha y)^{N_1 + \cdots + N_u}$ divides the form of the maximum degree of the implicit equation defining the input curve.*

Proof. This result is obtained using Lemma 2 in Blasco and Pérez-Díaz (2014a). □

From the previous theorem, we easily obtain the following corollary.

Corollary 3. *Let $\mathcal{P}(t) = (p_1(t) : p_2(t) : p_3(t))$ such that $p_3(t) = \prod_{i=1}^{d} p_{3i}(t)$ and for every root μ of $p_{3i}(t)$, it holds that $p_2(\mu)/p_1(\mu) = \alpha_i$. It holds that the form of maximum degree of the implicit equation defining the input curve is given by $\prod_{i=1}^{d}(x - \alpha_i y)^{\deg(p_{3i})}$ and the infinity points are $p_i = (1 : \alpha_i : 0)$ for $i = 1, \ldots, d$.*

Theorem 7. *In the conditions stated at the beginning of the subsection, it holds that $p = (1 : \alpha : 0)$ is a point of multiplicity $\sum_{j=1}^{u}(N_j - k_{1j})$ in the asymptote and in the input curve.*

Proof. Observe that each branch provides $N_j - k_{1j}$ to the multiplicity of the point p. In fact, if $N_j - k_{1j} \geq 2$ since

$$\frac{\partial^i}{\partial^i t}\left(\frac{p_2}{p_1}\right)(0) = 0, \quad \text{for} \quad i = 1, \ldots, N_j - k_{1j} - 1$$

and

$$\frac{\partial^{N_j - k_{1j}}}{\partial^{N_j - k_{1j}} t}\left(\frac{p_2}{p_1}\right)(0) \neq 0$$

(see Theorem 4), we get that $a_{N_j - i} = 0$ for $i = 1, \ldots, N_j - k_{1j} - 1$ and $a_{k_{1j}} \neq 0$. Furthermore, we also get that

$$G(t) = \gcd(p_2(t) - \alpha p_1(t), (t - s_j)^{N_j}\overline{p}(t)) = \prod_{j=1}^{u}(t - s_j)^{N_j - k_{1j}}$$

which implies that $\deg(G) = \sum_{j=1}^{u}(N_j - k_{1j})$. Hence, we conclude that $p = (1 : \alpha : 0)$ is a point of multiplicity $\sum_{j=1}^{u}(N_j - k_{1j})$. □

From the previous theorem and using Theorem 2 in Pérez-Díaz (2018), we obtain the following corollary.

Corollary 4. *Let $p = (1 : \alpha : 0)$ and infinity point having multiplicity $\sum_{j=1}^{u}(N_j - k_{1j}) \geq 2$. Then, p is a singular point. Furthermore, it is non–ordinary if and only if one of the following statements holds:*

1. *There exists at least a root $s_i \in \mathbb{K}$ of $p_3(t)$ of multiplicity $N_i \geq 2$.*
2. *If $N_i = 1$, $i = 1, \ldots, u$ and there exists at least two roots $s_0, s_1 \in \mathbb{K}$ such that*

$$p_1(s_1)(p_2'(s_0) - \alpha p_2'(s_1)) = (p_1'(s_0) - \alpha p_1'(s_1))p_2(s_1), \;\; \alpha = p_3'(s_0)/p_3'(s_1).$$

Proof. 1. Taking into account the proof of Theorem 7 and by applying Theorem 2, statement 1, in Pérez-Díaz (2018), we get that $p = (1 : \alpha : 0)$ is a non-ordinary singularity of multiplicity $\sum_{j=1}^{u}(N_j - k_{1j})$ in the asymptote and also in the input curve since

$$\frac{\partial^i}{\partial^i t}\left(\frac{p_2}{p_1}\right)(0) = 0, \quad \text{for} \quad i = 1, \ldots, N_j - k_{1j} - 1$$

and

$$\frac{\partial^{N_j - k_{1j}}}{\partial^{N_j - k_{1j}} t}\left(\frac{p_2}{p_1}\right)(0) \neq 0$$

(see Theorem 4).

2. We apply Theorem 2, statement 2, in Pérez-Díaz (2018). □

Finally, we consider the following particular case which is the common situation for the real applications which in general the input parametrization has some previous perturbations. This theorem is obtained from the previous results.

Theorem 8. *Let $\mathcal{P}(t) = (p_1(t) : p_2(t) : \prod_{j=1}^{u}(t - s_j)\overline{p}(t))$ be such that $\overline{p}(s_j) \neq 0$ and $p_2(s_j)/p_1(s_j) = \alpha$ for $j = 1, \ldots, u$ and $s_i \neq s_j$ for every $i \neq j$ and $i, j \in \{1, \ldots, u\}$. It holds that:*

1. *there exist u different branches associated to $p = (1 : \alpha : 0)$, and each branch has one leaf.*
2. *$p = (1 : \alpha : 0)$ is a point of multiplicity u since $G(t) = \gcd(p_2(t) - \alpha p_1(t), \prod_{j=1}^{u}(t - s_j)\overline{p}(t)) = \prod_{j=1}^{u}(t - s_j)$, i.e. $\deg(G) = u$.*

3. *The asymptotes are*

$$\mathcal{Q}_j(t)=(t,a_{1j}t+a_{0j}),\quad j=1,\dots,u$$

where

$$a_{1j}=\alpha=\frac{p_2}{p_1}(s_j),\quad a_{0j}=\frac{\partial}{\partial t}\left(\frac{p_2}{p_1}\right)(s_j)\frac{p_1(s_j)}{\prod_{i=1,\,i\neq j}^{u}(s_j-s_i)\overline{p}(s_j)},\quad j=1,\dots,u.$$

4. $(x-\alpha y)^u$ *divides the form of the maximum degree of the implicit equation defining the input curve.*

Example 7. Let $\mathcal{C}$ be the plane curve introduced in Example 6 defined by the parametrization

$$\mathcal{P}(s)=\left(\frac{s^4-s^3+5s^2+2s+1}{s^4(s-1)(s-2)},\frac{1/27(-108-216s+27s^4-367s^3-2440s^2)}{s^4(s-1)(s-2)}\right)\in\mathbb{R}(s)^2.$$

Using Theorems 5, 6 and 7 and Corollaries 3 and 4, we get the following properties concerning the infinity points.

First, we observe that $(1:-388/27:0)$ is an infinity point that has two branches each with ramification index equal to $N_1=N_2=1$, and $(1:-4:0)$ is an infinity point that has one branch with ramification index equal to $N_3=4$ (Theorem 5). In the first two branches, we have that $k_{11}=k_{12}=1$, but in the third one we have that $k_{13}=2$. Therefore, the degree of the asymptotes are 1, 1 and $4/2=2$, respectively. Additionally, one gets that the form of maximum degree of the implicit equation is $(x+388/27y)^2(x+4y)^4$ (see Theorem 6 and Corollary 3).

Finally, we also get that $(1:-388/27:0)$ is an ordinary singularity of multiplicity $\sum_{j=1}^{2}(N_j-k_{1j})=(1-0)+(1-0)=2$, and $(1:-4:0)$ is a non-ordinary singularity of multiplicity $(N_3-k_{13})=(4-2)=2$ (Theorem 7 and Corollary 4).

Using the results presented in this subsection and also Theorem 4, in the following example we show how we can construct all the families of parametric curves having some given asymptotes (see also Remark 5).

Example 8. Let us construct all the plane curves $\mathcal{C}$ having the asymptotes

$$\widetilde{\mathcal{Q}}_1(t)=(t,\,-12/61t+53/122),\quad \widetilde{\mathcal{Q}}_2(t)=(t^3,\,4t^3-89/6-7\cdot 2^{1/3}t).$$

From $\widetilde{\mathcal{Q}}_1$ we deduce that $(1:-12/61:0)$ is an infinity point that is simple and has only one branch and $N_1=1$. Furthermore, from $\widetilde{\mathcal{Q}}_2$ we deduce that $(1:4:0)$ is an infinity point of multiplicity 2 that has only one branch with $N_2=3$. Furthermore $k_{11}=k_{12}=1$ and the degree of the asymptotes are 1 and 3, respectively. Additionally, one gets that the form of maximum degree of the implicit equation is $(x+12/61y)(x-4y)^3$. Therefore, $\deg(\mathcal{P})=4$, where $\mathcal{P}$ denotes all the proper parametrizations having these asymptotes.

First, one may assume w.l.o.g. that $\mathcal{P}(s)=\left(\frac{p_1(s)}{p_3(s)},\frac{p_2(s)}{p_3(s)}\right)$, where $p_3(s)=s^3(s-\tau)$, $\tau\neq 0$, and

$$p_i(t)=\beta_{i4}s^4+\beta_{i3}s^3+\beta_{i2}s^2+\beta_{i1}s+\beta_{i0},\qquad p_i(0)p_i(\tau)\neq 0,\ \ i=1,2.$$

First, we impose that for the root of the denominator $s=0$, we obtain the asymptote $\widetilde{\mathcal{Q}}_2$. For this purpose, we consider $\ell(s):=\frac{p_2}{p_1}$ and $r(s):=\frac{p_1^{1/3}}{(s-\tau)^{1/3}}$, and we consider the equalities

$$\begin{aligned}
&a_3=\ell(0)=4\\
&a_2=\tfrac{\partial\ell}{\partial t}(0)r(0)=0,\\
&a_1=\tfrac{1}{2!}\cdot\frac{\partial}{\partial t}\left(\frac{\partial\ell(t)}{\partial t}\cdot r(t)^2\right)(0)=-7\cdot 2^{1/3},\\
&a_0=\tfrac{1}{3!}\cdot\frac{\partial^2}{\partial t^2}\left(\frac{\partial\ell(t)}{\partial t}\cdot r(t)^3\right)(0)=-89/6.
\end{aligned}$$

Second, we impose that for the root of the denominator $s=\tau$, we obtain the asymptote $\widetilde{\mathcal{Q}}_1$. For this purpose, let $r(s):=\frac{p_1}{s^3}$, and we consider the equalities

$$\begin{aligned}
&a_1=\ell(\tau)=-12/61\\
&a_0=\tfrac{\partial\ell}{\partial t}(\tau)r(\tau)=53/122.
\end{aligned}$$

All the intersection common points to the six curves defined by these equations and satisfying that $\mathcal{P}$ is a parametrization provide the following families of parametrizations satisfying that the asymptotes are given by $\widetilde{\mathcal{Q}}_i,\ i=1,2$. For instance, we impose that $\tau=2$:

$p_1(s)=b_{14}s^4+b_{13}s^3-3413/384s^2-1003/768s^2b_{11}+427/768Is^2b_{11}\sqrt{3}+427/384Is^2\sqrt{3}-b_{13}s^2+b_{11}s+1,$

$p_2(s)=4-4b_{13}s^2-12/61b_{14}s^4+4b_{13}s^3+4b_{11}s-235/732s^4b_{11}-7/3Is^3b_{11}\sqrt{3}+7/12Is^4b_{11}\sqrt{3}-245/96Is^2\sqrt{3}-4651/732s^4+82/3s^3-2741/96s^2+7/3s^3b_{11}-1003/192s^2b_{11}-64/61s^4b_{13}+427/192Is^2b_{11}\sqrt{3}-7/12Is^4\sqrt{3}+7/3Is^3\sqrt{3}.$

5. Conclusion

The main result of this paper, Theorem 4, provides a simple formula based on the computation of derivatives to determine the infinity branch corresponding to an infinity point of an input parametric curve. As a consequence, one may determine the generalized asymptotes of an input curve by only computing some simple derivatives of functions constructed from the given parametrization. So, we avoiding the laborious computation of Puiseux series and an alternative method that can be extended to curves in the n–dimensional space or even to non-algebraic curves is developed. Thus, the present paper yields a remarkable improvement of the methodology developed in some previous papers as Blasco and Pérez-Díaz (2015, 2020).

From these results, we present some applications related to the computation of the ramification index and the degree of the asymptote, the infinity form and the multiplicity and character of the infinity points. As a consequence, we show how to construct all the families of parametric curves having some given asymptotes.

As a future work, we aim to extend the notion of g-asymptote to the study of the asymptotic behavior of algebraic surfaces. We look for surfaces which approach a given one of higher degree, when "moving to infinity", that is, when some of the coordinates take infinitely large values. The ideas introduced in this paper might provide the foundations for efficient methods that allow us to compute those "asymptotic surfaces".

CRediT authorship contribution statement

The authors contributed equally to this work.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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