Applied Mathematics and Computation 421 (2022) 126945

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# Inner Bohemian inverses

Eunice Y.S. Chan[a], Robert M. Corless[b], Laureano González-Vega[c], J. Rafael Sendra[d,*], Juana Sendra[c]

[a] School of Medicine, The Chinese University of Hong Kong, Shenzhen, China
[b] School of Mathematical and Statistical Sciences, Western University, Canada
[c] Department of Quantitative Methods, CUNEF Universidad, Spain
[d] Universidad de Alcalá, Dpto. de Física y Matemáticas, Alcalá de Henares, Madrid, Spain

A R T I C L E   I N F O

*Article history:*
Received 30 June 2021
Revised 3 November 2021
Accepted 11 January 2022

*Keywords:*
Bohemian matrices
Rhapsodic matrices
Inner inverses
Generalized inverses

A B S T R A C T

In this paper, for certain type of structured $\{0, 1, -1\}$–matrices, we give a complete description of the inner Bohemian inverses over any population containing the set $\{0, 1, -1\}$. In addition, when the population is exactly $\{0, 1, -1\}$, we provide explicit formulas for the number of inner Bohemian inverses of these type of matrices.



## 1. Introduction

This paper combines two different areas within the field of matrix analysis and applications, and more specifically in the interrelationship between them. On the one hand, we work with Bohemian matrices and, on the other, with generalized inverse matrices. Therefore, in this introduction, each of these two subfields of matrix analysis will be discussed first, and then we will focus on the interconnection of both areas that, ultimately, is where this paper is most concerned. However, as a first, and quick, orientation to the topics let us recall that: (1) a Bohemian matrix is a matrix whose entries belong to a prescribed subset of a field, (2) an inner inverse of an $m \times n$ matrix $A$ over a ring is an $n \times m$ matrix $X$, over the same ring, such that $AXA = A$. As commented below, in this introduction, Bohemians appear in many applications, and inner inverses play an important role in the representation of many other generalized inverses.

One of the working fronts of this paper is the study of matrices whose entries belong to a fixed subset, generally bounded, of a ring; in practice, it is usually the ring $\mathbb{Z}$ of the integer numbers. We are also concerned with *structured* Bohemian matrices. Particular instances of these types of matrices are Metzler matrices (see Briat [5]), Bernoulli matrices (see Tikhomirov [43]), Hadamard matrices (see Horadam [21]), etc. In 2015, Steven Thornton and Rob Corless, in a poster at ECCAD at the Fields Institute, refered to these matrices as “Bohemian” matrices; the name of Bohemian matrix is a mnemonic for BOunded HEight Matrix of Integers. So, a matrix is called Bohemian if its entries come from a fixed, usually finite and discrete set, called the population. However, the study of this type of matrix has a much more extensive history, though not under that name. As source references in this topic one can mention, among others, Taussky [39], and Taussky [40], where matrices with integer entries are studied.

* Corresponding author.
*E-mail addresses:* eunicechan@cuhk.edu.cn (E.Y.S. Chan), rafael.sendra@uah.es (J.R. Sendra).

https://doi.org/10.1016/j.amc.2022.126945
0096-3003/

This type of matrix offers multiple areas of working, among others: the development of theoretical properties, the study of computational issues, and their applications (see e.g. Chan [7], Sendra [30], Thornton [42] for a panoramic view of the field). Concerning the theoretical aspects one may mention the analysis of properties that relies on algebraic properties that could endow the Bohemians, with a fixed population, with some algebraic structure. In this sense, the study of *inverse* Bohemian matrices plays a fundamental role. In this direction, it is worth mentioning the works [8,10,16,17], and section 5.3.3 in Thornton [42], as well as indicating that the motivation of this current work resides ultimately in these types of theoretical questions. Furthermore, many authors have addressed the study of characteristic polynomials, the eigenvalues, the determinants, etc, of these matrices (see e.g. Chan [7], Chan et al. [8, 9], Corless [11], Corless and Thornton [13, 14], Fasi and Porzio [18], Feng and Fan [19], Thornton [42]) as well as to the closely related problem of studying the zeros of polynomials with coefficients belonging to a fixed population (see e.g. Borwein and Jorgenson [3], Borwein and Pinner [4], Christensen [12,23,29], Odlyzko and Ponnen [27]).

To approach the computational problems related to Bohemian matrices with finite population, since the set of analysis is then finite, one may consider brute-force computation. However, in practice, the cardinality of the set under analysis is vast: typically the cardinality grows faster than exponentially with the dimension. A deeper theoretical analysis of the problem is required; our contribution in this paper is a clear example of this assertion. Nevertheless, an important aspect associated to this situation is the design and development of experiments of reasonable size that can generate conjectures to be studied afterwards. One may check some such conjectures at the Characteristic Polynomial Database [41]; see also [26].

Concerning applications of Bohemian matrices, one may mention Metzler matrices appearing in sign-pattern matrices (see e.g. Briat [5], Hall and Li [20]), signal processing, when using Bernoulli matrices (see e.g. Lu et al. [24]), or error correcting codes when working with Hadamard matrices (see e.g. Horadam [21]), etc.

The other working front of the paper is the field of generalized inverses. Probably, the most famous generalized inverses are the Moore–Penrose inverse, and the Drazin inverse (see e.g. Ben-Israel and Greville [2], Campbell and Meyer [6], Drazin [15], Rao [28], Wang et al. [46]). Nevertheless, the family of generalized inverses is much larger than just these two types and is usually introduced by a collection of conditions: the Penrose axioms. To be more precise, let $\mathbb{K}$ be a field and $\varphi$ an involutory automorphism over $\mathbb{K}$; that is, $\varphi \circ \varphi$ is the identity (see e.g. Sendra and Sendra [33], Stanimirović et al. [35]); the underlying idea of the involutory automorphism is to have a generalized notion, of the concept of conjugation in $\mathbb{C}$, to be applied over other fields. Then, for a given $m \times n$ matrix $A = (a_{ij})$, over $\mathbb{K}$, we consider the axioms

$$(1) \quad AXA = A \qquad (2) \quad XAX = X \qquad (3) \quad AX = X^*A^* \qquad (4) \quad XA = A^*X^*$$

where $X$ is an $n \times m$ matrix over $\mathbb{K}$, and where $A^*$ (respectively $X^*$) denotes the transpose matrix of $\varphi(A) = (\varphi(a_{ij}))$. If $X$ satisfies the four axioms, $X$ is called the Moore–Penrose inverse; for the Drazin inverse, two additional conditions are required (see e.g. Drazin [15], Sendra and Sendra [31, 32] for further details). For $S \subset \{1, 2, 3, 4\}$, in the literature, the notation $A\{S\}$ is used to represent the set of all matrices $X$ satisfying the conditions in $S$. In addition, one can also analyze generalized inverses with a prescribed kernel and/or range space (see Ke et al. [22], Stanimirović et al. [36, 38]).

In this paper, we are interested in the set $A\{1\}$. Matrices in $A\{1\}$ are called inner inverses of $A$. There are important reasons to focus our attention on inner inverses. First, it seems to be the simplest case because $A\{1\}$ has the geometric structure of a linear affine variety (see [37]). In addition, due to the representation theorem of Urquhart (see Stanimirović et al. [37], Urquhart [44, 45]), many generalized inverses can be expressed or represented by means of inner inverses.

Let us denote by $\mathscr{B}_{m\times n}(\mathbb{P})$ the set of all $m \times n$ Bohemian matrices with population $\mathbb{P}$. In general, given a non-singular $A \in \mathscr{B}_{n\times n}(\mathbb{P})$, it holds that $A^{-1} \notin \mathscr{B}_{n\times n}(\mathbb{P})$. So, the natural question of which nonsingular matrices have Bohemian inverse w.r.t. the same population arises. Such matrices are called rhapsodic[1]. For instance, if $\mathbb{P}$ is a subfield of $\mathbb{K}$, all non-singular matrices in $\mathscr{B}_{n\times n}(\mathbb{P})$ are rhapsodic. Another well known set of examples of this type of matrix are the unimodular matrices, that is, matrices in $\mathscr{B}_{n\times n}(\mathbb{Z})$ whose determinant is $\pm 1$. Other examples are the Mandelbrot matrices (see Theorem 6 in Chan et al. [8]). In [42], the rhapsodic problem is analyzed introducing a weaker requirement, namely, $A^{-1}$ is similar to a Bohemian matrix. Also, in Martínez-Rivera [25], in the frame of the conjecture of Barrett, Butler and Hall (see Barrett et al. [1]), the author studies inverses of *equimodular* matrices, which are square matrices which entries have all the same modulus; an equimodular matrix may well be Bohemian.

Now, let us take $A \in \mathscr{B}_{m\times n}(\mathbb{P})$ not necessarily invertible or rectangular. The new natural step is to study the *rhapsodic* behavior of the generalized inverses; that is, to analyze whether the generalized inverses of $A$ belong to $\mathscr{B}_{n\times m}(\mathbb{P})$. A very first step in this direction can be found in Chu et al. [10] where, for a very special type of latin square, they study when the Moore–Penrose inverse is again a latin square. Besides the interest of a wider analysis of the Moore–Penrose case, the next step is to study other generalized inverses. In this paper, we focus on inner inverses. To approach the problem of computing the inner Bohemian inverses of a given Bohemian matrix, one may try to use brute-force computation. That is, for a given finite population, compute all Bohemian matrices and then check those that are inner inverses of the given one. However,Tables 1–3 show that the brute-force strategy is not feasible already for quite modest dimensions.

In this paper, we present a complete analysis for some special types of Bohemian matrices, namely full matrices and well–settled matrices (see Definitions 1 and 2); we observe that full matrices are particular instances of equimodular matrices (see Martínez-Rivera [25]). More precisely, the contributions of this paper are:

[1] Yes, this is based on a joke. Several people still find this joke funny, because “Number theory is the Queen of Mathematics.”

**Table 1**
Case of full matrices of type I with $\mathbb{P} = \{0, \pm 1\}$ size $n \times (n-1)$.

| $n \times (n-1)$ | #(Inner Bohemians) | #(Bohemians)= $3^{n(n-1)}$ |
|---|---|---|
| $2 \times 1$ | 2 | 9 |
| $3 \times 2$ | 126 | 729 |
| $4 \times 3$ | 69.576 | 531.441 |
| $5 \times 4$ | 363.985.680 | 3.486.784.401 |
| $6 \times 5$ | 17.812.283.544.870 | 205.891.132.094.649 |
| $7 \times 6$ | 806.9792.560.277.356.314 | 109.418.989.131.512.359.209 |
| $8 \times 7$ | 33.609.055.109.399.933.461.665.528 | 523.347.633.027.360.537.213.511.521 |

**Table 2**
Case of full matrices of type II with $\mathbb{P} = \{0, \pm 1\}$ where the first block is $n \times (n-1)$ and the second is $n \times (n-2)$.

| $n \times ((n-1)+(n-2))$ | # (Inner Bohemians) | # (Bohemians)= $3^{n(2n-3)}$ |
|---|---|---|
| $3 \times (2+1)$ | 2.907 | 19.683 |
| $4 \times (3+2)$ | 363.985.680 | 3.486.784.401 |
| $5 \times (4+3)$ | 4.024.604.728.349.450 | 50.031.545.098.999.707 |
| $6 \times (5+4)$ | 3.800.557.141.293.418.496.841.798 | 58.149.737.003.040.059.690.390.169 |

**Table 3**
Case of well–settled matrices, where $\mathbb{P} = \{0, \pm 1\}$, and with two diagonal block of full matrices of type I and orders $n \times (n-1)$ each.

| $2n \times (2n-2)$ | # (Inner Bohemians) | # (Bohemians)= $3^{4n(n-1)}$ |
|---|---|---|
| $n = 2 \to 4 \times 2$ | 36 | 6561 |
| $n = 3 \to 6 \times 4$ | 315.630.756 | 282.429.536.481 |
| $n = 4 \to 8 \times 6$ | 26.357.375.491.548.319.296 | 79.766.443.076.872.509.863.3611 |

1. A complete description of the inner inverses of full matrices over any field (see Theorem 1).
2. A complete description of the inner Bohemian inverses of full matrices for any population from any field (see Theorem 2).
3. A complete description of the inner Bohemian inverses of well–settled matrices for any population from any field (see Theorems 3–5).
4. For the population $\mathbb{P} = \{0, \pm 1\}$ we give exact formulas for the number of Bohemian matrices of full and well–settled matrices, respectively (see Corollaries 2 and 3).

The paper is structured as follows. In Section 2 we introduce the notation, the basic definition and establish some technical lemmas. In Section 3 the case of full matrices is analyzed, and finally in Section 4 the results are extended to the well–settled case. We conclude the paper with a section devoted to conclusions and open problems.

## 2. Notation and preliminaries

We start by fixing some notation and terminology that will be used throughout this paper. In the sequel, $\mathscr{R}$ is a commutative integral domain with unit, and $\mathbb{K}$ a field containing $\mathscr{R}$. One may think of $\mathscr{R}$ as $\mathbb{Z}$ and of $\mathbb{K}$ as either $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{K}^{m\times n}$ denotes the set of $m \times n$ matrices over $\mathbb{K}$. When necessary, for $A \in \mathbb{K}^{m\times n}$, we will write $A_{mn}$ to indicate that the matrix $A$ is $m \times n$. For $A \in \mathbb{K}^{m\times n_1}$ and $B \in \mathbb{K}^{m\times n_2}$, we denote by $(A \quad B) \in \mathbb{K}^{m\times (n_1+n_2)}$ the matrix obtained by attaching $B$ at the right of $A$. Similarly, for $A \in \mathbb{K}^{m_1 n}$ and $B \in \mathbb{K}^{m_2 n}$ we denote by

$$\begin{pmatrix} A \\ B \end{pmatrix} \in \mathbb{K}^{(m_1+m_2)\times n}$$

the matrix obtained by attaching $B$ below $A$.

Furthermore, we denote by $\mathbf{1}$ the matrix with 1 in all its positions, by $\mathbf{0}$ the zero matrix, and by I is the identity matrix. Similarly, we will write $-\mathbf{1}$ to denote the matrix with $-1$ in all its positions. Moreover, we will write $\left( \pm\mathbf{1} \mid \mp\mathbf{1} \right)$ to refer simultaneously to the matrices $\left( \mathbf{1} \mid -\mathbf{1} \right)$, $\left( -\mathbf{1} \mid \mathbf{1} \right)$; analogously if the attachment is done vertically.

In addition, if $\mathscr{S} \subset \mathbb{K}^{m\times n}$, and $P, Q$ are matrices of suitable orders, we will denote by $P\mathscr{S}Q$ the set

$$P\mathscr{S}Q = \{PSQ \mid S \in \mathscr{S}\}$$

Similarly, we will use the notation $P\mathscr{S}$ or $\mathscr{S}Q$. Furthermore, for $\lambda \in \mathbb{K}$, we will write

$$\lambda\mathscr{S} = \{\lambda S \mid \lambda \in \mathscr{S}\},$$

and

$$\mathscr{S}^T = \{S^T \mid S \in \mathscr{S}\},$$

where $S^T$ means the transpose matrix of $S$.

Now, we recall the two main notions to be used in the paper, namely, Bohemian matrix and inner inverse. Let $\mathbb{P} \subset \mathscr{R}$ be a fixed population. A $\mathbb{P}$-Bohemian matrix (or simply a Bohemian matrix if there is no ambiguity) is a matrix whose entries belong to $\mathbb{P}$. Furthermore, we denote by $\mathscr{B}_{m\times n}(\mathbb{P})$ the set of all $m \times n$ $\mathbb{P}$-Bohemian matrix. That is

$$\mathscr{B}_{m\times n}(\mathbb{P}) = \left\{(a_{ij}) \in \mathbb{K}^{m\times n} \mid a_{ij} \in \mathbb{P}\right\}.$$

Alternatively, $\mathscr{B}_{m\times n}(\mathbb{P}) = \mathbb{P}^{m\times n}$.

Let $A \in \mathbb{K}^{m\times n}$. An inner inverse of $A$ is a matrix $X \in \mathbb{K}^{n\times m}$ such that $AXA = A$. We introduce the notation $A\{1\}$ to represent the set of all inner matrices of $A$. That is

$$A\{1\} = \left\{X \in \mathbb{K}^{n\times m} \mid AXA = A\right\}.$$

Finally, we introduce the notion of inner Bohemian inverse. Let $A \in \mathscr{B}_{m\times n}(\mathbb{P})$. We say that $X \in \mathbb{K}^{n\times m}$ is a inner Bohemian inverse of $A$ if $X \in \mathscr{B}_{n\times m}(\mathbb{P})$ and $AXA = A$, that is, if $X \in \mathscr{B}_{m\times n}(\mathbb{P}) \cap A\{1\}$. We will denote by $A_{\mathscr{B}(\mathbb{P})}\{1\}$ the set of all $\mathbb{P}$–inner Bohemian inverses of the $\mathbb{P}$-Bohemian matrix $A$.

The main goal of this paper is to analyze the set $A_{\mathscr{B}(\mathbb{P})}\{1\}$ for $A \in \mathscr{B}_{m\times n}(\mathbb{P})$. Just to show that we are not facing a trivial question, we observe that even though $A\{1\} \neq \emptyset$ (see Stanimirović et al. [37]), the set $A_{\mathscr{B}(\mathbb{P})}\{1\}$ may be empty. For instance, for

$$A := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathscr{B}_{2\times 3}(\{0,1\}),$$

it holds that

$$A\{1\} = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & -1 \\ x_1 & x_2 \end{pmatrix} \text{ with } x_i \in \mathbb{K} \right\}.$$

Therefore $A_{\mathscr{B}(\{0,1\})}\{1\} = \emptyset$, since $-1 \notin \mathbb{P} = \{0,1\}$. Furthermore, if $A = \mathbf{0} \in \mathbb{K}^{m\times n}$ then $A\{1\} = \mathbb{K}^{n\times m}$ and $A_{\mathscr{B}(\mathbb{P})}\{1\} = \mathscr{B}_{n\times m}(\mathbb{P})$.

We finish this section with some technical lemmas on inner inverses that will be used in the next sections.

**Lemma 1.** *Let $A \in \mathbb{K}^{m\times n}$, let $P \in \mathbb{K}^{m\times m}$ and $Q \in \mathbb{K}^{n\times n}$ be non-singular matrices, and let $\lambda \in \mathbb{K} \setminus \{0\}$. It holds that*

1. $(A\{1\})^T = A^T\{1\}$.
2. $(\lambda A)\{1\} = \dfrac{1}{\lambda}(A\{1\})$.
3. $Q(PAQ)\{1\}P = A\{1\}$.

**Proof.**

(1) $X \in (A\{1\})^T \Leftrightarrow X = Y^T$ with $Y \in A\{1\} \Leftrightarrow X = Y^T$ and $AYA = A \Leftrightarrow X = Y^T$ and $A^TY^TA^T = A^T \Leftrightarrow X \in A^T\{1\}$.

(2) $X \in (\lambda A)\{1\} \Leftrightarrow \lambda AX\lambda A = \lambda A \Leftrightarrow A(\lambda X)A = A \Leftrightarrow \lambda X \in A\{1\} \Leftrightarrow X \in \frac{1}{\lambda}(A\{1\})$.

(3) Let $X \in Q(PAQ)\{1\}P$. Then, there exists $Y \in (PAQ)\{1\}$ such that $X = QYP$; in particular $(PAQ)Y(PAQ) = PAQ$. So, $AXA = AQYPA = P^{-1}(PAQYPAQ)Q^{-1} = P^{-1}(PAQ)Q^{-1} = A$. Thus, $X \in A\{1\}$. Conversely, let $X \in A\{1\}$. Then, $AXA = A$. $X$ can be expressed as $X = QQ^{-1}XP^{-1}P$, and $PAQ(Q^{-1}XP^{-1})PAQ = PAXAQ = PAQ$. Thus $Q^{-1}XP^{-1} \in (PAQ)\{1\}$ and hence $X \in Q(PAQ)\{1\}P$.

□

**Lemma 2.** *Let $A = \left( B_{m\,n_1} \mid \mathbf{0}_{m\,n_2} \right) \in \mathbb{K}^{m\times(n_1+n_2)}$. Then*

$$A\{1\} = \left\{ \begin{pmatrix} X_{n_1\,m} \\ Y_{n_2\,m} \end{pmatrix} \middle| \; X_{n_1\,m} \in B_{m\,n_1}\{1\}, Y_{n_2\,m} \in \mathbb{K}^{n_2\times m} \right\}.$$

**Proof.** We observe that

$$\begin{pmatrix} B_{m\,n_1} & \mathbf{0}_{m\,n_2} \end{pmatrix} \begin{pmatrix} X_{n_1\,m} \\ Y_{n_2\,m} \end{pmatrix} \begin{pmatrix} B_{m\,n_1} & \mathbf{0}_{m\,n_2} \end{pmatrix} = \begin{pmatrix} B_{m\,n_1}X_{n_1\,m}B_{m\,n_1} & \mathbf{0}_{m\,n_2} \end{pmatrix}.$$

Now, the result follows by taking into account that, in order to be an inner inverse of $A$, the previous matrix has to be equal to $\left( B_{m\,n_1} \mid \mathbf{0}_{m\,n_2} \right)$. □

## 3. Inner Bohemian inverses of full matrices

In the section we analyze the inner Bohemian inverses of certain types of Bohemian matrices. In the next section, we extend the results to a larger class of Bohemian matrices. We start by introducing the notion of a “full” matrix.

**Definition 1.** We say that $A \in \mathbb{K}^{m\times n}$ is full if $A$ has one of the following forms

(1) Full matrix of type I: $A = \left(\pm\mathbf{1}_{m\,n}\right)$.

(2) Full matrix of type II: $A = \left( \begin{array}{c|c} \pm\mathbf{1}_{m\,n_1} & \mp\mathbf{1}_{m\,n_2} \end{array} \right)$.
(3) Full matrix of type III: $A = \left( \begin{array}{c|c} \pm\mathbf{1}_{m\,n_1} & \mathbf{0}_{m\,n_2} \end{array} \right)$.
(4) Full matrix of type IV: $A = \left( \begin{array}{c|c|c} \pm\mathbf{1}_{m\,n_1} & \mp\mathbf{1}_{m\,n_2} & \mathbf{0}_{m\,n_3} \end{array} \right)$.

**Remark 1.** We observe that, because of Lemma 2, it is enough to analyze inner Bohemian inverses of full matrices of type I and II. Furthermore, by Lemma 1, we may restrict our study to the cases $\left(\mathbf{1}_{m\,n}\right)$, and $\left( \begin{array}{c|c} \mathbf{1}_{m\,n_1} & -\mathbf{1}_{m\,n_2} \end{array} \right)$.

Furthermore, the transpose matrices of full matrices will also be covered. Moreover, taking into account Lemma 1 (5) with $P$ and $Q$ suitable permutation matrices, one deduces that the assumption on the position of the each of the blocks does not affect our analysis. So, in the sequel, without loss of generality, we limit our study to the types I and II.

We start by analyzing the inner inverses of full matrices. For this purpose, we first fix some notation.

*Notation:* For $X \in \mathscr{B}_{n\times m}(\mathbb{P})$, with $\mathbb{P} \subset \mathbb{K}$, we introduce the notation $S(X)$ = sum of all the entries of $X$.

In the next technical lemma we analyze the effect of the left-right multiplication of a matrix $X$ by full matrices.

**Lemma 3.**

1. *Let* $X \in \mathbb{K}^{n\times r}$. *Then* $\mathbf{1}_{m\,n} X \mathbf{1}_{r\,s} = S(X)\,\mathbf{1}_{m\,s}$.
2. *Let* $X \in \mathbb{K}^{n\times r}$. *Then* $\mathbf{1}_{m\,n} X \left( \begin{array}{c|c} \mathbf{1}_{r\,s_1} & -\mathbf{1}_{r\,s_2} \end{array} \right) = S(X) \left( \begin{array}{c|c} \mathbf{1}_{m\,s_1} & -\mathbf{1}_{m\,s_2} \end{array} \right)$.
3. *Let* $X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}$, *with* $X^i \in \mathbb{K}^{n_i\times r}$. *Then,*
$$\left( \begin{array}{c|c} \mathbf{1}_{m\,n_1} & -\mathbf{1}_{m\,n_2} \end{array} \right) X \left( \begin{array}{c|c} \mathbf{1}_{r\,s_1} & -\mathbf{1}_{r\,s_2} \end{array} \right) = \left(S(X^1) - S(X^2)\right) \left( \begin{array}{c|c} \mathbf{1}_{m\,s_1} & -\mathbf{1}_{m\,s_2} \end{array} \right)$$
4. *Let* $X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}$, *with* $X^i \in \mathbb{K}^{n_i\times r}$. *Then,*
$$\left( \begin{array}{c|c} \mathbf{1}_{m\,n_1} & -\mathbf{1}_{m\,n_2} \end{array} \right) X \mathbf{1}_{r\,s} = \left(S(X^1) - S(X^2)\right) \mathbf{1}_{m\,s}$$

**Proof.** Let us prove the first statement. Let $C_i$ denote de sum of the entries of the $i$th column of $X$. Then,
$$\mathbf{1}_{m\,n} X \mathbf{1}_{r\,s} = \begin{pmatrix} C_1 & \dots & C_r \\ \vdots & & \vdots \\ C_1 & \dots & C_r \end{pmatrix} \mathbf{1}_{r\,s} = \left(\sum_{i=1}^{r} C_i\right) \mathbf{1}_{m\,s} = S(X)\,\mathbf{1}_{m\,s}.$$
Statement (2) follows similarly.

To prove the third statement, let $C_i^k$ denote the sum of the entries of the $i$th column of $X^k$, and let $\alpha_k = C_k^1 - C_k^2$. Let $A = \left( \begin{array}{c|c} \mathbf{1}_{m\,n_1} & -\mathbf{1}_{m\,n_2} \end{array} \right)$ and $B = \left( \begin{array}{c|c} \mathbf{1}_{r\,n_1} & -\mathbf{1}_{r\,n_2} \end{array} \right)$. Then,
$$\begin{aligned} A\,X\,B &= \begin{pmatrix} C_1^1 - C_1^2 & \dots & C_r^1 - C_r^2 \\ \vdots & & \vdots \\ C_1^1 - C_1^2 & \dots & C_r^1 - C_r^2 \end{pmatrix} B \\ &= \begin{pmatrix} \alpha_1 & \dots & \alpha_r \\ \vdots & & \vdots \\ \alpha_1 & \dots & \alpha_r \end{pmatrix} B \\ &= \left( \begin{matrix} \underbrace{\begin{matrix} \sum_{k=1}^m \alpha_k & \dots & \sum_{k=1}^m \alpha_k \\ \vdots & & \vdots \\ \sum_{k=1}^m \alpha_k & \dots & \sum_{k=1}^m \alpha_k \end{matrix}}_{m\times s_1} & \underbrace{\begin{matrix} -\sum_{k=1}^r \alpha_k & \dots & -\sum_{k=1}^r \alpha_k \\ \vdots & & \vdots \\ -\sum_{k=1}^r \alpha_k & \dots & -\sum_{k=1}^r \alpha_k \end{matrix}}_{m\times s_2} \end{matrix} \right) \\ &= \left(\sum_{k=1}^r \alpha_k\right) B = \left(S\left(X^1\right) - S\left(X^2\right)\right) B. \end{aligned}$$
Statement (4) follows similarly. □

The next result characterizes the inner inverses of full matrices over $\mathbb{K}$.

**Theorem 1** (Inner inverses of full matrices)**.**

1. *If* $A = \left(\mathbf{1}_{mn}\right) \in \mathbb{K}^{m\times n}$,
$$A\{1\} = \left\{ \begin{pmatrix} 1 - \sum_{k=1}^{nm-1} \lambda_k & \lambda_1 & \dots & \lambda_{m-1} \\ \lambda_m & \lambda_{m+1} & \dots & \lambda_{2m-1} \\ \vdots & \vdots & & \vdots \\ \lambda_{(n-1)m} & \lambda_{(n-1)m+1} & \dots & \lambda_{nm-1} \end{pmatrix},\ \lambda_i \in \mathbb{K} \right\}.$$

2. *If* $A = \left( \begin{array}{c|c} \mathbf{1}_{mn_1} & -\mathbf{1}_{mn_2} \end{array} \right) \in \mathbb{K}^{m\times n}$,

$$A\{1\} = \left\{ \begin{pmatrix} 1-\sum_{k=1}^{n_1m-1}\lambda_k + \sum_{k=1}^{n_2m}\mu_k & \lambda_1 & \cdots & \lambda_{m-1} \\ \lambda_m & \lambda_{m+1} & \cdots & \lambda_{2m-1} \\ \vdots & \vdots & & \vdots \\ \lambda_{(n_1-1)m} & \lambda_{(n_1-1)m+1} & \cdots & \lambda_{n_1m-1} \\ \mu_1 & \mu_2 & \cdots & \mu_m \\ \vdots & \vdots & & \vdots \\ \mu_{(n_2-1)m+1} & \mu_{(n_2-1)m+2} & \cdots & \mu_{n_2m} \end{pmatrix}, \ \lambda_i, \mu_j \in \mathbb{K} \right\}.$$

**Proof.** We prove statement (2). Statement (1) follows analogously. Let $\mathcal{A}$ denote the right hand side set in the statement. A direct manipulation shows that $\mathcal{A} \subset A\{1\}$. On the other hand, $\mathcal{A}$ can be seen as a linear affine subspace of $\mathbb{K}^{mn}$ of dimension $mn-1$. Since $A\{1\}$, as a linear affine space, has dimension $mn - \text{rank}(A)^2 = mn - 1$ (see Stanimirović et al. [37]), we get the equality of both sets. □

The next theorem states the structure of the inner Bohemian inverses of full matrices.

**Theorem 2** (Inner Bohemian inverses of full matrices)**.**

1. *Let* $\mathbb{P} \subset \mathbb{K}$*, such that* $1 \in \mathbb{P}$*, and let* $A = \mathbf{1}_{mn} \in \mathbb{K}^{m\times n}$*. Then*

$$(A)_{\mathscr{B}(\mathbb{P})}\{1\} = \{X \in \mathscr{B}_{n\times m}(\mathbb{P}) \,|\, S(X) = 1\}.$$

2. *Let* $\mathbb{P} \subset \mathbb{K}$*, such that* $\pm 1 \in \mathbb{P}$*, and let* $A = \left( \begin{array}{c|c} \mathbf{1}_{mn_1} & -\mathbf{1}_{mn_2} \end{array} \right) \in \mathbb{K}^{m\times n}$*. Then*

$$A_{\mathcal{B}(\mathbb{P})}\{1\} = \left\{ \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} \in \mathbb{K}^{n\times m} \,\middle|\, \begin{array}{c} X^i \in \mathcal{B}_{n_i\times m}(\mathbb{P}),\ i \in \{1,2\}, \\ S(X^1) = 1 + S(X^2) \end{array} \right\}$$

**Proof.** (1) If $X \in \{X \in \mathscr{B}_{n\times m}(\mathbb{P}) \,|\, S(X) = 1\}$, by Lemma 3 (1), one has that $\mathbf{1}_{mn}X\mathbf{1}_{mn} = S(X)\,\mathbf{1}_{mn} = \mathbf{1}_{mn}$, and hence $X \in (\mathbf{1}_{mn})_{\mathscr{B}(\mathbb{P})}\{1\}$. Conversely, let $X \in (\mathbf{1}_{mn})_{\mathscr{B}(\mathbb{P})}\{1\}$. Then, $X$ can be expressed as in Theorem 1 (1), and clearly $S(X) = 1$. Therefore, $X$ belongs to the set on the right side of the statement.

(2) follows similarly using Lemma 3 (2) and Theorem 1 (2). □

Using the previous theorem we can deduce the cardinality of the inner Bohemian inverses of full matrices for some populations.

**Corollary 1.** *Let* $1 \in \mathbb{P} \subset \mathbb{N}$.

1. *If* $0 \notin \mathbb{P}$*, then* $\#((\mathbf{1}_{mn})_{\mathscr{B}(\mathbb{P})}\{1\}) = 0$*, unless* $m = n = 1$ when $\#((\mathbf{1}_{mn})_{\mathscr{B}(\mathbb{P})}\{1\}) = 1$.
2. *If* $0 \in \mathbb{P}$*,* $\#((\mathbf{1}_{mn})_{\mathscr{B}(\mathbb{P})}\{1\}) = mn$*.*

**Proof.** Let $X \in (\mathbf{1}_{mn})_{\mathscr{B}(\mathbb{P})}\{1\}$. By Theorem 2 (1), it holds that $S(X) = 1$. Since $0 \notin \mathbb{P}$, and since $\mathbb{P}$ does not contain negative numbers, then: if $mn > 1$, it holds that $S(X) = mn > 1$, and hence $(\mathbf{1}_{mn})_{\mathscr{B}(\mathbb{P})}\{1\} = \emptyset$; if $mn = 1$ then the only inner inverse is the matrix (1). This proves (1). For statement (2), using Theorem 2 (1), the inner Bohemian inverses of $(\mathbf{1}_{mn})$ are precisely the matrices of the canonical basis of $\mathcal{C}^{n\times m}$; this proves (2). □

In the following, we analyze the number of inner Bohemian inverses of full matrices. We start our analysis with the population $\mathbb{P} = \{0, \pm 1\} \subset \mathbb{K}$; note that the case $\mathbb{P} = \{0, 1\}$ is already covered by the previous corollary.

**Corollary 2.** *Let* $\mathbb{P} = \{0, \pm 1\}$.

1. $\#((\mathbf{1}_{mn})_{\mathscr{B}(\mathbb{P})}\{1\}) = \displaystyle\sum_{s=0}^{\lfloor \frac{nm-1}{2} \rfloor} \binom{nm}{s}\binom{nm-s}{s+1}$.
2. $\#\left(\left( \begin{array}{c|c} \mathbf{1}_{mn_1} & -\mathbf{1}_{mn_2} \end{array} \right)_{\mathscr{B}(\mathbb{P})}\{1\}\right) =$

$$\sum_{r_2=0}^{n_2m}\sum_{s_2=0}^{n_2m}\sum_{s_1=0}^{\lfloor \frac{(n_1+n_2)m-1}{2} - r_2 \rfloor} \binom{n_1m}{s_1}\binom{n_2m}{s_2}\binom{n_2m-s_2}{r_2}\binom{n_1m-s_1}{1+r_2+s_1-s_2}.$$

**Proof.** For a matrix $X$ we denote by $N^+(X)$, and by $N^-(X)$, the number of entries of $X$ equal to 1 and equal to $-1$, respectively.

Let us prove (1). By Theorem 2

$$(\mathbf{1}_{mn})_{\mathscr{B}(\mathbb{P})}\{1\} = \{X \in \mathscr{B}_{n\times m}(\mathbb{P}) \,|\, S(X) = 1\} = \left\{X \in \mathscr{B}_{n\times m}(\mathbb{P}) \,|\, N^+(X) = N^-(X) + 1\right\}.$$

Let $s$ represent the possible values of $N^-(X)$, with $X \in \mathscr{B}_{n\times m}(\mathbb{P})$, Clearly, $s \in \{0, \ldots, nm\}$. The number of matrices $X \in \mathscr{B}_{n\times m}(\mathbb{P})$ such that $N^+(X) = s+1$ is

$$\binom{nm-s}{s+1}.$$

Moreover there exist $\binom{nm}{s}$ possible matrices $X \in \mathscr{B}_{n\times m}(\mathbb{P})$ such that $N^-(X) = s$. On the other hand, we have that $N^+(X) + s \leq nm$ and $N^+(x) = s+1$. Thus, $0 \leq s \leq \lfloor \frac{nm-1}{2} \rfloor$. This proves statement (1).

Let us prove (2). By Theorem 2, the inner inverses are of the form

$$\begin{pmatrix} X^1 \\ X^2 \end{pmatrix}$$

where $X^i \in \mathscr{B}_{n_i\times m}(\mathbb{P}),\ i \in \{1,2\}$, and $N^+(X^1) - N^-(X^1) = 1 + (N^+(X^2) - N^-(X^2))$. Let $r_i$ and $s_i$ denote $N^+(X^i)$ and $N^-(X^i)$, respectively. Then, the cardinality of the set of inner Bohemian inverses is

$$\sum_{(r_1,r_2,s_1,s_2)\in \mathrm{I}} \binom{n_1 m}{s_1}\binom{n_2 m}{s_2}\binom{n_2 m - s_2}{r_2}\binom{n_1 m - s_1}{r_1},$$

where

$$\mathrm{I} = \{(r_1, r_2, s_1, s_2) \in \mathbb{N}^4 \mid r_i, s_i \in \{0, \ldots, n_i m\}, i \in \{1,2\}, \text{ and } r_1 - s_1 = 1 + r_2 - s_2\}.$$

Clearly, $r_1 = 1 + r_2 + s_1 - s_2$, and using that $0 \leq r_i + s_i \leq n_i m$, one deduces that $0 \leq s_1 \leq \lfloor \frac{(n_1+n_2)m-1}{2} - r_2 \rfloor$. This ends the proof. □

**Remark 2.** One may observe that the sequence $\{\#((\mathbf{1}_{m\,1})_{\mathscr{B}(\mathbb{P})}\{1\})\}_{m\in\mathbb{N}}$ (see Corollary 2 (1) taking $n=1$) appears as the sequence $\{a_m\}_{m\in\mathbb{N}}$ in OEIS A005717 (see Sloane [34]). Furthermore, the sequence $\{\#((\mathbf{1}_{m\,n})_{\mathscr{B}(\mathbb{P})}\{1\})\}_{m\in\mathbb{N}}$, for a fixed $n > 1$, appears as the subsequence $\{a_{mn+1}\}_{m\in\mathbb{N}}$.

In Tables 1 and 2 we show the number of inner Bohemian inverses of some full matrices of several dimensions. We observe that the third column in each table gives the number of corresponding Bohemian matrices with population $\{0, \pm 1\}$, which is $3^{n(n-1)}$ and $3^{n(2n-3)}$, respectively. On the other hand, we note that the second column of Table 1 appears as the subsequence $\{a_{n(n-1)+1}\}_{n\in\mathbb{N}}$ of the sequence $\{a_m\}_{m\in\mathbb{N}}$ in OEIS A005717 (see Remark 2 and Sloane [34]). Note that the numbers in the second column of the Tables 1 and 2 come from Corollary 2 (1), and Corollary 2 (2), respectively.

## 4. Inner Bohemian inverses of well–settled matrices

In this section, we analyze the inner inverses of a wider class of Bohemian matrices. We start by introducing the notion of a "well–settled" matrix.

**Definition 2.** We say that $A \in \mathbb{K}^{m\times n}$ is well–settled if, after multiplying by suitable permutation matrices $P, Q$, it holds that $PAQ$ is of the form

$$PAQ = \left(\begin{array}{c|c|c} M_{p_1 q_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_{p_s q_s} \end{array}\right) \tag{4.1}$$

where each $M_{p_i q_i}$ is full (see Definition 1).

We say that a well–settled matrix is pure if all matrices $M_{p_i q_i}$ are of the same type, either $\pm\mathbf{1}$ or $\left(\begin{array}{c|c} \pm\mathbf{1} & \mp\mathbf{1} \end{array}\right)$; otherwise, we say that the matrix is mixed.

The following matrices are examples of well–settled matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

However,

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is not well–settled, while

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

is, because

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

**Remark 3.** From the theoretical point of view, reasoning as in Remark 1, we may restrict our analysis, without loss of generality, to well–settled matrices whose involved full matrices are of the form $\mathbf{1}$ or $\left( \begin{array}{c|c} \mathbf{1} & -\mathbf{1} \end{array} \right)$. Moreover, using Lemma 1 (3), we may also assume that $P$ and $Q$ are the corresponding identity matrices.

From the computational point of view, the natural question of deciding whether a given Bohemian matrix is well–settled arises. We do not deal in this paper with this problem. Of course, since the number of possible applicable permutation matrices is finite, one may consider a brute-force approach to solve this problem.

In Section 3, Theorems 1 and 2, we have seen that inner Bohemian inverses of full matrices are related to the formula $S(X^1) = 1 + S(X^2)$. We will see that a similar phenomenon happens for well–settled matrices. But, first, we need to introduce the notion of "balanced" matrix that will play an important role in the following.

**Definition 3.** Let $X \in \mathbb{K}^{r\times s}$ and let $r_1, r_2 \in \mathbb{Z}$ be such that $r_1 > 0, r_2 \geq 0$ and $r_1 + r_2 = r$. We say that $X$ is $(r_1, r_2)$–balanced if $X$ can be expressed as

$$X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix},$$

where $X^i \in \mathbb{K}^{r_i\times s}$, and $S(X^1) = S(X^2)$.

**Remark 4.** We observe that if $X$ is $(r_1, 0)$–balanced means that $S(X) = 0$.

The following lemma, which is a direct consequence of Lemma 3, shows the relation between balanced matrices with the right-left multiplication of a matrix by full matrices.

**Lemma 4.**

1. *If* $\mathbf{1}_{mn} X \mathbf{1}_{rs} = \mathbf{0}_{ms}$, *with* $X \in \mathbb{K}^{n\times r}$, *then* $X$ *is* $(n, 0)$*–balanced.*
2. *If* $\mathbf{1}_{mn} X \left( \begin{array}{c|c} \mathbf{1}_{rs_1} & -\mathbf{1}_{rs_2} \end{array} \right) = \mathbf{0}_{m\,(s_1+s_2)}$, *with* $X \in \mathbb{K}^{n\times r}$, *then* $X$ *is* $(n, 0)$*–balanced.*
3. *If* $\left( \begin{array}{c|c} \mathbf{1}_{m\,n_1} & -\mathbf{1}_{m\,n_2} \end{array} \right) X \left( \begin{array}{c|c} \mathbf{1}_{r\,s_1} & -\mathbf{1}_{r\,s_2} \end{array} \right) = \mathbf{0}_{m\,(s_1+s_2)}$, *with* $X \in \mathbb{K}^{(n_1+n_2)\times r}$, *then* $X$ *is* $(n_1, n_2)$*–balanced.*
4. *If* $\left( \begin{array}{c|c} \mathbf{1}_{m\,n_1} & -\mathbf{1}_{m\,n_2} \end{array} \right) X \mathbf{1}_{rs} = \mathbf{0}_{m\,s}$, *with* $X \in \mathbb{K}^{(n_1+n_2)\times r}$, *then* $X$ *is* $(n_1, n_2)$*–balanced.*

**Proof.** It follows from Lemma 3. □

The next two lemmas extend the previous result to pure well–settled matrices.

**Lemma 5.** *Let*

$$A = \left( \begin{array}{c|c|c} \epsilon_1 \mathbf{1}_{p_1 q_1} & \dots & \mathbf{0}_{p_1 q_s} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{p_s q_1} & \dots & \epsilon_s \mathbf{1}_{p_s q_s} \end{array} \right) \in \mathbb{K}^{m\times n}, \text{ and } X = \left( \begin{array}{c|c|c} X^{11}_{q_1 p_1} & \dots & X^{1s}_{q_1 p_s} \\ \vdots & \ddots & \vdots \\ X^{s1}_{q_s p_1} & \dots & X^{ss}_{q_s p_s} \end{array} \right) \in \mathbb{K}^{n\times m},$$

*where* $\epsilon_k \in \{-1, 1\}$. *If* $AXA = \mathbf{0}_{mn}$, *then for all* $i, j \in \{1, \dots, s\}$ $X^{ij}_{q_i, p_j}$ *is* $(q_i, 0)$*–balanced.*

**Proof.** We observe that

$$AXA = \left( \begin{array}{c|c|c} \epsilon_1^2 \mathbf{1}_{p_1 q_1} X^{11}_{q_1 p_1} \mathbf{1}_{p_1 q_1} & \dots & \epsilon_1 \epsilon_s \mathbf{1}_{p_1 q_1} X^{1s}_{q_1 p_s} \mathbf{1}_{p_s q_s} \\ \vdots & \ddots & \vdots \\ \epsilon_s \epsilon_1 \mathbf{1}_{p_s q_s} X^{s1}_{q_s p_1} \mathbf{1}_{p_1 q_1} & \dots & \epsilon_s^2 \mathbf{1}_{p_s q_s} X^{ss}_{q_s p_s} \mathbf{1}_{p_s q_s} \end{array} \right)$$

So, the equality $AXA = 0$ implies that each of the above products, of the form $\mathbf{1}_{p_i q_j} X^{jk}_{q_j p_k} \mathbf{1}_{p_k q_k}$, is the zero matrix. Therefore, the result follows from Lemma 4. □

**Lemma 6.** *Let* $V_{p_i q_i} = \left( \begin{array}{c|c} \mathbf{1}_{p_i q_{i1}} & -\mathbf{1}_{p_i q_{i2}} \end{array} \right)$, *where* $q_i = q_{i1} + q_{i2}$, *and let*

$$A = \left( \begin{array}{c|c|c} \epsilon_1 V_{p_1 q_1} & \dots & \mathbf{0}_{p_1 q_s} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{p_s q_1} & \dots & \epsilon_s V_{p_s q_s} \end{array} \right) \in \mathbb{K}^{m\times n}, \; X = \left( \begin{array}{c|c|c} X^{11}_{q_1 p_1} & \dots & X^{1s}_{q_1 p_s} \\ \vdots & \ddots & \vdots \\ X^{s1}_{q_s p_1} & \dots & X^{ss}_{q_s p_s} \end{array} \right) \in \mathbb{K}^{n\times m},$$

*where* $\epsilon_k \in \{-1, 1\}$. *If* $AXA = \mathbf{0}_{mn}$ *then, for all* $i, j \in \{1, \dots, s\}$, $X^{ij}_{q_i, p_j}$ *is* $(q_{i1}, q_{i2})$*–balanced.*

**Proof.** We observe that

$$AXA=\left(\begin{array}{c|c|c} \epsilon_1^2 V_{p_1q_1}X^{11}_{q_1p_1}V_{p_1q_1} & \cdots & \epsilon_1\epsilon_s V_{p_1q_1}X^{1s}_{q_1p_s}V_{p_sq_s}\\ \vdots & \ddots & \vdots\\ \epsilon_s\epsilon_1 V_{p_sq_s}X^{s1}_{q_sp_1}V_{p_1q_1} & \cdots & \epsilon_s^2 V_{p_sq_s}X^{ss}_{q_sp_s}V_{p_sq_s}\end{array}\right)$$

So, the equality $AXA=0$ implies that each of the above products, of the form $V_{p_iq_j}X^{jk}_{q_jp_k}V_{p_kq_k}$, is the zero matrix. Therefore, the result follows from Lemma 4. □

We analyze next the inner Bohemian inverses of $\pm\mathbf{1}$–pure well–settled matrices.

**Theorem 3.** *(Inner Bohemian inverses of $\pm\mathbf{1}$–pure well–settled matrices)*
*Let $\mathbb{P}\subset\mathbb{K}$, such that $\pm1,0\in\mathbb{P}$, and let*

$$A=\left(\begin{array}{c|c|c} \epsilon_1\mathbf{1}_{p_1q_1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \ddots & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \epsilon_s\mathbf{1}_{p_sq_s}\end{array}\right)\in\mathbb{K}^{m\times n}$$

*where $\epsilon_k\in\{-1,1\}$. It holds that*

$$A_{\mathscr{B}(\mathbb{P})}\{1\}=\left\{\left(\begin{array}{c|c|c} B^{11}_{q_1p_1} & \cdots & B^{1s}_{q_1,p_s}\\ \vdots & \ddots & \vdots\\ B^{s1}_{q_sp_1} & \cdots & B^{ss}_{q_sp_s}\end{array}\right)\in\mathbb{K}^{n\times m}\ \middle|\ \begin{array}{l} B^{ii}_{q_ip_i}\in(\epsilon_i\mathbf{1}_{p_iq_i})_{\mathscr{B}(\mathbb{P})}\{1\}\\ B^{ij}_{q_ip_j}\ \text{is}(q_j,0)\text{-balanced if} i\neq j\end{array}\right\}.$$

**Proof.** Let $X\in\mathscr{B}_{n\times m}(\mathbb{P})$ be expressed as

$$X=\left(\begin{array}{c|c|c} X^{11}_{q_1p_1} & \cdots & X^{1s}_{q_1p_s}\\ \vdots & \ddots & \vdots\\ X^{s1}_{q_sp_1} & \cdots & X^{ss}_{q_sp_s}\end{array}\right)\in\mathbb{K}^{n\times m},$$

Then, the equality $AXA=A$ can be written as

$$\left(\begin{array}{c|c|c} (\epsilon_1\mathbf{1}_{p_1q_1})X^{11}_{q_1p_1}(\epsilon_1\mathbf{1}_{p_1q_1}) & \cdots & \epsilon_1\epsilon_s\mathbf{1}_{p_1q_1}X^{1s}_{q_1p_s}\mathbf{1}_{p_sq_s}\\ \vdots & \ddots & \vdots\\ \epsilon_s\epsilon_1\mathbf{1}_{p_sq_s}X^{s1}_{q_sp_1}\mathbf{1}_{p_1q_1} & \cdots & (\epsilon_s\mathbf{1}_{p_sq_s})X^{ss}_{q_sp_s}(\epsilon_s\mathbf{1}_{p_sq_s})\end{array}\right)=\left(\begin{array}{c|c|c} \epsilon_1\mathbf{1}_{p_1q_1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \ddots & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \epsilon_s\mathbf{1}_{p_sq_s}\end{array}\right)$$

We have that $(\epsilon_i\mathbf{1}_{p_iq_i})X^{ii}_{q_ip_i}(\epsilon_i\mathbf{1}_{p_iq_i})=\epsilon_i\mathbf{1}_{p_iq_i}$. So, $X^{ii}_{q_ip_i}\in(\epsilon_i\mathbf{1}_{p_iq_i})_{\mathscr{B}(\mathbb{P})}\{1\}$. For $i\neq j$, we have that $\mathbf{1}_{p_iq_i}X^{ij}_{q_ip_j}\mathbf{1}_{p_jq_j}=\mathbf{0}_{p_iq_j}$. So, by Lemma 4, if $i\neq j$, $X^{ij}_{q_ip_j}$ is $(q_j,0)$-balanced. □

We analyze now the inner Bohemian matrices of $(\pm\mathbf{1}\ \mp\mathbf{1})$-pure well–settled matrices.

**Theorem 4 (Inner Bohemian inverses of. $(\pm\mathbf{1}\ \mp\mathbf{1})$-*pure well–settled matrices)***
*Let $\mathbb{P}\subset\mathbb{K}$, such that $\pm1,0\in\mathbb{P}$, let $V_{p_iq_i}=\left(\begin{array}{c|c}\mathbf{1}_{p_iq_{i1}} & -\mathbf{1}_{p_iq_{i2}}\end{array}\right)$, where $p_{i1}+q_{i2}=q_i$, and let*

$$A=\left(\begin{array}{c|c|c} \epsilon_1 V_{p_1q_1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \ddots & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \epsilon_s V_{p_sq_s}\end{array}\right)\in\mathbb{K}^{m\times n},$$

*where $\epsilon_k\in\{-1,1\}$. It holds that*

$$A_{\mathbb{P}}\{1\}=\left\{\begin{pmatrix} B^{11}_{q_1p_1} & \cdots & B^{1s}_{q_1,p_s}\\ \vdots & \ddots & \vdots\\ B^{s1}_{q_sp_1} & \cdots & B^{ss}_{q_sp_s}\end{pmatrix}\in\mathbb{K}^{n\times m}\ \middle|\ \begin{array}{c} B^{ii}_{q_ip_i}\in(\epsilon_iV_{p_iq_i})_{\mathcal{B}(\mathbb{P})}\{1\}\\ B^{ij}_{q_ip_j}\ \text{is}\ (q_{i1},q_{i2})-\text{balanced if}\ i\neq j\end{array}\right\}.$$

**Proof.** The proof is analogous to the proof of Theorem 3 using Lemma 6. □

In the next theorem we analyze the case of mixed well–settled matrices.

**Theorem 5.** *(Inner Bohemian inverses of mixed well–settled matrices)*
*Let $\mathbb{P}\subset\mathbb{K}$, such that $\pm1,0\in\mathbb{P}$, and let*

$$A=\left(\begin{array}{c|c|c} M_{p_1q_1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \ddots & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & M_{p_sq_s}\end{array}\right)\in\mathbb{K}^{m\times n}$$

*be a mixed well–settled matrix. Then,* $A_{\mathscr{B}(\mathbb{P})}\{1\}$ *is the set*

$$\left\{ \begin{pmatrix} B^{11}_{q_1 p_1} & \dots & B^{1s}_{q_1, p_s} \\ \vdots & \ddots & \vdots \\ B^{s1}_{q_s p_1} & \dots & B^{ss}_{q_s p_s} \end{pmatrix} \in \mathbb{K}^{n\times m} \;\middle|\; B^{ij}_{q_i p_j} \text{is} \begin{cases} B^{ii}_{q_i p_i} \in (M_{p_i q_i})_{\mathcal{B}(\mathbb{P})}\{1\} \\ (q_i, 0) - \text{balanced} \\ \text{if } M_{p_i q_i} = \pm \mathbf{1}_{p_i q_i} \\ (q_{i1}, q_{i2}) - \text{balanced} \\ \text{if } M_{p_i q_i} = \pm \begin{pmatrix} \mathbf{1}_{p_i q_{i1}} & -\mathbf{1}_{p_i q_{i2}} \end{pmatrix}. \end{cases} \right\}.$$

**Proof.** Let $X \in \mathscr{B}_{n\times m}(\mathbb{P})$ be expressed as in the statement of Lemma 5. Then, the equality $AXA = A$ can be written as

$$\left( \begin{array}{c|c|c} M_{p_1 q_1} X^{11}_{q_1 p_1} M_{p_1 q_1} & \dots & M_{p_1 q_1} X^{1s}_{q_1 p_s} M_{p_s q_s} \\ \vdots & \ddots & \vdots \\ M_{p_s q_s} X^{s1}_{q_s p_1} M_{p_1 q_1} & \dots & M_{p_s q_s} X^{ss}_{q_s p_s} M_{p_s q_s} \end{array} \right) = \left( \begin{array}{c|c|c} M_{p_1 q_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_{p_s q_s} \end{array} \right)$$

We have that $M_{p_i q_i} X^{ii}_{q_i p_i} M_{p_i q_i} = M_{p_i q_i}$. So, $X^{ii}_{q_i p_i} \in (M_{p_i q_i})_{\mathscr{B}(\mathbb{P})}\{1\}$. For $i \neq j$, we have that $M_{p_i q_i} X^{ij}_{q_i p_j} M_{p_j q_j} = \mathbf{0}_{p_i q_j}$. So, by Lemma 4, if $i \neq j$,

$$X^{ij}_{q_i p_j} \text{is} \begin{cases} (q_i, 0) - \text{balanced} & \text{if } M_{p_i q_i} = \mathbf{1}_{p_i q_i} \text{ and } M_{p_j q_j} = \mathbf{1}_{p_j q_j} \\ (q_i, 0) - \text{balanced} & \text{if } M_{p_i q_i} = \mathbf{1}_{p_i q_i} \text{ and } M_{p_j q_j} = \begin{pmatrix} \mathbf{1}_{p_j q_{j1}} & -\mathbf{1}_{p_j q_{j2}} \end{pmatrix} \\ (q_{i1}, q_{i2}) - \text{balanced} & \text{if } M_{p_i q_i} = \begin{pmatrix} \mathbf{1}_{p_i q_{i1}} & -\mathbf{1}_{p_i q_{i2}} \end{pmatrix} \text{ and } M_{p_j q_j} = \mathbf{1}_{p_j q_j} \\ (q_{i1}, q_{i2}) - \text{balanced} & \text{if } M_{p_i q_i} = \begin{pmatrix} \mathbf{1}_{p_i q_{i1}} & -\mathbf{1}_{p_i q_{i2}} \end{pmatrix} \text{ and } M_{p_j q_j} = \begin{pmatrix} \mathbf{1}_{p_j q_{j1}} & -\mathbf{1}_{p_j q_{j2}} \end{pmatrix} \end{cases}$$

□

In the following, we analyze the cardinality of the set of inner Bohemian matrices of well–settled matrices, when the population is $\{0, \pm 1\}$. Let us first state some technical properties.

**Lemma 7.**

1. $\# \{X \in \mathscr{B}_{p\times q}(\{0, \pm 1\}) \,|S(X) = 0\} = \displaystyle\sum_{i=0}^{pq} \binom{pq}{i} \binom{pq - i}{i}.$
2. $\# \left\{X \in \mathscr{B}_{(p_1+p_2)\times q}(\{0, \pm 1\}) \;\middle|\; X \text{is} (p_1, p_2) - \text{balanced}\right\} =$
   $\displaystyle\sum_{k=-qt}^{qt} \sum_{i=0}^{qT} \left( \sum_{j=-qt}^{qt} \binom{qt}{j} \binom{qt - j}{k + j} \right) \binom{qT}{i} \binom{qT - i}{k + i},$
   *where* $t = \min\{p_1, p_2\}$ *and* $T = \max\{p_1, p_2\}$.

**Proof.** (1) Let $i \in \{0, \dots, pq\}$ denote the number of 1's in $X$. For a fixed $i \in \{0, \dots, pq\}$ there are $\binom{pq}{i}$ different matrices $X$ with exactly $i$ entries equal to 1. For each of these matrices $X$, since $S(X) = 0$, the number of $-1$'s has to be $pq - i$. So, for each of these matrices we have $\binom{pq-i}{i}$ ways of choosing the matrix entries being equal to $-1$. So, the cardinality of the set is

$$\sum_{i=0}^{pq} \binom{pq}{i} \binom{pq - i}{i}.$$

(2) We assume w.l.o.g. that $t = p_1$ and $T = p_2$, and let $X$ be expressed as

$$X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}$$

where $X^i$ is a $p_i \times q$ matrix. Let $k = S(X^1)$. Clearly, $k \in \{-tq, \dots, tq\}$. Moreover, since $t \leq T$, for each $X^1 \in \mathscr{B}_{p_1\times q}(\{0, \pm 1\})$ there exist $\alpha(k) > 0$ matrices $X^2 \in \mathscr{B}_{p_2\times q}(\{0, \pm 1\})$, such that $S(X^1) = S(X^2)$. Let us compute $\alpha(k)$. Let $i \in \{0, \dots, qT\}$ denote the number of $-1$'s in $X^2$. Then, since $S(X^2) = k$, then the number of 1's in $X^2$ has to be $k + i$. Now, we have that

$$\alpha(k) = \sum_{i=0}^{Tq} \binom{qT}{i} \binom{qT - i}{k + i}.$$

Furthermore, the number of matrices $X^2$ such that $S(X^2) = k$ for a fixed $k$, is

$$\sum_{j=-qT}^{j=qT} \binom{qT}{j} \binom{qT - j}{k + j}.$$

Combining all the previous, the result follows. □

Using the previous results we can now give formulas for the cardinality of the $\{0, \pm 1\}$–inner Bohemian matrices of well–settled matrices. For this purpose we introduce the following notation

1. $F_1(m,n) = \#((\mathbf{1}_{mn})_{\mathscr{B}(\mathbb{P})}\{1\})$ (see Corollary 2 (1)).
2. $F_2(m,n_1,n_2) = \#\left(\left(\begin{array}{c|c} \mathbf{1}_{mn_1} & -\mathbf{1}_{mn_2} \end{array}\right)_{\mathscr{B}(\mathbb{P})}\{1\}\right)$ (see Corollary 2 (2)).
3. $F_3(p,q) = \#\{X \in \mathscr{B}_{p\times q}(\{0,\pm1\})\,|S(X)=0\}$ (see Lemma 7 (1)).
4. $F_4(p_1,p_2,q) = \#\left\{X \in \mathscr{B}_{(p_1+p_2)\times q}(\{0,\pm1\}) \,\middle|\, X\text{is}(p_1,p_2)-\text{balanced}\right\}$ (see Lemma 7 (2)).

In this situation, one has the next result.

**Corollary 3.**

1. *Let*

$$A = \left(\begin{array}{c|c|c} \epsilon_1\mathbf{1}_{p_1q_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \epsilon_s\mathbf{1}_{p_sq_s} \end{array}\right) \in \mathbb{K}^{m\times n}$$

*where $\epsilon_k \in \{-1,1\}$. The cardinality of $A_{\mathscr{B}(\{0,\pm1\})}\{1\}$ is*

$$\prod_{k=1}^{s} F_1(p_k,q_k) \prod_{\substack{k_1,k_2\in\{1,\ldots,s\}\\ k_1\neq k_2}} F_3(p_{k_1},p_{k_2}).$$

2. *Let $V_{p_iq_i} = \left(\begin{array}{c|c} \mathbf{1}_{p_iq_{i1}} & -\mathbf{1}_{p_iq_{i2}} \end{array}\right)$, where $q_{i1}+q_{i2}=q_i$, and let*

$$A = \left(\begin{array}{c|c|c} \epsilon_1 V_{p_1q_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \epsilon_s V_{p_sq_s} \end{array}\right) \in \mathbb{K}^{m\times n},$$

*where $\epsilon_k \in \{-1,1\}$. The cardinality of $A_{\mathscr{B}(\{0,\pm1\})}\{1\}$ is*

$$\prod_{i=1}^{s} F_2(p_i,q_{i1},q_{i2}) \prod_{\substack{k_1,k_2\in\{1,\ldots,s\}\\ k_1\neq k_2}} F_4(q_{k_11},q_{k_12},p_{k_2})$$

3. *Let*

$$A = \left(\begin{array}{c|c|c} M_{p_1q_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_{p_sq_s} \end{array}\right) \in \mathbb{K}^{m\times n}$$

*be a mixed well–settled matrix. Let $I,J \subset \mathbb{N}$ with $I\cup J = \{1,\ldots,s\}$, $I\cap J = \emptyset$, and such that $M_{p_iq_i} = \pm\mathbf{1}$ if $i\in I$ and $M_{p_iq_i} = \pm\left(\begin{array}{c|c} \mathbf{1}_{p_iq_{i1}} & -\mathbf{1}_{p_iq_{i2}} \end{array}\right)$ if $i \in J$. Then, the cardinality of $A_{\mathscr{B}(\{0,\pm1\})}\{1\}$ is*

$$\prod_{i\in I} F_1(p_i,q_i) \prod_{j\in J} F_2(p_i,q_{i1},q_{i2}) \prod_{\substack{k_1,k_2\in I\\ k_1\neq k_2}} F_3(q_{k_1},p_{k_2}) \prod_{\substack{k_1,k_2\in J\\ k_1\neq k_2}} F_4(q_{k_11},q_{k_12},p_{k_2})$$

$$\prod_{k_1\in I,k_2\in J} F_3(q_{k_1},p_{k_2}) \prod_{k_1\in J,k_2\in I} F_4(q_{k_11},q_{k_12},p_{k_2})$$

**Proof.** Statement (1a) follows from Theorem 3 (1) and Corollary 2. Statement (1b) follows from Theorem 3 (2), Corollary 2, and Lemma 7 (1). Statement (2) follows from Theorem 4, Corollary 2, and Lemma 7 (2). Statement (3) follows from Theorem 5, Corollary and Lemma 7. □

We finish this section with a table (see Table 3) where we see how large the cardinality of the inner Bohemian inverses is. The third column corresponds to the number of $2n\times(2n-2)$ Bohemian matrices with population $\{0,\pm1\}$, namely, $3^{4n(n-1)}$. Note that the numbers in the second column of the Table 3 come from Corollary 3 (1).

## 5. Conclusions

In this paper we study the problem of describing the set of all inner Bohemian inverses of a given Bohemian matrix. For this purpose, as a first step in this line of research, we consider certain types of matrices, which we call full matrices (see Definition 1), as well as diagonal block matrices of full matrices (see Definition 2). One may distinguish the following main contributions of our paper:

1. For the class of full matrices we give a complete description of the set of all inner Bohemian inverses independently of the population (see Theorem 2). The key for proving this result is Theorem 1. More precisely, if $A \in \mathbb{K}^{m\times n}$, $A\{1\}$ can be seen as a linear affine space of dimension $mn - \operatorname{rank}(A)$ (see e.g. Stanimirović et al. [37]). Therefore, if $\operatorname{rank}(A) = 1$, which is the case of a full matrix, $A\{1\}$ is a hyperplane. In addition, due to the particular form of the full matrices, Theorem 1 provides a parametric representation, and indeed the implicit representation, of this space. Using this knowledge, we derive the description of the inner Bohemian set.
2. For the class of well–settled matrices we give a complete description of the set of all inner Bohemian inverses independently of the population (see Theorem 2–5). The key idea for this result is the notion of balanced matrix (see Definition 3) in combination with the fact that the inner inverse of the block-diagonal matrix has a good behavior.
3. For the particular case of the population $\{0, 1, -1\}$, we give exact formulas for the cardinality of the inner Bohemian sets of full and well–settled matrices (see Corollaries 2 and 3). The idea here is the diophantine study of the implicit linear equation of the hyperplane of the inner matrices and of the balanced matrices.

In addition, this paper opens a wide area of open problems to be studied. One may mention, among others:

1. The computation of the cardinality of the inner Bohemian inverses set for other populations different from $\{0, 1, -1\}$; this would imply the study of other linear diophantine equations.
2. The study of inner Bohemian inverses of different type; one may start with rank 1 Bohemian matrices to preserve the dimension of the linear space.
3. Extension of these results to other generalized inverses, such as outer inverses, i.e. matrices satisfying the second axiom, namely $XAX = X$; this would imply to work with affine algebraic varieties instead of linear varieties.
4. Extension of these results to generalized inverses with prescribed image/range.

## Acknowledgments

R.M. Corless is partially supported by NSERC, grant number 2020-06438. R.M. Corless, L. González-Vega, J.R. Sendra and J. Sendra are partially supported by the grant PID2020-113192GB-I00 (Mathematical Visualization: Foundations, Algorithms and Applications) from the Spanish MICINN.

## References

[1] W. Barrett, S. Butler, H.T. Hall, Symmetric 0-1 matrices with inverses having two distinct values and constant diagonal, Linear Algebra Appl. 491 (2016) 41–55.
[2] A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, second ed., Springer, New York, 2003.
[3] P. Borwein, L. Jorgenson, Visible structures in number theory, Am. Math. Mon. 108 (2001) 897–910.
[4] P. Borwein, C. Pinner, Polynomials with $\{0, +1, -1\}$ coefficients and a root close to a given point, Can. J. Math. 49 (1997) 887–915.
[5] C. Briat, Sign properties of Metzler matrices with applications, Linear Algebra Appl. 515 (2017) 53–86.
[6] S.L. Campbell, C.D. Meyer, Generalized inverses of linear transformations, Series: Classics in Applied Mathematics, SIAM, 2009.
[7] E.Y.S. Chan, A comparison of solution methods for mandelbrot-like polynomials, Electronic Thesis and Dissertation Repository, The University of Western Ontario, 2016. https://ir.lib.uwo.ca/etd/4028
[8] E.Y.S. Chan, R.M. Corless, L. Gonzalez-Vega, J.R. Sendra, J. Sendra, Algebraic linearizations of matrix polynomials, Linear Algebra Appl. 563 (2019) 373–399.
[9] E.Y.S. Chan, R.M. Corless, L. Gonzalez-Vega, J.R. Sendra, J. Sendra, S.E. Thornton, Upper Hessenberg and Toeplitz Bohemians, Linear Algebra Appl. 601 (2020) 72–100.
[10] K.L. Chu, S.W. Drury, S.W. Drury, Magic Moore–Penrose inverses and philatelic magic squares with special emphasis on the daniels-zlobec magic square, Croatian Operational Research Review (CRORR), vol. 2, 2011.
[11] R.M. Corless, Generalized companion matrices in the lagrange basis, in: Proceedings EACA, Universidad de Cantabria, Santander, Spain, 2004, pp. 317–322.
[12] D. Christensen, Plots of roots of polynomials with integer coefficients, http://jdc.math.uwo.ca/roots/. Accessed: 2016-06–25.
[13] R.M. Corless, S.E. Thornton, Visualizing eigenvalues of random matrices, ACM Commun. Comput. Algebra 50 (2016) 35–39, doi:10.1145/2930964.2930969.
[14] R.M. Corless, S.E. Thornton, The Bohemian eigenvalue project, ACM Commun. Comput. Algebra 50 (2016) 158–160.
[15] M.P. Drazin, A class of outer generalized inverses, Linear Algebra Appl. 436 (2012) 1909–1923.
[16] S.M. Elzaid, On the magic square and inverse, Int. J. Math. Educ. Sci. Technol. 36 (6) (2005) 673–680.
[17] J. Guyker, Magic squares with magic inverses, Int. J. Math. Educ. Sci. Technol. (2007) 683–688.
[18] M. Fasi, G.M.N. Porzio, Determinants of normalized Bohemian upper Hessenberg matrices, Electron. J. Linear Algebra 36 (2020) 352–366. ISSN 1081-3810. A publication of the International Linear Algebra Society
[19] J. Feng, H. Fan, A study on the determinants of some Hessenberg Toeplitz Bohemians, Preprints 2020, 202008027210.20944/preprints202008.0272.v1.
[20] F.J. Hall, Z. Li, Sign pattern matrices, in: L. Hogben (Ed.), Handbook of Linear Algebra, Chapman and Hall/CRC, 2013. (ch. 42)
[21] K.J. Horadam, Hadamard Matrices and Their Applications, Princeton University Press, 2007.
[22] Y. Ke, J. Chen, P.S. Stanimirović, M. Ćirić, Characterizations and representations of outer inverse for matrices over a ring, Linear Multilinear Algebra (2019) 1–22.
[23] J.E. Littlewood, On polynomials $\sum^n \pm z^m, \sum^n e^{\alpha_m i} z^m, z = e^{\theta i}$, J. Lond. Math. Soc. 41 (1966) 367–376.
[24] W. Lu, W. Li, K. Kpalma, J. Ronsin, Compressed sensing performance of random Bernoulli matrices with high compression ratio, IEEE Signal Process. Lett., Inst. Electr. Electron. Eng. 22 (8) (2015) 1074–1078.
[25] X. Martínez-Rivera, On 01 matrices whose inverses have entries of the same modulus, Linear Algebra Appl. 622 (2021) 333–354.
[26] N.J. Calkin, E.Y.S. Chan, R.M. Corless, Some facts and conjectures about mandelbrotpolynomials, Maple Trans. 1 (2021). Article 1
[27] A.M. Odlyzko, B. Ponnen, Zeros of polynomials with 0,1 coefficients, in: B. Salvy (Ed.), Algorithms Seminar, vol. 2130, 1993, pp. 169–172.
[28] K.P.S.B. Rao, Theory of generalized inverses over commutative rings, Algebra, Logics and Applications, vol. 17, Taylor & Francis, 2005.
[29] H.J. Ryser, Matrices of zeros and ones, Bull. Amer. Math. Soc. 66 (6) (1960) 442–464.
[30] J. Sendra, Bohemian matrices: past, present and future, in: R. Corless, J. Gerhard, I. Kotsireas (Eds.), Maple in Mathematics Education and Research. Communications in Computer and Information Science, CCIS, vol 1125, Springer Nature, Switzerland AG, 2020, pp. 3–16, doi:10.1007/978-3-030-41258-6.

[31] J.R. Sendra, J. Sendra, Gröbner basis computation of Drazin inverses with multivariate rational function entries, J. Comput. Appl. Math. 259 (2015) 450–459.
[32] J.R. Sendra, J. Sendra, Symbolic computation of Drazin inverses by specializations, J. Comput. Appl. Math. 301 (2016) 201–212.
[33] J.R. Sendra, J. Sendra, Computation of Moore–Penrose generalized inverses of matrices with meromorphic function entries, Appl. Math. Comput. 313 (2017) 355–366.
[34] N.J.A. Sloane, The on-line encyclopedia of integer sequences, Published electronically at https://oeis.org.
[35] P.S. Stanimirović, R. Behera, J.K. Sahoo, D. Mosic, J.R. Sendra, J. Sendra, A. Lastra, Computing tensor generalized inverses via specialization and rationalization, Rev. Real Acad. Cienc. Exactas, Fís. Nat. Ser. A 115 (116) (2021) 1–16.
[36] P.S. Stanimirović, M. Ćirić, A. Lastra, J.R. Sendra, J. Sendra, Representations and symbolic computation of generalized inverses over fields, Appl. Math. Comput. 406 (2021) 126287.
[37] P.S. Stanimirović, M. Ćirić, A. Lastra, J.R. Sendra, J. Sendra, Representations and geometrical properties of generalized inverses over fields, Linear and Multilinear Algebra (2021) 10.1080/03081087.2021.1985420 (accepted).
[38] P.S. Stanimirović, M. Ćirić, I. Stojanović, D. Gerontitis, Conditions for existence, representations and computation of matrix generalized inverses, Complexity (2017). Article ID 6429725
[39] O. Taussky, Matrices of rational integers, Bull. Am. Math. Soc. 66 (1960) 327–345.
[40] O. Taussky, Some computational problems involving integral matrices, J. Res. Natl. Bur. Stand. B 65 (1961) 15–17.
[41] S.E. Thornton, The characteristic polynomial database, available at http://bohemianmatrices.com/cpdb.
[42] S.E. Thornton, Algorithms for Bohemian matrices, Electronic Thesis and Dissertation Repository. The University of Western Ontario, 2019. https://ir.lib.uwo.ca/etd/6069/
[43] K. Tikhomirov, Singularities of random Bernoulli matrices, Ann. Math. 191 (2) (2020) 593–634.
[44] N.S. Urquhart, Computation of generalized inverse matrices which satisfy specified conditions, SIAM Rev. 10 (1968) 216–218.
[45] N.S. Urquhart, The nature of the lack of uniqueness of generalized inverse matrices, SIAM Rev. 11 (2) (1969) 268–271.
[46] G. Wang, Y. Wei, S. Qiao, Generalized inverses: theory and computations, Developments in Mathematics, vol. 53, Singapore: Springer; Beijing: Science Press, 2018.