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Computing tensor generalized inverses via specialization and rationalization

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Abstract

In this paper, we introduce the notion of outer generalized inverses, with predefined range and null space, of tensors with rational function entries equipped with the Einstein product over an arbitrary field, of characteristic zero, with or without involution. We assume that the involved tensor entries are rational functions of unassigned variables or rational expressions of functional entries. The research investigates the replacements in two stages. The lower-stage replacements assume replacements of unknown variables by constant values from the field. The higher-order stage assumes replacements of functional entries by unknown variables. This approach enables the calculation on tensors over meromorphic functions to be simplified by analogous calculations on matrices whose elements are rational expressions of variables. In general, the derived algorithms permit symbolic computation of various generalized inverses which belong to the class of outer generalized inverses, with prescribed range and null space, over an arbitrary field of characteristic zero. More precisely, we focus on a few algorithms for symbolic computation of outer inverses of matrices whose entries are elements of a field of characteristic zero or a field of meromorphic functions in one complex variable over a connected open subset of $\mathbb{C}$. Illustrative numerical results validate the theoretical results.

Keywords: Tensor; Einstein product; Tensors of functions; outer inverse; Meromorphic functions; Symbolic computation;

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1 Basic facts and motivation

Tensors have been used in numerous research areas, for example in signal processing, machine learning [13, 31], or chemistry [18]. Theoretical investigations have considered many aspects of the tensor theory as eigenvalues of tensors [5, 25], spectral analysis of hypergraphs [10, 11], or tensor eigenvalues [7, 8]. Also, popular applications of tensors include: solving tensor equations [3] and solving multilinear systems [38]. The monograph [24] surveyed diverse applications of tensors in scientific computation. Perturbation bounds of tensor eigenvalues and singular values were investigated in [9]. The tensor Moore-Penrose inverse

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was used in tensor regression analysis [13] as well as in finding least-squares solutions to tensor equations [17].

The central motivation for our research dates back from the results derived in the matrix case. In [28], the authors proposed an algorithm to reduce the computation of the Drazin inverse over certain computable fields, whose entries are rational functions of finitely many transcendental elements over a complex field, into the computation of the Drazin inverse of matrices with rational elements. As a consequence, a symbolic algorithm to compute the Drazin inverse of matrices, whose entries are elements of a finite transcendental field extension of a computable field, was derived. The main idea is replacing the involved functions by new variables. Then, the generalized inverses computation with rational functional entries is reduced to the case of matrices with rational expressions. In addition, a Gröbner basis approach for computing the Drazin inverse has been proposed in [6, 30]. The treatment of generalized inverses with rational function entries has also been extended to the case of the Moore–Penrose inverse, by introducing a suitable involutory automorphism (see [29]). This analysis has also been extended to the case of (B, C)-generalized inverses of matrices with rational functional entries (see [34]). The underlining common main idea of all these methods is to define, and analyze, a suitable polynomial that controls all denominators appearing in all steps of the algorithmic treatment of each of the cases commented above. This polynomial is then used to guarantee the reduction of the problem to the case of matrices with rational functions entries instead of rational functional entries.

In this paper, we see how the ideas above, mainly those in [29], [30], [34], can be extended to the case of tensors. For this purpose, in the sequel, we fix $(\mathbb{K}, \varphi)$, where $\mathbb{K}$ is a field of characteristic zero, and φ is an involutory automorphism of $\mathbb{K}$ that will generalize to $\mathbb{K}$ the notion of conjugation in $\mathbb{C}$. We recall that an automorphism φ is involutory if $\varphi \circ \varphi$ is the indentity map; see [29] for further details. When $\mathbb{K} = \mathbb{C}$ we take φ as the conjugation of complex numbers. The polynomial ring over $\mathbb{K}$ will be denoted by $\mathbb{K}[\mathbf{x}]$, where $\mathbf{x} = (x_1, \ldots, x_p)$ are unknown variables, while $\mathbb{K}(\mathbf{x})$ denotes the field of rational functions; note that φ can be naturally extended to an involutory automorphism over $\mathbb{K}(\mathbf{x})$, see e.g. [29]. In addition, let $\mathbb{K}^{m\times n}$ denote the set of all $m \times n$ matrices over $\mathbb{K}$. Throughout the paper, when we refer to $\mathbb{K}$, one must take into account that $\mathbb{K}$ can be replaced by $\mathbb{K}(\mathbf{x})$, or by any field (algebraic or transcendental) extension of $\mathbb{K}$. Nevertheless, in some parts of the paper, when we need to emphasize that we are working with rational functions we will use the corresponding notation, namely $\mathbb{K}(\mathbf{x})$.

An order $N > 0$ tensor $\mathcal{A} = (a_{i_1,i_2,\ldots,i_N})_{1\le i_j \le I_j}, (j = 1, \ldots, N)$ is a multidimensional array with $\mathfrak{I} = I_1 \cdots I_N$ entries, where $I_1, \ldots, I_N$ are positive integers. Let us denote by $\mathbb{K}^{I_1\times\cdots\times I_N}$ the set of the order N-dimensional $I_1 \times \cdots \times I_N$ tensors over $\mathbb{K}$.

Given $\mathcal{A} = (a_{i_1,\ldots,i_M,j_1,\ldots,j_N}) \in \mathbb{K}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$, we introduce the transpose of $\mathcal{A}$, denoted by $\mathcal{A}^{\mathrm{T}}$, as the tensor $\mathcal{B} = (b_{j_1,\ldots,j_N,i_1,\ldots,i_M}) \in \mathbb{K}^{J_1\times\cdots\times J_N\times I_1\times\cdots\times I_N}$, where $b_{j_1,\ldots,j_N,i_1,\ldots,i_M} = a_{i_1,\ldots,i_M,j_1,\ldots,j_N}$. Similarly, we denote by $\varphi(\mathcal{A})$ the tensor $(\varphi(a_{i_1,\ldots,i_M,j_1,\ldots,j_N})) \in \mathbb{K}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$. Also, $\varphi(\mathcal{A})^{\mathrm{T}}$ is the φ-conjugate transpose of $\mathcal{A}$, that we will denote by $\mathcal{A}^*$.

The tensor Einstein product is defined by the associative operation $*_N$ via

$$(\mathcal{A} *_N \mathcal{B})_{i_1\ldots i_N j_1\ldots j_M} = \sum_{k_1\ldots k_N} a_{i_1\ldots i_N k_1\ldots k_N} b_{k_1\ldots k_N j_1\ldots, j_M}, \tag{1.1}$$

where $\mathcal{A} \in \mathbb{K}^{I_1\times\cdots\times I_N\times K_1\times\cdots\times K_N}$, $\mathcal{B} \in \mathbb{K}^{K_1\times\cdots\times K_N\times J_1\times\cdots\times J_M}$, and $\mathcal{A} *_N \mathcal{B} \in \mathbb{K}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_M}$.

The null spaces and the ranges of tensors were introduced in [14]. Here, we recall the notion for arbitrary fields.

Definition 1.1. [14] *For $\mathcal{T} \in \mathbb{K}^{I_1\times\cdots\times I_N\times K_1\times\cdots\times K_N}$, the range $\mathbb{R}(\mathcal{T})$ and the null space $\mathbb{N}(\mathcal{T})$ of $\mathcal{T}$ are defined as*

$$\mathbb{R}(\mathcal{T}) = \{\mathcal{Y} \in \mathbb{K}^{I_1\times\cdots\times I_N} : \ \mathcal{Y} = \mathcal{T} *_N \mathcal{X}, \ \mathcal{X} \in \mathbb{K}^{K_1\times\cdots\times K_N}\},$$

$$\mathbb{N}(\mathcal{T}) = \{\mathcal{Z} \in \mathbb{K}^{K_1\times\cdots\times K_N} : \mathcal{T} *_N \mathcal{Z} = \mathcal{O}\},$$

where $\mathcal{O}$ is a proper zero tensor. For the case of a matrix M we denote by $\mathcal{R}(M)$ and $\mathcal{N}(M)$ the range and kernel of M, respectively.

The following additional notation will be used to improve and clarify our presentation:

$$M(k) = M_1 \times \cdots \times M_k, \quad i(k) = i_1, \ldots, i_k,$$

where M_s,i_s are positive integers for all $1 \le s \le k$. Using these notations, we can represent the tensor $\mathcal{A} \in \mathbb{K}^{M_1\times\cdots\times M_m\times N_1\times\cdots\times N_n}$ and its elements by $\mathcal{A}^{M(m)\times N(n)}$, and $a_{i(m),j(n)}$, respectively. This paper is devoted to the development of computational algorithms for the determination of various types of outer inverses of the tensor $\mathcal{A} \in \mathbb{K}^{M(m)\times N(n)}$.

In the next definition, we restate the notion of the tensor Moore-Penrose inverse.

Definition 1.2. *A tensor $\mathcal{Z} \in \mathbb{K}^{N(n)\times M(m)}$ which satisfies*

$$\begin{array}{llll} (1^T) & \mathcal{T} *_n \mathcal{Z} *_m \mathcal{T} = \mathcal{T}, & (2^T) & \mathcal{Z} *_m \mathcal{T} *_n \mathcal{Z} = \mathcal{Z}, \\ (3^T) & (\mathcal{T} *_n \mathcal{Z})^* = \mathcal{T} *_n \mathcal{Z}, & (4^T) & (\mathcal{Z} *_m \mathcal{T})^* = \mathcal{Z} *_m \mathcal{T}, \end{array}$$

is called the Moore-Penrose (M-P) inverse of $\mathcal{T} \in \mathbb{K}^{M(m)\times N(n)}$, and is denoted by $\mathcal{T}^{\dagger}$.

It is well-know that the Moore-Penrose inverse of a matrix, over the field of complex numbers, always exists and is unique. For the case of tensors, in [37] the authors prove that this inverse also exists and is unique when $\mathbb{K} = \mathbb{C}$. However, for the general case of arbitrary fields, with an endowed involutory automorphism, the situation is not so direct. In [29], it is shown that the existence and uniqueness of the Moore-Penrose inverse depend on the field, and on the involutory automorphism. Moreover, in [29] conditions for the existence are shown and the notion of Moore-Penrose field is introduced, which is a field such that all matrices with entries in this field have Moore-Penrose inverse. Using similar reasonings as those in [29] the existence and uniqueness of Moore-Penrose inverses of tensors over an arbitrary field can be analyzed. Nevertheless, we do not delve into this topic here.

In Definition 1.3 we recall the notion of Drazin inverse of a tensor. For this purpose, the concept of Drazin index is introduced by using the notion of rank of a tensor via its unfolding as a matrix (see [20] or Subsection 2.1).

Definition 1.3. *The Drazin inverse $\mathcal{T}^{\mathrm{D}}$ of a square even-order tensor $\mathcal{T} \in \mathbb{K}^{N(n)\times N(n)}$ of index* $\mathrm{ind}(\mathcal{A}) = k$ *is the tensor $\mathcal{Z} \in \mathbb{K}^{N(n)\times N(n)}$ which satisfies (2^T) in conjunction with*

$$\begin{array}{llll} (1^{Tk}) & \mathcal{T}^{k+1} *_n \mathcal{Z} = \mathcal{T}^k, & (5^T) & \mathcal{T} *_n \mathcal{Z} = \mathcal{Z} *_n \mathcal{T}. \end{array}$$

In the case $\mathrm{ind}(\mathcal{A}) = 1$*, the Drazin inverse $\mathcal{T}^{\mathrm{D}}$ becomes the group inverse $\mathcal{T}^{\#}$.*

Tensor inversion, as well as solutions of tensor equations, were investigated in [3, 38, 21]. The definition of the M-P inverse introduced in [1, 37] was generalized to arbitrary order complex tensors in [20, 32]. Various properties and representations of the tensor M-P inverse were derived in [1, 37], and of the tensor weighted M-P inverse were investigated in [14]. Further representations of various outer generalized inverses for complex tensors were given in [32]. In [23], the authors investigated the reverse-order law of arbitrary tensors. The tensor Drazin inverse of even-order square tensors, with the underlying Einstein product, was studied in [2, 15], while the Drazin inverse solution of the singular tensor equation $\mathcal{A} * \mathcal{X} = \mathcal{B}$, such that $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_k\times I_1\times\cdots\times I_k}$ is singular and $\mathcal{B}, \mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_k}$ was, studied in [15]. Sun *et al.* in [38] defined the $\{i\}$-inverse $(i = 1, 2, 5)$ as well as the group inverse of tensors. The core, the core-EP, the DMP, and the CMP tensor generalized inverses were investigated in [12, 26, 40].

If the equation (i^{T}) of the above equations $(1^T) - (4^T)$ holds, then $\mathcal{Z}$ is called an $\{i\}$-inverse of $\mathcal{T}$, and it is denoted by $\mathcal{T}^{(i)}$. The set of all $\{i\}$-inverses of $\mathcal{T}$ is denoted by $\mathcal{T}\{i\}$. Particularly, $\{2\}$-inverses (resp. $\{1\}$-inverses) are known as outer (resp. inner) inverses. A tensor $\mathcal{Z} \in \mathcal{T}\{S\}$, whose range and kernel satisfy $\mathbb{R}(\mathcal{Z}) = \mathbb{R}(\mathcal{B})$ (resp. $\mathbb{N}(\mathcal{Z}) = \mathbb{N}(\mathcal{C})$), is denoted by $\mathcal{T}^{(S)}_{\mathbb{R}(\mathcal{B}),*}$ (resp. $\mathcal{T}^{(S)}_{*,\mathbb{N}(\mathcal{C})}$). Consequently, $\mathcal{Z} \in \mathcal{T}\{S\}$, satisfying both $\mathbb{R}(\mathcal{Z}) = \mathbb{R}(\mathcal{B})$ and $\mathbb{N}(\mathcal{Z}) = \mathbb{N}(\mathcal{C})$, is termed as $\mathcal{T}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}$.

Recently, Ji and Wei in [16] considered conditions for the existence and various representations of the outer inverse of even-order tensors under the Einstein tensor product, and studied appropriate folding and unfolding operations. Several representations of outer inverses, as well as an efficient computational procedure for their determination, were presented in [27]. It is a known fact that outer inverses involve the Moore-Penrose inverse, the Drazin, and the group inverse: $\mathcal{T}^{(2)}_{\mathbb{R}(\mathcal{A}^*),\mathbb{N}(\mathcal{A}^*)} = \mathcal{T}^{\dagger}$, $\mathcal{T}^{(2)}_{\mathbb{R}(\mathcal{A}),\mathbb{N}(\mathcal{A})} = \mathcal{T}^{\#}$, and $\mathcal{T}^{(2)}_{\mathbb{R}(\mathcal{A}^k),\mathbb{N}(\mathcal{A}^k)} = \mathcal{T}^{\mathrm{D}}$. This fact is an additional motivation for investigating outer inverses.

The main results of this manuscript can be highlighted as follows.

(i) Continuing the results from [16], in this paper we give additional representations for outer inverses of tensors with known range and/or null space over fields of characteristic zero. The representations of outer inverses in [16] are derived on the basis of the group inverse or using the tensor full rank factorization with the underlying folding and unfolding operations. Our intention in the current research is to obtain generalizations of known Urquhart representations for constant complex matrices (see [39]). In this way, it is possible to derive not only outer inverses with given both the range and null space, but also representations of outer inverses with only prescribed range or only prescribed null space. Also, representations of $\{1, 2\}$-inverses of tensors can be derived.

(ii) A new approach in symbolic computation of outer generalized inverses of tensors over an arbitrary field, of characteristic zero, is proposed. The entries of the tensors are defined as rational functions of unknown variables or as rational expressions of functional entries. The research is aimed to two-stage replacements. In the first stage, it is possible to consider replacements of functional entries by unknown variables, while the second stage assumes replacements of unknown variables by constant values from the field.

(iii) Our next topic is the symbolic computation of generalized inverses of tensors with functional entries over a field of characteristic zero, with or without involution. The computation of outer inverses over certain computable fields is reduced to the simpler computation of outer inverses with rational expressions of unknown variables as entries. The main idea can be defined in three global steps: (a) replace each function by a variable; (b) perform the necessary computations, and (c) replace, in the result, the unknown variables by their original function pair. These algorithms generalize the corresponding results from [28, 29, 34].

(iv) Known characterizations, representations, and algorithms for generating the M-P and the Drazin inverse, as well as the group inverse of tensors, are derived as particular cases.

We would like to emphasize that the importance of the use of symbolic computation techniques lies, not only on the fact that the exact determination of generalized inverses of tensors is feasible, as mentioned in (ii), but also on the fact that it allows to translate a purely functional problem into an algebraic problem that can be treated, afterwards, via either symbolic or numerical techniques (see (iii) above).

The global organization by sections is as follows. The basic facts and motivation are presented in Section 1. Section 2 is aimed to the computational aspects of generalized inverses of tensors. Some preliminary results are given first, and then the representations, as well as the computation of outer inverses via specializations, are presented in three subsections. The computation of generalized inverses of tensors with functional entries is considered in Section 3. Some concluding remarks and directions for further research are discussed in Section 4.

2 Computing generalized inverses of tensors via specializations

This section is the first attempt to develop algorithms for symbolic computation of outer inverses of tensors over an arbitrary field.

2.1 Preliminary results

Matricization (also known as *unfolding* or *flattening*) is the transformation, denoted by $\mathrm{Mat}(\mathcal{A})$, that transforms a tensor into a matrix. A tensor $\mathcal{A}$ can be unfolded into an appropriate matrix A in different ways. In order to propose an effective procedure for the matricization, we apply the reshaping operation, denoted as *rsh*, which was originated in [32], and implemented by means of the built-in *Matlab* function `reshape` (see Def. 3.1, and Example 3.1 in [32]). The *rsh* transformation is defined as the function

$$\mathrm{rsh}:\ \mathbb{K}^{M(m)\times N(n)} \longrightarrow \mathbb{K}^{\mathfrak{M}\times\mathfrak{N}},$$

where the integers $\mathfrak{M},\mathfrak{N}$ are defined by

$$\mathfrak{M} = M_1\cdots M_m,\quad \mathfrak{N} = N_1\cdots N_n. \tag{2.1}$$

More precisely, we generalize the reshape function over an arbitrary field $\mathbb{K}$ as follows.

Definition 2.1. *The reshaping operation transforms the tensor $\mathcal{A}\in\mathbb{K}^{M(m)\times N(n)}$ into the matrix $A\in\mathbb{K}^{\mathfrak{M}\times\mathfrak{N}}$ by means of the built-in Matlab function* `reshape` *as in the following expression:*

$$\begin{array}{cccc}\mathrm{rsh}: & \mathbb{K}^{M(m)\times N(n)} & \longrightarrow & \mathbb{K}^{\mathfrak{M}\times\mathfrak{N}}\\ & \mathcal{A} & \longmapsto & A:=\mathrm{rsh}\,(\mathcal{A}) = \mathrm{reshape}(\mathcal{A},\mathfrak{M},\mathfrak{N}).\end{array} \tag{2.2}$$

The inverse of the rsh *matricization will be termed as tensorization and defined by the mapping*

$$\begin{array}{cccc}\mathrm{rsh}^{-1}: & \mathbb{K}^{\mathfrak{M}\times\mathfrak{N}} & \longrightarrow & \mathbb{K}^{M(m)\times N(n)}\\ & A & \longmapsto & \mathcal{A}:=\mathrm{rsh}^{-1}(A) = \mathrm{reshape}(A,M_1,\ldots,M_m,N_1,\ldots N_n).\end{array} \tag{2.3}$$

The matricization operator

$$\Phi_{\mathfrak{J}\mathfrak{K}}:\mathbb{K}^{J(n)\times K(n)}\to\mathbb{K}^{\mathfrak{J}\times\mathfrak{K}},\ \mathfrak{J}:=J_1\cdots J_n,\mathfrak{K}:=K_1\cdots K_n$$

was defined in [3],[41, Definition 2.3] by $\Phi_{\mathfrak{J}\mathfrak{K}}(\mathcal{A})=A$ component-wise as

$$(\mathcal{A})_{j_1,\ldots,j_n,k_1,\ldots,k_n}\to A_{s,t}, \tag{2.4}$$

where $\mathcal{A}\in\mathbb{K}^{J(n)\times K(n)}$, and $A\in\mathbb{K}^{\mathfrak{J}\times\mathfrak{K}}$. The relationships between the subscripts s,t and $j_1,\ldots,j_n$, $k_1,\ldots,k_n$ are given by

$$s = j_n+\sum_{p=1}^{n-1}\left((j_p-1)\prod_{q=p+1}^{n}J_q\right),\quad t = k_n+\sum_{p=1}^{n-1}\left((k_p-1)\prod_{q=p+1}^{n}K_q\right).$$

Generalizing this definition, it is possible to present the matricization operator *rsh*

$$\mathrm{rsh}:\mathbb{K}^{J(m)\times K(n)}\to\mathbb{K}^{\mathfrak{J}\times\mathfrak{K}},\ \mathfrak{J}:=J_1\cdots J_m,\mathfrak{K}:=K_1\cdots K_n,$$

defined in (2.2), for arbitrary order tensors as

$$s = j_m+\sum_{p=1}^{m-1}\left((j_p-1)\prod_{q=p+1}^{m}J_q\right),\quad t = k_n+\sum_{p=1}^{n-1}\left((k_p-1)\prod_{q=p+1}^{n}K_q\right).$$

It is also important to mention that a bijection ϕ to unfold (matricize) an arbitrary-order tensor into a matrix was proposed in [20]. Similar folding and unfolding operations on even-order tensors are defined and considered in [16].

Lemma 2.1 claims that the folding *rsh*, and its inverse unfolding posses properties of homomorphisms.

Lemma 2.1. [16, 32] *Let $\mathcal{T}_1\in\mathbb{K}^{M(m)\times N(n)}$, $\mathcal{T}_2\in\mathbb{K}^{N(n)\times L(l)}$, let the integers $\mathfrak{M},\mathfrak{N}$ be as in* (2.1)*, and $\mathfrak{L}=L_1\cdots L_l$. It holds that*

$$\mathrm{rsh}\,(\mathcal{T}_1 *_n \mathcal{T}_2) = \mathrm{rsh}\,(\mathcal{T}_1)\,\mathrm{rsh}\,(\mathcal{T}_2) = T_1T_2,\quad \mathrm{rsh}^{-1}(T_1T_2) = \mathrm{rsh}^{-1}(T_1) *_n \mathrm{rsh}^{-1}(T_2) = \mathcal{T}_1 *_n \mathcal{T}_2. \tag{2.5}$$

In [16], the notion of rank of a tensor $\mathcal{A}$ is introduced as the dimension of $\mathbb{R}(\mathcal{A})$. Moreover, using the homomorphic behavior of rsh (see Lemma 2.1), in [16] the authors prove that $\dim(\mathbb{R}(\mathcal{A})) = \text{rank}(\text{rsh}(\mathcal{A}))$. On the other hand, in [32] the same notion of tensorial rank is introduced by means of a matrix factorization of $\text{rsh}(\mathcal{A})$, and the same result as above is obtained. Since the previous commented reasonings are valid over any field, Definition 2.2 extends naturally the concept for tensors over arbitrary fields.

Definition 2.2. *Let the positive integers m, n, and $M_1, \ldots, M_m$, $N_1, \ldots, N_n$, be given, and let $\mathfrak{M}, \mathfrak{N}$ satisfy* (2.1). *Let $\mathcal{T} \in \mathbb{K}^{M(m)\times N(n)}$ and $T = \text{rsh}(\mathcal{T}) \in \mathbb{K}^{\mathfrak{M}\times\mathfrak{N}}$. Then the reshaping rank of $\mathcal{T}$, denoted by* $\text{rshrank}(\mathcal{T})$*, is defined as* $\text{rshrank}(\mathcal{T}) = \text{rank}(T) = \text{rank}(\text{rsh}(\mathcal{T}))$.

Algorithm 1 is an extension of the algorithm given in [32] from constant tensors to tensors with entries over a computable field.

Algorithm 1 Computation of $\text{rshrank}(\mathcal{A})$

Input: Arbitrary integers m, n, $M_1, \ldots, M_m$, $N_1, \ldots, N_n$, and $\mathcal{A} \in \mathbb{K}^{M(m)\times N(m)}$.

1: Compute $\mathfrak{M}, \mathfrak{N}$ as in (2.1).
2: Matricize $\mathcal{A}$ into the matrix A by the *rsh* operator (2.2). This matricization is termed as $A = \text{rsh}(\mathcal{A}) \in \mathbb{K}^{\mathfrak{M}\times\mathfrak{N}}$, and can be implemented by the *Matlab* standard function `reshape`.
3: Compute $r := \text{rank}(A)$.
4: Return the output $\text{rshrank}(\mathcal{A}) = r$.

Algorithm 2 presents an algorithm for computing $\mathcal{A}^{(1)}$, where $rref[\mathcal{A}|\, I_{\mathfrak{M}}]$ denotes the row reduced echelon form of $[\mathcal{A}|\, I_{\mathfrak{M}}]$.

Algorithm 2 Computation of $\mathcal{A}^{(1)}$

Input: Integers m, n, integers $M_1, \ldots, M_m$, $N_1, \ldots, N_n$, and $\mathcal{A} \in \mathbb{K}^{M(m)\times N(n)}$.

1: Compute $\mathfrak{M}$ and $\mathfrak{N}$ satisfying (2.1).
2: Reshape the tensor $\mathcal{A} \in \mathbb{K}^{M(m)\times N(n)}$ as $\text{rsh}(\mathcal{A}) = A \in \mathbb{K}^{\mathfrak{M}\times\mathfrak{N}}$:
$$A = \text{reshape}\,(\mathcal{A}, \mathfrak{M}, \mathfrak{N}) = \text{rsh}(\mathcal{A}).$$
3: $r \leftarrow \text{rshrank}(\mathcal{A})$.
4: $B \leftarrow rref[A|I_{\mathfrak{M}}]$.
5: $E \leftarrow$ last $\mathfrak{M}$ columns of B.
6: Find a permutation matrix P satisfying $EAP = \begin{bmatrix} I_r & K \\ O & O \end{bmatrix}$.
7: Generate a random matrix $L \in \mathbb{K}^{(\mathfrak{N}-r)\times(\mathfrak{M}-r)}$.
8: Perform the reshaping operations
$$\text{rsh}^{-1}(E) = \mathcal{E} \in \mathbb{K}^{M(m)\times M(m)}, \quad \text{rsh}^{-1}(P) = \mathcal{P} \in \mathbb{K}^{N(n)\times N(n)}, \quad \text{rsh}^{-1}\left(\begin{bmatrix} I_r & O \\ O & L \end{bmatrix}\right) = \mathcal{Q}.$$
9: Compute the output
$$\mathcal{X} := \mathcal{P} *_n \mathcal{Q} *_m \mathcal{E} \in +\mathbb{K}^{N(n)\times M(m)}.$$

2.2 Representation of outer generalized inverses of tensors.

This subsection is devoted to the representation of outer generalized inverses of tensors which entries are elements of $\mathbb{K}$.

The results of Theorem 2.1 generalize the known representations for constant complex matrices in [39]. The following results will be useful in its verification. Lemma 2.2 and Theorem 3.1 in [39] are stated for the field $\mathbb{C}$, however they are easily extendable to the case of an arbitrary field $\mathbb{K}$ of characteristic zero as follows.

Proposition 2.1. [32, Lemma 2.2] *Let $\mathcal{X} \in \mathbb{K}^{N(n)\times M(m)}$, $\mathcal{B} \in \mathbb{K}^{N(n)\times K(k)}$, and $\mathcal{C} \in \mathbb{K}^{L(l)\times M(m)}$ be given tensors. Then*

(1) $\mathbb{R}(\mathcal{X}) \subseteq \mathbb{R}(\mathcal{B})$ *if and only if* $\mathcal{X} = \mathcal{B} *_k \mathcal{U}$, *for some* $\mathcal{U} \in \mathbb{K}^{K(k)\times M(m)}$.

(2) $\mathbb{N}(\mathcal{X}) \supseteq \mathbb{N}(\mathcal{C})$ *if and only if* $\mathcal{X} = \mathcal{V} *_l \mathcal{C}$, *for some* $\mathcal{V} \in \mathbb{K}^{N(n)\times L(l)}$.

(3) $\mathbb{R}(\mathcal{X}) \subseteq \mathbb{R}(\mathcal{B})$, *and* $\mathbb{N}(\mathcal{X}) \supseteq \mathbb{N}(\mathcal{C})$ *if and only if* $\mathcal{X} = \mathcal{B} *_k \mathcal{U} *_l \mathcal{C}$, *for some* $\mathcal{U} \in \mathbb{K}^{K(k)\times L(l)}$.

Proposition 2.2. [32, Theorem 3.1] *If $\mathcal{X} \in \mathbb{K}^{M(m)\times N(n)}$, and $\mathcal{Y} \in \mathbb{K}^{N(n)\times L(l)}$ are two tensors, then*

(1) $\mathbb{R}(\mathcal{X} *_n \mathcal{Y}) = \mathbb{R}(\mathcal{X}) \Longleftrightarrow \mathrm{rshrank}(\mathcal{X} *_n \mathcal{Y}) = \mathrm{rshrank}(\mathcal{X})$,

(2) $\mathbb{N}(\mathcal{X} *_n \mathcal{Y}) = \mathbb{N}(\mathcal{Y}) \Longleftrightarrow \mathrm{rshrank}(\mathcal{X} *_n \mathcal{Y}) = \mathrm{rshrank}(\mathcal{Y})$.

Theorem 2.1. *Let $\mathcal{A} \in \mathbb{K}^{M(m)\times N(n)}$, $\mathcal{B} \in \mathbb{K}^{N(n)\times K(k)}$, $\mathcal{C} \in \mathbb{K}^{L(l)\times M(m)}$, and assume that $(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})^{(1)}$ is a fixed but arbitrary element of $(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})\{1\}$. Then*

$$\mathcal{X} := \mathcal{B} *_k (\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})^{(1)} *_l \mathcal{C} \in \mathbb{K}^{N(n)\times M(m)} \tag{2.6}$$

satisfies the following statements.

(1) $\mathcal{X} = \mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}$ *if and only if* $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{B})$.

(2) $\mathcal{X} = \mathcal{A}^{(2)}_{*,\mathbb{N}(\mathcal{C})}$ *if and only if* $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{C})$.

(3) $\mathcal{X} = \mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}$ *if and only if* $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{B}) = \mathrm{rshrank}(\mathcal{C})$.

(4) $\mathcal{X} = \mathcal{A}^{(1,2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}$ *if and only if* $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{B}) = \mathrm{rshrank}(\mathcal{C}) = \mathrm{rshrank}(\mathcal{A})$.

Proof. (1) ($\Longrightarrow$) Let $\mathcal{X}$ be as in (2.6) satisfying that $\mathcal{X} *_m \mathcal{A} *_n \mathcal{X} = \mathcal{X}$, and $\mathbb{R}(\mathcal{X}) = \mathbb{R}(\mathcal{B})$. Then there exists $U \in \mathbb{K}^{K(k)\times L(l)}$ satisfying $\mathcal{X} = \mathcal{B} *_k \mathcal{U} *_l \mathcal{C}$. Also, since $\mathbb{R}(\mathcal{X}) \supseteq \mathbb{R}(\mathcal{B})$, applying Proposition 2.1 (1), one deduces the existence of $\mathcal{W} \in \mathbb{K}^{M(m)\times K(k)}$ such that $\mathcal{B} = \mathcal{X} *_m \mathcal{W}$. The last two facts further imply that

$$\mathcal{B} = \mathcal{X} *_m \mathcal{W} = \mathcal{X} *_m \mathcal{A} *_n \mathcal{X} *_m \mathcal{W} = \mathcal{X} *_m \mathcal{A} *_n \mathcal{B} = \mathcal{B} *_k \mathcal{U} *_l \mathcal{C} *_m \mathcal{A} *_n \mathcal{B}.$$

Using $\mathbb{N}(\mathcal{B}) \subseteq \mathbb{N}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})$, in conjunction with $\mathcal{B} *_k \mathcal{U} *_l \mathcal{C} *_m \mathcal{A} *_n \mathcal{B} = \mathcal{B}$ for some $\mathcal{U} \in \mathbb{K}^{K(k)\times L(l)}$, it follows that $\mathbb{N}(\mathcal{B}) \subseteq \mathbb{N}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) \subseteq \mathbb{N}(\mathcal{B} *_k \mathcal{U} *_l \mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathbb{N}(\mathcal{B})$, and hence $\mathbb{N}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathbb{N}(\mathcal{B})$, which implies, by Proposition 2.2, $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{B})$.

($\Longleftarrow$) The assumption $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{B})$ implies that $\mathcal{B} = \mathcal{B} *_k (\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})^{(1)} *_l \mathcal{C} *_m \mathcal{A} *_n \mathcal{B}$. Then $\mathcal{X} = \mathcal{B} *_k (\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})^{(1)} *_l \mathcal{C} = \mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}$. Indeed $\mathcal{X} *_m \mathcal{A} *_n \mathcal{X} = \mathcal{X}$ immediately follows. Now, using $\mathcal{X} = \mathcal{B} *_k (\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})^{(1)} *_l \mathcal{C}$ and $\mathcal{B} = \mathcal{B} *_k (\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})^{(1)} *_l \mathcal{C} *_m \mathcal{A} *_n \mathcal{B} = \mathcal{X} *_m \mathcal{A} *_n \mathcal{B}$, it is possible to verify that $\mathbb{R}(\mathcal{X}) = \mathbb{R}(\mathcal{B})$. Therefore, $\mathcal{X} \in \mathcal{A}\{2\}_{\mathbb{R}(\mathcal{B})}$, where $\mathcal{A}\{2\}_{\mathbb{R}(\mathcal{B})}$ denotes a subset of outer inverses $\mathcal{A}\{2\}$ with range $\mathbb{R}(\mathcal{B})$.

The other parts of the proof can be verified similarly. □

Lemma 2.2. *Let m, n, $M_1, \ldots, M_m$, $N_1, \ldots, N_n$ be positive integers, and let $\mathfrak{M}$, $\mathfrak{N}$ be as in* (2.1). *Let $\mathcal{A} \in \mathbb{K}^{M(m)\times N(n)}$, and $A := \mathrm{rsh}(\mathcal{A}) \in \mathbb{K}^{\mathfrak{M}\times\mathfrak{N}}$. Then it holds that*

(1) $(\mathrm{rsh}(\mathcal{A}))^{(1)} = A^{(1)} = \mathrm{rsh}(\mathcal{A}^{(1)})$.

(2) $\left(\mathrm{rsh}^{-1}(A)\right)^{(1)} = \mathcal{A}^{(1)} = \mathrm{rsh}^{-1}(A^{(1)})$.

(3) $(\mathrm{rsh}(\mathcal{A}))^{(2)} = A^{(2)} = \mathrm{rsh}(\mathcal{A}^{(2)})$.

(4) $\left(\mathrm{rsh}^{-1}(A)\right)^{(2)} = \mathcal{A}^{(2)} = \mathrm{rsh}^{-1}(A^{(2)})$.

Proof. (1) Let $\mathcal{A}^{(1)} \in \mathcal{A}\{1\}$ be arbitrary. Then,

$$\mathrm{rsh}(\mathcal{A}) = \mathrm{rsh}\left(\mathcal{A} *_n \mathcal{A}^{(1)} *_m \mathcal{A}\right).$$

An application of Lemma 2.1 leads to

$$\mathrm{rsh}(\mathcal{A}) = \mathrm{rsh}(\mathcal{A})\,\mathrm{rsh}(\mathcal{A}^{(1)})\,\mathrm{rsh}(\mathcal{A}).$$

Thus $\mathrm{rsh}(\mathcal{A}^{(1)}) = (\mathrm{rsh}(\mathcal{A}))^{(1)} = A^{(1)}$. The verification of the other statements is analogous. □

Theorem 2.2 is a generalization of [16, Theorem 2.4] in three directions: it is valid for tensors over arbitrary fields, it is valid for outer inverses with only given range or kernel, and also for $\{1,2\}$-inverses.

Theorem 2.2. *Let $\mathcal{A} \in \mathbb{K}^{M(m)\times N(n)}$, $\mathcal{B} \in \mathbb{K}^{N(n)\times K(k)}$, $\mathcal{C} \in \mathbb{K}^{L(l)\times M(m)}$. Then $\mathcal{X}$, defined as in (2.6), is an outer inverse of $\mathcal{A}$ with the following properties about the reshaping.*

(1) *If* $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{B})$, *then* $\mathcal{X} = \mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*} = \mathrm{rsh}^{-1}\left(\mathrm{rsh}(\mathcal{A})^{(2)}_{\mathcal{R}(\mathrm{rsh}(\mathcal{B})),*}\right)$.

(2) *If* $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{C})$, *then* $\mathcal{X} = \mathcal{A}^{(2)}_{*,\mathbb{N}(\mathcal{C})} = \mathrm{rsh}^{-1}\left(\mathrm{rsh}(\mathcal{A})^{(2)}_{*,\mathcal{N}(\mathrm{rsh}(\mathcal{C}))}\right)$.

(3) *If* $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{B}) = \mathrm{rshrank}(\mathcal{C})$, *then*

$$\mathcal{X} = \mathrm{rsh}(\mathcal{A})^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})} = \mathrm{rsh}^{-1}\left(\mathrm{rsh}(\mathcal{A})^{(2)}_{\mathcal{R}(\mathrm{rsh}(\mathcal{B})),\mathcal{N}(\mathrm{rsh}(\mathcal{C}))}\right).$$

(4) *If* $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{B}) = \mathrm{rshrank}(\mathcal{C}) = \mathrm{rshrank}(\mathcal{A})$, *then*

$$\mathcal{X} = \mathcal{A}^{(1,2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})} = \mathrm{rsh}^{-1}\left(\mathrm{rsh}(\mathcal{A})^{(1,2)}_{\mathcal{R}(\mathrm{rsh}(\mathcal{B}),\mathcal{N}(\mathrm{rsh}(\mathcal{C}))}\right).$$

Proof. By Theorem 2.1, we have that $\mathcal{X} = \mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}$. In order to prove the other equality, we apply Lemma 2.2 to get

$$\mathrm{rsh}(\mathcal{X}) = \mathrm{rsh}(\mathcal{B})\left(\mathrm{rsh}(\mathcal{C})\mathrm{rsh}(\mathcal{A})\mathrm{rsh}(\mathcal{B})\right)^{(1)}\mathrm{rsh}(\mathcal{C}). \tag{2.7}$$

Furthermore, taking into account Definition 2.2 and Lemma 2.2, we have that

$$\mathrm{rank}(\mathrm{rsh}(\mathcal{B})) = \mathrm{rshrank}(\mathcal{B}) = \mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rank}(\mathrm{rsh}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})) = \mathrm{rank}(\mathrm{rsh}(C)\mathrm{rsh}(A)\mathrm{rsh}(B)).$$

Using Urquhart's formula over fields (see Theorem 3.1 and Theorem 3.5 in [33]), and taking into account (2.7), one has that

$$\mathrm{rsh}(\mathcal{A})^{(2)}_{\mathcal{R}(\mathrm{rsh}(\mathcal{B})),*} = \mathrm{rsh}(\mathcal{B})(\mathrm{rsh}(\mathcal{C})\mathrm{rsh}(\mathcal{A})\mathrm{rsh}(\mathcal{B}))^{(1)}\mathrm{rsh}(\mathcal{C}) = \mathrm{rsh}(\mathcal{X}) = \mathrm{rsh}(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}).$$

Finally, taking the inverse rsh^{-1} one gets the result.

The other claims follow analogously. □

2.3 Specializations on tensors whose entries are rational expressions

In this subsection we focus on generalized tensor inverses which entries are rational functions. Therefore, throughout this subsection we will work over the rational function field $\mathbb{K}(\mathbf{x})$. Thus, the entries of the tensors are fractions in which the numerator and/or the denominator are polynomials over $\mathbb{K}$. Moreover, we investigate the behavior of these generalized inverses when the unknown variables x_i, included in $\mathbf{x} = (x_1, \ldots, x_p)$, are substituted by some field elements $\mathbf{c} = (c_1, \ldots, c_p) \in \mathbb{K}^p$. Such replacement, when the denominator of the rational function does not vanish at $\mathbf{c}$, is termed as *specialization* and will be denoted by $\mathbf{x} \leftrightsquigarrow \mathbf{c}$.

The elements $a_{i(m),j(n)}$ of $\mathcal{A} \in \mathbb{K}(\mathbf{x})^{M(m)\times N(n)}$ are rational expressions, and can be represented as

$$a_{i(m),j(n)} = \frac{\mathrm{num}(a_{i(m),j(n)})}{\mathrm{den}(a_{i(m),j(n)})},$$

where the *greatest common divisor* of the numerator and the denominator is 1. Then, we introduce the notation $\mathrm{den}(\mathcal{A}) = \mathrm{lcm}\{\mathrm{den}(a_{i(m),j(n)})\}$, where lcm stands for the *least common multiple*. The side effects of each specialization $\mathbf{x} \leftrightsquigarrow \mathbf{c}$ are the possibility that $\mathbf{c}$ becomes a pole of $\mathcal{A}$ as well as the possibility that $\mathcal{A}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}$ decreases the reshaping rank of $\mathcal{A}$. Following the definition of a pole from [42], we say that $\mathbf{c}$ is a pole of $\mathcal{A}$ if at least one element $\frac{\mathrm{num}(a_{i(m),j(n)})}{\mathrm{den}(a_{i(m),j(n)})}$ has a pole at $\mathbf{c}$, i.e., $\mathrm{den}(a_{i(m),j(n)})(\mathbf{c}) = 0$. Similarly, following the definition of a zero from [42], $\mathbf{c} \in \mathbb{K}^p$ is a zero of $\mathcal{A}$ if $\mathrm{rshrank}(\mathcal{A}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}) < \mathrm{rshrank}(\mathcal{A})$. Note that, taking into account that $\mathrm{rshrank}(\mathcal{A}) = \mathrm{rank}(\mathrm{rsh}(\mathcal{A}))$, one can apply Theorem 5.1 in [34] to $\mathrm{rsh}(\mathcal{A})$ to derive a polynomial $F \in \mathbb{K}[\mathbf{x}]$, such that if $F(\mathbf{c}) \neq 0$ then $\mathbf{c}$ is not a zero of $\mathcal{A}$.

In the following, we restrict $\mathbb{K}$ in order to avoid appearances of poles and zeros. For a specified $\mathcal{A}(\mathbf{x}) \in \mathbb{K}(\mathbf{x})^{M(m)\times N(n)}$, let $\mathbb{K}(\mathbf{x})_{\mathcal{A}}$ denote the restriction of $\mathbb{K}(\mathbf{x})$ defined as

$$\mathbb{K}(\mathbf{x})_{\mathcal{A}} := \{\mathbf{c} \in \mathbb{K}^p : \ \mathrm{den}(\mathcal{A})(\mathbf{c}) \neq 0 \ \},$$

and

$$\mathbb{K}(\mathbf{x})_{\mathcal{A},\mathrm{rshrank}(\mathcal{A})} := \left\{\mathbf{c} \in \mathbb{K}(\mathbf{x})_{\mathcal{A}} : \ \mathrm{rshrank}(\mathcal{A}) = \mathrm{rshrank}(\mathcal{A}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}})\right\} \subset \mathbb{K}(\mathbf{x})_{\mathcal{A}}.$$

Under these restrictions, for a given $\mathcal{A} \in \mathbb{K}(\mathbf{x})^{M(m)\times N(n)}$, and for any $\mathbf{c} \in \mathbb{K}(\mathbf{x})_{\mathcal{A}}$, we introduce the map *Spec* that transforms a tensor with rational function entries into another tensor with rational fractions entries:

$$\begin{array}{rccc} \mathrm{Spec}: & \mathbb{K}(\mathbf{x})^{M(m)\times N(n)} & \longrightarrow & \mathbb{K}^{M(m)\times N(n)} \\ & \mathcal{A} = (a_{i(m),j(n)}(\mathbf{x})) & \longmapsto & \mathrm{Spec}(\mathcal{A}) := \mathcal{A}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = (a_{i(m),j(n)}(\mathbf{c})). \end{array}$$

Theorem 2.3 gives characterizations of $\{2\}$-inverses of tensors, with known range and null space, under specializations. It shows that under certain rank restrictions, and restrictions on $\mathbb{K}(\mathbf{x})_{\mathcal{A},\mathrm{rshrank}(\mathcal{A})}$, the specialization $\mathbf{x} \leftrightsquigarrow \mathbf{c}$ of the outer inverse of $\mathcal{A}$ with range $\mathbb{R}(\mathcal{B})$ and/or null space $\mathbb{N}(\mathcal{C})$ is equal to the outer inverse of $\mathcal{A}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}$ with range $\mathbb{R}(\mathcal{B}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}})$ and/or null space $\mathbb{N}(\mathcal{C}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}})$. These restrictions are caused by the fact that $\mathrm{rshrank}(\mathcal{A}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}) \leq \mathrm{rshrank}(\mathcal{A})$ as well as that $\mathbf{c}$ can be a pole.

The following lemma will be necessary.

Lemma 2.3. *Let $\mathcal{A} \in \mathbb{K}(\mathbf{x})^{M(m)\times N(n)}$, and $\mathbf{c} \in \mathbb{K}^p$. If an inner inverse $\mathcal{A}^{(1)}$ satisfies*

$$\mathrm{den}(\mathcal{A})(\mathbf{c}) \cdot \mathrm{den}(\mathcal{A}^{(1)})(\mathbf{c}) \neq 0,$$

then

$$(\mathcal{A}^{(1)})_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(\mathcal{A}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}\right)^{(1)}.$$

Proof. The hypothesis $\mathrm{den}(\mathcal{A})(\mathbf{c})\,\mathrm{den}(\mathcal{A}^{(1)})(\mathbf{c}) \neq 0$ implies that $\mathbf{c}$ is not a pole of any entry of $\mathcal{A}$ or $\mathcal{A}^{(1)}$. Therefore the specializations $\mathcal{A}_{\mathbf{x}\leftrightsquigarrow\mathbf{c}}$ and $(\mathcal{A}^{(1)})_{\mathbf{x}\leftrightsquigarrow\mathbf{c}}$ are well defined. In addition, it holds that

$$\mathcal{A}_{\mathbf{x}\leftrightsquigarrow\mathbf{c}} = (\mathcal{A} *_n \mathcal{A}^{(1)} *_m \mathcal{A})_{\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \mathcal{A}_{\mathbf{x}\leftrightsquigarrow\mathbf{c}} *_n (\mathcal{A}^{(1)})_{\mathbf{x}\leftrightsquigarrow\mathbf{c}} *_m \mathcal{A}_{\mathbf{x}\leftrightsquigarrow\mathbf{c}}.$$

So, $(\mathcal{A}^{(1)})_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(\mathcal{A}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}\right)^{(1)}$, and the proof is completed. □

The next theorem generalizes Theorem 5.2 in [34] to the case of tensors.

Theorem 2.3. *Let $\mathcal{A} \in \mathbb{K}(\mathbf{x})^{M(m)\times N(n)}$, $\mathcal{B} \in \mathbb{K}(\mathbf{x})^{N(n)\times K(k)}, \mathcal{C} \in \mathbb{K}(\mathbf{x})^{L(l)\times M(m)}$, let*

$$\mathcal{X} := \mathcal{B} *_k (\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})^{(1)} *_l \mathcal{C} \in \mathbb{K}(\mathbf{x})^{N(n)\times M(m)},$$

and let

$$\mathbf{c} \in \mathbb{K}(\mathbf{x})_{\mathcal{C}*_m\mathcal{A}*_n\mathcal{B},\mathrm{rshrank}(\mathcal{C}*_m\mathcal{A}*_n\mathcal{B})} \cap \mathbb{K}(\mathbf{x})_{\mathcal{A},\mathrm{rshrank}(\mathcal{A})} \cap \mathbb{K}(\mathbf{x})_{\mathcal{B},\mathrm{rshrank}(\mathcal{B})} \cap \mathbb{K}(\mathbf{x})_{\mathcal{C},\mathrm{rshrank}(\mathcal{C})} \cap \mathbb{K}(\mathbf{x})_{(\mathcal{C}*_m\mathcal{A}*_n\mathcal{B})^{(1)}}. \tag{2.8}$$

Then, the following statements holds

(1) *If* $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{B})$ *it holds that*

$$\mathcal{X}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(\mathcal{A}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}\right)^{(2)}_{\mathbb{R}(\mathcal{B}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}),*}.$$

(2) *If* $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{C})$ *it holds that*

$$\mathcal{X}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(\mathcal{A}^{(2)}_{*,\mathbb{N}(\mathcal{C})}\right)_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(\mathcal{A}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}\right)^{(2)}_{*,\mathbb{N}(\mathcal{C}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}})}.$$

(3) *If* $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{B}) = \mathrm{rshrank}(\mathcal{C})$ *it holds that*

$$\mathcal{X}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}\right)_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(\mathcal{A}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}\right)^{(2)}_{\mathbb{R}(\mathcal{B}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}),\mathbb{N}(\mathcal{C}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}})}. \tag{2.9}$$

(4) *If* $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{B}) = \mathrm{rshrank}(\mathcal{C}) = \mathrm{rshrank}(\mathcal{A})$ *it holds that*

$$\mathcal{X}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(\mathcal{A}^{(1,2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}\right)_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(\mathcal{A}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}\right)^{(1,2)}_{\mathbb{R}(\mathcal{B}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}),\mathbb{N}(\mathcal{C}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}})}.$$

Proof. We give the details of the proof of statement (1); the reasoning for the other cases is analogous. First, using the hypothesis and Theorem 2.1 (1) we get that

$$\mathcal{X} = \mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}.$$

By (2.8), we have that

$$\mathbf{c} \in \mathbb{K}(\mathbf{x})_{\mathcal{B},\mathrm{rshrank}(\mathcal{B})} \cap \mathbb{K}(\mathbf{x})_{(\mathcal{C}*_m\mathcal{A}*_n\mathcal{B})^{(1)}} \cap \mathbb{K}(\mathbf{x})_{\mathcal{C},\mathrm{rshrank}(\mathcal{C})}.$$

So $\mathbf{c}$ is not a pole of any of the entries of $\mathcal{B}$ or $(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})^{(1)}$ or $\mathcal{C}$. Therefore, the specialization $\mathcal{X}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}$ is well defined. Thus,

$$\mathcal{X}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}. \tag{2.10}$$

Now, by Lemma 2.3, it holds that

$$\left((\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})^{(1)}\right)_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left((\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}\right)^{(1)} = \left(\mathcal{C}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} *_m \mathcal{A}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} *_n \mathcal{B}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}\right)^{(1)}. \tag{2.11}$$

On the other hand, by (2.8), it holds that

$$\mathrm{rshrank}(\mathcal{B}) = \mathrm{rshrank}(\mathcal{B}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}), \text{ and } \mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}((\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}).$$

Therefore

$$\begin{aligned}
\mathrm{rshrank}(\mathcal{B}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}) = \mathrm{rshrank}(\mathcal{B}) &= \mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) \\
&= \mathrm{rshrank}((\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}) \\
&= \mathrm{rshrank}(\mathcal{C}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} *_m \mathcal{A}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} *_n \mathcal{B}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}).
\end{aligned}$$

Now, Theorem 2.1 (1) implies

$$\mathcal{X}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(\mathcal{A}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}\right)^{(2)}_{\mathbb{R}(\mathcal{B}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}),*}.$$

□

Remark 2.1. *Symbolic computation applied to matrix algebra provides the exact rank of a tensor, while the result produced by numeric calculations can be different because of the presence of round-off errors. Also, the result can be different depending on the various numerical methods used for the rank computation, see* [36]. *But such difficulties in the implementation of Theorem 2.3 in subsequent algorithms are unavoidable.*

2.4 Algorithms and examples

Algorithms 3, 4, and 5 give computational frameworks for various generalized inverses of tensors. Algorithm 3 is designed to compute $\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}$.

Algorithm 3 Computation of $\mathcal{X} := \mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}$.

Input: Positive integers m, n, k, l, and tensors $\mathcal{A} \in \mathbb{K}(\mathbf{x})^{M(m)\times N(n)}$, $\mathcal{B} \in \mathbb{K}(\mathbf{x})^{N(n)\times K(k)}$, $\mathcal{C} \in \mathbb{K}(\mathbf{x})^{L(l)\times M(m)}$.

Compute rshrank$(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})$ and rshrank$(\mathcal{B})$ using Algorithm 1.
if rshrank$(\mathcal{B})$ = rshrank$(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})$ **then**
  Compute $\mathcal{Y} := (\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})^{(1)}$ using Algorithm 2.
  Compute $\mathcal{X} := \mathcal{B} *_k \mathcal{Y} *_l \mathcal{C}$.
  return $\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*} = \mathcal{X}$.
end if

Algorithm 3 is tested in Example 2.1.

Example 2.1. Let $\mathbf{c} = (1, 2, 3, 4, 5, 6, 7)$, and $\mathbf{z} = (z_1, z_2, z_3, z_4, z_5, z_6, z_7)$ with $z_i \neq 0$ for $1 \leq i \leq 7$. Consider the tensor $\mathcal{A} \in (\mathbb{R}(z_1, z_2, z_3, z_4))^{(3\times3)\times(4\times4)}$ with entries

$$\mathcal{A}(:,:,1,1) = \begin{bmatrix} 0 & z_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \mathcal{A}(:,:,2,1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \mathcal{A}(:,:,3,1) = \begin{bmatrix} 0 & z_3 & 0 \\ 0 & 0 & z_3 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{A}(:,:,4,1) = \mathcal{A}(:,:,3,2) = \mathcal{A}(:,:,1,3) = \mathcal{A}(:,:,3,3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{A}(:,:,1,2) = \begin{bmatrix} 0 & z_3 & 0 \\ 0 & 0 & 0 \\ z_2 & 0 & 0 \end{bmatrix},\ \mathcal{A}(:,:,2,2) = \begin{bmatrix} 0 & 0 & 0 \\ z_2 & 0 & 0 \\ 0 & z_4 & 0 \end{bmatrix},\ \mathcal{A}(:,:,4,2) = \begin{bmatrix} z_3/z_6 & 0 & 0 \\ 0 & 0 & 0 \\ z_1 & 0 & z_5/z_7 \end{bmatrix},$$

$$\mathcal{A}(:,:,2,3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & z_5 \\ 0 & 0 & 0 \end{bmatrix},\ \mathcal{A}(:,:,4,3) = \begin{bmatrix} 0 & z_2 & 0 \\ z_7 & z_3 & 0 \\ 0 & z_4 & 0 \end{bmatrix},\ \mathcal{A}(:,:,1,4) = \begin{bmatrix} 0 & z_6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{A}(:,:,2,4) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & z_2 & 0 \end{bmatrix},\ \mathcal{A}(:,:,3,4) = \begin{bmatrix} 0 & z_5/z_3 & 0 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{bmatrix},\ \mathcal{A}(:,:,4,4) = \begin{bmatrix} z_3 & 0 & z_2 \\ 0 & 0 & z_1 \\ 0 & 0 & z_5 \end{bmatrix},$$

the tensor $\mathcal{B} \in (\mathbb{R}(z_1, z_2, z_3, z_4))^{4\times4\times3\times3}$ defined as

$$\mathcal{B}(:,:,1,1) = \begin{bmatrix} 0 & z_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ \mathcal{B}(:,:,2,1) = \begin{bmatrix} 0 & z_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{B}(:,:,3,1) = \mathcal{B}(:,:,1,2) = \mathcal{B}(:,:,3,2) = \mathcal{B}(:,:,1,3) = \mathcal{B}(:,:,2,3) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{B}(:,:,2,2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & z_1 + z_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ \mathcal{B}(:,:,3,3) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $\mathcal{C} \in (\mathbb{R}(z_1, z_2, z_3, z_4))^{4\times4\times3\times3}$ with entries

$$\mathcal{C}(:,:,1,1) = \begin{bmatrix} z_1/z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & z_6 & 0 & 0 \end{bmatrix},$$

$$\mathcal{C}(:,:,2,1) = \mathcal{C}(:,:,1,3) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathcal{C}(:,:,3,1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ z_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ z_2 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{C}(:,:,1,2) = \begin{bmatrix} 0 & z_3/z_5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathcal{C}(:,:,2,2) = \begin{bmatrix} z_1+z_2+z_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & z_4 & 0 & 0 \end{bmatrix},$$

$$\mathcal{C}(:,:,3,2) = \begin{bmatrix} 0 & z_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{C}(:,:,2,3) = \begin{bmatrix} z_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_5 \end{bmatrix}, \ \mathcal{C}(:,:,3,3) = \begin{bmatrix} z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using Algorithm 2, we compute $\mathcal{Y} := (\mathcal{C} *_2 \mathcal{A} *_2 \mathcal{B})^{(1)} \in (\mathbb{R}(z_1, z_2, z_3, z_4))^{(3\times3)\times(4\times4)}$, where

$$\mathcal{Y}(:,:,1,1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -z_3/(z_1 z_2^2 z4(z_3^2+z_5^2)) & 0 \\ 0 & 0 & z_3/(z_1 z_2(z_3^2+z_5^2)) \end{bmatrix},$$

$$\mathcal{Y}(:,:,2,1) = \begin{bmatrix} z_4/(z_2^2+z_4^2)^2 & 0 & 0 \\ z_4^2/(z_2 * (z_2^2+z_4^2)^2) & -z_3^2/(z_2^3 z_5(z_2^2+z_4^2)) & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{Y}(:,:,3,1) = \mathcal{Y}(:,:,2,2) = \mathcal{Y}(:,:,3,2) = \mathcal{Y}(:,:,4,2) = \mathcal{Y}(:,:,1,3) = \mathcal{Y}(:,:,2,3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{Y}(:,:,3,3) = \mathcal{Y}(:,:,4,3) = \mathcal{Y}(:,:,1,4) = \mathcal{Y}(:,:,2,4) = \mathcal{Y}(:,:,3,4) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{Y}(:,:,4,1) = \begin{bmatrix} z_2/(z_2^2+z_4^2)^2 & 0 & 0 \\ z_4/(z_2^2+z_4^2)^2 & -z_3^2/(z_2^2 z_4 z5(z_2^2+z_4^2)) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathcal{Y}(:,:,1,2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/(z_2^2 z_4) & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{Y}(:,:,4,4) = \begin{bmatrix} ,0 & 0 & 0 \\ 0 & -z_5/(z_1 z_2^2 z_4(z_3^2+z_5^2)) & 0 \\ 0 & 0 & z_5/(z_1 z_2(z_3^2+z_5^2)) \end{bmatrix}.$$

Now, one can verify that

- $\text{den}(\mathcal{B})(\mathbf{c}) \cdot \text{den}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})(\mathbf{c}) \cdot \text{den}\,(\mathcal{Y})\,(\mathbf{c}) \cdot \text{den}(\mathcal{C})(\mathbf{c}) \neq 0$,
- $\text{rshrank}(\mathcal{C} *_2 \mathcal{A} *_2 \mathcal{B}) = \text{rshrank}(\mathcal{B}) = \text{rshrank}((\mathcal{C} *_2 \mathcal{A} *_2 \mathcal{B})_{|\mathbf{z} \rightsquigarrow \mathbf{c}}) = \text{rshrank}(\mathcal{B}_{|\mathbf{z} \rightsquigarrow \mathbf{c}}) = 3$, and
- $\text{rshrank}(\mathcal{C}) = \text{rshrank}(\mathcal{C}_{|\mathbf{z} \rightsquigarrow \mathbf{c}}) = 5$.

Therefore, taking into account Theorem 2.3 (1) and Algorithm 2, the computed outer inverse of $\mathcal{A}$ is equal to

$$\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*} = \mathcal{B} *_2 (\mathcal{C} *_2 \mathcal{A} *_2 \mathcal{B})^{(1)} *_2 \mathcal{C} \in (\mathbb{R}(z_1, z_2, z_3, z_4))^{(4\times 4)\times(3\times 3)},$$

with elements

$$\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)(:,:,1,1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -z_3/(z_2^2 z_4(z_3^2+z_5^2)) & 0 & 0 \\ 0 & -(z_3(z_1+z_3))/(z_2^3 z_4(z_3^2+z_5^2)) & 0 & -z_3/(z_2(z_3^2+z_5^2)) \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)(:,:,2,1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)(:,:,1,3),$$

$$\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)(:,:,3,1) = \begin{bmatrix} 0 & 1/z_2 & 0 & 0 \\ 0 & -z_3^2/(z_2^2 z_4 z_5) & 0 & 0 \\ 0 & -(z_3^2(z_1+z_3))/(z_2^3 z_4 z_5) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)(:,:,1,2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & z_3/(z_2 z_4 z_5) & 0 & 0 \\ 0 & (z_3(z_1+z_3))/(z_2^2 z_4 z_5) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)(:,:,2,2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -(z_3(z_1+z_2+z_3))/(z_1 z_2 z_4(z_3^2+z_5^2)) & 0 & 0 \\ 0 & -(z_3(z_1+z_3)(z_1+z_2+z_3))/(z_1 z_2^2 z_4(z_3^2+z_5^2)) & 0 & (z_3(z_1+z_2+z_3))/(z_1(z_3^2+z_5^2)) \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)(:,:,3,2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/z_4 & 0 & 0 \\ 0 & (z_1+z_3)/(z_2 z_4) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)(:,:,2,3) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1/(z_1 z_2 z_4) & 0 & 0 \\ 0 & -(z_1+z_3)/(z_1 z_2^2 z_4) & 0 & 1/z_1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)(:,:,3,3) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -z_3/(z_1 z_4(z_3^2+z_5^2)) & 0 & 0 \\ 0 & -(z_3(z_1+z_3))/(z_1 z_2 z_4(z_3^2+z_5^2)) & 0 & (z_2 z_3)/(z_1(z_3^2+z_5^2)) \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Further, the outer inverse at $\mathbf{z}=\mathbf{c}$ is $\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)_{|_{\mathbf{x}\leftrightsquigarrow\mathbf{c}}} = \left(\mathcal{B} *_2 (\mathcal{C} *_2 \mathcal{A} *_2 \mathcal{B})^{(1)} *_2 \mathcal{C}\right)_{|_{\mathbf{z}\leftrightsquigarrow\mathbf{c}}}$, where

$$\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)_{|_{\mathbf{x}\leftrightsquigarrow\mathbf{c}}}(:,:,1,1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3/544 & 0 & 0 \\ 0 & -3/272 & 0 & 3/68 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)_{|_{\mathbf{x}\leftrightsquigarrow\mathbf{c}}}(:,:,2,1) = \left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)_{|_{\mathbf{x}\leftrightsquigarrow\mathbf{c}}}(:,:,1,3) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)_{|_{\mathbf{x}\leftrightsquigarrow\mathbf{c}}}(:,:,3,1) = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 0 & -9/80 & 0 & 0 \\ 0 & -9/40 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)_{|_{\mathbf{x}\leftrightsquigarrow\mathbf{c}}}(:,:,1,2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3/40 & 0 & 0 \\ 0 & 3/20 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)_{|_{\mathbf{x}\leftrightsquigarrow\mathbf{c}}}(:,:,2,2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -9/136 & 0 & 0 \\ 0 & -9/68 & 0 & 9/17 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)_{|_{\mathbf{x}\leftrightsquigarrow\mathbf{c}}}(:,:,3,2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)_{|_{\mathbf{x}\leftrightsquigarrow\mathbf{c}}}(:,:,1,3) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)_{|_{\mathbf{x}\leftrightsquigarrow\mathbf{c}}}(:,:,2,3) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1/8 & 0 & 0 \\ 0 & -1/4 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)_{|_{\mathbf{x}\leftrightsquigarrow\mathbf{c}}}(:,:,3,3) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3/136 & 0 & 0 \\ 0 & -3/68 & 0 & 3/117 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By Theorem 2.3 (1), we can verify that $\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}\right)_{|_{\mathbf{x}\leftrightsquigarrow\mathbf{c}}} = \left(\mathcal{A}_{|_{\mathbf{x}\leftrightsquigarrow\mathbf{c}}}\right)^{(2)}_{\mathbb{R}(\mathcal{B}|_{\mathbf{x}\leftrightsquigarrow\mathbf{c}}),*}$.

The following algorithm deals with the computation of $\mathcal{A}^{(2)}_{*,\mathbb{N}(\mathcal{C})}$.

Algorithm 4 Computation of $\mathcal{X} := \mathcal{A}^{(2)}_{*,\mathbb{N}(\mathcal{C})}$.

Input: Positive integers m, n, k, l, and tensors $\mathcal{A} \in \mathbb{K}(\mathbf{x})^{M(m)\times N(n)}$, $\mathcal{B} \in \mathbb{K}(\mathbf{x})^{N(n)\times K(k)}$, $\mathcal{C} \in \mathbb{K}(\mathbf{x})^{L(l)\times M(m)}$.

1: Compute rshrank($\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}$) and rshrank($\mathcal{C}$) using Algorithm 1.
2: **if** rshrank($\mathcal{C}$) = rshrank($\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}$) **then**
3: Compute $\mathcal{Y} := (\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})^{(1)}$ using Algorithm 2.
4: Compute $\mathcal{X} := \mathcal{B} *_k \mathcal{Y} *_l \mathcal{C}$.
5: **return** $\mathcal{A}^{(2)}_{*,\mathbb{N}(\mathcal{C})} = \mathcal{X}$.
6: **end if**

The following example illustrates Algorithm 4.

Example 2.2. Let $\mathbf{c} = (1,2,3,4)$, and $\mathbf{z} = (z_1, z_2, z_3, z_4)$ with $z_i \neq 0$ for $1 \leq i \leq 4$. Consider the tensor $\mathcal{A} \in (\mathbb{R}(z_1, z_2, z_3, z_4))^{(2\times2)\times(2\times2)}$ with entries

$$\mathcal{A}(:,:,1,1)=\begin{bmatrix} z_3 & z_2\\ z_1^2 & 1\end{bmatrix},\ \mathcal{A}(:,:,2,1)=\begin{bmatrix} 0 & z_2\\ 0 & z_1+z_2\end{bmatrix},$$

$$\mathcal{A}(:,:,1,2)=\begin{bmatrix} 0 & 3\\ z_1+z_2 & z_3+z_4\end{bmatrix},\ \mathcal{A}(:,:,2,2)=\begin{bmatrix} 0 & 0\\ z_4^2 & z_1+z_3+z_4\end{bmatrix}.$$

The tensor $\mathcal{B}\in(\mathbb{R}(z_1,z_2,z_3,z_4))^{(2\times2)\times(2\times2)}$ is determined by

$$\mathcal{B}(:,:,1,1)=\begin{bmatrix} 0 & z_1+z_3+z_4\\ z_2 & 0\end{bmatrix},\ \mathcal{B}(:,:,2,1)=\begin{bmatrix} z_4 & 0\\ 0 & z_3\end{bmatrix},\ \mathcal{B}(:,:,1,2)=\begin{bmatrix} 0 & z_1\\ 0 & z_1^2\end{bmatrix},\ \mathcal{B}(:,:,2,2)=\begin{bmatrix} 0 & 0\\ 0 & 0\end{bmatrix},$$

and $\mathcal{C}\in(\mathbb{R}(z_1,z_2,z_3,z_4))^{(2\times2)\times(2\times2)}$ is defined as

$$\mathcal{C}(:,:,1,1)=\begin{bmatrix} z_4 & z_2\\ 0 & 0\end{bmatrix},\mathcal{C}(:,:,2,1)=\begin{bmatrix} 0 & 0\\ 0 & 0\end{bmatrix},\mathcal{C}(:,:,1,2)=\begin{bmatrix} 0 & 0\\ z_1+z_3 & z_2\end{bmatrix},\mathcal{C}(:,:,2,2)=\begin{bmatrix} 0 & 0\\ 0 & 0\end{bmatrix}.$$

With the help of Algorithm 2, we compute $\mathcal{Y}:=(\mathcal{C}*_2\mathcal{A}*_2\mathcal{B})^{(1)}$ which entries are

$$\mathcal{Y}(:,:,1,1)=\begin{bmatrix} 0 & 0\\ 0 & 1\end{bmatrix},\mathcal{Y}(:,:,2,1)=\begin{bmatrix} 0 & 2\\ 0 & 0\end{bmatrix},$$

$$\mathcal{Y}(:,:,1,2)=\begin{bmatrix} -1/(z_2^2z_3) & 0\\ 1/(z_2z_3z_4) & -z_4/z_2\end{bmatrix},\mathcal{Y}(:,:,2,2)=\begin{bmatrix} 1/z_2^3 & -2(z_1+z_3)/z_2\\ 0 & 0\end{bmatrix}.$$

Now, we can easily verify that

- $\mathrm{den}(\mathcal{B})(\mathbf{c})\cdot\mathrm{den}(\mathcal{C}*_m\mathcal{A}*_n\mathcal{B})(\mathbf{c})\cdot\mathrm{den}\left((\mathcal{C}*_m\mathcal{A}*_n\mathcal{B})^{(1)}\right)(\mathbf{c})\cdot\mathrm{den}(\mathcal{C})(\mathbf{c})=(z_2^3z_3z_4)_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}=96\neq0,$
- $\mathrm{rshrank}(\mathcal{C}*_2\mathcal{A}*_2\mathcal{B})=\mathrm{rshrank}(\mathcal{C})=\mathrm{rshrank}((\mathcal{C}*_2\mathcal{A}*_2\mathcal{B})_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}})=\mathrm{rshrank}(\mathcal{C}_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}})=2,$ and
- $\mathrm{rshrank}(\mathcal{B})=\mathrm{rshrank}(\mathcal{B}_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}})=3.$

Applying Agorithm 4, we evaluate the outer inverse at $\mathbf{z}=\mathbf{c}$, $\left(\mathcal{A}^{(2)}_{*,\mathbb{N}(\mathcal{C})}\right)_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}=\left(\mathcal{B}*_2(\mathcal{C}*_2\mathcal{A}*_2\mathcal{B})^{(1)}*_2\mathcal{C}\right)_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}$, as follows

$$\left(\mathcal{A}^{(2)}_{*,\mathbb{N}(\mathcal{C})}\right)_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}(:,:,1,1)=\begin{bmatrix} -1/6 & 0\\ 1/12 & 0\end{bmatrix},\ \left(\mathcal{A}^{(2)}_{*,\mathbb{N}(\mathcal{C})}\right)_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}(:,:,2,1)=\begin{bmatrix} 0 & 0\\ 0 & 0\end{bmatrix},$$

$$\left(\mathcal{A}^{(2)}_{*,\mathbb{N}(\mathcal{C})}\right)_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}(:,:,1,2)=\begin{bmatrix} 1/4 & 0\\ 0 & 0\end{bmatrix},\ \left(\mathcal{A}^{(2)}_{*,\mathbb{N}(\mathcal{C})}\right)_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}(:,:,2,2)=\begin{bmatrix} 0 & 0\\ 0 & 0\end{bmatrix}.$$

This example can also be used to validate Theorem 2.3 (2), that is $\left(\mathcal{A}^{(2)}_{*,\mathbb{N}(\mathcal{C})}\right)_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}=\left(\mathcal{A}_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}\right)^{(2)}_{*,\mathbb{N}(\mathcal{C}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}})}$.

The computation of $\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}$ is explained in Algorithm 5.

Algorithm 5 Computation of $\mathcal{X}:=\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}$

.

Input: Positive integers m,n,k,l, and tensors $\mathcal{A}\in\mathbb{K}(\mathbf{x})^{M(m)\times N(n)}$, $\mathcal{B}\in\mathbb{K}(\mathbf{x})^{N(n)\times K(k)}$, $\mathcal{C}\in\mathbb{K}(\mathbf{x})^{L(l)\times M(m)}$.

Compute $\mathrm{rshrank}(\mathcal{C}*_m\mathcal{A}*_n\mathcal{B})$, $\mathrm{rshrank}(\mathcal{B})$ and $\mathrm{rshrank}(\mathcal{C})$ using Algorithm 1.
if $\mathrm{rshrank}(\mathcal{C})=\mathrm{rshrank}(\mathcal{C}*_m\mathcal{A}*_n\mathcal{B})=\mathrm{rshrank}(\mathcal{B})$ **then**
  Compute $\mathcal{Y}:=(\mathcal{C}*_m\mathcal{A}*_n\mathcal{B})^{(1)}$ using Algorithm 2.
  Compute $\mathcal{X}:=\mathcal{B}*_k\mathcal{Y}*_l\mathcal{C}$.
  return $\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}=\mathcal{X}$.
end if

Algorithm 5 is used in Example 2.3 to evaluate the specialization in the outer inverse $\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}\right)_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}$.

Example 2.3. Let $\mathbf{c} = (1, 2, 3, 4)$, and $\mathbf{z} = (z_1, z_2, z_3, z_4)$ with $z_i \neq 0$ for $1 \leq i \leq 4$. Suppose that the input is $\mathcal{A} \in (\mathbb{R}(z_1, z_2, z_3, z_4))^{(2\times 2)\times(2\times 2)}$ with entries

$$\mathcal{A}(:,:,1,1) = \begin{bmatrix} z_1 & z_2 \\ 0 & z_3 \end{bmatrix}, \mathcal{A}(:,:,2,1) = \begin{bmatrix} 0 & 0 \\ z_3 & 0 \end{bmatrix}, \mathcal{A}(:,:,1,2) = \begin{bmatrix} 0 & z_1 \\ 0 & 0 \end{bmatrix}, \mathcal{A}(:,:,2,2) = \begin{bmatrix} 0 & z_3 \\ 0 & z_4 \end{bmatrix},$$

$\mathcal{B} \in (\mathbb{R}(z_1, z_2, z_3, z_4))^{(2\times 2)\times(2\times 2)}$ with entries

$$\mathcal{B}(:,:,1,1) = \begin{bmatrix} z_1 & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{B}(:,:,2,1) = \begin{bmatrix} 0 & 0 \\ 0 & z_4 \end{bmatrix}, \mathcal{B}(:,:,1,2) = \begin{bmatrix} z_1 & 0 \\ 0 & z_3 \end{bmatrix}, \mathcal{B}(:,:,2,2) = \begin{bmatrix} 0 & z_3 \\ 0 & 0 \end{bmatrix},$$

and $\mathcal{C} \in (\mathbb{R}(z_1, z_2, z_3, z_4))^{(2\times 2)\times(2\times 2)}$ is defined as

$$\mathcal{C}(:,:,1,1) = \begin{bmatrix} z_4 & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{C}(:,:,2,1) = \begin{bmatrix} z_3 & 0 \\ z_1 & 0 \end{bmatrix}, \mathcal{C}(:,:,1,2) = \begin{bmatrix} 0 & 0 \\ z_3 & 0 \end{bmatrix}, \mathcal{C}(:,:,2,2) = \begin{bmatrix} z_2 & 0 \\ 0 & z_1 \end{bmatrix}.$$

Then, we can easily verify that $\mathrm{rshrank}(\mathcal{C} *_2 \mathcal{A} *_2 \mathcal{B}) = \mathrm{rshrank}(\mathcal{C}) = \mathrm{rshrank}(\mathcal{B}) = 3$, $\mathrm{rshrank}((\mathcal{C} *_2 \mathcal{A} *_2 \mathcal{B})_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}) = \mathrm{rshrank}(\mathcal{B}_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}) = \mathrm{rshrank}(\mathcal{C}_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}) = 3$, and

$$\mathrm{den}(\mathcal{B})(\mathbf{c}) \cdot \mathrm{den}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})(\mathbf{c}) \cdot \mathrm{den}\left((\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})^{(1)}\right)(\mathbf{c}) \cdot \mathrm{den}(\mathcal{C})(\mathbf{c}) = 45 \neq 0.$$

By Algorithm 5, we evaluate $\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})} = \mathcal{B} *_2 (\mathcal{C} *_2 \mathcal{A} *_2 \mathcal{B})^{(1)} *_2 \mathcal{C}$, where

$$\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}(:,:,1,1) = \begin{bmatrix} \frac{1}{z_1} & \frac{z_3^2 - z_2 z_4}{z_1^2 z_4} \\ 0 & -\frac{z_3}{z_1 z_4} \end{bmatrix}, \mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}(:,:,2,1) = \begin{bmatrix} \frac{z_3}{z_1 z_4} & \frac{z_1^2 z_4^2 + z_3^4 - z_2 z_3^2 z_4}{z_1^2 z_3 z_4^2} \\ 0 & -\frac{z_3^2}{z_1 z_4^2} \end{bmatrix},$$

$$\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}(:,:,1,2) = \begin{bmatrix} 0 & \frac{1}{z_1} \\ 0 & 0 \end{bmatrix}, \mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}(:,:,2,2) = \begin{bmatrix} 0 & -\frac{z_3}{z_1 z_4} \\ 0 & \frac{1}{z_4} \end{bmatrix}.$$

Further, the outer inverse at $\mathbf{z} = \mathbf{c}$ is $\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C}}\right)_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}} = \left(\mathcal{B} *_2 (\mathcal{C} *_2 \mathcal{A} *_2 \mathcal{B})^{(1)} *_2 \mathcal{C}\right)_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}$, where

$$\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}\right)_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}(:,:,1,1) = \begin{bmatrix} 1 & 1/4 \\ 0 & -3/4 \end{bmatrix}, \left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}\right)_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}(:,:,2,1) = \begin{bmatrix} 3/4 & 25/48 \\ 0 & -9/16 \end{bmatrix},$$

$$\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}\right)_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}(:,:,1,2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}\right)_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}(:,:,2,2) = \begin{bmatrix} 0 & -3/4 \\ 0 & 1/4 \end{bmatrix}.$$

Using Theorem 2.3 (3), we can verify that $\left(\mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}\right)_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}} = \left(\mathcal{A}_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}\right)^{(2)}_{\mathbb{R}(\mathcal{B}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}),\mathbb{N}(\mathcal{C}_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}})}$.

3 Computing tensor generalized inverses with functional entries

This section is devoted to the case of tensors whose entries are rational functional expressions. Let $\Omega \subset \mathbb{C}$ be a connected open subset of the field of complex numbers, and let $\mathrm{Mer}(\Omega)$ denote the set of all meromorphic functions over Ω. Let $\mathcal{F} = \{f_1(z), \ldots, f_p(z)\} \subset \mathrm{Mer}(\Omega)$. We denote by $\mathbf{f}$ the p-tuple $\mathbf{f} = (f_1(z), \ldots, f_p(z))$. We consider the field extension $\mathbb{C}(\mathcal{F})$, that is, the field containing all rational expressions involving the elements in $\mathbf{f}$. More precisely,

$$\mathbb{C}(\mathcal{F}) = \left\{ \frac{P(\mathbf{f})}{Q(\mathbf{f})} \;\middle|\; P, Q \in \mathbb{C}[w_1, \ldots, w_p] \text{ and } Q(\mathbf{f}) \neq 0. \right\}$$

In the sequel we fix $\mathcal{F}$, and hence $\mathbf{f}$.

Let us consider the set

$$\mathbb{C}(\mathbf{x})_{\mathcal{F}}^{m\times n} = \{A \in \mathbb{C}(\mathbf{x})^{m\times n} \,|\, \mathrm{den}(A)(\mathbf{f}) \neq 0\}.$$

The transformation *Rat*, that converts a matrix with functional elements into a matrix with entries given as a fraction of polynomials, was proposed in [29]. It is defined as

$$\begin{array}{rccl} \mathrm{Rat}: & \mathbb{C}(\mathcal{F})^{m\times n} & \longrightarrow & \mathbb{C}(\mathbf{x})_{\mathcal{F}}^{m\times n} \\ & A = (a_{i,j}(\mathbf{f})) & \longmapsto & \mathrm{Rat}(A) = A_{|\mathbf{f}\leftrightsquigarrow\mathbf{x}} = (a_{i,j}(\mathbf{x})), \end{array}$$

and its inverse map is

$$\begin{array}{rccl} \mathrm{Func}: & \mathbb{C}(\mathbf{x})_{\mathcal{F}}^{m\times n} & \longrightarrow & \mathbb{C}(\mathcal{F})^{m\times n} \\ & A = (a_{i,j}(\mathbf{x})) & \longmapsto & \mathrm{Func}(A) = A_{|\mathbf{x}\leftrightsquigarrow\mathbf{f}} = (a_{i,j}(\mathbf{f})). \end{array}$$

The following result was originated in [29] (see Section 5 in [29] for the definition of the involutory automorphism used for meromorphic functions and for the notion of adjunction in the Moore-Penrose inverse; see also Lemma 5.3 in [34]). But first, we recall the notion of algebraic dependence. Let $K \subset L$ be two fields, and $\{\alpha_1,\dots,\alpha_n\} \subset L$. $\{\alpha_1,\dots,\alpha_n\}$ are called algebraically dependent over K if there exists a non-zero polynomial $P \in K[w_1,\dots,w_n]\setminus\{0\}$ such that $P(\alpha_1,\dots,\alpha_n)=0$; otherwise, we say that $\{\alpha_1,\dots,\alpha_n\}$ are algebraically independent over K. For instance, let us consider the complex functions $F_1(z)=\sin(z), F_2(z)=\cos(z)$ as elements of $\mathrm{Mer}(\mathbb{C})$. Then, $\{F_1(z),F_2(z)\}$ is algebraically dependent over $\mathbb{C}$, because taking the polynomial $P(w_1,w_2) := w_1^2+w_2^2-1 \in \mathbb{C}[w_1,w_2]$ it holds that $P(F_1(z),F_2(z))=0$.

Proposition 3.1. [29] *Let* $A \in \mathbb{C}(\mathcal{F})^{m\times n}$. *If*

$$\mathrm{den}(\mathrm{Rat}(A)^{\dagger})(\mathbf{f}) \neq 0,$$

then $A^{\dagger}$ *exists and*

$$(\mathrm{Rat}(A)^{\dagger})_{|\mathbf{x}\leftrightsquigarrow\mathbf{f}} = A^{\dagger}.$$

In this section we extend the concepts and results for the case of tensors with functional entries. Using the rationalization, as well as the reshape function, we discuss the symbolic computation of generalized inverses of tensors with entries in $\mathbb{C}(\mathcal{F})$ as follows. We consider the analogous function *Rati* that transforms a tensor with functional entries into another tensor with fractional elements. For this purpose, first, we introduce the set

$$\mathbb{C}(\mathbf{x})_{\mathcal{F}}^{M(m)\times N(n)} = \{\mathcal{X} \in \mathbb{C}(\mathbf{x})^{M(m)\times N(n)} \,|\, \mathbf{den}(\mathcal{X})(\mathbf{f}) \neq 0\}.$$

In general, the functions included in $\mathcal{F}$ could be algebraically dependent. In that case $\mathbb{C}(\mathbf{x})_{\mathcal{F}}^{M(m)\times N(n)} \subsetneq \mathbb{C}(\mathbf{x})^{M(m)\times N(n)}$. In this situation, the map *Rati* is defined as

$$\begin{array}{rccl} \mathrm{Rati}: & \mathbb{C}(\mathcal{F})^{M(m)\times N(n)} & \longrightarrow & \mathbb{C}(\mathbf{x})_{\mathcal{F}}^{M(m)\times N(n)} \\ & \mathcal{A} = (a_{i(m),j(n)}(\mathbf{f})) & \longmapsto & \mathrm{Rati}(\mathcal{A}) = \mathcal{A}_{|\mathbf{f}\leftrightsquigarrow\mathbf{x}} = (a_{i(m),j(n)}(\mathbf{x})). \end{array}$$

Clearly, the transformation *Spec* (see Subsection 2.3) is not invertible. On the other hand, the rationalization transformation *Rati* is invertible. More precisely, the inverse mapping, termed as *functionalization*, is defined as:

$$\begin{array}{rccl} \mathrm{Functi}: & \mathbb{C}(\mathbf{x})_{\mathcal{F}}^{M(m)\times N(n)} & \longrightarrow & \mathbb{C}(\mathcal{F})^{M(m)\times N(n)} \\ & \mathcal{A} = (a_{i(m),j(n)}(\mathbf{x})) & \longmapsto & \mathrm{Functi}(\mathcal{A}) = \mathcal{A}_{|\mathbf{x}\leftrightsquigarrow\mathbf{f}} = (a_{i(m),j(n)}(\mathbf{f})). \end{array}$$

Note that

$$\forall \mathcal{A} \in \mathbb{C}(\mathcal{F})_{\mathcal{F}}^{M(m)\times N(n)}, \mathrm{Rati}(\mathrm{Functi}(\mathcal{A})) = \mathcal{A}.$$

Summarizing, the maps introduced above provide the commutative diagram

$$\begin{array}{ccc} \mathbb{C}(\mathcal{F})^{\mathfrak{M}\times\mathfrak{N}} & \underset{\text{Func}}{\overset{\text{Rat}}{\rightleftarrows}} & \mathbb{C}(\mathbf{x})_{\mathcal{F}}^{\mathfrak{M}\times\mathfrak{N}} \\ \mathrm{rsh}_{\mathcal{F}}\uparrow & & \uparrow\mathrm{rsh}_{\mathbf{x}} \\ \mathbb{C}(\mathcal{F})^{M(m)\times N(n)} & \underset{\text{Functi}}{\overset{\text{Rati}}{\rightleftarrows}} & \mathbb{C}(\mathbf{x})_{\mathcal{F}}^{M(m)\times N(n)}. \end{array}$$

Note that, in (2.2), the map *rsh* was introduced for an arbitrary field $\mathbb{K}$. Here, since we are dealing with two different fields, mainly $\mathbb{C}(\mathcal{F})$ and $\mathbb{C}(\mathbf{x})$, we use, respectively, the notation $\mathrm{rsh}_{\mathcal{F}}$ and $\mathrm{rsh}_{\mathbf{x}}$ to distinguish between them. The following result follows immediately from the previous diagram.

Proposition 3.2. *Let* $\mathcal{A} \in \mathbb{C}(\mathcal{F})^{M(m)\times N(n)}$*, and* $A \in \mathbb{C}(\mathbf{x})_{\mathcal{F}}^{\mathfrak{M}\times\mathfrak{N}}$*. It holds:*

(1) $\mathrm{rsh}_{\mathbf{x}}(\mathrm{Rati}(\mathcal{A})) = \mathrm{Rat}(\mathrm{rsh}_{\mathcal{F}}(\mathcal{A}))$.

(2) $\mathrm{rsh}_{\mathcal{F}}^{-1}(\mathrm{Func}(A)) = \mathrm{Functi}(\mathrm{rsh}_{\mathbf{x}}^{-1}(A))$.

Using the commutativity of the diagram above, or equivalently, Proposition 3.2, we conclude the following result.

Corollary 3.1. *If* $\mathcal{A} \in \mathbb{C}(\mathcal{F})^{M(m)\times N(n)}$*, then* $\mathcal{A}^{(1)} \neq \emptyset$*.*

Proof. Let $M = \mathrm{Func}(\mathrm{rsh}_{\mathbf{x}}(\mathrm{Rati}(\mathcal{A}))) \in \mathbb{C}(\mathcal{F})^{\mathfrak{M}\times\mathfrak{N}}$. Since $\mathbb{C}(\mathcal{F})$ is a field, by Lemma 2.1 in [33], we know that $M^{(1)}$ exists. Now, the result follows applying $\mathrm{rsh}_{\mathcal{F}}^{-1}$ and Lemma 2.1. □

Lemma 3.1 improves the results of Proposition 3.1 and Corollary 3.1, and will be applied to prove Theorem 3.1.

Lemma 3.1. *Let* $\mathcal{A} \in \mathbb{C}(\mathcal{F})^{M(m)\times N(n)}$*, with* $\mathbf{x} = (x_1, \ldots, x_p)$*, and let* $\mathbf{f} = (f_1(z), \ldots, f_p(z))$*. If an inner inverse* $\mathcal{A}^{(1)}$ *satisfies*

$$\mathrm{den}(\mathcal{A})(\mathbf{f})\,\mathrm{den}(\mathcal{A}^{(1)})(\mathbf{f}) \neq 0,$$

then

(1) $(\mathcal{A}^{(1)})_{|\mathbf{f}\leftrightsquigarrow\mathbf{x}} = \left(\mathcal{A}_{|\mathbf{f}\leftrightsquigarrow\mathbf{x}}\right)^{(1)}$.

(2) $\left((\mathcal{A}^{(1)})_{|\mathbf{f}\leftrightsquigarrow\mathbf{x}}\right)_{|\mathbf{x}\leftrightsquigarrow\mathbf{f}} = \mathcal{A}^{(1)}$.

Proof. Statement (1) can be verified similarly as in Lemma 2.3. In order to prove (2), we rewrite the statement as $\mathrm{Rati}(\mathcal{A}^{(1)}) = (\mathrm{Rati}(\mathcal{A}))^{(1)}$. So, taking the inverse of Rati, we get

$$\mathcal{A}^{(1)} = \mathrm{Functi}(\mathrm{Rat}(\mathcal{A}^{(1)})) = \mathrm{Functi}((\mathrm{Rati}(\mathcal{A}))^{(1)}) = \left((\mathcal{A}^{(1)})_{|\mathbf{f}\leftrightsquigarrow\mathbf{x}}\right)_{|\mathbf{x}\leftrightsquigarrow\mathbf{f}}.$$

□

Using Theorem 2.1, and the previous results, the following result can be proved.

Theorem 3.1. *Let* $\mathcal{A} \in \mathbb{C}(\mathcal{F})^{M(m)\times N(n)}$, $\mathcal{B} \in \mathbb{C}(\mathcal{F})^{N(n)\times K(k)}$, $\mathcal{C} \in \mathbb{C}(\mathcal{F})^{L(l)\times M(m)}$ *be such that*

$$\mathrm{den}(\mathrm{Rati}(\mathcal{A}))(\mathbf{f}) \cdot \mathrm{den}(\mathrm{Rati}(\mathcal{B}))(\mathbf{f}) \cdot \mathrm{den}(\mathrm{Rati}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})^{(1)}))(\mathbf{f}) \cdot \mathrm{den}(\mathrm{Rati}(\mathcal{C}))(\mathbf{f}) \neq 0. \tag{3.1}$$

Let $\mathcal{X} = \mathcal{B} *_k (\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})^{(1)} *_l \mathcal{C}$*. Then, it holds that*

(1) *If* $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{B})$*, and* $\mathrm{rshrank}(\mathrm{Rati}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})) = \mathrm{rshrank}(\mathrm{Rati}(\mathcal{B}))$*, then*

$$\mathcal{X} = \left((\mathcal{A}_{|\mathbf{f}\leftrightsquigarrow\mathbf{x}})^{(2)}_{\mathbb{R}(\mathcal{B}_{|\mathbf{f}\leftrightsquigarrow\mathbf{x}}),*}\right)_{|\mathbf{x}\leftrightsquigarrow\mathbf{f}} = \mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),*}.$$

(2) *If* $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{C})$ *and* $\mathrm{rshrank}(\mathrm{Rati}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})) = \mathrm{rshrank}(\mathrm{Rati}(\mathcal{C}))$*, then*

$$\mathcal{X} = \left((\mathcal{A}_{|\mathbf{f}\leftrightsquigarrow\mathbf{x}})^{(2)}_{*,\mathbb{N}(\mathcal{C}_{|\mathbf{f}\leftrightsquigarrow\mathbf{x}})}\right)_{|\mathbf{x}\leftrightsquigarrow\mathbf{f}} = \mathcal{A}^{(2)}_{*,\mathbb{N}(\mathcal{C})}.$$

(3) *If* $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{B}) = \mathrm{rshrank}(\mathcal{C})$ *and* $\mathrm{rshrank}(\mathrm{Rati}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})) = \mathrm{rshrank}(\mathrm{Rati}(\mathcal{B})) = \mathrm{rshrank}(\mathrm{Rati}(\mathcal{C}))$, *then*

$$\mathcal{X} = \left((\mathcal{A}_{|\mathbf{f} \rightsquigarrow \mathbf{x}})^{(2)}_{\mathbb{R}(\mathcal{B}_{|\mathbf{f} \rightsquigarrow \mathbf{x}}),\mathbb{N}(\mathcal{C}_{|\mathbf{f} \rightsquigarrow \mathbf{x}})} \right)_{|\mathbf{x} \rightsquigarrow \mathbf{f}} = \mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}.$$

(4) *If* $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{B}) = \mathrm{rshrank}(\mathcal{C}) = \mathrm{rshrank}(\mathcal{A})$ *and* $\mathrm{rshrank}(\mathrm{Rati}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})) = \mathrm{rshrank}(\mathrm{Rati}(\mathcal{B})) = \mathrm{rshrank}(\mathrm{Rati}(\mathcal{C})) = \mathrm{rshrank}(\mathrm{Rati}(\mathcal{A}))$, *then*

$$\mathcal{X} = \left((\mathcal{A}_{|\mathbf{f} \rightsquigarrow \mathbf{x}})^{(1,2)}_{\mathbb{R}(\mathcal{B}_{|\mathbf{f} \rightsquigarrow \mathbf{x}}),\mathbb{N}(\mathcal{C}_{|\mathbf{f} \rightsquigarrow \mathbf{x}})} \right)_{|\mathbf{x} \rightsquigarrow \mathbf{f}} = \mathcal{A}^{(1,2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}.$$

Proof. We prove statement (3). The other cases follow similarly. Since $\mathrm{rshrank}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B}) = \mathrm{rshrank}(\mathcal{B}) = \mathrm{rshrank}(\mathcal{C})$, by Theorem 2.1, we get that

$$\mathcal{X} = \mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}.$$

On the other hand, since $\mathrm{rshrank}(\mathrm{Rati}(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})) = \mathrm{rshrank}(\mathrm{Rati}(\mathcal{B})) = \mathrm{rshrank}(\mathrm{Rati}(\mathcal{C}))$, by Theorem 2.1, one has that

$$\mathrm{Rati}(\mathcal{B}) *_k (\mathrm{Rati}(\mathcal{C}) *_m \mathrm{Rati}(\mathcal{A}) *_n \mathrm{Rati}(\mathcal{B}))^{(1)} *_l \mathrm{Rati}(\mathcal{C}) = (\mathrm{Rati}(\mathcal{A}))^{(2)}_{\mathbb{R}(\mathrm{Rati}(\mathcal{B})),\mathbb{N}(\mathrm{Rati}(\mathcal{C}))}.$$

By Lemma 3.1, we have that

$$\mathrm{Rati}(\mathcal{B}) *_k \mathrm{Rati}((\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})^{(1)}) *_l \mathrm{Rati}(\mathcal{C}) = (\mathrm{Rati}(\mathcal{A}))^{(2)}_{\mathbb{R}(\mathrm{Rati}(\mathcal{B})),\mathbb{N}(\mathrm{Rati}(\mathcal{C}))}.$$

Hence

$$\mathcal{X} = \mathrm{Rati}(\mathcal{B} *_k (\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})^{(1)} *_l \mathcal{C}) = (\mathrm{Rati}(\mathcal{A}))^{(2)}_{\mathbb{R}(\mathrm{Rati}(\mathcal{B})),\mathbb{N}(\mathrm{Rati}(\mathcal{C}))}.$$

Applying the inverse function Functi we get

$$\mathcal{X} = \mathrm{Functi}\left((\mathrm{Rati}(\mathcal{A}))^{(2)}_{\mathbb{R}(\mathrm{Rati}(\mathcal{B})),\mathbb{N}(\mathrm{Rati}(\mathcal{C}))} \right) = \left((\mathcal{A}_{|\mathbf{f} \rightsquigarrow \mathbf{x}})^{(2)}_{\mathbb{R}(\mathcal{B}_{|\mathbf{f} \rightsquigarrow \mathbf{x}}),\mathbb{N}(\mathcal{C}_{|\mathbf{f} \rightsquigarrow \mathbf{x}})} \right)_{|\mathbf{x} \rightsquigarrow \mathbf{f}}.$$

This concludes the proof. □

As a consequence of Theorem 3.1, we derive Algorithm 6, that computes outer inverses of tensors with rational functional entries. The main idea of the algorithm is based on the fact that the specialization and functionalization are inverse operations. Moreover, symbolic computation on rational entries of unknown variables is much simpler than the analogous computation with rational entries involving functions. So Algorithm 6 is based on three main steps:

- replace each new function by a new variable,
- perform necessary computations on matrices with rational expressions, and
- replace unknown variables in the result obtained in the previous step by their original functional pair.

The detailed algorithm is given as follows.

Algorithm 6 Computation of outer inverses of tensors with functional entries.

Input: Subset $\mathcal{F} = \{f_1(\mathbf{z}), \ldots, f_p(\mathbf{z})\} \subset \mathrm{Mer}(\Omega)$ of self-adjoint functions, and $\mathcal{A} \in \mathbb{C}(\mathcal{F})^{M(m)\times N(n)}$, $\mathcal{B} \in \mathbb{C}(\mathcal{F})^{N(n)\times K(k)}$, $\mathcal{C} \in \mathbb{C}(\mathcal{F})^{L(l)\times M(m)}$.

1: Compute $A = \mathrm{rsh}(\mathcal{A}),\ B = \mathrm{rsh}(\mathcal{B}),\ C = \mathrm{rsh}(\mathcal{C})$.
2: Compute $Y := \mathrm{Rat}(B)\,(\mathrm{Rat}(C)\mathrm{Rat}(A)\mathrm{Rat}(B))^{(1)}\,\mathrm{Rat}(C)$ using, for instance, Algorithm 5.
3: Set $X = Y_{|\mathbf{x} \rightsquigarrow \mathbf{f}}$.
4: Compute $\mathcal{X} = \mathrm{rsh}^{-1}(X)$.
5: Return $\mathcal{X} = \mathcal{A}^{(2)}_{\mathbb{R}(\mathcal{B}),\mathbb{N}(\mathcal{C})}$.

Example 3.1. Let $\mathcal{A} = (a_{ijkl}) \in (\mathbb{C}(\mathcal{F}))^{(2\times 2)\times(2\times 2)}$ with entries

$$a_{ij11} = \begin{bmatrix} 0 & \frac{i\sin(z)}{\cos(z)+i\sin(z)} \\ 0 & 0 \end{bmatrix},\ a_{ij12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ a_{ij21} = \begin{bmatrix} 0 & 0 \\ \frac{\cos(z)\sin(z)+i\sin(z)}{\cos(z)+i\sin(z)} & 0 \end{bmatrix}, a_{ij22} = \begin{bmatrix} \frac{i\sin(z)}{\cos(z)-i\sin(z)} & 0 \\ 0 & 0 \end{bmatrix},$$

in conjunction with the tensors $\mathcal{B} = (b_{ijkl}) \in (\mathbb{C}(\mathcal{F}))^{(2\times 2)\times(2\times 2)}$ with entries

$$b_{ij11} = \begin{bmatrix} 0 & \frac{i\sin(z)}{\cos(z)+i\sin(z)} \\ 0 & 0 \end{bmatrix},\ b_{ij12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = b_{ij22},\ b_{ij21} = \begin{bmatrix} \frac{i\sin(z)}{\cos(z)-i\sin(z)} & 0 \\ 0 & \frac{i\sin(z)}{\cos(z)-i\sin(z)} \end{bmatrix},$$

and $\mathcal{C} = (c_{ijkl}) \in (\mathbb{C}(\mathcal{F}))^{(2\times 2)\times(2\times 2)}$ with entries

$$c_{ij11} = \begin{bmatrix} \frac{i\sin(z)}{\cos(z)+i\sin(z)} & 0 \\ \frac{\cos(z)\sin(z)+i\sin(z)}{\cos(z)-i\sin(z)} & 0 \end{bmatrix},\ c_{ij12} = \begin{bmatrix} 0 & \frac{ie^z}{\cos(z)+i\sin(z)} \\ 0 & 0 \end{bmatrix},\ c_{ij21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = c_{ij22}.$$

For the computation of the inner inverse in Step 2 of Algorithm 6, we will use the Moore-Penrose approach. For this purpose, we observe that the functions appearing in the example are self-adjoint (see Def. 7 in [29]).

After the replacement $\mathbf{f} \rightsquigarrow \mathbf{x}$, given by $\{\cos(z) \to x_1, \sin(z) \to x_2, e^z \to x_3\}$ we obtain the tensors $\mathrm{Rati}(\mathcal{A})$, $\mathrm{Rati}(\mathcal{B})$, and $\mathrm{Rati}(\mathcal{C})$ with rational entries, respectively:

$$\mathrm{Rati}(a_{ij11}) = \begin{bmatrix} 0 & \frac{ix_2}{x_1+ix_2} \\ 0 & 0 \end{bmatrix},\ \mathrm{Rati}(a_{ij12}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ \mathrm{Rati}(a_{ij21}) = \begin{bmatrix} 0 & 0 \\ \frac{x_1x_2+ix_2}{x_1+ix_2} & 0 \end{bmatrix}, \mathrm{Rati}(a_{ij22}) = \begin{bmatrix} \frac{ix_2}{x_1-ix_2} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathrm{Rati}(b_{ij11}) = \begin{bmatrix} 0 & \frac{ix_2}{x_1+ix_2} \\ 0 & 0 \end{bmatrix},\ \mathrm{Rati}(b_{ij12}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathrm{Rati}(b_{ij22}),\ \mathrm{Rati}(b_{ij21}) = \begin{bmatrix} \frac{ix_2}{x_1-ix_2} & 0 \\ 0 & \frac{ix_2}{x_1-ix_2} \end{bmatrix},$$

and

$$\mathrm{Rati}(c_{ij11}) = \begin{bmatrix} \frac{ix_2}{x_1+ix_2} & 0 \\ \frac{x_1x_2+ix_2}{x_1-ix_2} & 0 \end{bmatrix},\ \mathrm{Rati}(c_{ij12}) = \begin{bmatrix} 0 & \frac{ix_3}{x_1+ix_2} \\ 0 & 0 \end{bmatrix},\ \mathrm{Rati}(c_{ij21}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathrm{Rati}(c_{ij22}).$$

Now $\mathcal{Y} := \mathrm{Rati}(\mathcal{B}) *_k (\mathrm{Rati}(\mathcal{C}) *_m \mathrm{Rati}(\mathcal{A}) *_n \mathrm{Rati}(\mathcal{B}))^\dagger *_l \mathrm{Rati}(\mathcal{C})$ is defined by the entries

$$y_{ij11} = \begin{bmatrix} \frac{-(x_2(x_1^2+2)(x_2+ix_1))}{(x_1^2x_2^2+2x_2^2+x_3^2)} & 0 \\ 0 & \frac{-(x_2(x_1^2+2)(x_2+ix_1))}{(x_1^2x_2^2+2x_2^2+x_3^2)} \end{bmatrix},\ y_{ij21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = y_{ij22},$$

$$y_{ij12} = \begin{bmatrix} \frac{-(x_3^2(ix_1-x_2))}{(x_1^2x_2^3+2x_2^3+x_2x_3^2)} & 0 \\ 0 & \frac{-(x_3^2(ix_1-x_2))}{(x_1^2x_2^3+2x_2^3+x_2x_3^2)} \end{bmatrix}.$$

One can verify that $\mathcal{X} = \mathcal{Y}_{|\mathbf{x} \rightsquigarrow \mathbf{f}}$ which entries are

$$x_{ij11} = \begin{bmatrix} \frac{-(\sin(z)(\sin(z)+i\cos(z))(\cos(z)^2+2))}{(e^{2z}+2\sin(z)^2+\cos(z)^2\sin(z)^2)} & 0 \\ 0 & \frac{-(\sin(z)(\sin(z)+i\cos(z))(\cos(z)^2+2))}{(e^{2z}+2\sin(z)^2+\cos(z)^2\sin(z)^2)} \end{bmatrix},\ x_{ij21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = x_{ij22},$$

$$x_{ij12} = \begin{bmatrix} \frac{-(e^{2z}(i\cos(z)-\sin(z)))}{(2\sin(z)^3+e^{2z}\sin(z)+\cos(z)^2\sin(z)^3)} & 0 \\ 0 & \frac{-(e^{2z}(i\cos(z)-\sin(z)))}{(2\sin(z)^3+e^{2z}\sin(z)+\cos(z)^2\sin(z)^3)} \end{bmatrix}.$$

The condition (3.1) is satisfied, which implies that $\mathcal{X} = \mathcal{Y}_{|\mathbf{x} \rightsquigarrow \mathbf{f}} = \mathcal{B} *_k (\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})^\dagger *_l \mathcal{C}$.

4 Conclusion

We investigate outer generalized inverses, with prescribed range and null space, of tensors with rational function entries equipped with the Einstein product over an arbitrary field of characteristic zero. The involved tensor entries are defined as either rational entries of unassigned variables or as rational expressions with functional entries. Properties of generalized inverses with appropriate replacements, termed as specialization, rationalization, and functionalization are considered. The research investigates replacements in two stages. The lower-stage replacements assume replacements of unknown variables by constant values from the field. The higher-order stage assumes replacements of functional entries by unknown variables. This replacement is invertible, and enables the simplification of the computation of tensor generalized inverses, over meromorphic functions, by the analogous calculations on matrices whose elements are rational functions. Derived algorithms are designed for computing the tensor generalized inverses over an arbitrary field, of characteristic zero, symbolically. Several numerical examples worked out to validate the results in the matrix, and in the tensor case.

Possible further research includes mainly the following topics.

1. Solve tensor equations required to find the inner inverses $(\mathcal{C} *_m \mathcal{A} *_n \mathcal{B})^{(1)}$.
    (a) One possibility is to use the Recurrent Neural Network (RNN) approach to solve the tensor equations by generalizing the algorithms developed in [35], from the matrix to the tensor case. Such possibility is supported by the recent application of neural network techniques to solving the Sylvester tensor equation in [22].
    (b) Another approach is to use the possibility of symbolic software packages in order to solve the required tensor equations symbolically.
    (c) A third possibility is the application of iterative algorithms for solving some tensor equations proposed in [41].
2. Develop an effective algorithm for computing outer tensor inverses using the (tensor and matrix) QR decomposition.
3. Our goal is to consider symbolic entries as time-varying functions, which enables the development of the corresponding applicable algorithms on time-varying tensors. Then, it would be possible to compute tensor generalized inverses using dynamical systems and RNN methods.
4. It is interesting to study weighted generalized inverses of tensors [4, 14, 19].

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