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Representations and geometrical properties of generalized inverses over fields

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Abstract

In this paper, as a generalization of Urquhart's formulas, we present a full description of the sets of inner inverses and (B, C)-inverses over an arbitrary field. In addition, identifying the matrix vector space with an affine space, we analyze geometric properties of the main generalized inverse sets. We prove that the set of inner inverses, and the set of (B, C)-inverses, form affine subspaces and we study their dimensions. Furthermore, under some hypotheses, we prove that the set of outer inverses is not an affine subspace but it is an affine algebraic variety. We also provide lower and upper bounds for the dimension of the outer inverse set.

Keywords: outer inverses; inner inverses; Moore-Penrose inverse; (B, C) inverse; affine subspaces; Urquhart's formula.

Mathematics Subject Classification: 15A09

1 Introduction

Generalized inverses for a given $m \times n$ matrix A over a field are those $n \times m$ matrices X, over the same field, satisfying some of the, so called, Penrose equations that appear in (2.1) and (2.2) in Section 2. Depending on the required conditions from (2.1) and (2.2), different generalized inverses appear. Some of the most *famous* generalized inverses are the Moore-Penrose inverse, the Drazin inverse, Core inverses, group inverses, or the inner and outer inverses (see e.g. [1], [15], [16], [17], [33]). In addition, one may introduce a fixed $n \times k$ matrix B and/or a fixed $n \times l$ matrix C and ask the generalized inverse to have the same range as B and/or the same kernel as C; see (2.3) in Section 2. In this way, the (B, C)-inverses appear (see e.g. [2], [5], [13], [23], [24], [34], [36], [37], [38] for further details).

These matrices have turned to be important in applications (see e.g. [8], [9], [12], [14], [32]) and many authors have addressed the study of this type of matrices, both, from the theoretical point of view (see e.g. [1], [6], [15], [33]) and from the computational point of view (see e.g. [4] [7], [9], [11], [18], [19], [20], [21], [22], [25], [26], [27], [28]). Additionally, some authors have also studied the so called representation of generalized inverses (see e.g. [5], [13], [26], [34], [36], [37]). Intuitively, the idea is to analyze the existence, to characterize, and to study relationships among different types of pseudoinverses. One may say that the starting point of this branch is the well known Urquhart's formula, where outer inverses with prescribed

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null and/or range spaces, and expressible of certain particular form, are related to inner inverses (see [30] and [31]).

In this paper, we work over an arbitrary field $\mathbb{K}$, and we focus on two different aspects: the representation problem and the analysis of geometric problems. More precisely, we present a generalization of Urquhart's formulas showing that all elements in the corresponding generalized inverse sets are expressible as Urquhart Theorem requires. As a consequence, we provide a full description of the sets of (B, C)-inverses (see Theorem 3.2). The main reference for our reasonings is [13]. In the second part of the paper, using the derived description of the (B, C)-inverse sets, we study some geometric properties. For this purpose, we consider the matrix vector space $\mathbb{K}^{m\times n}$, respectively $\mathbb{K}^{n\times m}$, as an affine space. This allows us to see a matrix as a point in an affine space, and each set of generalized inverses as an affine subset. Therefore, the natural question on whether the generalized inverse sets are affine (non necessarily linear) varieties appears and, if so, the investigation on their main geometric properties. The knowledge of the geometric properties provides important information for the theoretical analysis and for the computational aspects, since they describe how simple or complex is each set of generalized inverses. To our knowledge, this paper is the first one dealing with geometric classifications and geometric properties of generalized inverses.

The study of geometric properties starts with the case of inner inverses. With this interpretation of matrices as points in an affine space, it holds that $A\{1\}$ is a linear affine subspace of dimension $mn - \text{rank}(A)^2$ (see Theorem 4.1). Observe that this result is a re-interpretation, for the case of $\mathbb{K} = \mathbb{C}$, of Corollary 1 in [1], page 52; see also [31] and [29]. For the outer inverses, when $\mathbb{K}$ is not isomorphic to $\mathbb{Z}_2$ and A is not zero matrix, we prove that $A\{2\}$ is not an affine subspace but it is an affine algebraic variety different of the whole space (see Lemma 4.1). In addition, we state lower bounds for $\dim(A\{2\})$ (see Theorem 4.3) and we prove that, when A is not the zero matrix, the $\dim(A\{2\}) > 0$ which implies that $A\{2\}$ has infinitely many elements. In addition, the illustrating examples show that, in general, $A\{2\}$ is reducible with components of different dimensions. (B, C)-inverses are also studied. More precisely, we state that outer inverses with predefined null space and/or range are linear subspaces and we provide lower and upper bounds for the dimension (see Theorem 4.2). Moreover, for the case of full-column rank (for B) and of full-row rank (for C), we give the precise dimension of the corresponding affine subspaces. These results certify that, in general, the computation of inner inverses and (B, C)-inverses can be approached directly by means of linear algebra techniques. However, the computation and description of $A\{2\}$ require, in general, techniques from elimination theory as Gröbner bases.

The global structure of the paper is as follows. In Section 2 we recall the basic definitions and start with the analysis of some basic properties. In Section 3 we recall Urquhart Theorem and we discuss some of its possible generalizations. Section 4 is devoted to the geometric study of generalized inverses. We finish the paper with a brief section where the main conclusions are summarized.

Notation. Throughout this paper we will use the following notation. Let $\mathbb{K}$ be a field. We denote by $\mathbb{K}^{m\times n}$ (resp. by $\mathbb{K}_r^{m\times n}$) the ring of $m \times n$ matrices over $\mathbb{K}$ (resp. the set of $m \times n$ matrices over $\mathbb{K}$ of rank r). As usual, I and $\mathbf{0}$ denote, respectively, the unit matrix and the zero matrix of an appropriate order. The transpose of A is denoted by A^{T}. Furthermore, we denote by $\mathcal{R}(A)$, $\text{rank}(A)$ and $\mathcal{N}(A)$ the range space, the rank, and the null space of $A \in \mathbb{K}^{m\times n}$, respectively. Further, $\mathbb{Z}_2$ denotes the finite field with two elements.

2 Generalized inverses over arbitrary fields

This section is devoted to introduce the basic notions of generalized inverses and (B, C)-inverses over an arbitrary field. In addition, we discuss some properties as the existence of these type of matrices.

Let $A \in \mathbb{K}^{m\times n}$ be a fixed matrix. The problem of pseudoinverses computation leads to the determination of matrices $X \in \mathbb{K}^{n\times m}$ satisfying some of the, so called, Penrose equations [1]

[1]For equations (3) and (4), we assume that $\mathbb{K}$ is endowed with an involutory automorphism φ so that, for a matrix M over

$$(1)\quad AXA = A \qquad (2)\quad XAX = X \qquad (3)\quad (AX)^* = AX \qquad (4)\quad (XA)^* = XA. \tag{2.1}$$

in combination with the equations

$$(1^k)\quad A^{k+1}X = A^k \quad k \geq \mathrm{ind}(A), \qquad (5)\quad AX = XA, \tag{2.2}$$

where $\mathrm{ind}(A) = \min \left\{ j |\ \mathrm{rank}(A^j) = \mathrm{rank}(A^{j+1}) \right\}$, under the assumption that A is square. Condition (1^k) is equivalent indeed to ask that $A^{\mathrm{ind}(A)+1}X = A^{\mathrm{ind}(A)}$ (see e.g. [37], Proposition 1).

For a given $A \in \mathbb{K}^{m\times n}$ and $\mathcal{S} \subseteq \{1, 2, 3, 4, 1^k, 5\}$, we denote by $A\{\mathcal{S}\}$ the set of all $X \in \mathbb{K}^{n\times m}$ satisfying the equation (i) for each $i \in \mathcal{S}$. An arbitrary matrix from $A\{\mathcal{S}\}$ is called an $\{\mathcal{S}\}$-inverse of A and it is designated as $A^{(\mathcal{S})}$. Similarly, $A\{\mathcal{S}\}_s$ denotes all $\{\mathcal{S}\}$-inverses of A of rank s.

Matrices in $A\{1\}$ are called *inner inverses* of A. Let see that, in our case, inner inverses always exist.

Lemma 2.1. *The set of inner inverses of $A \in \mathbb{K}^{m\times n}$ satisfies that $A\{1\} \neq \emptyset$.*

Proof. First, assume that $A \in \mathbb{K}^{m\times m}$, for some $m \geq 1$. Then, using that a field is a regular (Von Neumann) ring, see e.g. [10] pg. 110, one deduces that $\mathbb{K}^{m\times m}$ is regular (see Theorem 24, pag 114 in [10]), and hence $A\{1\} \neq \emptyset$.

In the general case, i.e. $A \in \mathbb{K}^{m\times n}$, let us consider the case $m < n$ first. In this situation, let

$$\overline{A} = \begin{pmatrix} A \\ \mathbf{0} \end{pmatrix}$$

be the $n \times n$ matrix obtained by attaching $n - m$ zero rows below A, and let Y be the $n \times n$ inner inverse of $\overline{A}$. Then, it holds that the $n \times m$ principal submatrix of Y is an inner matrix of A.

If $m > n$, then one may consider the transpose matrix to conclude the result. □

Matrices in $A\{2\}$ are called *outer inverses* of A.

Lemma 2.2. *The set of outer inverses of $A \in \mathbb{K}^{m\times n}$ satisfies $A\{2\} \neq \emptyset$. Moreover, $A\{2\}$ has more than one element if and only if $A \neq \mathbf{0}$.*

Proof. Observe that $\mathbf{0} \in A\{2\}$ for every A. So, $A\{2\} \neq \emptyset$. Furthermore, $\mathbf{0}\{2\} = \{\mathbf{0}\}$. In addition, if $A = (a_{i,j}) \neq \mathbf{0}$, every nonzero entry $a_{i_0,j_0} \neq 0$ generates an outer inverse as follows: $X = (x_{i,j})$ is defined as $x_{j_0,i_0} = 1/a_{i_0,j_0}$ and $x_{i,j} = 0$ otherwise. □

In Section 4, one can find more information on $A\{2\}$.

The set $A\{1, 2, 3, 4\}$ relates to the well-known case of the *Moore-Penrose inverse*. When $\mathbb{K} = \mathbb{C}$, $A\{1, 2, 3, 4\}$ contains exactly one element, namely the Moore-Penrose inverse of A, that we denote by $A^\dagger$. If $\mathbb{K}$ is not the field of the complex numbers or φ is another involutory automorphism, $A\{1, 2, 3, 4\}$ could be either empty or contain exactly one element. Those fields $(\mathbb{K}, \varphi)$ such that $A\{1, 2, 3, 4\} \neq \emptyset$ for every $A \in \mathbb{K}^{m\times n}$, are called Moore-Penrose fields (see [21] for further details).

If $m = n$, $A\{2, 1^k, 5\}$ corresponds to the *Drazin inverse* of A, that we denote by A^D. Moreover, if $\mathrm{ind}(A) = 1$, the Drazin inverse is the group inverse, denoted by $A^\#$. Taking into account Theorem 9 in

$\mathbb{K}$, M^* denotes the transpose of the matrix $\varphi(M)$ (see [21] for further details). If $\mathbb{K}$ is a subfield of the field $\mathbb{C}$ of the complex numbers, φ is assumed to be the usual complex number conjugation.

[17], since $\operatorname{rank}(A^{\operatorname{ind}(A)}) = \operatorname{rank}(A^{\operatorname{ind}(A)+1})$ and $A^{2\operatorname{ind}(A)+1}\{1\} \neq \emptyset$ (see above), one deduces that $A\{2, 1^k, 5\}$ contains always exactly one element, namely the Drazin inverse of A.

Now, we consider $\{S\}$-inverses with prescribed range (resp. prescribed kernel). So, in addition to $A \in \mathbb{K}^{m\times n}$, let us fix $B \in \mathbb{K}^{n\times k}$ and $C \in \mathbb{K}^{\ell\times m}$. Then, we introduce the sets

$$\begin{aligned} A\{S\}_{\mathcal{R}(B),*} &= \{X \in A\{S\} \,|, \mathcal{R}(X) = \mathcal{R}(B)\}, \\ A\{S\}_{*,\mathcal{N}(C)} &= \{X \in A\{S\} \mid \mathcal{N}(X) = \mathcal{N}(C)\}, \\ A\{S\}_{\mathcal{R}(B),\mathcal{N}(C)} &= A\{S\}_{\mathcal{R}(B),*} \cap A\{S\}_{*,\mathcal{N}(C)}. \end{aligned} \tag{2.3}$$

We observe that if $X \in A\{S\}$ then $X \in \mathbb{K}^{n\times m}$, and hence $\mathcal{R}(X) \subset \mathbb{K}^n$ and $\mathcal{N}(X) \subset \mathbb{K}^m$. Moreover, $B \in \mathbb{K}^{n\times k}$ and $C \in \mathbb{K}^{\ell\times m}$. So, $\mathcal{R}(B) \subset \mathbb{K}^n$ and $\mathcal{N}(C) \subset \mathbb{K}^m$. Therefore, the above equality of vector subspaces make sense.

Using the same notation criterium as above, the elements in $A\{S\}_{\mathcal{R}(B),*}$ will be represented as $A^{(S)}_{\mathcal{R}(B),*}$; similarly, for $A^{(S)}_{*,\mathcal{N}(C)}$ and $A^{(S)}_{\mathcal{R}(B),\mathcal{N}(C)}$. This type of inverses are known as *(B, C)-inverses*.

A complete analysis on the existence of (B, C)-inverses for matrices over a ring is presented in [13]. The results in [13] can be applied to our case, namely, the case of matrices over a field. For this purpose, we first observe that $A\{1\} \neq \emptyset$ (see Lemma 2.1). Thus, for every $m, n \in \mathbb{N}$, and for every field $\mathbb{K}$, one has that $\mathbb{K}^{m\times n}$ is regular. In this situation, the following results follow immediately.

1. Applying Theorem 2.3. (iv) in [13], we get that $A\{2\}_{\mathcal{R}(B),*} \neq \emptyset \iff \operatorname{rank}(B) = \operatorname{rank}(AB)$ (compare to Corollary 2.5. in [13] when $\mathbb{K} = \mathbb{C}$).

2. Applying Theorem 2.4. (v) in [13], we get that $A\{1, 2\}_{\mathcal{R}(B),*} \neq \emptyset \iff \operatorname{rank}(B) = \operatorname{rank}(AB) = \operatorname{rank}(A)$ (compare to Corollary 2.5. in [13] when $\mathbb{K} = \mathbb{C}$).

3. We observe that every field is a regular ring. So, we have that every field is a right FP-injective ring (see Def. 2.6. in [13]). Thus, applying Theorem 2.10. (iv) in [13], we get that $A\{2\}_{*,\mathcal{N}(C)} \neq \emptyset \iff \operatorname{rank}(C) = \operatorname{rank}(CA)$ (compare to Corollary 2.12. in [13] when $\mathbb{K} = \mathbb{C}$).

4. Similarly, applying Theorem 2.11. (v) in [13], we get that we get that $A\{1, 2\}_{*,\mathcal{N}(C)} \neq \emptyset \iff \operatorname{rank}(C) = \operatorname{rank}(CA) = \operatorname{rank}(A)$ (compare to Corollary 2.12. in [13] when $\mathbb{K} = \mathbb{C}$).

For further results on the existence of generalized inverses with prescribed range and kernel, see Section 3.

For $\mathbb{K} = \mathbb{C}$, the Moore-Penrose inverse $A^\dagger$, the weighted Moore-Penrose inverse $A^\dagger_{M,N}$, the Drazin inverse A^{D} and the group inverse $A^\#$ can be derived using appropriate choices of the matrices B and C (see, e.g. [33]):

$$\begin{aligned} A^\dagger = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}, &\quad A^\dagger_{M,N} = A^{(2)}_{\mathcal{R}(A^\sharp),\mathcal{N}(A^\sharp)}, \quad \text{where } A^\sharp = N^{-1}A^*M \\ A^D = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)}, &\quad k \geq \operatorname{ind}(A), \quad A^\# = A^{(2)}_{\mathcal{R}(A),N(A)}, \quad \operatorname{ind}(A) = 1. \end{aligned} \tag{2.4}$$

3 Representations of generalized inverses over arbitrary fields

We start this section recalling the well known Urquhart formula for generalized inverses. The Urquhart characteriztion was originally introduced in [30], and later continued in [33, Theorem 1.3.7, pg. 28] and [1, Theorem 13, pag. 72]. The formula is usually presented for matrices over $\mathbb{C}$, but one can check that the proof is indeed valid over any field. For completeness reasons, we restate it here for an arbitrary field $\mathbb{K}$.

Theorem 3.1. [Urquhart formula] *Let $A \in \mathbb{K}^{m\times n}$, $B \in \mathbb{K}^{n\times k}$, $C \in \mathbb{K}^{l\times m}$ and $X := B(CAB)^{(1)}C$. Then:*

(1) $X \in A\{1\} \Longleftrightarrow \operatorname{rank}(CAB) = \operatorname{rank}(A)$.

(2) $X \in A\{2\}_{\mathcal{R}(B),\star} \Longleftrightarrow \operatorname{rank}(CAB) = \operatorname{rank}(B)$.

(3) $X \in A\{2\}_{\star,\mathcal{N}(C)} \Longleftrightarrow \operatorname{rank}(CAB) = \operatorname{rank}(C)$.

(4) $X \in A\{2\}_{\mathcal{R}(B),\mathcal{N}(C)} \Longleftrightarrow \operatorname{rank}(CAB) = \operatorname{rank}(B) = \operatorname{rank}(C)$.

(5) $X \in A\{1,2\}_{\mathcal{R}(B),\mathcal{N}(C)} \Longleftrightarrow \operatorname{rank}(CAB) = \operatorname{rank}(B) = \operatorname{rank}(C) = \operatorname{rank}(A)$.

For properly selected matrices A, B, C as in Theorem 3.1, Urquhart formula provides sufficient and necessary condition for a matrix X of the form

$$X = BMC, \quad M \in (CAB)\{1\} \tag{3.1}$$

to be an element in different sets of generalized inverses. However, it does not ensure whether all matrices in each generalized inverse set are necessarily of the form (3.1). In Theorem 3.2 we show that indeed this is the case. In the following we will use some results from [13]. For this purpose, we recall that $\mathbb{K}^{m\times n}$ is a regular (Von Neumann) ring (see Lemma 2.1), and that every field is a FP-injective ring.

Theorem 3.2. [Generalized Urquhart formula] *Let $A \in \mathbb{K}^{m\times n}$ and $X \in \mathbb{K}^{n\times m}$. It holds that*

(1) *The next statements are equivalent:*

  (i) $X \in A\{1\}$.

  (ii) *There exist $k, l \in \mathbb{N}$, and $B \in \mathbb{K}^{n\times k}$, $C \in \mathbb{K}^{l\times m}$ such that $X = B(CAB)^{(1)}C$, and* $\operatorname{rank}(CAB) = \operatorname{rank}(A)$.

(2) *Let $B \in \mathbb{K}^{n\times k}$. Then the following statements are equivalent*

  (i) $X \in A\{2\}_{\mathcal{R}(B),*}$.

  (ii) $\operatorname{rank}(AB) = \operatorname{rank}(B)$ *and there exists $M \in (AB)\{1\}$ such that $X = BM$.*

  (iii) *There exist $C \in \mathbb{K}^{l\times m}$, with $l \in \mathbb{N}$, and $M \in (CAB)\{1\}$ such that* $\operatorname{rank}(CAB) = \operatorname{rank}(B)$ *and $X = BMC$.*

(3) *Let $C \in \mathbb{K}^{l\times m}$. Then the next assertions are mutually equivalent:*

  (i) $X \in A\{2\}_{*,\mathcal{N}(C)}$.

  (ii) $\operatorname{rank}(CA) = \operatorname{rank}(C)$ *and there exists $M \in (CA)\{1\}$ such that $X = MC$.*

  (iii) *There exist $B \in \mathbb{K}^{n\times k}$, with $k \in \mathbb{N}$, and $M \in (CAB)\{1\}$ such that* $\operatorname{rank}(CAB) = \operatorname{rank}(C)$ *and $X = BMC$.*

(4) *Let $B \in \mathbb{K}^{n\times k}$ and $C \in \mathbb{K}^{l\times m}$. Then the following statements are equivalent:*

  (i) $X \in A\{2\}_{\mathcal{R}(B),\mathcal{N}(C)}$.

  (ii) $\operatorname{rank}(CAB) = \operatorname{rank}(B) = \operatorname{rank}(C)$ *and there exists $M \in (CAB)\{1\}$ such that $X = BMC$.*

(5) *Let $B \in \mathbb{K}^{n\times k}$ and $C \in \mathbb{K}^{l\times m}$. The subsequent statements are equivalent*

  (i) $X \in A\{1,2\}_{\mathcal{R}(B),\mathcal{N}(C)}$.

  (ii) $\operatorname{rank}(CAB) = \operatorname{rank}(B) = \operatorname{rank}(C) = \operatorname{rank}(A)$ *and there exists $M \in (CAB)\{1\}$ such that $X = BMC$.*

Proof.

(1) (i)⇒(ii). Let $X \in A\{1\}$. Let $k = n$, $l = m$, and let $B = I_n$ and $C = I_m$. Then, X can be expressed in the form $X = B(CAB)^{(1)}C$ exactly as $X = A^{(1)} = I_n(I_m A I_n)^{(1)} I_m$. Moreover, $\text{rank}(CAB) = \text{rank}(A)$.
(ii)⇒(i). It follows from Theorem 3.1 (1).

(2) (i)⇒(ii). Let $X \in A\{2\}$ and $\mathcal{R}(X) = \mathcal{R}(B)$. Then, Theorem 2.3, statement(i) in [13] holds, and hence using the same theorem one gets that X can be represented as $B(AB)^{(1)}$, for some $(AB)^{(1)} \in (AB)\{1\}$. Furthermore, since $X \in A\{2\}_{\mathcal{R}(B),\star}$, by Theorem 2.3 (iv) in [13], one has that $\mathcal{N}(B) = \mathcal{N}(AB)$. Thus $\text{rank}(AB) = \text{rank}(B)$.
(ii)⇒ (iii) Follows from the particular settings $l = m$ and $C = I_m$.
(iii)⇒(i). It follows from Theorem 3.1 (2).

(3) (i)⇒ (ii) Let $X \in A\{2\}$ satisfy $\mathcal{N}(X) = \mathcal{N}(C)$. Then, as every field is a FP-injective ring, one can apply Theorem 2.10 (i) in [13] to get that X can be represented as $(CA)^{(1)}C$ for some $(CA)^{(1)} \in (CA)\{1\}$. Moreover, since $X \in A\{2\}_{\star,\mathcal{N}(C)}$, by Theorem 2.10 in [13], one has that $\mathcal{R}(C) = \mathcal{R}(CA)$. Thus $\text{rank}(CA) = \text{rank}(C)$.
(ii)⇒ (iii) from taking $k = n$ and $C = I_n$.
(iii)⇒(i). It follows from Theorem 3.1 (3).

(4) (i)⇒(ii). It follows from Theorem 2.15. (i),(vi) in [13].
(ii)⇒(i). It follows from Theorem 3.1 (4).

(5) Let $X \in A\{1,2\}_{\mathcal{R}(B),\mathcal{N}(C)}$. Then, $X \in A\{2\}_{\mathcal{R}(B),\mathcal{N}(C)}$. So, by the statement (4)(i), we get that $X = B(CAB)^{(1)}C$, and using Theorem 2.13, (i) and (iv), in [13], we conclude $\text{rank}(CAB) = \text{rank}(B) = \text{rank}(C) = \text{rank}(A)$.
(ii)⇒(i). It follows from Theorem 3.1 (5).

□

Remark 3.1. *The main differences between Theorem 3.1 and Theorem 3.2 can be highlighted as follows.*

1. *Theorem 3.1 considers $X := B(CAB)^{(1)}C$ and then gives corresponding statements about X. On the other hand, Theorem 3.2 gives characterizations of the set of inner and outer inverses.*

2. *Theorem 3.1 does not give answer to the following question: is it possible to find inner or outer inverses in the form which is different than $B(CAB)^{(1)}C$? On the other hand, Theorem 3.2 reveals that every inner and outer matrix can be represented in the form $B(CAB)^{(1)}C$.*

3. *Theorem 3.1 gives statements about $A \in \mathbb{K}_r^{m\times n}$, $B \in \mathbb{K}^{n\times k}$, $C \in \mathbb{K}^{l\times m}$ and $X = B(CAB)^{(1)}C$ of fixed dimensions. On the other hand, Theorem 3.2 considers B, C of variable orders $n \times k$ and $l \times m$, respectively.*

Theorem 3.2 shows that the elements in $A\{2\}_{\mathcal{R}(B),\star}, A\{2\}_{\star,\mathcal{N}(C)}, A\{2\}_{\mathcal{R}(B),\mathcal{N}(C)}$ and $A\{1,2\}_{\mathcal{R}(B),\mathcal{N}(C)}$ can be expressed in terms of inner inverses. More precisely, one gets that

$$\begin{cases} A\{2\}_{\mathcal{R}(B),\star} = \bigcup_{l\in\mathbb{N}} \left\{BMC \,|\, C \in \mathbb{K}^{l\times m} \text{ with } \text{rank}(CAB) = \text{rank}(B) \text{ and } M \in (CAB)\{1\}\right\} \\ A\{2\}_{\star,\mathcal{N}(C)} = \bigcup_{k\in\mathbb{N}} \left\{BMC \,|\, B \in \mathbb{K}^{n\times k} \text{ with } \text{rank}(CAB) = \text{rank}(C) \text{ and } M \in (CAB)\{1\}\right\}. \end{cases} \tag{3.2}$$

Let M, U, V matrices of suitable orders. In the sequel, we use the following notation

$$UM\{1\} = \{UR \,|\, R \in M\{1\}\},\ \ M\{1\}V = \{RV \,|\, R \in M\{1\}\},\ \ UM\{1\}V = \{URV \,|\, R \in M\{1\}\}. \tag{3.3}$$

In this situation, the next theorem, which is a direct consequence of the results in [13], gives a closer description of these sets of generalized inverses.

Theorem 3.3. *Consider $A \in \mathbb{K}^{m\times n}$, $B \in \mathbb{K}^{n\times k}$, $C \in \mathbb{K}^{l\times m}$. It holds that*

(1) $A\{2\}_{\mathcal{R}(B),\star} = \begin{cases} B\,(AB)\{1\}, & \textit{if } \operatorname{rank}(AB) = \operatorname{rank}(B) \\ \emptyset, & \textit{otherwise} \end{cases}$

(2) $A\{2\}_{\star,\mathcal{N}(C)} = \begin{cases} (CA)\{1\}\,C, & \textit{if } \operatorname{rank}(CA) = \operatorname{rank}(C) \\ \emptyset, & \textit{otherwise} \end{cases}$

(3) $A\{2\}_{\mathcal{R}(B),\mathcal{N}(C)} = \begin{cases} B\,(CAB)\{1\}C, & \textit{if } \operatorname{rank}(CAB) = \operatorname{rank}(C) = \operatorname{rank}(B) \\ \emptyset, & \textit{otherwise} \end{cases}$

(4) $\#(A\{2\}_{\mathcal{R}(B),\mathcal{N}(C)}) = \begin{cases} 1, & \textit{if } \operatorname{rank}(CAB) = \operatorname{rank}(C) = \operatorname{rank}(B) \\ 0, & \textit{otherwise} \end{cases}$

(5) $A\{1,2\}_{\mathcal{R}(B),\mathcal{N}(C)} = \begin{cases} A\{2\}_{\mathcal{R}(B),\mathcal{N}(C)}, & \textit{if } \operatorname{rank}(CAB) = \operatorname{rank}(C) = \operatorname{rank}(B) = \operatorname{rank}(A) \\ \emptyset, & \textit{otherwise.} \end{cases}$

Proof. Statement (1) follows from Theorem 2.3. in [13] and Lemma 2.1.
Statement (2) follows from Theorem 2.10. in [13].
Statement (3) follows from Theorem 3.2 (4)(i).
Statement (4) follows from Theorem 2.15. in [13].
Statement (5): by Theorem 2.13. and Theorem 2.15 in [13], one gets that $A\{1,2\}_{\mathcal{R}(B),\mathcal{N}(C)} \neq \emptyset \iff \operatorname{rank}(CAB) = \operatorname{rank}(A) = \operatorname{rank}(C) = \operatorname{rank}(B)$. Moreover, if $A\{1,2\}_{\mathcal{R}(B),\mathcal{N}(C)} \neq \emptyset$ using $A\{1,2\}_{\mathcal{R}(B),\mathcal{N}(C)} \subset A\{2\}_{\mathcal{R}(B),\mathcal{N}(C)}$ and statement (3) and (4) above, one gets the result. □

Theorem 3.3 simplifies the description in (3.2) by taking $l = m$ and $C = I_m$ for $A\{2\}_{\mathcal{R}(B),\star}$, and $k = n$ and $B = I_n$ for $A\{2\}_{\star,\mathcal{N}(C)}$. In the following, we generalize the result by showing that C and B can be taken as full-column and full-row rank matrices, respectively.

The aim of Theorem 3.4 is to show that $A\{2\}_{*,\mathcal{N}(C)}$ can be generated using even a full-row rank rectangular matrix B over the complex numbers $\mathbb{C}$.

Theorem 3.4. *Let $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{l\times m}$ be given and $B \in \mathbb{C}^{n\times k}$ be of full row rank. If $\operatorname{rank}(CAB) = \operatorname{rank}(C)$, then*

$$A\{2\}_{*,\mathcal{N}(C)} = B(CAB)\{1\}C. \tag{3.4}$$

Proof. The proof is based on the reverse order law for inner inverses proposed in Corollary 2.9 from [35]. In our case, we consider the matrices $(CA) \in \mathbb{C}^{l\times n}$ and $B \in \mathbb{C}^{n\times k}_n$. Since the conditions of part (iii) of Corollary 2.9 from [35] hold, it follows that

$$B\{1\}(CA)\{1\} \subseteq (CAB)\{1\}.$$

where $B\{1\}(CA)\{1\} = \{R\,S \mid R \in B\{1\}, S \in (CA)\{1\}\}$. But, the right inverse $B_R^{-1} = B^*(BB^*)^{-1}$ belongs to $B\{1\}$. Consequently,

$$B_R^{-1}(CA)\{1\} \subseteq B\{1\}(CA)\{1\} \subseteq (CAB)\{1\}$$

which implies

$$(CA)\{1\}C = BB_R^{-1}(CA)\{1\}C \subseteq B(CAB)\{1\}C.$$

On the other hand, $A\{2\}_{*,\mathcal{N}(C)} = (CA)\{1\}C$ implies $B(CAB)\{1\}C \subseteq (CA)\{1\}C$, which completes the proof. □

The dual result is obtained in Theorem 3.5 and shows that $A\{2\}_{\mathcal{R}(B),*}$ can be generated using even a full-column rank rectangular matrix C.

Theorem 3.5. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$ be given and $C \in \mathbb{C}^{l\times m}_m$ be of full column rank. If $\operatorname{rank}(CAB) = \operatorname{rank}(B)$, it follows that*

$$A\{2\}_{\mathcal{R}(B),*} = B(CAB)\{1\}C. \tag{3.5}$$

Remark 3.2. *The proofs in Theorems 3.4 and 3.5 uses* [35]*, which is stated for complex matrices, and cannot be applied directly to the case of arbitrary field. These proofs use, for instance, the claim: if B is full row rank the BB^{T} is non-singular. However, this is not true for every field. For instance,*

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in (\mathbb{Z}_2)^{2\times 3}$$

is full row rank but

$$BB^{\mathrm{T}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is singular. So, the verification of these statements in arbitrary filed requires an alternative proof.

In the following we show how to generalize Theorems 3.4 and 3.5 to the case of arbitrary fields. For this purpose, we start with the following technical lemma.

Lemma 3.1.

(1) *Let $W \in \mathbb{K}^{n\times l}$ and let $B \in \mathbb{K}^{n\times k}$ be a full row rank. Then, there exists $M \in \mathbb{K}^{k\times l}$ such that $W = BM$.*

(2) *Let $W \in \mathbb{K}^{l\times n}$ and let $C \in \mathbb{K}^{k\times n}$ be a full column rank. Then, there exists $M \in \mathbb{K}^{l\times k}$ such that $W = MC$.*

Proof. (1) Equivalent vector form of the matrix equation $W = BM$ can be derived using the property $\mathrm{vect}(JKL) = \left(L^{\mathrm{T}} \otimes J\right)\mathrm{vect}(K)$, where vect denotes the vectorization operator. So,

$$\mathrm{vect}(W) = (I_l \otimes B)\,\mathrm{vect}(M) \in \mathbb{K}^{nl}.$$

As a consequence, we consider the linear system of equations

$$(I_l \otimes B)\,X = \mathrm{vect}(W), \quad X \in \mathbb{K}^{kl}. \tag{3.6}$$

Let $(I_l \otimes B \mid \mathrm{vect}(W)) \in \mathbb{K}^{nl\times(kl+1)}$ denote the augmented matrix of the linear system (3.6). Then,

$$nl = \mathrm{rank}(I_l \otimes B) \leq \mathrm{rank}\left((I_l \otimes B \,|\, \mathrm{vect}(W))\right) \leq nl.$$

So, $\mathrm{rank}(I_l \otimes B) = \mathrm{rank}((I_l \otimes B \mid \mathrm{vect}(W)))$. Thus, the linear system is consistent and hence M exists, namely $\mathrm{vect}(M)$ is any solution to (3.6).

(2) Follows from (1) taking transposes. □

Theorem 3.6.

(1) *Let $A \in \mathbb{K}^{m\times n}$, $B \in \mathbb{K}^{n\times k}$, and let $C \in \mathbb{K}^{l\times m}$ be full column rank. Then,*

$$A\{2\}_{\mathcal{R}(B),\star} = \begin{cases} B(CAB)\{1\}\,C, & \textit{if } \mathrm{rank}(CAB) = \mathrm{rank}(B) \\ \emptyset, & \textit{otherwise.} \end{cases}$$

(2) *Let $A \in \mathbb{K}^{m\times n}$, $C \in \mathbb{K}^{l\times m}$, and let $B \in \mathbb{K}^{n\times k}$ be full row rank. Then,*

$$A\{2\}_{\star,\mathcal{N}(C)} = \begin{cases} B(CAB)\{1\}\,C, & \textit{if } \mathrm{rank}(CAB) = \mathrm{rank}(C) \\ \emptyset, & \textit{otherwise.} \end{cases}$$

Proof. We first observe that, since C is of full-column rank, then $\mathrm{rank}(CAB) = \mathrm{rank}(AB)$. By Theorem 3.3 (1) we have that $A\{2\}_{\mathcal{R}(B),\star} = B(AB)\{1\}$ iff $\mathrm{rank}(CAB) = \mathrm{rank}(AB) = \mathrm{rank}(B)$. Let us see that $B(BA)\{1\} = B(CAB)\{1\}C$. Let $Z \in B(AB)\{1\}$. Then, Z can be written as $Z = BW$ where $(AB)W(AB) = AB$. Therefore

$$(CAB)W(AB) = CAB.$$

By Lemma 3.1, W can be factorized as $W = MC$. Thus

$$(CAB)M(CAB) = CAB$$

and $Z = BMC$. So, $M \in (CAB)\{1\}$. This implies that $Z \in B(CAB)\{1\}C$. The other inclusion follows from (3.2) and Theorem 3.3 (1). The proof of statement (2) is analogous. □

4 Geometrical structure of generalized inverses

In this section, we study the geometric structure of some of the generalized inverses sets. For this purpose, in this section, we consider the vector space $\mathbb{K}^{n\times m}$ as an affine space. Therefore, a matrix $A\in\mathbb{K}^{n\times m}$ is seen as a point, and hence it makes sense to analyze geometric properties of subsets, formed by generalized inverse matrices, in this affine space.

The results in the previous section show that all the elements in $A\{2\}_{\mathcal{R}(B),\star}$, $A\{2\}_{\star,\mathcal{N}(C)}$ and $A\{2\}_{\mathcal{R}(B),\mathcal{N}(C)}$ are expressible in terms of inner inverses. This motivates to start our study with the set $A\{1\}$ of inner inverses.

Theorem 4.1. *The set $A\{1\}$ is a linear affine subspace of $\mathbb{K}^{n\times m}$ and $\dim(A\{1\}) = mn - \operatorname{rank}(A)^2$.*

Proof. By Lemma 2.1, we know that $A\{1\}\neq\emptyset$. Let $A^{(1)}\in A\{1\}$, then $A\{1\}$ can be expressed as (see[2] [31] or Corollary 1 in [1] p. 52)

$$A\{1\} = \{A^{(1)} + Z - A^{(1)}AZAA^{(1)} \mid Z\in\mathbb{K}^{n\times m}\}.$$

Therefore, $A\{1\}$ is an affine subspace of $\mathbb{K}^{m\times n}$. The dimension of $A\{1\}$, as affine subspace, is the dimension of the solution space of the associated linear system to the matrix equality $AXA = A$. On the other hand, the vectorization of the equation $AXA = A$ is $\operatorname{vect}(AXA) = \left(A^{\mathrm{T}}\otimes A\right)\operatorname{vect}(X) = \operatorname{vect}(A)$. Therefore, it holds that the matrix equation $A\,X\,A = A$ is equivalent to the linear system $\left(A^{\mathrm{T}}\otimes A\right)\operatorname{vect}(X) = \operatorname{vect}(A)$ whose coefficient matrix is the Kronecker product $A^{\mathrm{T}}\otimes A$. We know by Lemma 2.1 that the system is compatible, and $\operatorname{rank}(A^{\mathrm{T}}\otimes A) = \operatorname{rank}(A^{\mathrm{T}})\operatorname{rank}(A) = \operatorname{rank}(A)^2$ (see e.g. [29]). Since the linear system includes nm variables and the rank of the matrix of the system is $\operatorname{rank}(A)^2$, the dimension of the solution space is $mn - \operatorname{rank}(A)^2$. $\square$

Remark 4.1. *Based on the proof of Theorem 4.1, the set $A\{1\}$ is expressible as*

$$A\{1\} = A^{(1)} + \mathcal{V}(A\{1\}),$$

where

$$\mathcal{V}(A\{1\}) = \{Z - A^{(1)}AZAA^{(1)} \mid Z\in\mathbb{K}^{n\times m}\}$$

is the transition vector space *of $A\{1\}$ satisfying $A\,\mathcal{V}(A\{1\})\,A = \{\mathbf{O}\}$.*

Remark 4.2.

1. *The minimum dimension of $A\{1\}$ is achieved when A has full rank. In this case,*

$$\dim(A\{1\}) = mn - \min\{m,n\}^2 = \min\{m,n\}\left(\max\{m,n\} - \min\{m,n\}\right).$$

2. $\dim(A\{1\}) = 0$ *iff* $m = n = \operatorname{rank}(A)$ *iff A is square and non-singular. In this case, $A\{1\} = \{A^{-1}\}$.*

3. *The maximum dimension of $A\{1\}$ is achieved when $A = \mathbf{0}$. In this case, $A\{1\} = \mathbb{K}^{n\times m}$.*

Example 4.1. In this example we illustrate the affine structure of $A\{1\}$ corresponding to the inner inverses of 1×2 matrices. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \end{pmatrix} \in \mathbb{K}^{1\times 2}.$$

If $A = \mathbf{0}$ then $A\{1\} = \mathbb{K}^{1\times 2}$, which dimension is 2. Let $A\neq\mathbf{0}$; solving the equation $A\,X\,A = A$ one gets

$$A\{1\} = \begin{cases} \begin{pmatrix} 0 \\ \frac{1}{a_{12}} \end{pmatrix} + \left\{\lambda\begin{pmatrix} 1 \\ 0 \end{pmatrix} \,\middle|\, \lambda\in\mathbb{K}\right\}, & \text{if } a_{11} = 0 \\[2ex] \begin{pmatrix} \frac{1}{a_{11}} \\ 0 \end{pmatrix} + \left\{\lambda\begin{pmatrix} a_{12} \\ a_{11} \end{pmatrix} \,\middle|\, \lambda\in\mathbb{K}\right\}, & \text{if } a_{11}\neq 0. \end{cases}$$

Note that $\dim(A\{1\}) = 1$, and applying Theorem 4.1 we get that $\dim(A\{1\}) = 1\cdot 2 - 1 = 1$.

[2] In [31] the field is not explicitly stated, and in [1] the proof is for complex matrices. However, one can check that the proof is valid over any field.

Now, we analyze the set $A\{2\}$ of outer inverses. $A\{2\}$, by definition, can be seen as the zero-set of finitely many polynomial equations (see the Penrose condition (2) in Section 2). Therefore, $A\{2\}$ is an affine algebraic variety in $\mathbb{K}^{n\times m}$. A natural question is whether this affine algebraic variety is linear. We observe that $\mathbf{0}\{2\} = \{\mathbf{0}\}$ and $(1_{\mathbb{Z}_2})\{2\} = \mathbb{Z}_2$, where $(1_{\mathbb{Z}_2})$ is the unit 1×1 matrix over $\mathbb{Z}_2$; note that all the elements of $\mathbb{Z}_2$ satisfy the equation $x^2 = x$. So, in these two cases we get an affine subspace. However, as the Lemma 4.1 shows, in general, $A\{2\}$ is not an affine subspace.

Lemma 4.1. *Let $\mathbb{K}$ not be isomorphic to $\mathbb{Z}_2$. It holds that*

1. *If $A \neq \mathbf{0}$, then $A\{2\}$ is not an affine subspace of $\mathbb{K}^{n\times m}$.*
2. $\emptyset \neq A\{2\} \subsetneq \mathbb{K}^{n\times m}$.

Proof. (1) Since $\mathbf{0} \in A\{2\}$, one has that $A\{2\}$ is an affine subspace iff $A\{2\}$ is a vector subspace. Since $A \neq \mathbf{0}$, by Lemma 2.2, there exists $M \in A\{2\}$ with $M \neq \mathbf{0}$. Let $\lambda \in \mathbb{K}\setminus\{0,1\}$; note that $\mathbb{K}$ is not isomorphic to $\mathbb{Z}_2$ and hence λ exists. In this situation, $(\lambda M)A(\lambda M) = \lambda^2 M \neq \lambda M$. Therefore, $\lambda M \notin A\{2\}$. Thus, $A\{2\}$ is not an affine subspace.
(2) If $A = \mathbf{0}$, then $A\{2\} = \{\mathbf{0}\} \neq \mathbb{K}^{n\times m}$. Let $A = (a_{ij})$ with $a_{i_0j_0} \neq 0$. We consider $X = (x_{ij}) \in \mathbb{K}^{n\times m}$ such that $0 \neq x_{j_0i_0} \neq 1/a_{i_0j_0}$ and $x_{ij} = 0$ otherwise; note that this is possible because $\mathbb{K}$ is not isomorphic to $\mathbb{Z}_2$ and hence $\mathbb{K} \setminus \{0,1\} \neq \emptyset$. Then, the position (j_0, i_0) of the matrix $XAX - X$ is $x_{j_0i_0}(a_{i_0j_0}x_{j_0i_0} - 1) \neq 0$. So, $X \notin A\{2\}$, and therefore $A\{2\} \neq \mathbb{K}^{n\times m}$. □

In the following Example 4.2 we see that the outer inverses set of complex 1×2 matrices decomposes as a union of two affine linear subsets. Nevertheless, in Example 4.3 we see that the outer inverses set of complex 2×3 matrices decomposes as a union of two affine linear subspaces and non-linear affine varieties.

Example 4.2. Let A be as in the Example 4.1. If $A = \mathbf{0}$, then $A\{2\} = \{\mathbf{0}\}$. Let $A \neq \mathbf{0}$. Using Gröbner bases to solve the system of algebraic equations derived from the matrix equality $XAX = X$, we get that $A\{2\}$ decomposes as the union of a point, namely the null matrix, and a line. More precisely,

$$A\{2\} = \begin{cases} \{\mathbf{0}\} \cup \left\{ \begin{pmatrix} 0 \\ \frac{1}{a_{12}} \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,\middle|\, \lambda \in \mathbb{K} \right\}, & \text{if } a_{11} = 0 \\ \{\mathbf{0}\} \cup \left\{ \begin{pmatrix} \frac{1}{a_{11}} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -a_{12} \\ a_{11} \end{pmatrix} \,\middle|\, \lambda \in \mathbb{K} \right\}, & \text{if } a_{11} \neq 0. \end{cases}$$

Example 4.3. In this example, we analyze $A\{2\}$ for $A \in \mathbb{C}^{2\times 3}$. As above, we assume that $A \neq \mathbf{0}$. In the analysis of $A\{2\}$ via Gröbner bases, we distinguish several cases. Let us denote by δ_i the 2×2 minors of A, i.e.,

$$\delta_1 = a_{11}a_{22} - a_{12}a_{21},\ \delta_2 = a_{11}a_{23} - a_{13}a_{21},\ \delta_3 = a_{12}a_{23} - a_{13}a_{22}.$$

We decompose $\mathbb{C}^{2\times 3}$ as $\mathbb{C}^{2\times 3} = \Omega_1 \cup \cdots \cup \Omega_5$, where

$$\Omega_1 = \left\{(a_{ij}) \in \mathbb{C}^{2\times 3} \,\middle|\, a_{11}a_{22}\delta_1\delta_2 \neq 0\right\}, \Omega_2 = \left\{(a_{ij}) \in \mathbb{C}^{2\times 3} \,\middle|\, a_{11} = 0\right\}, \Omega_3 = \left\{(a_{ij}) \in \mathbb{C}^{2\times 3} \,\middle|\, a_{22} = 0\right\},$$
$$\Omega_4 = \left\{(a_{ij}) \in \mathbb{C}^{2\times 3} \,\middle|\, \delta_1 = 0\right\}, \Omega_5 = \left\{(a_{ij}) \in \mathbb{C}^{2\times 3} \,\middle|\, \delta_2 = 0\right\}.$$

In the following, we analyze the problem for the open set Ω_1. For the other cases, the close sets $\Omega_i, i = 2, \ldots, 5$, the reasoning is similar.

If $A \in \Omega_1$, then $A\{2\}$ decomposes as the union of 3 affine varieties (one point, one line, and a 3-dimensional affine algebraic variety of degree 3). More precisely,

$$A\{2\} = V_1 \cup V_2 \cup V_3.$$

Let us describe the varieties V_i.
To begin with $V_1 = \{\mathbf{0}\}$.
In addition, V_2 has degree 3 and dimension 3, and it represents the affine algebraic variety

$$V_2 = \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} \in \mathbb{C}^{3\times 2} \;\middle|\; \begin{array}{r} -x_{11}x_{22} + x_{12}x_{21} = 0 \\ x_{32}x_{11} - x_{12}x_{31} = 0 \\ x_{32}x_{21} - x_{22}x_{31} = 0 \\ a_{11}x_{11}x_{21} + a_{12}{x_{21}}^2 + a_{13}x_{21}x_{31} + a_{21}x_{11}x_{22} + a_{22}x_{21}x_{22} + a_{23}x_{22}x_{31} - x_{21} = 0 \\ a_{11}{x_{11}}^2 + a_{12}x_{11}x_{21} + a_{13}x_{11}x_{31} + a_{21}x_{11}x_{12} + a_{22}x_{11}x_{22} + a_{23}x_{12}x_{31} - x_{11} = 0 \\ a_{11}x_{11} + a_{12}x_{21} + a_{13}x_{31} + a_{21}x_{12} + a_{22}x_{22} + x_{32}a_{23} = 1 \end{array} \right\}.$$

Further, V_3 is the line (note that, since the entries of A are taken in Ω_1, the line V_3 does not pass through the zero matrix)

$$V_3 = \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} \in \mathbb{C}^{3\times 2} \;\middle|\; \begin{array}{rcl} a_{11}x_{31} + x_{32}a_{21} & = & 0 \\ \delta_1 x_{22} + \delta_2 x_{32} & = & a_{11} \\ a_{11}\delta_1 x_{21} - a_{21}\delta_2 x_{32} & = & -a_{11}a_{21} \\ \delta_1 x_{12} - \delta_3 x_{32} & = & a_{12} \\ a_{11}\delta_1 x_{11} + a_{21}\delta_3 x_{32} & = & a_{11}a_{22} \end{array} \right\}.$$

One may check that $V_i \cap V_j = \emptyset$ for $i \neq j$. Furthermore, since V_2 and V_3 have no singularities, one has that $A\{2\}$ is smooth.

Parametrization of the line V_3 indicates that the outer inverses of A included in V_3 are of the form

$$V_3 = \left\{ \frac{1}{\delta_1} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} -a_{21}\delta_3 & a_{11}\delta_3 \\ a_{21}\delta_2 & -a_{11}\delta_2 \\ -\delta_1 a_{21} & a_{11}\delta_1 \end{pmatrix} \;\middle|\; \lambda \in \mathbb{C} \right\}.$$

The variety V_2 is rational, and can be expressed parametrically as (note that, in general, a rational parametrization does not cover the whole variety and additional lower dimensional parametrizations are required; see [3])

$$V_2 = \Sigma_1 \cup \cdots \cup \Sigma_5,$$

where

$$\Sigma_1=\left\{\left(\begin{array}{cc} -\dfrac{\left(\mu^2 a_{23}+(\lambda\, a_{13}+\rho\, a_{22}-1)\,\mu+a_{12}\rho\,\lambda\right)\lambda}{(\lambda\, a_{11}+\mu\, a_{21})\,\mu} & \dfrac{-\lambda\,\mu\, a_{13}-a_{12}\rho\,\lambda-\mu^2 a_{23}-\mu\,\rho\, a_{22}+\mu}{\lambda\, a_{11}+\mu\, a_{21}} \\ \dfrac{\lambda\,\rho}{\mu} & \rho \\ \lambda & \mu \end{array}\right) \middle|\ \lambda,\mu,\rho\in\mathbb{C}\right\}$$

$$\Sigma_2=\left\{\left(\begin{array}{cc} -\dfrac{\lambda\, a_{21}}{a_{11}} & \lambda \\ \dfrac{((\mu\, a_{23}-1)\, a_{11}-a_{13}a_{21}\mu)\, a_{21}}{a_{11}\,(a_{11}a_{22}-a_{12}a_{21})} & \dfrac{-\mu\, a_{11}a_{23}+a_{13}a_{21}\mu+a_{11}}{a_{11}a_{22}-a_{12}a_{21}} \\ -\dfrac{\mu\, a_{21}}{a_{11}} & \mu \end{array}\right) \middle|\ \lambda,\mu\in\mathbb{C}\right\}$$

$$\Sigma_3=\left\{\left(\begin{array}{cc} -\dfrac{\lambda\,(\lambda\, a_{12}+\mu\, a_{22}-1)}{\lambda\, a_{11}+\mu\, a_{21}} & -\dfrac{\mu\,(\lambda\, a_{12}+\mu\, a_{22}-1)}{\lambda\, a_{11}+\mu\, a_{21}} \\ \lambda & \mu \\ 0 & 0 \end{array}\right) \middle|\ \lambda,\mu\in\mathbb{C}\right\}$$

$$\Sigma_4=\left\{\left(\begin{array}{cc} \dfrac{-\lambda\, a_{12}-\mu\, a_{13}+1}{a_{11}} & 0 \\ \lambda & 0 \\ \mu & 0 \end{array}\right) \middle|\ \lambda,\mu\in\mathbb{C}\right\}$$

$$\Sigma_5=\left\{\left(\begin{array}{cc} \dfrac{-\lambda\, a_{21}+1}{a_{11}} & \lambda \\ 0 & 0 \\ 0 & 0 \end{array}\right) \middle|\ \lambda,\mu\in\mathbb{C}\right\}.$$

In Theorem 4.2, applying Theorem 3.3 and Theorem 4.1, we show that outer inverses with prescribed subspaces are affine subspaces. For this purpose, we will use the notation as in Remark 4.1, and consider the following linear maps Φ_B and φ_C. Let $B\in\mathbb{K}^{n\times k}$ and $C\in\mathbb{K}^{l\times m}$, then we define

$$\begin{array}{rccc} \Phi_B: & \mathbb{K}^{k\times m} & \mapsto & \mathbb{K}^{n\times m}, \\ & M & \mapsto & B\,M \end{array} \qquad \begin{array}{rccc} \varphi_C: & \mathbb{K}^{n\times l} & \mapsto & \mathbb{K}^{n\times m} \\ & M & \mapsto & M\,C. \end{array}$$

Theorem 4.2. *Let $A\in\mathbb{K}^{m\times n}$, $B\in\mathbb{K}^{n\times k}$, $C\in\mathbb{K}^{l\times m}$, with $B\neq\mathbf{0}$ and $C\neq\mathbf{0}$. The following statements are valid.*

(1) *The assumption $\operatorname{rank}(AB)=\operatorname{rank}(B)$ initiates $A\{2\}_{\mathcal{R}(B),\star}$ is the affine subspace in $\mathbb{K}^{n\times m}$ defined by*

$$A\{2\}_{\mathcal{R}(B),\star}=B(AB)^{(1)}+\Phi_B(\mathcal{V}((AB)\{1\})). \tag{4.1}$$

Moreover,

$$m\operatorname{rank}(B)-\operatorname{rank}(B)^2\leq\dim(A\{2\}_{\mathcal{R}(B),\star})\leq\min\{m\operatorname{rank}(B)-1,\ mk-\operatorname{rank}(B)^2\}.$$

(2) *The assumption $\operatorname{rank}(CA)=\operatorname{rank}(C)$ implies that $A\{2\}_{\star,\mathcal{N}(C)}$ is the affine subspace in $\mathbb{K}^{n\times m}$ given in the form*

$$A\{2\}_{\star,\mathcal{N}(C)}=(CA)^{(1)}C+\varphi_C(\mathcal{V}((CA)\{1\})). \tag{4.2}$$

Moreover,

$$n\,\mathrm{rank}(C)-\mathrm{rank}(C)^2 \le \dim(A\{2\}_{\star,\mathcal{N}(C)}) \le \min\{n\,\mathrm{rank}(C)-1,\ nl-\mathrm{rank}(C)^2\}.$$

(3) *If* $\mathrm{rank}(CAB)=\mathrm{rank}(C)=\mathrm{rank}(B)$ *then* $A\{2\}_{\mathcal{R}(B),\mathcal{N}(C)}$ *is the zero-dimensional affine subspace*

$$A\{2\}_{\mathcal{R}(B),\mathcal{N}(C)} = \{B\,(CAB)^{(1)}C\}.$$

(4) *If* $\mathrm{rank}(CAB)=\mathrm{rank}(C)=\mathrm{rank}(B)=\mathrm{rank}(A)$, *then* $A\{1,2\}_{\mathcal{R}(B),\mathcal{N}(C)}$ *is the zero-dimensional affine subspace* $A\{2\}_{\mathcal{R}(B),\mathcal{N}(C)}$.

Proof. Statement (1): since $\Phi_B(\mathcal{V}((AB)\{1\}))$ is a vector subspace, applying Theorem 3.3 (1) and Remark 4.1, one gets that $A\{2\}_{\mathcal{R}(B),\star}$ is the affine subspace defined by (4.1). Let us prove now the claim on the dimension. Clearly, $\Phi_B(\mathcal{V}((AB)\{1\}))\subset \mathrm{Im}(\Phi_B)$. Let us see that the inclusion is strict. Indeed, let $M\in (AB)\{1\}\subset \mathbb{K}^{k\times m}$. Then, $\Phi_B(-M)=-BM\in \mathrm{Im}(\Phi_B)$. Let us assume that $-BM\in\Phi_B(\mathcal{V}((AB)\{1\}))$. Then, $\mathbf{0}=BM-BM\in A\{2\}_{\mathcal{R}(B),\star}$. But, $B\neq \mathbf{0}$ and hence $\mathcal{R}(B)\neq\mathcal{R}(\mathbf{0})$. Thus, $\Phi_B(\mathcal{V}((AB)\{1\}))\subsetneq \mathrm{Im}(\Phi_B)$. On the other hand, the matrix representation of the linear map Φ_B is $I_m\otimes B$. Therefore, $\dim(\mathrm{Im}(\Phi_B))=m\,\mathrm{rank}(B)$. So, $\dim(A\{2\}_{\mathcal{R}(B),\star})\le m\,\mathrm{rank}(B)-1$. Furthermore, from Theorem 4.1, we get

$$\dim(A\{2\}_{\mathcal{R}(B),\star})=\dim(\Phi_B(\mathcal{V}((AB)\{1\}))\le \dim(\mathcal{V}((AB)\{1\})=mk-\mathrm{rank}(AB)^2.$$

Now, using the hypothesis on the rank, we get that $\dim(A\{2\}_{\mathcal{R}(B),\star})\le mk-\mathrm{rank}(B)^2$.

Let $\Phi_{B,\mathcal{V}((AB)\{1\})}$ denote the restriction of Φ_B to $\mathcal{V}((AB)\{1\}$. Then $\mathrm{Ker}(\Phi_{B,\mathcal{V}((AB)\{1\})})\subset \mathrm{Ker}(\Phi_B)$. So,

$$\begin{array}{rcl}\dim(A\{2\}_{\mathcal{R}(B),\star}) &=& \dim(\Phi_B(\mathcal{V}((AB)\{1\}))\\ &=& \dim(\mathrm{Im}(\Phi_{B,V((AB)\{1\})}))\\ &=& \dim(\mathcal{V}((AB)\{1\}))-\dim(\mathrm{Ker}(\Phi_{B,\mathcal{V}((AB)\{1\})}))\\ &\ge& \dim(\mathcal{V}((AB)\{1\}))-\dim(\mathrm{Ker}(\Phi_B))\\ &=& mk-\mathrm{rank}(B)^2-(mk-\mathrm{rank}(I_m\otimes B))\\ &=& mk-\mathrm{rank}(B)^2-(mk-m\,\mathrm{rank}(B))\\ &=& m\,\mathrm{rank}(B)-\mathrm{rank}(B)^2.\end{array}$$

Statement (2) follows analogously.
Statment (3): it follows from Theorem 3.3 (3), (4).
Statment (4): it follows from Theorem 3.3 (5). □

Remark 4.3.

1. *Note that if the rank conditions in Theorem 4.2 do not hold, then the corresponding generalized inverses set is empty.*

2. *Observe that, if* $\mathrm{rank}(AB)=\mathrm{rank}(B)$, *then* $\mathrm{rank}(B)\le\min\{m,k,n\}$, *and hence the lower bound in Theorem 4.2 (1) is non-negative. Similarly for the lower bound in Theorem 4.2 (2).*

The following result is a consequence of Theorem 4.2.

Corollary 4.1. *Let* $A\in\mathbb{K}^{m\times n}$, $B\in\mathbb{K}^{n\times k}$, $C\in\mathbb{K}^{l\times m}$. *It holds that*

1. *If* $\mathrm{rank}(B)=1$ *or* B *is of full-column rank, and* $\mathrm{rank}(AB)=\mathrm{rank}(B)$, *then*

$$\dim\left(A\{2\}_{\mathcal{R}(B),\star}\right)=(m-\mathrm{rank}(B))\,\mathrm{rank}(B).$$

2. *If* $\mathrm{rank}(C)=1$ *or* C *is of full-row rank, and* $\mathrm{rank}(CA)=\mathrm{rank}(A)$, *then*

$$\dim\left(A\{2\}_{\star,\mathcal{N}(C)}\right)=(n-\mathrm{rank}(C))\,\mathrm{rank}(C).$$

We know that $A\{2\}$ is an affine algebraic variety that, in general, is not linear (see Lemma 4.1). In the following, we analyze its dimension. For this purpose, in the sequel, we assume that the field $\mathbb{K}$ is not isomorphic to $\mathbb{Z}_2$ and that $A \neq \mathbf{0}$. In this situation, Lemma 4.1 (2) implies that $\dim(A\{2\}) < nm$. The strategy to derive lower bounds is to find a matrix B, or C, satisfying the hypotheses in Theorem 4.2. Then, it holds that

$$\dim(A\{2\}) \geq \max\left\{\dim(A\{2\}_{\mathcal{R}(B),\star},\, \dim(A\{2\}_{\star,\mathcal{N}(C)}\right\}.$$

More precisely, we state the next theorem.

Theorem 4.3. *Let $\mathbb{K}$ not be isomorphic to $\mathbb{Z}_2$. If $A \neq \mathbf{0}$, it holds that*

$$\max\left\{\max\{m,n\}-1,\, \operatorname{rank}(A)(\max\{m,n\}-\operatorname{rank}(A))\right\} \leq \dim(A\{2\}) < nm.$$

Proof. The upper bound follows from Lemma 4.1 (2). We first prove that $\max\{m,n\}-1 \leq \dim(A\{2\})$. Indeed, since $A \neq \mathbf{0}$, there exists at least one entry of $A = (a_{ij})$ that is not zero. Say $a_{i_0 j_0} \neq 0$. Then, let $C \in \mathbb{K}^{m\times 1}$ be the row matrix corresponding to the i_0th vector of the canonical basis, and $B \in \mathbb{K}^n$ be the column matrix corresponding to the j_0 vector of the canonical basis. Then, CA is the i_0th row of A and BC is the j_0th column of A. So, by the construction, it holds that $\operatorname{rank}(CA) = \operatorname{rank}(C)$ and $\operatorname{rank}(AB) = \operatorname{rank}(B)$. The statement now follows from Theorem 4.2 (1), (2).

Next, we prove that $\operatorname{rank}(A)(\max\{m,n\}-\operatorname{rank}(A)) \leq \dim(A\{2\})$. We consider the factorization $PA = LU$ where P is an $m\times m$ permutation matrix, L is an $m\times m$ non-singular lower triangular matrix, and U is an $m\times n$ upper triangular matrix. Note that U can be expressed as

$$U = \begin{pmatrix} V \\ \mathbf{0} \end{pmatrix},$$

with $V \in \mathbb{K}^{\operatorname{rank}(A)\times n}$ and $\operatorname{rank}(V) = \operatorname{rank}(A)$. Now, if it is necessary, we right-multiply by a permutation $n\times n$ matrix Q such that

$$U\,Q = \begin{pmatrix} V_1 & V_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where $V_1 \in \mathbb{K}^{\operatorname{rank}(A)\times\operatorname{rank}(A)}$, $\operatorname{rank}(V_1) = \operatorname{rank}(A)$, and $V_2 \in \mathbb{K}^{\operatorname{rank}(A)\times(n-\operatorname{rank}(A))}$. Moreover, let

$$J = \begin{pmatrix} I_{\operatorname{rank}(A)} \\ \mathbf{0} \end{pmatrix} \in \mathbb{K}^{n\times\operatorname{rank}(A)}.$$

Then

$$AQJ = P^{\mathrm{T}}LUQJ = P^{\mathrm{T}}L\begin{pmatrix} V_1 & V_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\begin{pmatrix} I_{\operatorname{rank}(A)} \\ \mathbf{0} \end{pmatrix} = P^{\mathrm{T}}L\begin{pmatrix} V_1 \\ \mathbf{0} \end{pmatrix}.$$

Moreover, since $P^{\mathrm{T}}L$ is non-singular, $\operatorname{rank}(AQJ) = \operatorname{rank}(V_1) = \operatorname{rank}(A) = \operatorname{rank}(J)$. And since Q is invertible, $\operatorname{rank}(AQJ) = \operatorname{rank}(J) = \operatorname{rank}(QJ)$. Thus, by Theorem 3.3 (1), $\emptyset \neq A\{2\}_{\mathcal{R}(QJ),\star} \subset A\{2\}$. Moreover, by Theorem 4.2 (1), taking $B = QJ$ and using that $\operatorname{rank}(QJ) = \operatorname{rank}(J) = \operatorname{rank}(A)$, we get that

$$m\operatorname{rank}(A) - \operatorname{rank}(A)^2 \leq \dim(A\{2\}_{\mathcal{R}(J),\star}) \leq \dim(A\{2\}). \tag{4.3}$$

Now, we consider a factorization of the form $AQ = WR$ where Q is an $n\times n$ permutation matrix, R is an $n\times n$ non-singular upper triangular matrix, and W is an $m\times n$ lower triangular matrix. Repeating the above reasoning, left-multiplying if necessary by an $m\times m$ permutation matrix P, one deduces

$$n\operatorname{rank}(A) - \operatorname{rank}(A)^2 \leq \dim(A\{2\}_{\star,\mathcal{N}(J)}) \leq \dim(A\{2\}). \tag{4.4}$$

Now, from (4.3) and (4.4) one deduces the lower bound in the statement. □

Remark 4.4. *We observe that, in the proof of Theorem 4.3, the hypothesis on the field $\mathbb{K}$ is only used to prove the upper bound and not the lower bound.*

Corollary 4.2 shows that the only matrices with finitely many outer inverses are the zero matrix and the 1×1 matrices.

Corollary 4.2. *Let $A \in \mathbb{K}^{m\times n}$, $A \neq \mathbf{0}$. Then, $\dim(A\{2\}) > 0$ if and only if $mn > 1$.*

Proof. If $A = (a) \in \mathbb{K}^{1\times 1}$, then $A\{2\} = \{0, a\}$. On the other hand, if $mn > 1$, since $A \neq \mathbf{0}$, Theorem 4.3, we have that $\dim(A\{2\}) \geq \max\{m, n\} - 1 \geq 1$. $\square$

Let us illustrate the above results by means of an example.

Example 4.4. We consider the following matrices over $\mathbb{C}$:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ C = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

Since $\text{rank}(AB) = 1 = \text{rank}(B)$, by Theorem 3.3 (1) or by Theorem 4.2 (1), it holds that $A\{2\}_{\mathcal{R}(B),\star} \neq \emptyset$. Furthermore, since $(AB)\{1\}$ is the affine line

$$(AB)\{1\} = \left\{\begin{pmatrix} 1 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 \end{pmatrix} \,\middle|\, \lambda \in \mathbb{C}\right\},$$

we get that $A\{2\}_{\mathcal{R}(B),\star}$ is the affine line

$$A\{2\}_{\mathcal{R}(B),\star} = \left\{B \begin{pmatrix} 1 & 0 \end{pmatrix} + \lambda B \begin{pmatrix} 0 & 1 \end{pmatrix} \,\middle|\, \lambda \in \mathbb{C}\right\} = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, \lambda \in \mathbb{C}\right\}.$$

Note that $\dim(A\{2\}_{\mathcal{R}(B),\star}) = 1$ (compare to Theorem 4.2 (1) and Corollary 4.1 4.3 (1)). On the other hand, since $A\{2\}_{\mathcal{R}(B),\star} \subset A\{2\}$, let us compute $A\{2\}$ and see how this inclusion works. Using Gröbner basis, one gets that $A\{2\}$ decomposes as

$$A\{2\} = V_1 \cup \{\mathbf{0}\},$$

where

$$V_1 = \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} \in \mathbb{C}^{3\times 2} \,\middle|\, \begin{array}{r} x_{11} + x_{21} + x_{31} = 1 \\ -x_{21}x_{32} + x_{31}x_{22} = 0 \\ x_{21}x_{12} + x_{21}x_{22} + x_{21}x_{32} - x_{22} = 0 \\ x_{31}x_{12} + x_{21}x_{32} + x_{31}x_{32} - x_{32} = 0 \end{array} \right\}.$$

Moreover, $\dim(V_1) = 3$, $\text{degree}(V_1) = 3$, and $A\{2\}_{\mathcal{R}(B),\star} \subset V_1 \subset A\{2\}$. Therefore, $\dim(A\{2\}) = \dim(V_1) = 3$; observe that the lower bound given by Theorem 4.3 is 2.

Let us analyze now $A\{2\}_{\star,\mathcal{N}(C)}$. Since $\text{rank}(CA) = 1 = \text{rank}(C)$, by Theorem 3.3 (2) or by Theorem 4.2 (2), it holds that $A\{2\}_{\star,\mathcal{N}(C)} \neq \emptyset$. Furthermore, since $(CA)\{1\}$ is the affine plane

$$(CA)\{1\} = \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, \lambda, \mu \in \mathbb{C}\right\}$$

we get that $A\{2\}_{\star,\mathcal{N}(C)}$ is the affine plane

$$\begin{aligned} A\{2\}_{\star,\mathcal{N}(C)} &= \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} C + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} C + \mu \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} C \,\middle|\, \lambda, \mu \in \mathbb{C}\right\} \\ &= \left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 & -1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} + \mu \begin{pmatrix} -1 & -1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \,\middle|\, \lambda, \mu \in \mathbb{C}\right\}. \end{aligned}$$

Note that $\dim(A\{2\}_{\star,\mathcal{N}(C)}) = 2$ (compare to Theorem 4.2 (2) and Corollary 4.1 (2)). Furthermore, remark that $A\{2\}_{\star,\mathcal{N}(C)} \subset V_1 \subset A\{2\}$. On the other hand, it is observable that

$$A\{2\}_{\mathcal{R}(B),\mathcal{N}(C)} = A\{2\}_{\mathcal{R}(B),\star} \cap A\{2\}_{\star,\mathcal{N}(C)} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \right\} \in V_1.$$

Observe that $\text{rank}(CAB) = \text{rank}(A) = \text{rank}(B) = \text{rank}(C) = 1$, accrodingly with Theorem 3.3 (4) and Theorem 4.2 (4).

5 Conclusion

In this paper, we present a full description of the sets of (B, C)–inverses (see Theorem 3.2). In addition, identifying the matrix vector space with an affine space, we visualize each matrix as a point in an affine space, and we study the geometric structure of the most important generalized inverses sets. In particular, we prove that inner inverses and (B, C)-inverses form affine subspaces. Moreover, we analyze their dimension. Studying a particular example, we observe that the situation, for the set of outer inverses, is much more complicated and several components, that are not affine subspaces and with different dimensions, appear. Motivated by this example, we show that, when $\mathbb{K}$ is not isomorphic to $\mathbb{Z}_2$ and $A \neq \mathbf{0}$, the set $A\{2\}$ is not an affine subspace but it is an affine algebraic variety. We also analyze the dimension of this algebraic variety.

The knowledge of these geometric properties provides important information for the theoretical analysis and for the computational aspects, since they describe how simple or complex is each set of generalized inverses. In particular, these results certify that, in general, the computation of inner inverses and (B, C)-inverses can be approached directly by means of linear algebra techniques. However, the computation and description of $A\{2\}$ is more complicated and require, in general, other techniques as Gröbner bases. This may be an interesting topic for future research.

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