




Article

Covering Rational Surfaces with Rational Parametrization Images

Jorge Caravantes ^{1,*}, J. Rafael Sendra ¹, David Sevilla ² and Carlos Villarino ¹

¹ Department of Physics and Mathematics, The University of Alcalá, 28801 Alcalá de Henares (Madrid), Spain; rafael.sendra@uah.es (J.R.S.); carlos.villarino@uah.es (C.V.)

² Department of Mathematics, The University of Extremadura, 06800 Mérida (Badajoz), Spain; sevillad@unex.es

* Correspondence: jorge.caravantes@uah.es

Abstract: Let S be a rational projective surface given by means of a projective rational parametrization whose base locus satisfies a mild assumption. In this paper we present an algorithm that provides three rational maps $f, g, h : \mathbb{A}^2 \dashrightarrow S \subset \mathbb{P}^n$ such that the union of the three images covers S . As a consequence, we present a second algorithm that generates two rational maps $f, \tilde{g} : \mathbb{A}^2 \dashrightarrow S$, such that the union of its images covers the affine surface $S \cap \mathbb{A}^n$. In the affine case, the number of rational maps involved in the cover is in general optimal.

Keywords: rational surface; birational parametrization; surjective parametrization; surface cover; base points



Citation: Caravantes, J.; Sendra, J.R.; Sevilla, D.; Villarino, C. Covering Rational Surfaces with Rational Parametrization Images. *Mathematics* **2021**, *9*, 338. <https://doi.org/10.3390/math9040338>

Academic Editor: Gabriel Eduard Vilcu

Received: 12 January 2021

Accepted: 2 February 2021

Published: 8 February 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Rational parametrizations of algebraic varieties are an important tool in many geometric applications like those in computer aided design (see, e.g., [1,2]) or computer vision (see, e.g., [3,4]). Nevertheless, the applicability of this tool can be negatively affected if the parametrization is missing basic properties: for instance its injectivity, its surjectivity, or the nature of the ground field where the coefficients belong to; see, e.g., the introductions of the papers [5–7] for some illustrating examples of this phenomenon.

Let us mention here some illustrating situations of this phenomenon; for more detailed examples, we refer the reader, e.g., to the introductions of the papers [5–7]. Let us say that we are given a rational parametrization $\mathcal{P}(t)$ of a curve that describes the possible positions that a given robot may achieve. Now, for a position P in the affine space, we want to detect the value t_0 of the parameter t such that $\mathcal{P}(t_0) = P$. If the parametrization \mathcal{P} is not injective, the answer may contain unnecessary information, for instance several parameter values for the same goal, and hence the computation time may not be optimal. On the other hand, if the parametrization is not surjective the situation can be worse because the point P may be on the trajectory curve, but not be reachable via the given parametrization for any parameter value. In addition, if the parametrization has complex non-real coefficients, it would be unnecessarily complicated to detect the real point positions of the robot.

These types of problems have been addressed by different authors from the theoretical and computational points of view. For the optimality of the ground field we refer to [8–10] where the notions of hypercircle and ultraquadric were introduced to approach the problem. In the case of the injectivity, the answer is based on Lüroth's theorem for the case of curves (see, e.g., [11]) and on Castelnuovo's theorem for the case of surfaces (see, e.g., [12]). Computationally, the injectivity problem has also been studied and there are different approaches; for the case of curves, we refer to [13–15], and for the case of surfaces we refer to [14,16,17]. It is worth mentioning that, up to our knowledge, the determination of injective surface re-parametrizations is still an open problem.

In this paper we focus on the third problem, namely, the surjectivity of rational parametrizations. Surjective parametrizations, also called normal parametrizations, have

been studied extensively but many important questions, both theoretical and computational, still remain open. The case of curves, over algebraically closed fields of characteristic zero, is comprehensively understood, and one can always find a surjective, indeed also injective, affine parametrization of the affine algebraic curve (see, e.g., [11,18–20]). Furthermore, the case of curves defined over the field of the real numbers is also studied and characterized in [20].

The situation, as usually happens, turns to be much more complicated when dealing with surfaces. In [21,22] the first steps in this direction were given and answers for certain types of surfaces, like quadrics, were provided. Also, in [23] the relation of the polynomiality of parametrizations to the surjectivity was analyzed. Nevertheless, in [24] it is shown that there exist rational surfaces that cannot be parametrized birationally and surjectively. As a consequence of this fact, the question of whether every rational surface can be covered by the union of the images of finitely many birational parametrizations is of interest. The answer is positive and can be deduced from the results in [25]. In previous papers, the second, the third and the fourth authors have studied this problem for special types of surfaces. In [5] the unreachable points of parametrizations of surfaces of revolution are characterized. In [7] it is proved that ruled surfaces can be covered by using two rational parametrizations. In addition, in [6] an algorithm to cover an affine rational surface without based points at infinity with at most three parametrizations is presented.

In this paper we continue the research described above and we present two main algorithmic and theoretical results. Moreover, we provide an algorithm that, for any projective surface parametrization, generates a cover of the projective surface with three parametrizations, assuming that, either the base locus of the input is empty, or the Jacobian of the input parametrization, specialized at each base point, has rank two. As a consequence of this result, we also present an algorithm that, for a given affine parametrization whose projectivization satisfies the condition on the base points mentioned above, returns a cover of the affine surface with two affine parametrizations.

Taking into account the results in [24], the affine cover presented in this paper is, in general, optimal. Furthermore, it improves the results in [6] and extends the results in [7] to a much more general class of surfaces. With respect to the projective cover case, although theoretically interesting, we cannot ensure that the number of parametrizations involved in the cover is optimal for a generically large class of projective surfaces since, for instance, the whole projective plane can be covered with just two maps from the affine plane as the following example shows. We leave this theoretical question open as future research.

Example 1. Consider the following two maps:

$$\begin{aligned} f : \mathbb{A}^2 &\rightarrow \mathbb{P}^2 & g : \mathbb{A}^2 &\rightarrow \mathbb{P}^2 \\ (s, t) &\mapsto (1 : s : t) & (s, t) &\mapsto (s(1-s) : t : (1-s)^2) \end{aligned}$$

Then for any point $P := (x : y : z) \in \mathbb{P}^2$, if $x \neq 0$ then $P := f(\frac{y}{x}, \frac{z}{x})$; if $x = 0 \neq z$ then $P := g(0, \frac{y}{z})$; and if $x = z = 0$ then $P := g(1, y)$, $y \neq 0$. This means that we can cover the whole projective plane with just two maps from the affine plane.

The paper is structured as follows. In Section 2, we introduce some notation and we briefly recall some notions and results that will be used throughout the paper. In Section 3, we present the projective cover algorithm and in Section 4 we illustrate the result by means of some examples. In Section 5, we apply the results in Section 3 to derive the two affine parametrization cover algorithm.

2. Preliminaries

In this section, we briefly recall some concepts and results that will be used in the subsequent sections. We essentially recall some results on the fundamental locus of rational maps and some consequences and the characterization of zero dimensional ideals via

Gröbner bases. Throughout this paper \mathbb{K} is an algebraically closed field of characteristic zero, and \mathbb{P}^n the projective space over \mathbb{K} . Moreover, we denote by \mathbb{A}_i the affine space $\{(x_0 : \dots : x_n) \in \mathbb{P}^n, | x_i \neq 0\}$. In the examples, the field \mathbb{K} will be the field \mathbb{C} of the complex numbers.

Let X be an irreducible projective variety and let $f : X \dashrightarrow \mathbb{P}^n$ be a rational map. The *fundamental locus* of f is the algebraic set $F(f)$ of points to which f cannot be extended regularly. Any $P \in F(f)$ is called a *fundamental point* of f . The following theorem analyzes the dimension of the fundamental locus.

Theorem 1. (Lemma V.5.1 [12]) *Let X be a smooth irreducible projective variety and let $f : X \dashrightarrow \mathbb{P}^n$ be a rational map generically finite. The fundamental locus of f has codimension at least 2 in X .*

Corollary 1. *Let X be a smooth irreducible surface and f as in Theorem 1. $F(f)$ is either empty or zero dimensional.*

The traditional way for solving indeterminacies in algebraic geometry consists in blowing up fundamental points (see, e.g., IV.3.3 [26]) and composing with the corresponding map as the next theorem shows.

Theorem 2. (Example II.7.17.3 [12] or Theorem II.7 [27]) *Let X be a smooth surface. Let $f : X \dashrightarrow \mathbb{P}^n$ be a rational map. Then there exists a commutative diagram*

$$\begin{array}{ccc} & Y & \\ g \swarrow & & \searrow h \\ X & \overset{f}{\dashrightarrow} & \mathbb{P}^n \end{array}$$

where g is a composite of blowups involving fundamental points of f and h is a morphism.

A first consequence of Theorem 2 is the following.

Corollary 2. (Corollary 2.5 [24]) *Let X and f be as in Theorem 2. For any fundamental point P of f , $h(g^{-1}(P))$ is a connected finite union of rational curves.*

Remark 1. *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^n$ be as in Theorem 2, and let S be the Zariski closure of $f(\mathbb{P}^2)$ in \mathbb{P}^n . The complementary in S of the $f(\mathbb{A}^2)$ is, according to Theorem 2, contained in $h(g^{-1}(F(f) \cup L_\infty))$, where $L_\infty = \mathbb{P}^2 - \mathbb{A}^2$. Such a subset consists of some rational curves and, if f contracts L_∞ , a closed point (see Corollary 2.5 [24]).*

We end this section with a well-known result on elimination theory that will be used in Section 3.

Theorem 3. (Chapter 5, Theorem 6 [28]) *Let I be an ideal in $\mathbb{K}[x_1, \dots, x_n]$. Then, the following statements are equivalent:*

1. *The algebraic subset of \mathbb{K}^n defined by I is a finite set.*
2. *Let B be a Gröbner basis for I with respect to a fixed monomial ordering. Then, for each $1 \leq i \leq n$, there is some $m_i \in \mathbb{N}$ such that $x_i^{m_i}$ is the leading monomial of an element of B .*

3. Covering Projective Surfaces with Three Parametrizations

Throughout this section, let $S \subset \mathbb{P}^n$ be a rational projective surface and let

$$F = (F_0 : \dots : F_n) : \mathbb{P}^2 \dashrightarrow S \subset \mathbb{P}^n$$

be a (not necessarily birational) parametrization of S , given by $n + 1$ homogeneous coprime polynomials F_0, \dots, F_n where the nonzero polynomials have degree d . In addition, let the homogeneous ideal $I = (F_0, \dots, F_n)\mathbb{K}[x_0, x_1, x_2]$ be called the fundamental ideal associated to F .

Since the polynomials defining F are coprime, by Corollary 1, I defines a closed algebraic subset \mathcal{A} of \mathbb{P}^2 that is either empty or consists of a finite amount of points. If $\mathcal{A} = \emptyset$, then F defines a regular map and its restrictions to each of the three affine planes $\mathbb{A}_i = \{(x_0 : x_1 : x_2) \mid x_i \neq 0\}$ covering \mathbb{P}^2 define 3 charts that cover S , since the image of a projective variety by a regular map is always Zariski closed. Otherwise, say $\mathcal{A} = \{P_1, \dots, P_k\}$; we need to make the following assumption:

(*) If $\mathcal{A} \neq \emptyset$, then for every $P \in \mathcal{A} := \{P_1, \dots, P_k\}$ the Jacobian matrix of F at P has rank 2.

Note that (*) guarantees that I does not define multiple points (i.e. the base points of F are simple). We also assume without loss of generality that:

- (a) no P_j is in any of the lines $\{x_0 = 0\}$, $\{x_1 = 0\}$ and $\{x_2 = 0\}$,
- (b) no pair $\{P_i, P_j\}$, $i \neq j$, is aligned with any of the coordinate points $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$.

We observe that the real constraint lays in (*), since conditions (a) and (b) are satisfied after a general change of coordinates. In the following remark we discuss how these hypotheses can be computationally checked.

Remark 2.

1. Note that condition (*) implies that the point P_j is regular in the projective scheme defined by the ideal I . Then, the intersection multiplicity is 1 at every P_j . Now, if we consider the ideal J defined by the 3×3 minors of the Jacobian matrix of F , the following methods, among others, can be applied to test (*):
 - (i) Check whether the ideal $I + J$ is zero-dimensional.
 - (ii) Check whether $\sqrt{I + J} = (x_0, x_1, x_2)\mathbb{K}[x_0, x_1, x_2]$ (i.e., the irrelevant ideal) or, equivalently, whether $I + J$ contains a power of any of the variables.
 - (iii) By means of resultants and gcds, using the formulas in (Theorems 2 and 3 [29]) (see also [30]).
2. Checking (i) without explicitly determining P_1, \dots, P_k can be carried out by certifying that all the ideals generated by $\{F_0, \dots, F_n, x_i\}$ ($i = 0, 1, 2$) either are zero-dimensional or contain a power of the irrelevant ideal.
3. Checking (ii) can be done by computing the ideal bases B_1 and B_2 of Algorithm 3PatchSurface and checking whether they have adequate shape (see Proposition 1). However, it may be more efficient to check that, for all $i = 0, 1, 2$, the gcd of all resultants of couples (F_0, F_j) with respect to x_i is square free.

Now we consider the three affine planes \mathbb{A}_i defined above. According to (a) all P_j lie in the intersection of the three affine planes. In this situation, the strategy is as follows. We will work with the parametrization

$$f := F|_{\mathbb{A}_0} : \mathbb{A}_0 \dashrightarrow S$$

as defined in \mathbb{A}_0 , and we will blowup \mathbb{A}_1 and \mathbb{A}_2 at the base points of F to get new affine planes $\widetilde{\mathbb{A}}_1$ and $\widetilde{\mathbb{A}}_2$ with projections $\text{Bl}_1 : \widetilde{\mathbb{A}}_1 \rightarrow \mathbb{A}_1$ and $\text{Bl}_2 : \widetilde{\mathbb{A}}_2 \rightarrow \mathbb{A}_2$. Now, we introduce the compositions

$$g := F|_{\mathbb{A}_1} \circ \text{Bl}_1 : \widetilde{\mathbb{A}}_1 \rightarrow \mathbb{A}_1 \rightarrow S \text{ and } h := F|_{\mathbb{A}_2} \circ \text{Bl}_2 : \widetilde{\mathbb{A}}_2 \rightarrow \mathbb{A}_2 \rightarrow S,$$

and we prove that the union of the images of f , g and h is the whole S (see details in Proposition 1). During this process, we also need to keep track of what happens with the

infinity line of \mathbb{A}_0 , namely $L_\infty = \{(x_0 : x_1 : x_2) \mid x_0 = 0\}$. As a consequence, we derive the following Algorithm 1.

Algorithm 1: 3PatchSurface

Require: A map $F = (F_0 : \dots : F_n)$ defined by coprime homogeneous polynomials in $\mathbb{K}[x_0, x_1, x_2]$, where the nonzero polynomials have the same degree, parametrizing a Zariski dense subset of a projective surface $S \subset \mathbb{P}^n$, such that conditions $(*)$, (a) and (b) are satisfied.

Ensure: Two maps $G = (G_0 : \dots : G_n)$ and $H = (H_0 : \dots : H_n)$ defined by homogeneous polynomials in $\mathbb{K}[x_0, x_1, x_2]$ where the nonzero polynomials have the same degree (while in the same list), such that the union of the images of $F(1 : _ : _)$, $G(_ : 1 : _)$, $H(_ : _ : 1) : \mathbb{A}^2 \rightarrow S$ cover S .

- 1: **if** the radical of the homogeneous ideal $(F_0, \dots, F_n)\mathbb{K}[x_0, x_1, x_2]$ is irrelevant (i.e., F defines a regular morphism) **then**
 - 2: Return $G = F, H = F$.
 - 3: **end if**
 - 4: For the ideals $I_i = (F_0, \dots, F_n, x_i - 1)$ for $i = 1, 2$, compute reduced Gröbner bases $B_1 = \{x_2 - q_1(x_0), x_1 - 1, p_1(x_0)\}$ for I_1 and $B_2 = \{x_2 - 1, x_1 - q_2(x_0), p_2(x_0)\}$ for I_2 , with lexicographical order $x_0 < x_1 < x_2$.
 - 5: Set $k = \deg(p_i) > \deg(q_i)$ for whatever $i = 1, 2$.
 - 6: Set $\mathbf{P}_1(x_0 : x_1) = p_1(\frac{x_0}{x_1})x_1^k$ and $\mathbf{Q}_1(x_0 : x_1) = q_1(\frac{x_0}{x_1})x_1^{k-1}$.
 - 7: Put $\tilde{G} = (\tilde{G}_0 : \dots : \tilde{G}_n) = F(x_0x_1^k : x_1^{k+1} : x_1^2\mathbf{Q}_1(x_0 : x_1) + x_2\mathbf{P}_1(x_0 : x_1))$,
 - 8: Set $\mathbf{P}_2(x_0 : x_2) = p_2(\frac{x_0}{x_2})x_2^k$ and $\mathbf{Q}_2(x_0 : x_2) = q_2(\frac{x_0}{x_2})x_2^{k-1}$.
 - 9: Set $\tilde{H} = (\tilde{H}_0 : \dots : \tilde{H}_n) = F(x_0x_2^k : x_2^2\mathbf{Q}_2(x_0 : x_2) + x_1\mathbf{P}_2(x_0 : x_2) : x_2^{k+1})$
 - 10: Set $\hat{G} = \gcd(\tilde{G}_0, \dots, \tilde{G}_n)$, $\hat{H} = \gcd(\tilde{H}_0, \dots, \tilde{H}_n)$.
 - 11: Return $G = \tilde{G}/\hat{G}, H = \tilde{H}/\hat{H}$.
-

Remark 3. In Proposition 1 we show that the integer k , introduced in the algorithm, is exactly the number of base points, that is the cardinality of \mathcal{A} (see above), so we have not introduced equivocal notation.

In the following, we see that the output of Algorithm 3PatchSurface is correct (see Theorem 4). We also recall that the required conditions for the algorithm can be checked computationally according to Remark 2. We start by proving that Step 4 works properly, assuming that conditions $(*)$, (a) and (b) are satisfied. This is probably a well-known result in a more general setting but, up to the authors’ knowledge, there are no suitable references for the proof.

Proposition 1. Let $F = (F_0 : \dots : F_n)$, I_1 and I_2 be as in Algorithm 1. There exist $p_1, q_1 \in \mathbb{K}[x_2]$, $p_2, q_2 \in \mathbb{K}[x_1]$ such that $k = \deg(p_i) > \deg(q_i)$ for all $i = 1, 2$, and the reduced Gröbner basis B_1 and B_2 of I_1 and I_2 respectively, have the following shape:

$$B_1 = \{x_2 - q_1(x_0), x_1 - 1, p_1(x_0)\} \text{ and } B_2 = \{x_2 - 1, x_1 - q_2(x_0), p_2(x_0)\}.$$

Proof. Observe that both I_1 and I_2 define finite sets in \mathbb{A}_1 and \mathbb{A}_2 , respectively. By Theorem 3, this implies that, since B_1 and B_2 are Gröbner bases, with respect to the lexicographical order $x_0 < x_1 < x_2$, there is a polynomial in each B_i just involving x_0 . This is p_1 for I_1 and p_2 for I_2 . Due to $(*)$, (a) and (b) , each p_i defines k different parallel lines in \mathbb{A}_i , so its degree is k .

Applying again Theorem 3, there is another monic polynomial in $\mathbb{K}[x_0][x_1] \cap I_1$. Since the basis is reduced, and $x_1 - 1$ was originally among the generators of I_1 , this polynomial in $\mathbb{K}[x_0][x_1]$ for I_1 is precisely $x_1 - 1$.

Now, let q_i be the interpolation polynomial whose graph goes through all the base points $P_j \in \mathbb{A}_i$ of F . It does exist because (b) holds (so there are no two different P_j in the same vertical line) and its degree is at most $k - 1$. Then $x_1 - q_2(x_0)$ vanishes at every P_j so, by Hilbert’s Nullstellensatz, it belongs to $\sqrt{I_2}$. Since $\deg(p_2) = k > \deg(q_2)$, $x_1 - q_2(x_0)$ cannot be reduced by $p(x_0)$, so, since B_1 is reduced, this is the monic polynomial in $\mathbb{K}[x_0][x_1] \cap I_2$.

We apply an analogous argument and reduction by $x_1 - 1$ to deduce that the monic polynomial in $\mathbb{K}[x_0, x_1][x_2]$ for I_1 must be $x_2 - q_2(x_0)$, and we know that $x_2 - 1$ is a reduced monic polynomial for I_2 , so it is in B_2 .

Since B_1 and B_2 are reduced Gröbner bases with respect to the lexicographical order $x_0 < x_1 < x_2$ and the ideals they generate define, precisely, the fundamental locus, they are the reduced Gröbner bases of I_1 and I_2 . \square

Before continuing, we state a Lemma.

Lemma 1. *In the conditions of Algorithm 1, neither $G(_ : 1 : _)$ nor $H(_ : _ : 1)$ have affine base points.*

Proof. By the properties of the rational map $F : \mathbb{P}^2 \dashrightarrow \mathbb{P}^n$, defined by F , and the fact that a blow up is bijective outside its blown up points, we know that any base point of $G(_ : 1 : _) = F \circ \text{Bl}_1(_ : 1 : _)$ would be in one of the lines $\text{Bl}_1(P_j)$. Now, we fix $P_j = (1 : \alpha_j : \beta_j)$ and we then prove that $G(_ : 1 : _)$ has no base points in the line $\{x_0 = \frac{1}{\alpha_j}\} \subset \mathbb{A}_1$.

Since all $F_i(x_0 : 1 : x_2)$ vanish at P_j , they have $\alpha_j x_0 - 1$ as a common factor (note that $\alpha_j \neq 0$ because (a) holds). Then, since $G_i = \tilde{G}_i / \hat{G}$, we have that $G_i(x_0 : 1 : x_2)$ is a divisor of

$$\begin{aligned} \overline{G}_i(x_0 : 1 : x_2) &:= \frac{\tilde{G}_i(x_0 : 1 : x_2)}{x_0 - \frac{1}{\alpha_j}} = \frac{F_i(x_0 : 1 : q_1(x_0) + p_1(x_0)x_2)}{x_0 - \frac{1}{\alpha_j}} = \\ &= \frac{F_i(x_0 : 1 : q_1(x_0) + p_1(x_0)x_2) - F_i(\frac{1}{\alpha_j} : 1 : q_1(\frac{1}{\alpha_j}) + p_1(\frac{1}{\alpha_j})\frac{\beta}{\alpha_j})}{x_0 - \frac{1}{\alpha_j}} = \\ &= \frac{F_i(x_0 : 1 : q_1(x_0) + p_1(x_0)x_2) - F_i(\frac{1}{\alpha_j} : 1 : q_1(\frac{1}{\alpha_j}))}{x_0 - \frac{1}{\alpha_j}}. \end{aligned}$$

This means that

$$\begin{aligned} \overline{G}_i\left(\frac{1}{\alpha_j} : 1 : x_2\right) &= \\ &= \frac{\partial F_i}{\partial x_0}\left(\frac{1}{\alpha_j} : 1 : q_1\left(\frac{1}{\alpha_j}\right)\right) + \frac{\partial F_i}{\partial x_2}\left(\frac{1}{\alpha_j} : 1 : q_1\left(\frac{1}{\alpha_j}\right)\right) \left(\frac{\partial q_1}{\partial x_0}\left(\frac{1}{\alpha_j}\right) + \frac{\partial p_1}{\partial x_0}\left(\frac{1}{\alpha_j}\right)x_2\right) = \\ &= \begin{pmatrix} \frac{\partial F_i}{\partial x_0}(P_j) & \frac{\partial F_i}{\partial x_1}(P_j) & \frac{\partial F_i}{\partial x_2}(P_j) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ \frac{\partial q_1}{\partial x_0}\left(\frac{1}{\alpha_j}\right) + \frac{\partial p_1}{\partial x_0}\left(\frac{1}{\alpha_j}\right)x_2 \end{pmatrix}. \end{aligned}$$

On the other side, the vector $(1, \alpha_j, \beta_j)$ is also in the kernel of the Jacobian matrix of F at P_j , due to Euler’s formula for homogeneous polynomials: $x_0 \frac{\partial F}{\partial x_0} + x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2} = \deg(F)F$.

By (*), the Jacobian matrix of F has rank 2 at P_j , so $(1, \alpha_j, \beta_j)$ generates its kernel. Then, $(1, 0, \frac{\partial q_1}{\partial x_0}(\frac{1}{\alpha_j}) + \frac{\partial p_1}{\partial x_0}(\frac{1}{\alpha_j})x_2)$ is not in such kernel, since $\alpha_j \neq 0$. Therefore,

$$\overline{G}(\frac{1}{\alpha_j} : 1 : x_2) = \text{Jac}(F) \cdot \begin{pmatrix} 1 \\ 0 \\ \frac{\partial q_1}{\partial x_0}(\frac{1}{\alpha_j}) + \frac{\partial p_1}{\partial x_0}(\frac{1}{\alpha_j})x_2 \end{pmatrix} \neq 0$$

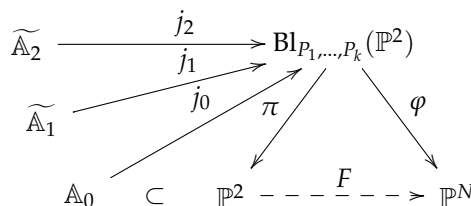
for all $x_2 \in \mathbb{C}$. Since all entries of $G(\frac{1}{\alpha_j} : 1 : x_2)$ are divisors of those of $\overline{G}(\frac{1}{\alpha_j} : 1 : x_2)$, then G is nonzero throughout the whole line $x_0 = \frac{1}{\alpha_j}$ in $\widetilde{\mathbb{A}}_1$.

Repeating the argument with H finishes the proof. \square

The next result states the correctness of Algorithm 1:

Theorem 4. *The three parametrizations F, G and H output by Algorithm 1 satisfy that the union of the images of $F(1 : _ : _)$, $G(_ : 1 : _)$ and $H(_ : _ : 1)$ covers S completely.*

Proof. We devote these first lines to sketch the proof. Keeping in mind Theorem 2, when (*), (a) and (b) hold, we construct the following diagram:



Here, $\text{Bl}_{P_1, \dots, P_k}(\mathbb{P}^2)$ is the blowup of the base points. To prove that the output of Algorithm 3PatchSurface works as expected, we need to show that

- $F(1 : _ : _) = \varphi \circ j_0$, $G(_ : 1 : _) = \varphi \circ j_1$ and $H(_ : _ : 1) = \varphi \circ j_2$.
- $j_0(\mathbb{A}_0) \cup j_1(\widetilde{\mathbb{A}}_1) \cup j_2(\widetilde{\mathbb{A}}_2) = \text{Bl}_{P_1, \dots, P_k}(\mathbb{P}^2)$.
- φ has no base points.

In this situation, we observe that, since φ has no base points, then it is a regular morphism, so $\varphi(\text{Bl}_{P_1, \dots, P_k}(\mathbb{P}^2))$ is an algebraic subset of \mathbb{P}^n . On the other side, F is dominant over S , so

$$S = \varphi(\text{Bl}_{P_1, \dots, P_k}(\mathbb{P}^2)) = F(\mathbb{A}_0) \cup G(\widetilde{\mathbb{A}}_1) \cup H(\widetilde{\mathbb{A}}_2),$$

and, hence, the theorem holds.

First of all, we define j_1 and j_2 . The blow up of all the base points of F is, locally, the blow up of the ideal I_i in \mathbb{A}_i with i being 1 or 2. Knowing the bases of I_1 and I_2 , we have (see Section 4.2 [26]):

$$\text{Bl}_{I_1}(\mathbb{A}_1) = \{((x_0 : 1 : x_2), (y_0 : y_1)) \mid p_1(x_0)y_0 = (x_2 - q_1(x_0))y_1\} \subset \mathbb{A}_1 \times \mathbb{P}^1,$$

$$\text{Bl}_{I_2}(\mathbb{A}_2) = \{((x_0 : x_1 : 1), (y_0 : y_1)) \mid p_2(x_0)y_0 = (x_1 - q_2(x_0))y_1\} \subset \mathbb{A}_2 \times \mathbb{P}^1.$$

While the way to glue the two open subsets is not interesting for the purpose of this proof, it is easy to see that $\text{Bl}_{I_1}(\mathbb{A}_1) \cup \text{Bl}_{I_2}(\mathbb{A}_2)$ is the whole $\text{Bl}_{\{P_1, \dots, P_k\}}(\mathbb{P}^2)$ minus the point that is the strict transform of $(1 : 0 : 0)$. We now consider the inclusions

$$\begin{aligned} j_1 : \quad \widetilde{\mathbb{A}}_1 &\rightarrow \text{Bl}_{I_1}(\mathbb{A}_1) \\ (x_0, x_2) &\mapsto ((x_0 : 1 : q_1(x_0) + x_2 p_1(x_0)), (x_2 : 1)) \end{aligned} \quad ,$$

$$\begin{aligned} j_2 : \quad \widetilde{\mathbb{A}}_2 &\rightarrow \text{Bl}_{I_2}(\mathbb{A}_2) \\ (x_0, x_1) &\mapsto ((x_0 : q_2(x_0) + x_1 p_2(x_0) : 1), (x_1 : 1)) \end{aligned} \quad .$$

If $\pi_i : \text{Bl}_i(\mathbb{A}_i) \rightarrow \mathbb{A}_i$ is the first projection in any of the cases, it is clear that $\text{Bl}_i = \pi_i \circ j_i$ is defined by

$$\text{Bl}_1(x_0, x_2) = (x_0, q_1(x_0) + x_2 p_1(x_0)) \text{ and } \text{Bl}_2(x_0, x_1) = (x_0, q_2(x_0) + x_1 p_2(x_0)).$$

Therefore, we have that

$$G(x_0 : 1 : x_2) = F(\text{Bl}_1(x_0, x_2)) \text{ and } H(x_0 : x_1 : 1) = F(\text{Bl}_2(x_0, x_1)).$$

On the other hand, $\widetilde{\mathbb{A}}_i$ covers the whole \mathbb{A}_i except the vertical lines through the base points. These affine lines are completely contained in \mathbb{A}_0 , since the infinity point is $(0 : 0 : 1)$ for \mathbb{A}_1 and $(0 : 1 : 0)$ for \mathbb{A}_2 . This means that, to show that $\widetilde{\mathbb{A}}_1, \widetilde{\mathbb{A}}_2$ and $\mathbb{A}_0 \setminus \{P_1, \dots, P_k\}$ —through j_1, j_2 and the blowup j_0 of the base points—cover $\text{Bl}_{\{P_1, \dots, P_k\}}(\mathbb{P}^2)$, we just need to prove that $\widetilde{\mathbb{A}}_1$ and $\widetilde{\mathbb{A}}_2$ cover the exceptional divisor. So we fix $P_j = (1 : \alpha_j : \beta_j)$ and we call $E_j \simeq \mathbb{P}^1$ the component of the exceptional divisor corresponding to P_j . Note that Bl_i covers a full neighborhood of P_j in \mathbb{A}_i minus the vertical line. This corresponds to the line joining P_j with $(0 : 0 : 1)$ for the case $i = 1$ and the line joining P_j with $(0 : 1 : 0)$ for the case $i = 2$. By condition (b), these lines do not contain other base points. These two lines represent two different directions at P_j (i.e., two different points in E_j). This means that $j_1(\{x_0 = \frac{1}{\alpha_j}\})$ covers all E_j except one point and that $j_2(\{x_0 = \frac{1}{\beta_j}\})$ is an affine line passing through that point.

The only task remaining is proving that φ has no base points. Such base points would be in the exceptional divisor, which is covered by $\widetilde{\mathbb{A}}_1$ and $\widetilde{\mathbb{A}}_2$, but Lemma 1 states that there are no base points of φ in $j_1(\widetilde{\mathbb{A}}_1) \cup j_2(\widetilde{\mathbb{A}}_2)$. \square

4. Examples

This section is devoted to illustrating Algorithm 3PatchSurface by examples. In all of them, and in those of the next section, we have used Sage [31] for the calculations. We start with a toy example in which we explicitly show that the three parametrizations cover the whole projective surface.

Example 2. Whitney’s umbrella has implicit equation $y_0 y_1^2 - y_2^2 y_3 = 0$. The usual parametrization $(x_0 : x_1 : x_2) \mapsto (x_0^2 : x_1 x_2 : x_0 x_1 : x_2^2)$ does not satisfy condition (a), so we change coordinates to get a new parametrization

$$F(x_0 : x_1 : x_2) = (x_2^2 - 2x_0 x_2 + x_0^2 : x_1^2 + x_1 x_2 - x_0 x_1 - x_0 x_2 : -x_1 x_2 + x_0 x_1 + x_0 x_2 - x_0^2 : x_1^2 + 2x_1 x_2 + x_2^2).$$

The only base point of the new parametrization is $(1 : -1 : 1)$. We then compute the bases

$$B_1 = \{x_2 + 1, x_1 - 1, x_0 + 1\} \text{ and } B_2 = \{x_2 - 1, x_1 + 1, x_0 - 1\}.$$

So we get:

$$\widetilde{G}(x_0 : x_1 : x_2) = (x_0 + x_1)G(x_0 : x_1 : x_2) \text{ and } \widetilde{H}(x_0 : x_1 : x_2) = (x_2 - x_0)H(x_0 : x_1 : x_2)$$

where

$$G = (x_1^3 - 2x_1^2 x_2 + x_1^2 x_0 + x_1 x_2^2 - 2x_1 x_2 x_0 + x_2^2 x_0 : x_1^2 x_2 - x_1 x_2 x_0 : x_1^3 - x_1^2 x_2 - x_1^2 x_0 + x_1 x_2 x_0 : x_1 x_2^2 + x_2^2 x_0)$$

and

$$H = (x_2^3 - x_2^2 x_0 : x_1^2 x_2 - x_1^2 x_0 + x_1 x_2^2 + x_0 x_1 x_2 : x_1 x_2^2 - x_0 x_1 x_2 + x_2^3 + x_0 x_2^2 : x_1^2 x_2 - x_0 x_1^2).$$

We now prove that the whole surface is covered by $F(1 : _ : _)$, $G(_ : 1 : _)$ and $H(_ : _ : 1)$. Let $A = (y_0 : y_1 : y_2 : y_3)$ be such that $y_0y_1^2 - y_2^2y_3 = 0$. Then:

1. if $y_0y_2(y_0y_1 + y_0y_2 - y_2^2) \neq 0$, then, taking $y_0 = 1$,

$$A = F\left(1 : \frac{y_2 + y_1 + y_2^2}{y_2 + y_1 - y_2^2} : \frac{-y_2 + y_1 - y_2^2}{y_2 + y_1 - y_2^2}\right)$$

Note here that $y_3 = \frac{y_1^2}{y_2^2}$, due to the equation of the umbrella and $y_0 = 1$.

2. if $y_0y_1 + y_0y_2 - y_2^2 = 0$ and $y_0y_2 \neq 0$, taking $y_0 = 1$, we get $y_1 = y_2^2 - y_2$. This equality transforms the equation of the umbrella in $y_3 = (y_2 - 1)^2$. Then, $A = G(0 : 1 : 1 - 1/y_2)$.
3. if $y_2 = 0$ and $y_0 \neq 0$, then, taking $y_0 = 1$,

$$A = F\left(1 : 1 : \frac{-1 + \sqrt{y_3}}{1 + \sqrt{y_3}}\right).$$

Note here that every A in this case is gotten twice. Even $(1 : 0 : 0 : 1)$ is both $F(1 : 1 : 0)$ and $H(0 : -1 : 1)$.

4. if $y_0 = 0$, the infinity hyperplane section of Whitney’s umbrella has two components: $y_2 = 0$ and $y_3 = 0$. Then:

$$A = (0 : y_1 : 0 : y_3) = F\left(1 : 0 : \frac{(y_3 + y_1)}{(y_3 - y_1)}\right)$$

when $y_3 - y_1 \neq 0$, and $A = (0 : 1 : 0 : 1) = G(0 : 1 : 1)$. Moreover,

$$A = (0 : y_1 : y_2 : 0) = G(-1 : y_1 + y_2 : y_1)$$

when $y_1 + y_2 \neq 0$. Finally, $A = (0 : -1 : 1 : 0) = H(1 : -1 : 1)$.

Observe that $F(1 : _ : _)$ covers the whole umbrella minus a couple of rational curves. Then, $G(_ : 1 : _)$ covers these curves minus just a point, that is covered by $H(_ : _ : 1)$.

The following example applies Algorithm 1 to a classic surface. While the computation time is not long, the output is too large, so we just sketch some computations.

Example 3. Let us consider the Clebsch cubic, given by the equation

$$z_0^3 + z_1^3 + z_2^3 + z_3^3 - (z_0 + z_1 + z_2 + z_3)^3 = 0.$$

A parametrization is defined by

$$\begin{aligned} &(-x_0^2x_1 + x_0^2x_2 + x_0x_1^2 - x_0x_2^2 : x_0^3 - x_0^2x_1 - x_0^2x_2 + x_1x_2^2 : \\ &\quad -x_0^3 + x_0^2x_1 + x_0^2x_2 - x_1^2x_2 : -x_0x_1^2 + x_0^2x_1 - x_1x_2^2). \end{aligned}$$

This parametrization, however, does not satisfy conditions (a) and (b). After the coordinate change in \mathbb{P}^2 given by

$$(x_0 : x_1 : x_2) \mapsto (x_0 + 3x_1 + 2x_2 : x_0 + x_1 + 3x_2 : -x_0 - x_1 + x_2),$$

we get the parametrization $F = (F_0 : F_1 : F_2 : F_3)$, where

$$\begin{aligned} F_0 &= 8x_2^3 + 8x_2^2x_1 + 8x_2^2x_0 - 18x_2x_1^2 - 12x_2x_1x_0 - 2x_2x_0^2 - 18x_1^3 - 30x_1^2x_0 - 14x_1x_0^2 - 2x_0^3, \\ F_1 &= -5x_2^3 - 17x_2^2x_1 - 9x_2^2x_0 + 19x_2x_1^2 + 14x_2x_1x_0 + 3x_2x_0^2 + 28x_1^3 + 30x_1^2x_0 + 12x_1x_0^2 + 2x_0^3, \\ F_2 &= -x_2^3 + 15x_2^2x_1 + 7x_2^2x_0 - 13x_2x_1^2 - 2x_2x_1x_0 + 3x_2x_0^2 - 26x_1^3 - 24x_1^2x_0 - 6x_1x_0^2, \end{aligned}$$

$$F_3 = -9x_2^3 + 6x_2^2x_1 + 18x_2x_1^2 + 4x_2x_1x_0 - 2x_2x_0^2 + 5x_1^3 + 5x_1^2x_0 - x_1x_0^2 - x_0^3.$$

One can check that the base points are $P_1 = (5 : 2 : -3)$, $P_2 = (11 : -2 : -5)$, $P_3 = (7 : -2 : -1)$, $P_4 = (1 : 2 : 1)$, $P_{5,6} = (\pm 2 - \sqrt{5} : 2 : \sqrt{5})$. This agrees with the well known fact that a cubic smooth surface is a plane blown up at 6 general points.

Now we take $x_1 = 1$ or $x_2 = 1$ and get the two Gröbner bases $B_i = \{x_i - 1, x_{3-i} - q_i(x_0), p_i(x_0)\}$, where $i \in \{1, 2\}$ and:

$$p_1(x_0) = x_0^6 + 8x_0^5 + \frac{21}{4}x_0^4 - 61x_0^3 - \frac{1077}{16}x_0^2 + \frac{239}{4}x_0 - \frac{385}{64},$$

$$q_1(x_0) = -\frac{393}{8360}x_0^5 - \frac{18511}{50160}x_0^4 - \frac{12941}{50160}x_0^3 + \frac{13537}{5280}x_0^2 + \frac{1123141}{401280}x_0 - \frac{21649}{14592},$$

$$p_2(x_0) = x_0^6 + \frac{178}{15}x_0^5 + \frac{199}{5}x_0^4 + \frac{916}{25}x_0^3 - \frac{2387}{75}x_0^2 - \frac{3926}{75}x_0 - \frac{77}{15},$$

$$q_2(x_0) = -\frac{7075}{11264}x_0^5 - \frac{212195}{33792}x_0^4 - \frac{221885}{16896}x_0^3 + \frac{49613}{16896}x_0^2 + \frac{612691}{33792}x_0 + \frac{2987}{3072}.$$

Performing substitutions

$$\tilde{G}(x_0 : 1 : x_2) = F(x_0 : 1 : q_1(x_0) + x_2p_1(x_0))$$

and

$$\tilde{H}(x_0 : x_1 : 1) = F(x_0 : q_2(x_0) + x_1p_2(x_0) : 1)$$

then dividing by the gcd of all entries in each case produces two parametrizations of degree 15, $G(x_0 : 1 : x_2)$ and $H(x_0 : x_1 : 1)$, with about 45 coefficients per polynomial and about 70 bits per coefficient, that cover, together with $F(1 : x_1 : x_2)$, the whole cubic.

5. The Affine Case

In this section, we slightly change our point of view and we consider the problem of covering a rational affine surface by means of the images of several affine parametrizations. So, in the sequel we consider that we are given F and S as in Section 5, and we deal with the problem of covering $S \cap \mathbb{A}^n$, where \mathbb{A}^n is the open subset of \mathbb{P}^n defined by the first variable not vanishing. Equivalently, one may consider that we are indeed given an affine parametrization and the affine surface that it defines. Nevertheless, to be consistent with the notation used throughout the paper, we will use the first notational statement of the problem.

In this section, we prove that to cover a rational affine surface, only two patches are necessary (see Theorem 5 and Corollary 3). The basic idea is as follows. The given parametrization $F(1 : x_1 : x_2)$ covers a constructible subset. The complement of such subset is contained in the image of L_∞ (that is, $F(0 : x_1 : x_2)$) and the base points, which is a finite union of affine rational curves (see Corollary 2) and, maybe, an isolated point corresponding to a contracted L_∞ . The parametrization $G(x_0 : 1 : x_2)$ of Algorithm 1, restricted to certain vertical lines, covers all such affine curves except at most one point. Since such curves also have a point at infinity, we want such point to be the image of the point at infinity of the parameter line. Based on these ideas, we derive Algorithm 2 that, when the original parametrization satisfies $(*)$, (a) and (b) , covers affine surfaces using just two parametrizations.

Algorithm 1: 2PatchForAffine

Require: A list $F = (F_0 : \dots : F_n)$ of coprime homogeneous polynomials of the same degree in $\mathbb{K}[x_0, x_1, x_2]$, parametrizing a Zariski dense subset of a projective surface $S \subset \mathbb{P}^n_{\mathbb{C}}$, such that conditions $(*)$, (a) and (b) are satisfied.

Ensure: A list $g' = (g'_0 : \dots : g'_n)$ of rational functions of two variables such that $F(1 : _ : _)$ and $g'(_, _)$ cover $S \cap \mathbb{A}^n$.

- 1: Compute $B_1 = \{x_2 - q_1(x_0), x_1 - 1, p_1(x_0)\}$ and $G = (G_0 : \dots : G_n)$ as in Algorithm 1.
- 2: **for** α root of $p_1(x_0)$ **or** $\alpha = 0$ **do**
- 3: **if** $\deg G_0(\alpha : 1 : x_2) < \max\{\deg G_i(\alpha : 1 : x_2) \mid i = 1, \dots, n\}$ **or** $G_0(\alpha : 1 : x_2)$ is constant **then**
- 4: Include α in set A and $\beta_\alpha := \infty$.
- 5: **else**
- 6: Include α in set B . Choose β_α among the roots of $G_0(\alpha : 1 : x_2)$.
- 7: **end if**
- 8: **end for**
- 9: Let $s(x_0)$ be a polynomial vanishing at all the $\alpha \in A$ and not vanishing at any of the $\alpha \in B$. See Remark 4 below, for suggestions on how to find one.
- 10: Choose $r(x_0)$, a polynomial such that $r(\alpha) = \beta_\alpha s(\alpha)$ for all $\alpha \in B$ and $r(\alpha) \neq 0$ for all $\alpha \in A$.
- 11: Find Bezout coefficients u and v such that $\gcd(r, s) = u \cdot r + v \cdot s$.
- 12: **return** $g'(x_0, x_2) = G\left(x_0 : 1 : \frac{r(x_0)x_2 + v(x_0)}{s(x_0)x_2 - u(x_0)}\right)$.

Remark 4. Let us comment some computational aspects of Algorithm 2.

1. For $s(x_0)$, one may proceed as follows: collect the coefficients of $G_0(x_0 : 1 : x_2)$ for x_2 except the one for x_2^0 , and compute the gcd, $d(x_0)$, of these coefficients; then $s(x_0) = \gcd(d(x_0), x_0 p_1(x_0))$.
2. Note that Algorithm 1 does not extend the field that is used to define F . However, Algorithm 2 needs to consider possibly algebraic coordinates for the points that need to be sent to the infinity in the parameter plane \mathbb{A}_1 . Example 4 shows that the application of the algorithm forces the usage of algebraic coefficients.

In order to state the correctness of Algorithm 2, we start with a technical lemma.

Lemma 2. In the setting of Algorithm 2, if no component of $G(\alpha : 1 : x_2)$ is constant, it holds that

1. If $\alpha \in A$, then $\lim_{x_2 \rightarrow \infty} G(\alpha : 1 : x_2)$ is a point at infinity.
2. If $\alpha \in B$, then $G(\alpha : 1 : \beta_\alpha)$ is a point at infinity.

Proof. Let α be an element of A . If $G_0(\alpha : 1 : x_0)$ is identically zero, then the result is obvious. Otherwise, the map $(\frac{G_1}{G_0}(\alpha : 1 : x_2), \dots, \frac{G_n}{G_0}(\alpha : 1 : x_2))$, from the affine plane to the surface, has degree strictly higher in the numerator of at least one of its components, because $\deg(G_0(\alpha : 1 : x_2)) < \max\{\deg(G_i(\alpha : 1 : x_2)) \mid i = 1, \dots, n\}$. So the limit, when x_2 tends to ∞ , is at infinity. Note that the full $(n + 1)$ -tuple $G(\alpha : 1 : x_2)$ is not constant, so the case of constant first entry $G_0(\alpha : 1 : x_2)$ satisfies the inequality too.

Now, let α be an element of B . Then, $G_0(\alpha : 1 : \beta_\alpha) = 0$, so the statement holds. \square

Let us, now, prove that Algorithm 2 works as expected.

Theorem 5. Let F, S and g' be as in Algorithm 2. Then $F(1 : x_1 : x_2)$ and $g'(x_0, x_2)$ cover the whole $S \cap \{y_0 \neq 0\}$.

Proof. Since the input of Algorithms 1 and 2 are the same, by Theorem 4, one deduces that $F(1 : x_1 : x_2)$, $G(x_0 : 1 : x_2)$ and $H(x_0 : x_1 : 1)$ cover the projective surface S .

Taking into account how G and H in Algorithm 1 are defined, any point in $\widetilde{\mathbb{A}}_1$ and $\widetilde{\mathbb{A}}_2$, not in L_∞ or in $\{p_i(x_0) = 0\}$, is sent by $G(x_0 : 1 : x_2)$ or $H(x_0 : x_1 : 1)$ into the image by F of a point in \mathbb{A}_0 . Therefore, we need to check that g' covers any affine point in $G(\{x_0 = 0\} \cup \{p_1(x_0) = 0\}) \cup H(\{x_0 = 0\} \cup \{p_2(x_0) = 0\})$.

Any component C of $G(\{x_0 = 0\} \cup \{p_1(x_0) = 0\}) \cup H(\{x_0 = 0\} \cup \{p_2(x_0) = 0\})$ is either a point or a rational curve covered by $G(\alpha : 1 : x_2) \cup H(\alpha' : x_1 : 1)$, where either $\alpha = \alpha' = 0$ or $(1 : \frac{1}{\alpha} : \frac{1}{\alpha'})$ is a base point of F . Moreover, such component is the Zariski closure of the image of the restriction $G|_{\{x_0=\alpha, x_1 \neq 0\}}$, which coincides with the Zariski closure of $g'|_{\{x_0=\alpha\}}$.

If C is just a point, then it is covered by $g'|_{\{x_0=\alpha\}}$. Otherwise, it is well known that any morphism defined in an open subset of a projective smooth curve can be extended regularly to the whole curve and that the image of a projective curve by a regular morphism is a Zariski closed subset. Then, we can extend $g'|_{\{x_0=\alpha\}}$ to the Zariski closure of the line where it is defined and we would cover completely C . This means that $g'|_{\{x_0=\alpha\}}$ covers all C minus, at most, just a point (the image of the infinity point of the affine line). However, this point is $G(\alpha : 1 : \beta_\alpha)$, which is at infinity by Lemma 2. Therefore, any point in $C \cap \{y_0 \neq 0\}$ is in $g'(\{x_0 = \alpha\}) \subset g'(\mathbb{A}^2)$. \square

Example 4. Consider the following projective transformation of the Veronese morphism:

$$F : \mathbb{P}^2 \rightarrow \mathbb{P}^5$$

$$(x_0 : x_1 : x_2) \mapsto (x_0^2 + x_1^2 + x_2^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2)$$

Since there are no base points, Algorithm 2 generates $G = F$, and then, for just $\alpha = 0$, it computes β_0 such that $\beta_0^2 + 1 = 0$. This means that β_0 must be imaginary, so it is not rational. The output

$$g'(x_0, x_2) = G\left(x_0 : 1 : \frac{ix_2 + 1}{x_2}\right)$$

has imaginary coefficients.

We observe that any choice of two rationally defined affine planes \mathbb{A}_0 and \mathbb{A}_1 of the projective plane will leave a point P in the projective plane over \mathbb{Q} out of the union, and then $F(P)$, which is not at infinity, will not be covered. Observe, however, that the surface is isomorphic to the projective plane, so one can compose the two maps appearing in Example 1 with the Veronese morphism to cover, not just the affine part, but the whole projective surface without extending the field

Example 5. Let us again consider the cubic of Example 3. We recall that

$$p_1(x_0) = x_0^6 + 8x_0^5 + \frac{21}{4}x_0^4 - 61x_0^3 - \frac{1077}{16}x_0^2 + \frac{239}{4}x_0 - \frac{385}{64}.$$

One can factor G_0 , as obtained in Example 3, and one of the factors is

$$200640x_2x_0^4 + 1203840x_2x_0^3 - 1304160x_2x_0^2 - 9329760x_2x_0 + 4827900x_2 - 9432x_0^3 - 55180x_0^2 + 56238x_0 + 388135.$$

Here, one can easily get x_2 as a rational function on x_0 , so we have a point going to infinity at each vertical line:

$$x_2 = \beta_{x_0} = \frac{9432x_0^3 + 55180x_0^2 - 56238x_0 - 388135}{200640x_0^4 + 1203840x_0^3 - 1304160x_0^2 - 9329760x_0 + 4827900}. \tag{1}$$

The set A is given by the common roots of p_1 and the denominator in (1), so

$$s(x_0) = \gcd(p_1, \text{denominator}(\beta_{x_0})) = 16x_0^4 + 96x_0^3 - 104x_0^2 - 744x_0 + 385.$$

We need a polynomial $r(x_0)$, coprime with $s(x_0)$, whose values at the roots of $\frac{x_0p_1(x_0)}{s(x_0)}$ equal $\beta_{x_0}s(x_0)$. Such roots are 0 and $1 \pm \frac{\sqrt{5}}{2}$, and the interpolating polynomial

$$r(x_0) = \frac{18511}{3135}x_0^2 - \frac{898}{209}x_0 - \frac{7057}{228}$$

works. We now need a Bezout identity $ur + vs = 1$, so we get

$$u(x_0) = -\frac{26198287461}{21693258568709}x_0^3 - \frac{1732666485971}{130159551412254}x_0^2 - \frac{487041584557}{12396147753548}x_0 - \frac{632041996387}{74376886521288}$$

and

$$v(x_0) = \frac{1510767910251}{3389825077278640}x_0 + \frac{357074564524303}{186537231395390304}$$

Then, we have that the images of $F(1 : x_1 : x_2)$ and $G\left(x_0 : 1 : \frac{r(x_0)x_2 + v(x_0)}{s(x_0)x_2 - u(x_0)}\right)$ cover the whole affine cubic.

In [24], it is proved that there exist affine surfaces that cannot be covered by means of a unique map from the affine plane. In fact, the surface in Example 5 is proved to be one of them. Now, the following corollary of Theorem 5 shows that, under hypotheses (*), (a), (b), one can always cover the affine surface with two affine parametrization images.

Corollary 3. *Let S be an affine surface such that there exists a parametrization $f : \mathbb{A}^2 \dashrightarrow S$ with a projectivization F satisfying (*), (a) and (b). Then S can be covered with just two parametrizations.*

In order to prove that two affine patches are enough we have had to impose, to the projectivization of the input affine parametrization, hypotheses (*), (a) and (b). If we do not impose (*), we cannot ensure this general result. However, it is interesting to observe that there are affine surfaces not satisfying (*) that can be covered by only one map. To create an example, it is enough to send the exceptional divisor to the infinity hyperplane together with the image of L_∞ .

Example 6. *Consider the transformation of the plane $t(x_0 : x_1 : x_2) = (x_0x_1x_2 : x_1^3 : x_0x_2^2)$ and let F be its composition with the degree-3 Veronese morphism $v_3(x_0 : x_1 : x_2) = (x_0x_1x_2 : x_0^3 : \dots : x_2^3)$. We can observe that $\mathbb{P}^2 - t(\mathbb{A}_0) = \{x_0x_1x_2 = 0\} = v_3^{-1}(\{y_0 = 0\})$. On the other hand, the fundamental locus is defined by the ideal $I = (x_0x_1x_2, x_1^3, x_0x_2^2)$, which is singular, so F does not satisfy (*), but $F(1, x_1, x_2)$ covers the whole affine surface.*

6. Conclusions

In the introduction we have commented the importance for applications of having surjective parametrizations of curves and surfaces. For the curve case, the problem can be solved by means of a single parametrization. For the surface case, the situation is much more complicated and, in general, one may need more than one parametrization to cover it. So the problem is reformulated by, on one hand, asking for the minimization of the number of parametrizations in the cover and, on the other, by requiring the actual computation of covers. There were previous results for some particular types of surfaces, specially of those whose construction is directly related to curves as ruled surfaces, revolution surfaces, etc.

In this paper we present theoretical results on the number of parametrizations that one may need to cover a surface, and we enlarge the class of surfaces where this approach is valid. More precisely, we present two algorithms, one for the projective case and the other for the affine case. In the affine case, we are able to cover any rational affine surface satisfying certain mild hypotheses on the base locus of the input parametrization, in a way that is optimal in the number of cover elements, namely, two. For the projective case, the answer provides three cover parametrizations. Two open problems are the extension of the results to the case where no condition of the base locus is imposed, and the optimality on the number of cover parametrizations in the projective case.

Author Contributions: All the stages of work have been participated by all the authors of the article. All authors have read and agreed to the published version of the manuscript.

Funding: The authors are partially supported by FEDER/Ministerio de Ciencia, Innovación y Universidades-Agencia Estatal de Investigación/MTM2017-88796-P (Symbolic Computation: new challenges in Algebra and Geometry together with its applications). J. Caravantes, J.R. Sendra and C. Villarino belong to the Research Group ASYNACS (Ref. CT-CE2019/683). D. Sevilla is a member of the research group GADAC and is partially supported by Junta de Extremadura and FEDER funds (group FQM024).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Farin, G.; Hoschek, J.; Kim, M.S. (Eds.) *Handbook of Computer Aided Geometric Design*; North-Holland: Amsterdam, The Netherlands, 2002; p. xxviii+820.
2. Hoschek, J.; Lasser, D. *Fundamentals of Computer Aided Geometric Design*; A.K. Peters: Wellesley, MA, USA; Berlin, Germany, 1993.
3. Agoston, M. *Computer Graphics and Geometric Modelling*; Computer Graphics and Geometric Modeling; Springer: Berlin/Heidelberg, Germany, 2005.
4. Marsh, D. *Applied Geometry for Computer Graphics and CAD*; Springer Undergraduate Mathematics Series; Springer: London, UK, 2005.
5. Sendra, J.R.; Villarino, C.; Sevilla, D. Missing sets in rational parametrizations of surfaces of revolution. *Comput. Aided Des.* **2015**, *66*, 55–61. [[CrossRef](#)]
6. Sendra, J.R.; Sevilla, D.; Villarino, C. Covering of surfaces parametrized without projective base points. In Proceedings of the International Symposium on Symbolic and Algebraic Computation, ISSAC '14, Kobe, Japan, 23–25 July 2014; Nabeshima, K., Nagasaka, K., Winkler, F., Szántó, Á., Eds.; ACM: New York, NY, USA, 2014; pp. 375–380. [[CrossRef](#)]
7. Sendra, J.R.; Sevilla, D.; Villarino, C. Covering rational ruled surfaces. *Math. Comput.* **2017**, *86*, 2861–2875. [[CrossRef](#)]
8. Sendra, J.R.; Winkler, F. Parametrization of algebraic curves over optimal field extensions. *J. Symb. Comput.* **1997**, *23*, 191–207.
9. Recio, T.; Sendra, J.R.; Tabera, L.F.; Villarino, C. Generalizing circles over algebraic extensions. *Math. Comp.* **2010**, *79*, 1067–1089. [[CrossRef](#)]
10. Andradas, C.; Recio, T.; Sendra, J.R.; Tabera, L.F. On the simplification of the coefficients of a parametrization. *J. Symb. Comput.* **2009**, *44*, 192–210. [[CrossRef](#)]
11. Sendra, J.R.; Winkler, F.; Pérez-Díaz, S. *Rational Algebraic Curves. A Computer Algebra Approach*; Algorithms and Computation in Mathematics; Springer: Berlin, Germany, 2008; Volume 22, p. x+267.
12. Hartshorne, R. *Algebraic Geometry (Graduate Texts in Mathematics, No. 52)*; Springer: New York, NY, USA; Heidelberg, Germany, 1977; p. xvi+496.
13. Alonso, C.; Gutiérrez, J.; Recio, T. Reconsidering algorithms for real parametric curves. *Appl. Algebra Engrg. Comm. Comput.* **1995**, *6*, 345–352. [[CrossRef](#)]
14. Pérez-Díaz, S. On the problem of proper reparametrization for rational curves and surfaces. *Comput. Aided Geom. Des.* **2006**, *23*, 307–323. [[CrossRef](#)]
15. Sederberg, T.W. Improperly parametrized rational curves. *Comput. Aided Geom. Des.* **1986**, *3*, 67–75. [[CrossRef](#)]
16. Pérez-Díaz, S. A partial solution to the problem of proper reparametrization for rational surfaces. *Comput. Aided Geom. Des.* **2013**, *30*, 743–759. [[CrossRef](#)]
17. Schicho, J. Rational parametrization of surfaces. *J. Symb. Comput.* **1998**, *26*, 1–29. [[CrossRef](#)]
18. Andradas, C.; Recio, T. Plotting missing points and branches of real parametric curves. *Appl. Algebra Engrg. Comm. Comput.* **2007**, *18*, 107–126. [[CrossRef](#)]
19. Rubio, R.; Serradilla, J.; Vélez, M.P. A note on implicitization and normal parametrization of rational curves. In Proceedings of the 2006 International Symposium on Symbolic and Algebraic Computation, Genoa, Italy, 9–12 July 2006; ACM Press: New York, NY, USA, 2006; pp. 306–309.
20. Sendra, J.R. Normal Parametrizations of Algebraic Plane Curves. *J. Symb. Comput.* **2002**, *33*, 863–885. [[CrossRef](#)]
21. Bajaj, C.L.; Royappa, A.V. Finite representations of real parametric curves and surfaces. *Internat. J. Comput. Geom. Appl.* **1995**, *5*, 313–326. [[CrossRef](#)]
22. Gao, X.S.; Chou, S.C. On the normal parameterization of curves and surfaces. *Internat. J. Comput. Geom. Appl.* **1991**, *1*, 125–136. [[CrossRef](#)]
23. Pérez-Díaz, S.; Sendra, J.R.; Villarino, C. A first approach towards normal parametrizations of algebraic surfaces. *Internat. J. Algebra Comput.* **2010**, *20*, 977–990. [[CrossRef](#)]

24. Caravantes, J.; Sendra, J.R.; Sevilla, D.; Villarino, C. On the existence of birational surjective parametrizations of affine surfaces. *J. Algebra* **2018**, *501*, 206–214. [[CrossRef](#)]
25. Bodnár, G.; Hauser, H.; Schicho, J.; Villamayor U., O. Plain varieties. *Bull. Lond. Math. Soc.* **2008**, *40*, 965–971. Available online: <https://academic.oup.com/blms/article-pdf/40/6/965/772642/bdn078.pdf> (accessed on 8 February 2021). [[CrossRef](#)]
26. Shafarevich, I.R.T. *Basic Algebraic Geometry 1: Varieties in Projective Space*, 2nd ed.; Translated from the 1988 Russian edition and with notes by Miles Reid; Springer: Berlin, Germany, 1994; p. xx+303.
27. Beauville, A. *Complex Algebraic Surfaces; London Mathematical Society Lecture Note Series*; Barlow, R., Shepherd-Barron, N.I., Reid, M., transed.; Cambridge University Press: Cambridge, UK, 1983; Volume 68, p. iv+132.
28. Cox, D.A.; Little, J.; O’Shea, D. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, 4th ed.; Undergraduate Texts in Mathematics; Springer: Cham, Switzerland, 2015; p. xvi+646. [[CrossRef](#)]
29. Cox, D.A.; Pérez-Díaz, S.; Sendra, J.R. On the base point locus of surface parametrizations: Formulas and consequences. *arXiv* **2020**, arXiv:2008.08009.
30. Pérez-Díaz, S.; Sendra, J.R. Computing Birational Polynomial Surface Parametrizations without Base Points. *Mathematics* **2020**, *8*, 2224. [[CrossRef](#)]
31. The Sage Developers. SageMath, the Sage Mathematics Software System (Version 9.0). 2020. Available online: <http://www.sagemath.org> (accessed on 8 February 2021).