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A connection between birational automorphisms of the plane and linear systems of curves*

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Abstract

In this paper, we prove that there exits a one-to-one correspondence between birational automorphisms of the plane and pairs of pencils of curves intersecting in a unique point. As a consequence, we show how to construct birational automorphisms of the plane of a certain degree d (fixed in advance) from some curves generating two linear systems of curves of degrees d and $\widetilde{d}$, where $\widetilde{d} = d - 2$ for $d > 2$, and $\widetilde{d} = 1$ otherwise. In addition, we also get the inverse of the birational automorphism constructed, and we show that its degree is obtained from the degree of the linear system of curves. As a special case, we show how these results can be stated to polynomial birational automorphisms of the plane.

Keywords: Birationality; Automorphism; Linear Systems of Curves;

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1 Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. The group of birational automorphisms of the projective space $\mathbb{P}^n$ over $\mathbb{K}$ is called the group of Cremona transformations of $\mathbb{P}^n$ or the Cremona group. There is an extensive classical literature about this group (see for instance, [1], [3], [4], [7], [8], [13], [14], [15], [17], [19], [20]).

It is well known that birational automorphisms of the parameter space $\mathbb{P}^1$ are the Möbius transformations. In the plane, by Noether's Theorem, if $\mathbb{K}$ is an algebraically closed field, each Cremona transformation can be expressed as a composition

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of quadratic transformations. The simplest examples of Cremona transformations may be given as linear-fractional transformations

$$\mathcal{P}(t_1, t_2) = \left(\frac{a_1 t_1 + b_1 t_2 + c_1}{a_2 t_1 + b_2 t_2 + c_2}, \frac{a_3 t_1 + b_3 t_2 + c_3}{a_4 t_1 + b_4 t_2 + c_4} \right).$$

Cremona transformations of $\mathbb{P}^1$ and $\mathbb{P}^2$ have been extensively used by algebraic geometers and in particular, they play an important role in the frame of the algebraic manipulations of curves and surfaces. For instance, they are effective in the reduction of singularities of curves to points with distinct tangents (see [34]). That is, a plane algebraic curve can be transformed by a Cremona transformation into a plane algebraic curve with ordinary multiple points. For surfaces, it is also shown that every algebraic surface can be transformed by a Cremona transformation into a surface having only ordinary multiple curves (for further details see [38]). In [9] and [10], the reduction of linear systems of plane curves by Cremona transformations is considered. There, one attempts to do a reduction by quadratic Cremona transformation which gives in turn a proof of the classical result that the Cremona transformations are generated by the quadratic ones (see [33]).

On the other side, it is well known that, in the case of algebraic curves, Möbius transformations preserve the degree of a given curve's parametrization. Hence the parametric degree (that is, the degree of a parametrization) is the same for all proper parametrizations. In the case of surfaces, the parametric degree is not preserved by Cremona transformations. However, in general, one can find a Cremona transformation that reduce the degree of a surface's parametrization (see [30]).

For these reasons, the problem of easily constructing Cremona transformations, by controlling its degree, is very important. In fact, several authors have dealt with this question recently (see e.g. [6] and [11]). Thus, given $d \in \mathbb{N}$, in this paper we are interested in constructing a birational automorphism of the plane $\mathcal{S}$ such that $\deg(\mathcal{S}) = d$. For this purpose, we construct a pencil of curves $\mathcal{V}_1$ of degree d (where singularities and simple points are known), and using *Algorithm for Pencil Parametrization* (see Section 2) we determine a linear subsystem $\mathcal{V}_2$ of dimension 1 of the system of adjoint curves to $\mathcal{V}_1$. From the curves generating $\mathcal{V}_1$ and $\mathcal{V}_2$, we obtain $\mathcal{S}$. From this construction, we get that the inverse of $\mathcal{S}$ is the unique non–constant intersection point of $\mathcal{V}_1$ and $\mathcal{V}_2$, and its degree is obtained from the degree of $\mathcal{V}_1$ and $\mathcal{V}_2$.

Reciprocally, we show how a given birational automorphism is related with a pair of pencils that intersect in a unique point. More precisely, we are given a birational automorphism $\mathcal{P}$, and we construct a pair of pencils $(\mathcal{V}_1, \mathcal{V}_2)$, from which we obtain a birational automorphism $\mathcal{S}$. We show that "up to composition with a polynomial De Jonquières transformation", $\mathcal{P}$ is *equivalent* to $\mathcal{S}$ (that is, $\mathcal{P} = \mathcal{J} \circ \mathcal{S}$, where $\mathcal{J}$ is a De Jonquières transformation). Therefore, in this paper, we prove that there exits a one-to-one correspondence between birational automorphisms of the plane and pairs of pencils $(\mathcal{V}_1, \mathcal{V}_2)$ intersecting in a unique point.

Finally, we also show how these results can be stated similarly for the case of birational automorphisms of the plane that are polynomial and thus, in particular, we construct polynomial birational automorphisms of the plane of a desired degree d. We remark that polynomial automorphisms have an additional interest in practical applications, since the non-existence of denominators avoids the possible unstable behavior, when the parameters take values close to the points of the curves defined by the denominators (see e.g. [26]).

For higher dimension there has also been a lot of research on the subject ([2], [5], [16], [18], [21], [22], [23], [27]), though the results obtained remain sporadic and, in general, there are no substantial advances with respect to the pioneering works in the knowledge either about the structure of arbitrary Cremona transformations themselves or about the structure of the group of Cremona transformations, even for $n = 3$. The results presented here attempt to open several ways that can be used to provide significant results concerning Cremona transformations for $n \geq 3$ (see Section 4).

The structure of the paper is as follows: in Section 2, we provide some preliminaries and previous results. For this purpose, two subsections are considered: in Subsection 2.1, we deal with the classical problem of parametrizing a plane curve over a subfield k of and algebraically closed field K of characteristic zero, and the definition of k-rational curve is introduced (see Definition 1). In Subsection 2.2, we specialize Subsection 2.1 to the case where the input curve is a pencil of curves, $k = \mathbb{K}(t)$ and $K = \overline{\mathbb{K}(t)}$ being $\mathbb{K}$ an algebraically closed field of characteristic zero. Section 3 is devoted to show a one-to-one correspondence between birational automorphisms of the plane and pairs of pencils intersecting in a unique point. For this purpose, three subsections are considered: in Subsection 3.1, we show how birational automorphisms of the plane of a desired degree d can be constructed from the curves generating two 1-dimensional systems of curves of degrees d and $\widetilde{d}$, where $\widetilde{d} = d-2$ for $d > 2$, and $\widetilde{d} = 1$ otherwise (see Theorem 1 and statement 2 in Corollary 2). In Subsection 3.2, we show how a given birational automorphism is related with a pair of pencils. More precisely, we are given a birational automorphism $\mathcal{P}$, and we construct a pair of pencils $(\mathcal{V}_1, \mathcal{V}_2)$ intersecting in a unique point, from where we obtain a birational automorphism $\mathcal{S}$. We show that "up to composition with a polynomial De Jonquières transformation", $\mathcal{P}$ is *equivalent* to $\mathcal{S}$ (see Theorem 2). In Subsection 3.3, we show how these results can be stated similarly to the case of birational polynomial automorphisms (see Proposition 1 and Theorem 3). We finish with a section with conclusions and open questions (Section 4).

2 Preliminaries and Previous Results

In this section, we introduce the notation and some previous algorithmic methods and results that will be used throughout the paper. In Subsection 2.1, we recall basic facts about parametrizations of rational curves. In particular, we consider K an algebraically closed field of characteristic zero and we treat briefly the classical problem of parametrizing an algebraic plane curve $\mathcal{C}^* \subset \mathbb{P}^2(K)$ over a subfield $k \subseteq K$. Afterwards, in Subsection 2.2, we specialize Subsection 2.1 to the case where $\mathcal{C}^*$ is a pencil of curves, $k = \mathbb{K}(t)$ and $K = \overline{\mathbb{K}(t)}$ (the algebraic closure of the field $\mathbb{K}(t)$) being $\mathbb{K}$ an algebraically closed field of characteristic zero.

2.1 Parametrization of a Rational Curve

Let K be an algebraically closed field of characteristic zero, and let $k \subseteq K$ a subfield. We denote by $\mathbb{A}^2(K)$, the affine plane embedded into the projective plane $\mathbb{P}^2(K)$ by identifying the point $(a, b) \in \mathbb{A}^2(K)$ with the point $(a : b : 1) \in \mathbb{P}^2(K)$. Hence, given an affine algebraic plane curve $\mathcal{C}$ over K, we denote by $\mathcal{C}^*$ the corresponding projective algebraic curve, i.e. the projective closure of $\mathcal{C}$ in $\mathbb{P}^2(K)$.

If the affine curve $\mathcal{C}$ is defined by a polynomial $f(x) \in K[x]$, $x = (x_1, x_2)$, the corresponding projective curve $\mathcal{C}^*$ is defined by the homogenization $F(X) \in K[X]$, $X = (x_1 : x_2 : x_3)$, of $f(x)$. Thus, $\mathcal{C}^* = \{(a : b : c) \in \mathbb{P}^2(K) \mid F(a, b, c) = 0\}$, and every point (a, b) on $\mathcal{C}$ corresponds to a point on $(a : b : 1)$ on $\mathcal{C}^*$, and every additional point on $\mathcal{C}^*$ is a point at infinity. Reciprocally, if $\mathcal{C}^*$ is a projective curve defined by the form $F(X) \in K[X]$, we denote by $\mathcal{C}$ the affine plane curve defined by $f(x) := F(x, 1)$ (that is, we dehomogenize w.r.t. the variable x_3). We also recall that $\deg(\mathcal{C}) = \deg_x(f) = \deg(\mathcal{C}^*) = \deg_X(F)$.

Throughout this subsection, we are interested in curves $\mathcal{C}^*$ that can be rationally parametrized over k which is equivalent to the existence of a rational parametrization of $\mathcal{C}^*$ with coefficients in k. In Definition 1, we introduce formally the notion of k-rationality of $\mathcal{C}^*$.

Definition 1 *Let $\mathcal{C}^*$ be the projective plane curve defined by an irreducible homogeneous polynomial $F(X) \in K[X]$. We say that $\mathcal{C}^*$ is* k-rational *if there are polynomials $Q_i(s) \in k[s]$, $i = 1, 2, 3$, inducing a non–constant map $\mathbb{P}^1 \to \mathbb{P}^2$ such that $F(Q_1 : Q_2 : Q_3) \equiv 0$ in $K[s]$. We say that $\mathcal{Q}^*(s) = (Q_1(s), Q_2(s), Q_3(s))$ is* a projective rational k-parametrization of $\mathcal{C}^*$ *or* a projective rational parametrization of $\mathcal{C}^*$ over k*.*

One may extend the notion of k-rationality to the corresponding affine curve, $\mathcal{C}$, defined by an irreducible polynomial $f(x) \in K[x]$. In this case, $\mathcal{C}$ is k-rational if there exists $\mathcal{Q}(s) \in k(s)^2 \setminus k^2$ such that $f(\mathcal{Q}(s)) = 0$. We refer to $\mathcal{Q}(s)$ as the rational k-parametrization of $\mathcal{C}$.

If $\mathcal{C}$ is defined by $\mathcal{Q}(s) = (q_1(s)/q(s), q_2(s)/q(s)) \in k(s)^2 \setminus k^2$, the corresponding projective curve $\mathcal{C}^*$ is defined by $\mathcal{Q}^*(s) = (q_1(s) : q_2(s) : q(s)) \in \mathbb{P}^2(k(s))$, and reciprocally. By abuse of notation, we equivalently say that $\mathcal{C}^*$ is k-rational or $\mathcal{C}$ is k-rational.

Observe that using the results of Section 4.5 in [32] (see Theorem 4.41), if $\mathcal{C}$ is k-rational then $f(x) \in k[x]$ (and $F(X) \in k[X]$).

In the following, for the sake of completeness, we consider a k-rational algebraic plane curve $\mathcal{C}$ of degree d defined by an irreducible polynomial $f(x) \in k[x]$, and we show how to compute a k-parametrization $\mathcal{Q}(s) \in k(s)^2 \setminus k^2$ of $\mathcal{C}$. We recall that $\mathcal{C}^*$ is rational if and only if

$$\text{genus}(\mathcal{C}^*) = \frac{1}{2}[(d-1)(d-2) - \sum_{p \in S} m_p(m_p - 1)] = 0,$$

where S is the set of all the singularities and neighboring singularities of $\mathcal{C}^*$ in $\mathbb{P}^2(K)$, and m_p denotes the multiplicity of the point $p \in S$. In addition, from Theorem 4.74 in [32], we have that if $\mathcal{C}^*$ is a rational curve implicitly defined by an irreducible polynomial $f(x) \in k[x]$, then $\mathcal{C}^*$ is parametrizable over k if and only if there exists a simple point on $\mathcal{C}^*$ with coordinates over k. In this subsection, we do not deal with the problem of deciding whether $\mathcal{C}^*$ is k-rational. Instead, we assume that $\mathcal{C}^*$ is k-rational and we show how to compute a k-parametrization of it. This will be enough for our purposes.

Thus, we present an algorithm that is essentially an application of the method developed in [32] (see Sections 4.7 and 4.8), where a rational parametrization of a given rational algebraic curve implicitly defined is computed (see statement 4 in Remark 1). The method described in [32] follows basically the approach in [31] and [36] and the idea is to use a linear system of curves such that for almost every curve in this system, all its intersections with $\mathcal{C}^*$, except one, are predetermined. The set of all these intersection points is the same one for every curve in the system, and the points in this set are called the "base points". Thus, if one computes the intersection points of $\mathcal{C}^*$ with a generic representative of the system, the expression of the unknown intersection point gives the parametrization of $\mathcal{C}^*$ in terms of the parameter defining the linear system.

More precisely, let $\mathcal{H}_{d-2}$ be the linear system of adjoint curves to $\mathcal{C}^*$ of degree $d-2$ (we assume that $d > 2$, otherwise we consider adjoint curves of degree 1; see statement 2 in Remark 1). That is, $\mathcal{H}_{d-2}$ is the linear system of curves of degree $d-2$ having each r–fold of $\mathcal{C}^*$ as a base point of multiplicity $r-1$; i.e. as a point of multiplicity at least $r-1$. Since $\mathcal{C}$ is k-rational, there exists a parametrization defined over k and the singularities of $\mathcal{C}$ can be decomposed as a finite union of families of conjugate parametric points over k such that all points in the same family have the same multiplicity and character (see Theorem 16 in [24]). Thus, $\mathcal{H}_{d-2}$ is computed without extending k (see Theorem 4.66 in [32]).

Under these conditions, the multiplicity of intersection of a curve in $\mathcal{H}_{d-2}$ and $\mathcal{C}^*$ at a base point of multiplicity $r-1$ is at least $r(r-1)$. Using that the genus of $\mathcal{C}^*$ is zero ($\mathcal{C}^*$ is k-rational), and taking into account Bézout's Theorem, one deduces that $(d-2)$ intersections of $\mathcal{C}^*$ and a generic element in $\mathcal{H}_{d-2}$ are not predetermined.

In this situation, one takes $(d-3)$ different simple points on $\mathcal{C}^*$, and determines the 1-dimensional linear subsystem $\mathcal{D}^*$ of $\mathcal{H}_{d-2}$ obtained when these simple points are required to be base points of multiplicity 1 (in the following, we refer to $\mathcal{D}^*$ as an adjoint pencil of $\mathcal{C}^*$). We remark that $\mathcal{D}^*$ depends on the choice of the simple points.

Note that if one takes these $(d-3)$ simple points of $\mathcal{C}$ in $\mathbb{P}^2(k)$, then the desired parametrization will be defined over k since $\mathcal{D}^*$ is computed without extending k (see Theorem 4.66 in [32]). Note that since we are assuming that $\mathcal{C}$ is k-rational, then $\mathcal{C}$ can be defined by a parametrization over k, and thus one can find infinitely many simple points in $\mathbb{P}^2(k)$. If the simple points are taken over K, then the output parametrization will be defined over K but we can not guarantee that it is over k (see Theorems 4.66 and 4.68, and Corollary 4.69 in [32]).

In this way, the number of predetermined intersections of $\mathcal{C}^*$ and $\mathcal{D}^*$ (counted with multiplicity) is $(d-1)(d-2)+(d-3) = d(d-2)-1$, i.e. only one intersection point is missing. Computing this free intersection point, one finds a projective rational k-parametrization of $\mathcal{C}^*$. We remark that taking into account the results presented in [32] (see Theorems 4.57, 4.58, 4.61 and 4.62), one has that the base points and the simple points impose independent linear conditions on $\mathcal{H}_{d-2}$ and thus, $\mathcal{D}^*$ always exists.

Summarizing these ideas, one has the following algorithm that computes a rational k-parametrization of a curve $\mathcal{C}$ of degree $d>2$ that is k-rational (for the case $d \leq 2$, see statement 2 in Remark 1). For the sake of simplicity, we assume that all singularities of $\mathcal{C}^*$ are ordinary (for a complete description see Sections 4.7 and 4.8 in [32]). The algorithm described as well as some illustrative examples can be found in [32] (see algorithm *Parametrization-by-Adjoints* in Section 4.7).

Algorithm for k-Parametrization.

Input: A k-rational plane curve $\mathcal{C}$ implicitly defined by a polynomial $f(x) \in k[x]$.
Output: A rational k-parametrization $\mathcal{Q}(s) \in k(s)^2 \setminus k^2$ of $\mathcal{C}$.

Step 1. Compute the singularities of $\mathcal{C}^*$ and their multiplicities.

Step 2. Determine the linear system $\mathcal{H}_{d-2}$ of adjoint curves of degree $(d-2)$.

Step 3. Compute $(d-3)$ different simple points of $\mathcal{C}^*$ in $\mathbb{P}^2(k)$.

Step 4. Determine the linear subsystem $\mathcal{D}^*$ of $\mathcal{H}_{d-2}$ by requiring that every simple point in Step 3 is a base point of multiplicity one (that is, we compute an adjoint pencil of $\mathcal{C}^*$). Let

$$G(X,s) = G_1(X) - sG_2(X) \in k(s)[X],\ \gcd(G_1, G_2) = 1$$

be the defining polynomial of $\mathcal{D}^*$.

Step 5. Return the k-parametrization $\mathcal{Q}(s) \in k(s)^2 \setminus k^2$ given by the solution in $\{x_1, x_2\}$ of the system defined by $\mathrm{pp}_s(\mathrm{Res}_{x_i}(f(x), g(x,s))) = 0$, $i = 1, 2$, where $g(x,s) := G(x,1,s)$ and $\mathrm{pp}_s(\ell(x,s))$ denotes the primitive part with respect to s of a polynomial $\ell(x,s)$.

Remark 1 *1. The above process can be applied similarly to linear systems of adjoint curves of degree $d-1$ or d (see Step 2). In this case, in order to reduce the systems of adjoints to subsystems of dimension* 1 *(see Step 3), one needs to compute* $(2d-3)$ *and* $(3d-3)$ *different simple points, respectively (see Sections 4.7 and 4.8 in [32]).*

2. *If $d \leq 2$, we use adjoints of degree* 1*. For $d = 1$ no additional simple points are needed. If $d = 2$, we need to compute one simple point (see Section 4.6 in [32]).*

3. *From Lemma 4.52 in [32], we get that the parametrization $\mathcal{Q}(s) \in k(s)^2 \setminus k^2$ is invertible (i.e. proper).*

4. *There are alternative parametrization methods such as [28] based on the construction of adjoints of high degree, or [35] where the anticanonical divisor is computed.*

2.2 Parametrization of a Pencil of Curves

In this subsection, we consider $\mathbb{K}$ as an algebraically closed field of characteristic zero, and we specialize Subsection 2.1 to the case where $\mathcal{C}^*$ is a pencil of curves, $k = \mathbb{K}(t)$ and $K = \overline{\mathbb{K}(t)}$ (the algebraic closure of the field $\mathbb{K}(t)$).

More precisely, let $G(X), H(X) \in \mathbb{K}[X]$ be coprime homogeneous polynomials, and we identify the pencil of curves over $\mathbb{K}$ generated by G and H with the projective *curve*

$$\mathcal{V}^* = \{(a:b:c) \in \mathbb{P}^2(\mathbb{K}(t)) \mid F(a,b,c) := G(a,b,c) - tH(a,b,c) = 0\},$$

defined over $\mathbb{K}(t)$. Note that F is irreducible since $\gcd(G,H) = 1$, and $\deg(\mathcal{V}^*) = \deg_X(F)$. We denote by $\mathcal{C}_i^*$, $i = 1, 2$, the projective curves defined by the polynomials G and H, respectively, and let $\mathcal{C}_i$, $i = 1, 2$, be the corresponding affine curves defined by $g(x) = G(x,1)$, and $h(x) = H(x,1)$, respectively.

Throughout this subsection, we are interested in curves $\mathcal{V}^*$ that can be rationally parametrized over $\mathbb{K}(t) \subset K$ which is equivalent to the existence of a rational parametrization of $\mathcal{V}^*$ with coefficients in $\mathbb{K}(t)$ (apply Definition 1 to $\mathcal{V}^*$, $k = \mathbb{K}(t)$ and $K = \overline{\mathbb{K}(t)}$).

Thus, we consider a $\mathbb{K}(t)$-rational curve $\mathcal{V}$ of degree d defined by an irreducible polynomial $f(x) = g(x) - th(x) \in \mathbb{K}(t)[x]$, and we show how to compute a $\mathbb{K}(t)$-parametrization $\mathcal{Q}(s) \in (\mathbb{K}(t))(s)^2 \setminus \mathbb{K}(t)^2$ of $\mathcal{V}$ by adapting the *Algorithm for k-Parametrization* to the case of $k = \mathbb{K}(t)$ and $K = \overline{\mathbb{K}(t)}$.

For this purpose, we first prove that the singularities of $\mathcal{V}^*$ lie in $\mathbb{P}^2(\mathbb{K})$ since they are exactly the intersection of the singularities of the curves defined by the equations $G = 0$ and $H = 0$. In addition, the multiplicity of a singularity $P \in \mathcal{V}^*$, is the minimum of the multiplicity of $P \in \mathcal{C}_1^*$ and the multiplicity of $P \in \mathcal{C}_2^*$.

Lemma 1 *It holds that the singularities of $\mathcal{V}^*$ lie in $\mathbb{P}^2(\mathbb{K})$. Furthermore, $m_P(\mathcal{V}^*) = \min\{m_P(\mathcal{C}_1^*), m_P(\mathcal{C}_2^*)\}$, where $m_P(\mathcal{D}^*)$ denotes the multiplicity of singularity at a point $P \in \mathcal{D}^*$.*

Proof. Let $P = (a_1 : a_2 : a_3)$ be a singularity of $\mathcal{V}^*$. Then, $F(P) = \frac{\partial F}{\partial x_k}(P) = 0$, $k = 1, 2, 3$, and $\mathcal{E}_k(P) = 0$, where $\mathcal{E}_k := G\frac{\partial H}{\partial x_k} - H\frac{\partial G}{\partial x_k} = 0$, $k = 1, 2, 3$. Therefore, $\mathrm{resultant}_{x_k}(\mathcal{E}_m, \mathcal{E}_n)(a_i, a_j) = 0$, for $m, n, k \in \{1, 2, 3\}$ with $m < n$ and $i < j$ ($i \neq k, j \neq k$) which implies that $P \in \mathbb{P}^2(\mathbb{K})$ since $\mathrm{resultant}_{x_k}(\mathcal{E}_m, \mathcal{E}_n)(x_i, x_j)$ is an homogeneous polynomial in $\mathbb{K}[x_i, x_j]$ (see e.g. Chapter 3 in [12]).

On the other side, taking into account the definition of multiplicity (see e.g. Definition 2.2 in [32]), the first part of the lemma which states that $P \in \mathbb{P}^2(\mathbb{K})$, and that

$$\frac{\partial^\ell F}{\partial x_1^{\ell_1}\partial x_2^{\ell_2}\partial x_3^{\ell_3}}(P) = \frac{\partial^\ell G}{\partial x_1^{\ell_1}\partial x_2^{\ell_2}\partial x_3^{\ell_3}}(P) - t\frac{\partial^\ell H}{\partial x_1^{\ell_1}\partial x_2^{\ell_2}\partial x_3^{\ell_3}}(P), \quad \ell_1 + \ell_2 + \ell_3 = \ell$$

where $\ell,\ \ell_j \in \mathbb{N}$, one deduces that $m_P(\mathcal{V}^*) = \min\{m_P(\mathcal{C}_1^*), m_P(\mathcal{C}_2^*)\}$. □

Taking into account Lemma 1, and that there exists a parametrization of $\mathcal{V}$ defined over $\mathbb{K}(t)$ ($\mathcal{V}$ is $\mathbb{K}(t)$-rational), if one finds $(d-3)$ different simple points of $\mathcal{V}$ in $\mathbb{P}^2(\mathbb{K}(t))$, then the output parametrization $\mathcal{Q}(s)$ will be defined over $\mathbb{K}(t)$; that is, $\mathcal{Q}(s) \in (\mathbb{K}(t))(s)^2 \setminus \mathbb{K}(t)^2$ (see Subsection 2.1). Note that since we are assuming that $\mathcal{V}$ is $\mathbb{K}(t)$-rational, then $\mathcal{V}$ can be defined by a parametrization over $\mathbb{K}(t)$, and thus one can find infinitely many points in $\mathbb{P}^2(\mathbb{K}(t))$ (for instance, one may check first whether there are simple points in $\mathbb{P}^2(\mathbb{K}) \subset \mathbb{P}^2(\mathbb{K}(t))$ by computing the intersection points of $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$). If the simple points are taken over K, then $\mathcal{Q}(s) \in K(s)^2 \setminus K^2$ but we could not ensure that $\mathcal{Q}$ is defined over $\mathbb{K}(t)$. Of course, one may apply different algorithms (see e.g. [28], [35], etc.) to determine the parametrization over $\mathbb{K}(t)$ (we recall that the parametrization exists since $\mathcal{V}$ is $\mathbb{K}(t)$-rational).

Summarizing these ideas, one has the following algorithm that computes a rational $\mathbb{K}(t)$-parametrization the curve $\mathcal{V}$ of degree $d > 2$ that is $\mathbb{K}(t)$-rational (for the case $d \leq 2$, see statement 2 in Remark 1). This algorithm is an special case of the *Algorithm* *for k-Parametrization* presented in Subsection 2.1. For the sake of simplicity, we assume that all singularities of $\mathcal{V}^*$ are ordinary.

Algorithm for Pencil Parametrization.

Input: A $\mathbb{K}(t)$-rational curve $\mathcal{V}$ implicitly defined by an irreducible polynomial $f(x) = g(x) - th(x) \in \mathbb{K}(t)[x]$.
Output: A rational $\mathbb{K}(t)$-parametrization $\mathcal{Q}(s) \in (\mathbb{K}(t))(s)^2 \setminus \mathbb{K}(t)^2$ of $\mathcal{V}$.

Step 1. Compute the singularities of $\mathcal{V}^*$ and their multiplicities. For this purpose, we apply Lemma 1.

Step 2. Determine the linear system $\mathcal{H}_{d-2}$ of adjoint curves of degree $(d-2)$. For this purpose, one considers a homogeneous polynomial $L(X)$ of degree $(d-2)$ with undetermined coefficients. For each singular point of multiplicity r_i, one requires that L and all its partial derivatives up to order $(r_i - 1)$ vanish at the singular point. This generates a linear system of equations in the undetermined coefficients of L. Solving it, and substituting in L, we get the defining polynomial of $\mathcal{H}_{d-2}$; let us call it again L.

Step 3. Compute $(d-3)$ different simple points of $\mathcal{V}^*$ in $\mathbb{P}^2(\mathbb{K}(t))$. Observe that first, one may check whether there are simple points in $\mathbb{P}^2(\mathbb{K}) \subset \mathbb{P}^2(\mathbb{K}(t))$ by computing the intersection points of $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$.

Step 4. Determine the linear subsystem $\mathcal{W}^*$ of $\mathcal{H}_{d-2}$ by requiring that every simple point in Step 3 is a base point of multiplicity one (that is, we compute an adjoint pencil of $\mathcal{V}^*$). This step can be approached as Step 2, i.e. by requiring that L vanishes at each simple point, solving the provided linear system and substituting the solution in L. Note that since $\dim(\mathcal{W}^*) = 1$, the defining polynomial of $\mathcal{W}^*$ can be expressed as

$$R(X,t) = M(X,t) - sN(X,t) \in \mathbb{K}(t,s)[X], \ \gcd(M,N) = 1.$$

Observe that since the singularities are in $\mathbb{P}^2(\mathbb{K})$ (see Lemma 1), if we may compute $(d-3)$ simple points in $\mathbb{P}^2(\mathbb{K})$, then $R(X) = M(X) - sN(X) \in \mathbb{K}(s)[X]$ (that is, R does not depend on t).

Step 5. Return the $\mathbb{K}(t)$-parametrization $\mathcal{Q}(s) \in (\mathbb{K}(t))(s)^2 \setminus \mathbb{K}(t)^2$ given by the solution in $\{x_1, x_2\}$ of the system defined by $\mathrm{pp}_s(\mathrm{Res}_{x_i}(f(x), r(x,t))) = 0$, $i = 1, 2$, where $r(x,t) := R(x, 1, t)$.

In the following, we illustrate *Algorithm for Pencil Parametrization* with an example.

Example 1 *Let $\mathcal{V}$ be the curve of degree $d = 4$ defined by the irreducible polynomial $f(x) = g(x) - th(x)$, where*

$$g(x) = x_1^2x_2^2 - x_1x_2^3 + 17x_1x_2^2 - 56x_1x_2 - 68x_1^2x_2 - 20x_1^3x_2 + x_2^3 + 128x_1^2 + 128x_1^3 + 32x_1^4 \in \mathbb{C}[x],$$
$$h(x) = 20x_1x_2^2 - \tfrac{992}{9}x_1x_2 - \tfrac{2000}{9}x_1^2x_2 - \tfrac{752}{9}x_1^3x_2 + x_2^4 + \tfrac{4352}{9}x_1^2 + \tfrac{4352}{9}x_1^3 + \tfrac{1088}{9}x_1^4 \in \mathbb{C}[x].$$

Note that the generators of $\mathcal{V}$ are the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ implicitly defined by the polynomials g and h, respectively.

We compute the singularities of $\mathcal{V}^*$, and we obtain the points in $\mathbb{C}^2$: $(-2, 0)$, $(0, 0)$, $(1, 4)$ of multiplicities $2, 2, 2$, respectively (see Step 1 of the algorithm). Note that $\mathrm{genus}(\mathcal{V}) = 0$. Now, we determine the linear system $\mathcal{H}_2$ of adjoint curves of degree $d-2 = 2$ (see Step 2). We get that $\mathcal{H}_2$ is defined by the polynomial

$$L(X) = 3a_{01}x_2x_3 + 3a_{02}x_2^2 - 8x_1a_{01}x_3 - 32x_1a_{02}x_3 - 8a_{11}x_1x_3 + 3a_{11}x_1x_2 - 4x_1^2a_{01} - 16x_1^2a_{02} - 4x_1^2a_{11}.$$

In order to determine the $\mathbb{C}(t)$-parametrization, we need $d - 3 = 1$ simple point of $\mathcal{V}^*$ in $\mathbb{P}^2(\mathbb{C}(t))$ (see Step 3). For this purpose, we first check whether there are simple points in $\mathbb{P}^2(\mathbb{C})$ by computing the intersection points of the curves $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$. We obtain three simple points. We consider one of these simple points: $(-1, 2) \in \mathbb{C}^2$.

Now, we compute an adjoint pencil of $\mathcal{V}^*$, $\mathcal{W}^*$, by requiring that the simple point $(-1, 2)$ is a base point of multiplicity one (see Step 4). We get that the defining polynomial of $\mathcal{W}$ is $r(x) = m(x) - sn(x)$, where

$$m(x) = -16x_1 - 8x_1^2 + x_2 + 5x_1x_2 \in \mathbb{C}[x], \qquad n(x) = -x_2^2 + 48x_1 - 14x_1x_2 + 24x_1^2 \in \mathbb{C}[x].$$

Observe that since the simple point is in $\mathbb{C}^2$, the generators of $\mathcal{W}$ are plane curves over $\mathbb{C}$ implicitly defined by the polynomials m and n.

Finally, we return the $\mathbb{C}(t)$-parametrization $\mathcal{Q}(s) \in (\mathbb{C}(t))(s)^2 \setminus \mathbb{C}(t)^2$ given by the solution in $\{x_1, x_2\}$ of the system defined by $\mathrm{pp}_s(\mathrm{Res}_{x_i}(f(x), r(x))) = 0$, $i = 1, 2$, where

$$\mathrm{pp}_s(\mathrm{Res}_{x_2}(f(x), r(x)))(x_1, t, s) = -162t - 162s - 810tx_1 + 270t^2 + 162x_1s - 297ts - 567s^2 + 3294t^2x_1 - 8289tx_1s + 1116st^2 + 1053s^2x_1 + 1116ts^2 + 34308t^2x_1s - 29790s^2tx_1 + 2025x_1s^3 + 130104s^2t^2x_1 - 43344s^3tx_1 + 648x_1s^4 + 212784s^3t^2x_1 - 17856s^4tx_1 + 123008s^4t^2x_1, \qquad \textit{and}$$

$$\mathrm{pp}_s(\mathrm{Res}_{x_1}(f(x), r(x)))(x_2, t, s) = 972 - 5832t + 6804s - 43344x_2s^3t - 17856x_2s^4t + 123008x_2s^4t^2 + 212784x_2s^3t^2 + 34308t^2sx_2 + 130104t^2s^2x_2 - 8289x_2st + 159216s^2t^2 - 71424s^3t + 123008s^3t^2 + 60336st^2 - 109224ts^2 + 7020t^2 + 4536s^3 + 13203s^2 + 162x_2s - 810x_2t - 45900ts - 29790x_2s^2t + 2025x_2s^3 + 648x_2s^4 + 3294x_2t^2 + 1053x_2s^2.$$

We remark that in Step 3 of the algorithm, the computed $(d-3)$ different simple points are in $\mathbb{P}^2(\mathbb{K}(t))$. First, one may check whether there are simple points in $\mathbb{P}^2(\mathbb{K}) \subset \mathbb{P}^2(\mathbb{K}(t))$ by computing the intersection points of $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$ (see Example 1). This can be done in many cases; however, sometimes we can not find *enough* different simple points in $\mathbb{P}^2(\mathbb{K})$, and we have to compute them in $\mathbb{P}^2(\mathbb{K}(t))$. Here, we can find enough different simple points if $\mathcal{V}$ is $\mathbb{K}(t)$-rational.

Example 2 *Let $\mathcal{V}$ be the curve of degree $d=3$ defined by $f(x)=g(x)-th(x)$, where $g(x)=-x_2^3+x_1^4 \in \mathbb{C}[x]$, and $h(x)=-x_2^4 \in \mathbb{C}[x]$. The generators of $\mathcal{V}$ are the curves $\mathcal{C}_1$, and $\mathcal{C}_2$ implicitly defined by the polynomials g and h, respectively.*

We compute the singularities of $\mathcal{V}^*$, and we obtain that $(0,0) \in \mathbb{C}^2$ is a singularity of multiplicity 3 (see Step 1 of the algorithm). Note that $\text{genus}(\mathcal{V})=0$. Now, we determine the linear system $\mathcal{H}_2$ of adjoint curves of degree $d-2=2$ (see Step 2). We get that $\mathcal{H}_2$ is defined by the polynomial $L(X)=a_{02}x_2^2+a_{11}x_1x_2+a_{20}x_1^2$.

In order to determine the $\mathbb{C}(t)$-parametrization, we need $d-3=1$ simple point of $\mathcal{V}^*$ in $\mathbb{P}^2(\mathbb{C}(t))$ (see Step 3). For this purpose, we first check whether there are simple points in $\mathbb{P}^2(\mathbb{C})$ by computing the intersection points of the curves $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$. We have that the curves $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$ only intersect at the singular point. Thus, we have to compute a simple point in $\mathbb{P}^2(\mathbb{C}(t))$. We obtain the simple point $(-1/(1+t^3), 1/(t+t^4)) \in \mathbb{C}(t)^2$.

Now, we compute an adjoint pencil of $\mathcal{V}^*$, $\mathcal{W}^*$, by requiring that the simple point is a base point of multiplicity one (see Step 4). We get that the defining polynomial of $\mathcal{W}$ is $r(x,t)=m(x,t)-sn(x,t)$, where $m(x,t)=-x_2^2t^2+x_1^2 \in \mathbb{C}(t)[x]$, and $n(x,t)=-x_2^2t-x_2x_1 \in \mathbb{C}(t)[x]$.

Finally, we return the $\mathbb{C}(t)$-parametrization $\mathcal{Q}(s) \in (\mathbb{C}(t))(s)^2 \setminus \mathbb{C}(t)^2$ given by the solution in $\{x_1,x_2\}$ of the system defined by $\text{pp}_s(\text{Res}_{x_i}(f(x),r(x,t)))=0$, $i=1,2$, where

$$\text{pp}_s(\text{Res}_{x_2}(f(x),r(x,t)))(x_1,t,s)=x_1t-t+s+x_1t^4-4x_1t^3s+6x_1t^2s^2-4x_1ts^3+x_1s^4, \text{ and}$$

$$\text{pp}_s(\text{Res}_{x_1}(f(x),r(x,t)))(x_2,t,s)=-1+x_2t+x_2t^4-4x_2t^3s+6x_2t^2s^2-4x_2ts^3+x_2s^4.$$

In this paper, we deal neither with the problem of deciding whether $\mathcal{V}$ is $\mathbb{K}(t)$-rational nor with the problem of computing simple points in $\mathbb{P}^2(\mathbb{K}(t))$ (see Step 3 of the algorithm). Instead, we construct a curve $\mathcal{V}$ of degree d where singularities and simple points are known, and we use *Algorithm for Pencil Parametrization* to compute a pencil of adjoints $\mathcal{W}$ (see Step 4 of the *Algorithm for Pencil Parametrization*). Afterwards, from the curves generating $\mathcal{V}$ and $\mathcal{W}$, we will get a birational automorphism of degree d (see Theorem 1, statement 2 in Corollary 2 and Proposition 1 in Section 3).

3 Construction of Birational Automorphisms of the Plane

In this section, we consider $d \in \mathbb{N}$, and we show how to construct birational automorphisms of the plane of degree d (see Theorem 1 and statement 2 in Corollary 2 in Subsection 3.1). The idea is to construct a $\mathbb{K}(t_1)$-rational *curve* $\mathcal{V}_1$, and from there we compute the adjoint pencil $\mathcal{V}_2$ (see Step 4 of the *Algorithm for Pencil Parametrization*). The birational automorphism (and also its inverse, see Corollary 1) is obtained from the curves generating $\mathcal{V}_1$ and $\mathcal{V}_2$.

Reciprocally, in Subsection 3.2, we are given a birational automorphism $\mathcal{P}$, and we construct a certain $\mathcal{V}_1$ being $\mathbb{K}(t_1)$-rational, and from there its adjoint pencil $\mathcal{V}_2$. In Theorem 2, we show the relation between $\mathcal{P}$ and the pair of pencils $(\mathcal{V}_1, \mathcal{V}_2)$.

From Subsections 3.1 and 3.2, we deduce that there exits a one-to-one correspondence between birational automorphisms of the plane and pairs of pencils $(\mathcal{V}_1, \mathcal{V}_2)$ intersecting in a unique point. As a consequence, we obtain a method that allows to construct birational automorphisms of the plane of a desired degree.

As a particular case, in Subsection 3.3, we show how these results can be stated for the case of birational polynomial automorphisms (see Proposition 1 and Theorem 3). Similarly as in the general case, as a consequence we obtain a method that allows to construct birational automorphisms of the plane that are polynomial (see Proposition 1).

3.1 From Pencils to Birational Automorphisms

In the following, we consider $G_{1,1}, G_{1,2} \in \mathbb{K}[X]$ coprime homogeneous polynomials, and the projective *curve*

$$\mathcal{V}_1^* = \{(a : b : c) \in \mathbb{P}^2(\mathbb{K}(t_1)) \mid F_1(a,b,c) := G_{1,1}(a,b,c) - t_1 G_{1,2}(a,b,c) = 0\},$$

defined over $\mathbb{K}(t_1)$ with $\deg(\mathcal{V}_1^*) = \deg_X(F_1) = d$. We denote by $\mathcal{C}_i^*$ the projective curve defined by the polynomial $G_{1,i}$, for $= 1, 2$. Using *Algorithm for Pencil Parametrization* (see Step 4), we compute the adjoint pencil of $\mathcal{V}_1^*$,

$$\mathcal{V}_2^* = \{(a : b : c) \in \mathbb{P}^2(\mathbb{K}(t_1, t_2)) \mid F_2(a,b,c,t_1) := G_{2,1}(a,b,c,t_1) - t_2 G_{2,2}(a,b,c,t_1) = 0\},$$

where $G_{2,1}, G_{2,2} \in \mathbb{K}(t_1)[X]$ are coprime homogeneous polynomials. We have that $\deg(\mathcal{V}_2^*) = \deg_X(F_2) = \widetilde{d}$, where $\widetilde{d} = d - 2$ for $d > 2$, and $\widetilde{d} = 1$ otherwise (see *Algorithm for Pencil Parametrization*, and statement 2 in Remark 1).

We recall that the corresponding affine curves, $\mathcal{V}_1$ and $\mathcal{V}_2$ are defined by the irreducible polynomials

$$f_1(x) := F_1(x, 1) = g_{1,1}(x) - t_1 g_{1,2}(x) \in \mathbb{K}(t_1)[x], \qquad \text{and}$$

$$f_2(x,t_1) := F_2(x,1,t_1) = g_{2,1}(x,t_1) - t_2 g_{2,2}(x,t_1) \in \mathbb{K}(t_1,t_2)[x].$$

The following theorem (which is the main theorem of this subsection) shows that the curves generating $\mathcal{V}_1$ and $\mathcal{V}_2$ define a birational automorphism $\mathcal{S}$. In Corollary 1, we prove some properties of the inverse of $\mathcal{S}$, and in Corollary 2 (statement 2) we show how to control the degree of the birational automorphism $\mathcal{S}$ constructed in this theorem.

Theorem 1 *Let $\mathcal{V}_1$ be a $\mathbb{K}(t)$-rational curve of degree d defined by an irreducible polynomial $f_1(x) = g_{1,1}(x) - t_1 g_{1,2}(x) \in \mathbb{K}(t_1)[x]$. Let $f_2(x,t_1) = g_{2,1}(x,t_1) - t_2 g_{2,2}(x,t_1) \in \mathbb{K}(t_1,t_2)[x]$ be an irreducible polynomial defining the adjoint pencil of $\mathcal{V}_1$, $\mathcal{V}_2$. Then,*

$$\mathcal{S}(x) = (g_1(x), g_2(x)), \quad \textit{where } g_1(x) = \frac{g_{1,1}(x)}{g_{1,2}(x)},\ g_2(x) = \frac{g_{2,1}(x, g_1(x))}{g_{2,2}(x, g_1(x))}$$

is birational.

Proof. Let $\mathcal{Q}(t_1,t_2) \in \mathbb{K}(t_1,t_2)^2 \setminus \mathbb{K}^2$ be the intersection point of the polynomials f_1 and f_2 (see Step 5 of *Algorithm for Pencil Parametrization*). Observe that $g_{1,2}(\mathcal{Q}) \neq 0$. Otherwise, $g_{1,1}(\mathcal{Q}) = 0$ (because $f_1(\mathcal{Q}) = 0$) and then, since $\gcd(g_{1,1}, g_{1,2}) = 1$, we would deduce that $\mathcal{Q} \in \mathbb{K}^2$ which is impossible. Thus, since $g_{1,2}(\mathcal{Q}) \neq 0$ and $f_1(\mathcal{Q}) = 0$, we get that $g_1(\mathcal{Q}) = t_1$.
On the other hand, one also has that $g_{2,2}(\mathcal{Q}, g_1(\mathcal{Q})) \neq 0$. Indeed, if $g_{2,2}(\mathcal{Q}, g_1(\mathcal{Q})) = g_{2,2}(\mathcal{Q}, t_1) = 0$ (note that $g_1(\mathcal{Q}) = t_1$), then $g_{2,1}(\mathcal{Q}, t_1) = 0$ (because $f_2(\mathcal{Q}) = 0$). Thus, $(\mathcal{Q}(t_1,t_2), t_1)$ parametrizes the surfaces defined by the polynomials $g_{2,i}(x_1,x_2,x_3)$ for $i = 1,2$ which is impossible (the polynomials $g_{2,i}$ are irreducible and $\gcd(g_{2,1}, g_{2,2}) \neq 1$). Hence, $g_{2,2}(\mathcal{Q}, g_1(\mathcal{Q})) \neq 0$ and then $g_2(\mathcal{Q}) = t_2$ (note that $f_2(\mathcal{Q}, g_1(\mathcal{Q})) = 0$). Therefore, we conclude that $\mathcal{S}(\mathcal{Q}(t_1,t_2)) = (t_1,t_2)$ which implies that $\mathcal{S}$ is birational. $\square$

In the following corollary, we prove that the output of *Algorithm for Pencil Parametrization* determines the inverse of the birational automorphism computed in Theorem 1. In addition, we show some formulae that provide the degree of this inverse. For this purpose, in the following we denote by $\mathrm{index}(\mathcal{M})$, the tracing index of a parametrization $\mathcal{M}$ (see Definition 4.24 in [32]). Intuitively speaking, $\mathrm{index}(\mathcal{M})$ is a natural number such that almost all points on the curve defined by $\mathcal{M}$ are generated, via $\mathcal{M}(t)$, by exactly $\mathrm{index}(\mathcal{M})$ parameter values. Thus, if $\mathrm{index}(\mathcal{M}) = 1$, we have that $\mathcal{M}$ is proper (i.e. an invertible parametrization). In order to compute $\mathrm{index}(\mathcal{M})$, one may apply the results in Section 4.3 in [32].

Corollary 1 *Let $\mathcal{Q}(t_1,t_2) = (q_1(t_1,t_2), q_2(t_1,t_2)) \in \mathbb{K}(t_1,t_2)^2 \setminus \mathbb{K}^2$ be the intersection point of $\mathcal{V}_1$ and $\mathcal{V}_2$. It holds that $\mathcal{Q}$ is the inverse of the birational automorphism $\mathcal{S}$. In addition,*

$$\deg_{t_2}(q_1) = \deg_{x_2}(h_1), \quad \deg_{t_2}(q_2) = \deg_{x_1}(h_1), \quad \textit{and}$$

$$\deg_{t_1}(q_1) = \deg_{x_2}(h_2)\mathrm{index}(\mathcal{Q}_2), \quad \deg_{t_1}(q_2) = \deg_{x_1}(h_2)\mathrm{index}(\mathcal{Q}_2)$$

where $h_1(x) := g_{1,1}(x) - t_1 g_{1,2}(x)$ and $h_2(x) := g_{2,1}(x, g_1(x)) - t_2 g_{2,2}(x, g_1(x))$.

Proof. From the proof of Theorem 1, we get that the inverse of $\mathcal{S}$ is given by the output of *Algorithm for Pencil Parametrization*; i.e. by the intersection point of $\mathcal{V}_1$ and $\mathcal{V}_2$. In addition, from this proof, we also get that

$$g_i(\mathcal{Q}) = t_i,\ i = 1, 2, \quad \text{where} \quad g_1(x) = \frac{g_{1,1}(x)}{g_{1,2}(x)},\ g_2(x) = \frac{g_{2,1}(x, g_1(x))}{g_{2,2}(x, g_1(x))}.$$

Then, $\mathcal{Q}_i(t_j)$, where $\mathcal{Q}_i(t_j) := \mathcal{Q}(t_1, t_2) \in (\mathbb{K}(t_i))(t_j)^2$, $i, j \in \{1, 2\}$, $i \neq j$ (that is, we see $\mathcal{Q}$ as a parametrization in the variable t_j with coefficients in $\mathbb{K}(t_i)$) is a parametrization of the curve over $\mathbb{K}(t_i)$ defined by $h_i(x)$. Thus, we apply the results in [32] (see Section 4.2), and we get that $\deg_{t_j}(q_1) = \deg_{x_2}(h_i)\mathrm{index}(\mathcal{Q}_i)$ and $\deg_{t_j}(q_2) = \deg_{x_1}(h_i)\mathrm{index}(\mathcal{Q}_i)$, for $i, j \in \{1, 2\}$, $i \neq j$. In addition, since $\mathrm{index}(\mathcal{Q}_1) = 1$ (see statement 3 in Remark 1), we get that

$$\deg_{t_2}(q_1) = \deg_{x_2}(h_1), \quad \deg_{t_2}(q_2) = \deg_{x_1}(h_1), \quad \text{and}$$
$$\deg_{t_1}(q_1) = \deg_{x_2}(h_2)\mathrm{index}(\mathcal{Q}_2), \quad \deg_{t_1}(q_2) = \deg_{x_1}(h_2)\mathrm{index}(\mathcal{Q}_2).$$

□

In the following corollary, we first show that the degree of $\mathcal{S}$ is not controlled; i.e. we do not know in advance the degree of the birational automorphism $\mathcal{S}$ although an upper bound is provided (see statement 1 in Corollary 2). However, if one may determine the needed simple points in $\mathbb{P}^2(\mathbb{K})$, then $f_2 \in \mathbb{K}(t_2)[x]$ and thus $\deg(\mathcal{S}) = d$ (see statement 2 in Corollary 2). Hence, the following corollary will be used to construct birational automorphisms of a desired degree d (see Example 2).

Corollary 2 *Let $\mathcal{V}_1$ be a $\mathbb{K}(t)$-rational curve of degree d defined by an irreducible polynomial $f_1(x) = g_{1,1}(x) - t_1 g_{1,2}(x) \in \mathbb{K}(t_1)[x]$. Then,*

1. *If $f_2(x, t_1) = g_{2,1}(x, t_1) - t_2 g_{2,2}(x, t_1) \in \mathbb{K}(t_1, t_2)[x]$ is an irreducible polynomial of degree $\widetilde{d}$ defining the adjoint pencil of $\mathcal{V}_1$, $\mathcal{V}_2$, and*

$$\mathcal{S}(x) = (g_1(x), g_2(x)),\ \ g_1(x) = \frac{g_{1,1}(x)}{g_{1,2}(x)},\ g_2(x) = \frac{g_{2,1}(x, g_1(x))}{g_{2,2}(x, g_1(x))}$$

   is the birational automorphism computed in Theorem 1, it holds that

$$\deg(g_1) = \deg_x(f_1) = d, \quad \deg(g_2) \leq \deg_x(f_2) + \deg_{t_1}(f_2)\deg(g_1) = \widetilde{d} + \deg_{t_1}(f_2)d.$$

   In addition, $\deg(\mathcal{S}) \leq \max\{d, \widetilde{d} + \deg_{t_1}(f_2)d\}$.

2. *If $d \geq 3$ and $\mathcal{V}_1$ has at least $(d-3)$ different simple points in $\mathbb{P}^2(\mathbb{K})$, then $f_2(x) = g_{2,1}(x) - t_2 g_{2,2}(x) \in \mathbb{K}(t_2)[x]$ is an irreducible polynomial defining the adjoint pencil $\mathcal{V}_2$ and*

$$\mathcal{S}(x) = (g_1(x), g_2(x)), \quad g_i(x) = \frac{g_{i,1}(x)}{g_{i,2}(x)},\ \ i = 1, 2$$

   is birational with $\deg(\mathcal{S}) = d$.

Proof.

1. Taking into account that $\deg_x(f_1) = d$ and $\deg_x(f_2) = \widetilde{d}$, where $\widetilde{d} = d - 2$ for $d > 2$, and $\widetilde{d} = 1$ otherwise, and that $\gcd(g_{i,1}, g_{i,2}) = 1$, $i = 1, 2$, we deduce that

$$\deg(g_1) = \deg_x(f_1) = d, \quad \deg(g_2) \leq \deg_x(f_2) + \deg_{t_1}(f_2)\deg(g_1) = \widetilde{d} + \deg_{t_1}(f_2)d.$$

Therefore, $\deg(\mathcal{S}) \leq \max\{d, \widetilde{d} + \deg_{t_1}(f_2)d\}$.

2. Since we may determine the needed simple points in $\mathbb{P}^2(\mathbb{K})$ (see Step 3 of *Algorithm for Pencil Parametrization*, and statement 2 in Remark 1), then $f_2 \in \mathbb{K}(t_2)[x]$ (see Step 4 of *Algorithm for Pencil Parametrization*) and hence, from Theorem 1, we deduce that

$$\mathcal{S}(x) = (g_1(x), g_2(x)), \quad g_i(x) = \frac{g_{i,1}(x)}{g_{i,2}(x)}, \;\; i = 1, 2$$

is birational. In addition, since $f_2 \in \mathbb{K}(t_2)[x]$, we get that $\deg_{t_1}(f_2) = 0$ which implies that $\deg(g_2) = \deg_x(f_2) = \widetilde{d} \leq d$ (note that $g_2(x) = g_{2,1}(x)/g_{2,2}(x)$). Therefore, $\deg(\mathcal{S}) = d$.

□

Remark 2 *Note that if $d = 2$, we get that $\mathcal{V}_1$ always has at least 1 simple point in $\mathbb{P}^2(\mathbb{K})$. Thus, no additional condition has to be imposed in statement 2 in Corollary 2.*

Remark 3 *The birational automorphism $\mathcal{S}$ computed in Theorem 1 and Corollary 2 (statement 2), does not parametrize a plane curve. Indeed, let us assume that $\mathcal{S}$ parametrizes a plane curve implicitly defined by an irreducible polynomial $p(x) \in \mathbb{K}[x] \setminus \mathbb{K}$. Thus, $p(\mathcal{S}) = 0$. This implies that $p(t_1, t_2) = p(\mathcal{S}(\mathcal{Q}(t_1, t_2))) = 0$ (from Corollary 1, we have that $\mathcal{Q} = \mathcal{S}^{-1}$) which is impossible.*

Taking into account Corollary 2 (statement 2), we deduce that if we compute a curve $\mathcal{V}_1$ of degree d in the conditions of Corollary 2 (statement 2), we may construct a birational automorphism $\mathcal{S}$ of degree d from the generators of $\mathcal{V}_1$, and the generators of the adjoint pencil of $\mathcal{V}_1$, $\mathcal{V}_2$.

Example 3 *Let us construct a $\mathbb{C}(t)$-rational curve, $\mathcal{V}_1$, of degree $d = 5$, defined by an irreducible polynomial of the form $f_1(x) = g_{1,1}(x) - t_1 g_{1,2}(x) \in \mathbb{C}(t_1)[x]$. For this purpose, we determine $\mathcal{V}_1$ having the singularities*

$$(0:0:1) \text{ of multiplicity 3}, \quad (-1:-1:1) \text{ of multiplicity 2},$$

$$(I:1:0) \text{ of multiplicity 2 and ,} \quad (-I:1:0) \text{ of multiplicity 2},$$

where we denote by I the imaginary unit. We observe that $\text{genus}(\mathcal{V}_1) = 0$*. In addition, since we are going to need $d-3=2$ different simple points (see Step 3 of Algorithm for Pencil Parametrization), we determine $\mathcal{V}_1$ by requiring that* $(2:5:1)$ *and* $(-1:3:1)$ *are simple points in $\mathcal{V}_1$. We obtain*

$$g_{1,1}(x) = -(11x_1^5 + 11x_1x_2^4 + 44x_1x_2^3 - 22x_1^4 + 22x_1^3x_2^2 + 44x_1^3x_2 - 1560x_1^3 + 2556x_1^2x_2 + 22x_2^4 - 476x_1x_2^2 - 476x_2^3) \in \mathbb{C}[x],$$

$$g_{1,2}(x) = 11x_1^4x_2 + 33x_1^3x_2 + 11x_1^2x_2^2 + 22x_1^2x_2^3 + 33x_1x_2^3 + 2769x_1^2x_2 - 457x_1x_2^2 + 11x_2^5 + 11x_1^4 - 1657x_1^3 - 611x_2^3 \in \mathbb{C}[x].$$

Now, let us compute $\mathcal{V}_2$. For this purpose, we determine a linear system of adjoint curves to $\mathcal{V}_1$ of degree $d-2=3$. We remark that this linear system has each $r-$fold point above as a point of multiplicity at least $r-1$ (see Step 2 of Algorithm for Pencil Parametrization). In addition, the simple points introduced above are required to be points of multiplicity at least 1 *(see Steps 3 and 4 of Algorithm for Pencil Parametrization). Under these conditions, we get that the pencil of adjoints $\mathcal{V}_2$ is defined by $f_2(x) = g_{2,1}(x) - t_2 g_{2,2}(x)$, where*

$$g_{2,1}(x) = -(11x_1^3 - 58x_1x_2 + 98x_1^2 + 11x_1x_2^2 - 18x_2^2) \in \mathbb{C}[x],$$

$$g_{2,2}(x) = 135x_1^2 + 11x_1^2x_2 - 46x_1x_2 + 11x_2^3 - 67x_2^2 \in \mathbb{C}[x].$$

Finally, from Corollary 2 (statement 2), we deduce that

$$\mathcal{S}(x) = (g_{1,1}(x)/g_{1,2}(x), g_{2,1}(x)/g_{2,2}(x)) \in \mathbb{C}(x)^2$$

is a birational automorphism of the plane of degree $d=5$. Note that the inverse of $\mathcal{S}$ can be obtained by computing the intersection point in $\mathbb{K}(t_1,t_2)^2 \setminus \mathbb{K}^2$ of $\mathcal{V}_1$ and $\mathcal{V}_2$ (see Corollary 1, and Step 5 of Algorithm for Pencil Parametrization).

3.2 From Birational Automorphisms to Pencils

In this subsection, we are given a birational automorphism $\mathcal{P}$, and we construct a $\mathbb{K}(t_1)$-rational curve $\mathcal{V}_1$, and from there its adjoint pencil $\mathcal{V}_2$. Thus, using Theorem 1 we get a new birational automorphism $\mathcal{S}$, and in Theorem 2 we show the relation between $\mathcal{P}$ and $\mathcal{S}$. In particular, we prove that $\mathcal{P}$ is *equivalent* to $\mathcal{S}$ in the sense that $\mathcal{P} = \mathcal{J} \circ \mathcal{S}$, where $\mathcal{J}$ is a De Jonquières transformation.

For this purpose, we first prove the following lemma.

Lemma 2 *Let $\mathcal{P}(x) = (p_1(x), p_2(x)) \in \mathbb{K}(x)^2 \setminus \mathbb{K}^2$, $p_i = p_{i,1}/p_{i,2}$,* $\gcd(p_{i,1}, p_{i,2}) = 1$*, $i = 1,2$ be a birational automorphism. Let $\mathcal{Z}_i$ be the curves defined by the polynomials $H_i(x) = p_{i,1}(x) - t_i\, p_{i,2}(x) \in \mathbb{K}(t_i)[x]$, for $i=1,2$. It holds that $\mathcal{Z}_i$, $i=1,2$, are $\mathbb{K}(t)$-rational.*

Proof. Since $\mathcal{P}$ is birational, there exists the inverse $\mathcal{M}(t_1,t_2) := \mathcal{P}^{-1}(t_1,t_2) \in \mathbb{K}(t_1,t_2)^2 \setminus \mathbb{K}^2$. Thus, $\mathcal{P}(\mathcal{M}(t_1,t_2)) = (t_1,t_2)$ which implies that $H_i(\mathcal{M}(t_1,t_2)) = 0$, $i \in \{1,2\}$. Then, $H_1(\mathcal{M}_1(t_2)) = H_2(\mathcal{M}_2(t_1)) = 0$, where $\mathcal{M}_1(t_2) := \mathcal{M}(t_1,t_2) \in (\mathbb{K}(t_1))(t_2)^2$ (that is, we see $\mathcal{M}$ as a parametrization in the variable t_2 with coefficients in $\mathbb{K}(t_1)$; similarly, $\mathcal{M}_2(t_1) := \mathcal{M}(t_1,t_2) \in (\mathbb{K}(t_2))(t_1)^2$ that is, we see $\mathcal{M}$ as a parametrization in the variable t_1 with coefficients in $\mathbb{K}(t_2)$). Note that $\mathcal{M}_1(t_2) \notin \mathbb{K}(t_1)^2$; indeed: if $\mathcal{M}_1(t_2) \in \mathbb{K}(t_1)^2$, then $\mathcal{M} = \mathcal{P}^{-1} \in \mathbb{K}(t_1)^2$ which implies that $\mathcal{P}^{-1}$ parametrizes a curve defined by a polynomial $h(x) \in \mathbb{K}[x]$. Thus, $h(\mathcal{P}^{-1}) = 0$ and hence, $h(\mathcal{P}^{-1}(\mathcal{P}(t_1,t_2))) = h(t_1,t_2) = 0$. This is impossible, and then $\mathcal{M}_1(t_2) \notin \mathbb{K}(t_1)^2$. Similarly $\mathcal{M}_2(t_1) \notin \mathbb{K}(t_2)^2$. Therefore, $\mathcal{M}_i$ is a rational parametrization of $\mathcal{Z}_i$ over $\mathbb{K}(t_i)$. Hence, $\mathcal{Z}_i$, $i=1,2$, are $\mathbb{K}(t)$-rational. $\square$

Given a birational automorphism $\mathcal{P}(x) = (p_1(x), p_2(x)) \in \mathbb{K}(x)^2 \setminus \mathbb{K}^2$, $p_i = p_{i,1}/p_{i,2}$, $\gcd(p_{i,1}, p_{i,2}) = 1$, $i=1,2$, from Lemma 2, we get that $\mathcal{Z}_1$ is $\mathbb{K}(t)$-rational. Thus, we may apply Theorem 1 to $\mathcal{Z}_1$, and we get the birational automorphism:

$$\mathcal{S}_1^{\mathcal{P}} = (p_1, g_2), \text{ where } g_2(x) = \frac{g_{2,1}(x, p_1(x))}{g_{2,2}(x, p_1(x))},$$

and $g_{2,1}(x,t_1) - t_2\, g_{2,2}(x,t_1)$ is the adjoint pencil of $\mathcal{Z}_1$. Similarly, since $\mathcal{Z}_2$ is $\mathbb{K}(t)$-rational (see Lemma 2), we get that (p_2, g_1) is birational, where $g_1(x) = g_{1,1}(x, p_2(x))/g_{1,2}(x, p_2(x))$, and $g_{1,1}(x,t_2) - t_1\, g_{1,2}(x,t_2)$ is the adjoint pencil of $\mathcal{Z}_2$ (we apply Theorem 1 to $\mathcal{Z}_2$). Instead of (p_2, g_1), we consider (g_1, p_2) that clearly is also birational. Then, we write

$$\mathcal{S}_2^{\mathcal{P}} = (g_1, p_2), \text{ where } g_1(x) = \frac{g_{1,1}(x, p_2(x))}{g_{1,2}(x, p_2(x))}.$$

We refer to $\mathcal{S}_1^{\mathcal{P}}$ and $\mathcal{S}_2^{\mathcal{P}}$ as the associated birational automorphisms to $\mathcal{P}$. We remark that $\mathcal{S}_i^{\mathcal{P}}$, $i=1,2$, does not parametrize a plane curve (see Remark 3).

In the following theorem, it is shown that $\mathcal{P}$ is *equivalent* to the associated birational automorphisms $\mathcal{S}_j^{\mathcal{P}}$, $j=1,2$, in the sense that $\mathcal{P} = \mathcal{J}_j \circ \mathcal{S}_j^{\mathcal{P}}$, where $\mathcal{J}_j$ is a De Jonquières transformation. We remind the reader that a De Jonquières transformation is defined as the Cremona transformation

$$\left(x_1, \frac{\alpha(x_1)x_2 + \beta(x_1)}{\gamma(x_1)x_2 + \delta(x_1)}\right), \qquad \alpha, \beta, \gamma, \delta \in \mathbb{K}[x_1], \quad \gcd(\alpha x_2 + \beta, \gamma x_2 + \delta) = 1$$

(note that $\gcd(\alpha x_2 + \beta, \gamma x_2 + \delta) = 1$ is equivalent to $\gcd(\alpha, \beta, \gamma, \delta) = 1$ and $\alpha\delta - \beta\gamma \neq 0$; see Chapter 4 in [32]). This transformation is most naturally interpreted as a birational transformation of $\mathbb{P}^1 \times \mathbb{P}^1$ which preserve projection onto one of the factors.

Theorem 2 *Let $\mathcal{P}(x) = (p_1(x), p_2(x)) \in \mathbb{K}(x)^2 \setminus \mathbb{K}^2$ be birational. The following statements hold:*

1. *$\mathcal{P}(x) = (x_1, B(x)) \circ \mathcal{S}_1^{\mathcal{P}}(x)$, where $B(x) = \frac{\alpha(x_1)x_2+\beta(x_1)}{\gamma(x_1)x_2+\delta(x_1)} \in \mathbb{K}(x) \setminus \mathbb{K}(x_1)$, and $\alpha, \beta, \gamma, \delta \in \mathbb{K}[x_1]$, $\gcd(\alpha x_2 + \beta, \gamma x_2 + \delta) = 1$.*

2. *$\mathcal{P}(x) = (A(x), x_2) \circ \mathcal{S}_2^{\mathcal{P}}(x)$, where $A(x) = \frac{m(x_2)x_1+n(x_2)}{u(x_2)x_1+v(x_2)} \in \mathbb{K}(x) \setminus \mathbb{K}(x_2)$, and $m, n, u, v \in \mathbb{K}[x_2]$, $\gcd(mx_1 + n, ux_1 + v) = 1$.*

Proof. Let us prove that statement 1 holds (one reasons similarly to prove that statement 2 holds). Since $\mathcal{S}_1^{\mathcal{P}}$ ($\mathcal{S}_2^{\mathcal{P}}$ to prove statement 2) is birational (see Theorem 1), we have that

$$R(x) = (R_1(x), R_2(x)) := \mathcal{P} \circ (\mathcal{S}_1^{\mathcal{P}})^{-1} \in \mathbb{K}(x)^2 \setminus \mathbb{K}^2$$

is birational, and it satisfies that $\mathcal{P} = R(\mathcal{S}_1^{\mathcal{P}})$. Let us show, that $R(x) = (x_1, B(x))$, where $B(x) = \frac{\alpha(x_1)x_2+\beta(x_1)}{\gamma(x_1)x_2+\delta(x_1)}$, and $\gcd(\alpha x_2 + \beta, \gamma x_2 + \delta) = 1$. For this purpose, first we prove that $R_1(x) = x_1$. We observe that since $\mathcal{P} = R(\mathcal{S}_1^{\mathcal{P}})$, in particular $p_1 = R_1(\mathcal{S}_1^{\mathcal{P}}) = R_1(p_1, g_2)$ and then, $\mathcal{S}_1^{\mathcal{P}}$ parametrizes the plane curve implicitly defined by the irreducible polynomial

$$h(x) := r_{1,1}(x) - r_{1,2}(x)x_1, \quad \text{where } R_1 = r_{1,1}/r_{1,2}, \text{ and } \gcd(r_{1,1}, r_{1,2}) = 1.$$

Since $\mathcal{S}_1^{\mathcal{P}}$ does not parametrize a plane curve (see Remark 3), we get that $h(x)$ is identically zero, which implies that $R_1(x) = x_1$. Thus, we have that

$$R(x) = (x_1, R_2(x)) \in \mathbb{K}(x)^2.$$

Now, we prove that $R_2(x) = \frac{\alpha(x_1)x_2+\beta(x_1)}{\gamma(x_1)x_2+\delta(x_1)}$, $\gcd(\alpha x_2 + \beta, \gamma x_2 + \delta) = 1$. For this purpose, taking into account that $R(x) = (x_1, R_2(x))$ and that R is birational, we get that $\deg_{x_2}(R_2) = 1$ (see Lemma 4.32 in [32]). Therefore, we conclude that (see Chapter 4 in [32])

$$R_2(x) = \frac{\alpha(x_1)x_2 + \beta(x_1)}{\gamma(x_1)x_2 + \delta(x_1)},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{K}[x_1]$ (we clean denominators if it is necessary), and $\gcd(\alpha, \beta, \gamma, \delta) = 1$, and $\alpha\delta - \beta\gamma \neq 0$. Finally, we observe that since $\mathcal{P} = R(\mathcal{S}_1^{\mathcal{P}})$, and in particular $p_2 = R_2(p_1, g_2)$, we get that $R_2 \in \mathbb{K}(x)$. Moreover $R_2 \notin \mathbb{K}(x_1)$, since $\deg_{x_2}(R_2) = 1$. □

Remark 4 *We note that:*

1. *The converse of Theorem 2 can be stated as follows: let $\mathcal{P}(x) = (x_1, B(x)) \circ \mathcal{S}(x)$, where $\mathcal{S}$ is a birational automorphism and B is a De Jonquières transformation defined as $B(x) = \frac{\alpha(x_1)x_2+\beta(x_1)}{\gamma(x_1)x_2+\delta(x_1)} \in \mathbb{K}(x) \setminus \mathbb{K}(x_1)$, $\alpha, \beta, \gamma, \delta \in \mathbb{K}[x_1]$, $\gcd(\alpha x_2 + \beta, \gamma x_2 + \delta) = 1$. Then, $\mathcal{P}$ is a birational automorphism. Indeed: we consider*

$$L(x) := \left(x_1, \frac{\delta(x_1)x_2 - \beta(x_1)}{-\gamma(x_1)x_2 + \alpha(x_1)}\right), \quad \text{where} \quad \gcd(\alpha x_2 + \beta, \gamma x_2 + \delta) = 1.$$

Note that L is well defined since if $\gamma = \alpha = 0$ then $\gcd(\alpha x_2 + \beta, \gamma x_2 + \delta) \neq 1$ (we recall that condition $\gcd(\alpha x_2 + \beta, \gamma x_2 + \delta) = 1$ is equivalent to $\gcd(\alpha, \beta, \gamma, \delta) = 1$ and $\alpha\delta - \beta\gamma \neq 0$; see Chapter 4 in [32]). In addition, we also have that $\gcd(\delta x_2 - \beta, -\gamma x_2 + \alpha) = 1$, and $(x_1, B(x)) \circ L(x) = (x_1, x_2)$. Thus, $(x_1, B(x))$ is birational and $L(x)$ is its inverse (see Lemma 1 in [25]). Hence, since $\mathcal{S}$ is also birational, from $\mathcal{P}(x) = (x_1, B(x)) \circ \mathcal{S}(x)$, we deduce that $\mathcal{P}$ is birational.

2. *Reasoning as above, we also get that if $\mathcal{P}(x) = (A(x), x_2) \circ \mathcal{M}(x)$, where $\mathcal{M}$ is a birational automorphism and A is a De Jonquières transformation, then $\mathcal{P}$ is a birational automorphism.*

3. *Observe that the birational automorphism $\mathcal{S}_i^{\mathcal{P}}$ depends on the choice of an adjoint pencil of $\mathcal{Z}_i$, for $i = 1, 2$. In fact, if one considers two different adjoint pencils of $\mathcal{Z}_1$ (similarly if one reasons for $\mathcal{Z}_2$), we get two birational automorphisms $\mathcal{S}_{1,1}^{\mathcal{P}}$ and $\mathcal{S}_{1,2}^{\mathcal{P}}$. By applying Theorem 2, we get that*

$$\mathcal{P}(x) = (x_1, B_1(x)) \circ \mathcal{S}_{1,1}^{\mathcal{P}}(x) = (x_1, B_2(x)) \circ \mathcal{S}_{1,2}^{\mathcal{P}}(x),$$

where B_1, B_2 are the De Jonquières transformations described in statement 1 in Theorem 2. Thus,

$$\mathcal{S}_{1,1}^{\mathcal{P}}(x) = (x_1, B_1(x))^{-1} \circ (x_1, B_2(x)) \circ \mathcal{S}_{1,2}^{\mathcal{P}}(x) = (x_1, D(x)) \circ \mathcal{S}_{1,2}^{\mathcal{P}}(x),$$

where D is a De Jonquières transformation. Thus, the birational automorphism $\mathcal{S}_1^{\mathcal{P}}$ is unique "up to a De Jonquières transformation" (similarly for $\mathcal{S}_2^{\mathcal{P}}$).

3.3 The Polynomial Case

In this subsection, we deal with the special and important case of birational automorphisms that are polynomial, and we show how the results obtained in Subsections 3.1 and 3.2 can be particularized for the case of birational polynomial automorphisms (see Proposition 1 and Theorem 3). More precisely, we prove that there exits a one-to-one correspondence between birational polynomial automorphisms of the plane and pairs of pencils $(\mathcal{V}_1, \mathcal{V}_2)$ intersecting in a unique point. As a consequence, given $d \in \mathbb{N}$, we show how to construct birational automorphisms, $\mathcal{S}$, of degree d, such that $\mathcal{S}(x) = (g_1(x),\, g_2(x)) \in \mathbb{K}[x]^2$.

For this purpose, throughout this subsection, we consider the $\mathbb{K}(t)$-rational *curve* $\mathcal{V}_1^*$ defined by an irreducible polynomial

$$F_1(X) = G_1(X) - t_1 x_3^d \in \mathbb{K}(t_1)[X], \quad \text{where } d = \deg(G_1) = \deg(\mathcal{V}_1^*).$$

In addition, let $\mathcal{V}_1$ be the corresponding affine curve defined by the irreducible polynomial

$$f_1(x) = g_1(x) - t_1 \in \mathbb{K}(t_1)[x], \quad g_1 \in \mathbb{K}[x] \setminus \mathbb{K}.$$

The singularities and neighboring singularities of $\mathcal{V}_1^*$ are points at infinity in $\mathbb{P}^2(\mathbb{K})$ since they are exactly the intersection of the singularities of the curves defined by the equations $G_1 = 0$ and $x_3^d = 0$ (see Lemma 1). In this particular case, the polynomial $L(X)$ computed by applying Step 2 of *Algorithm for Pencil Parametrization* has an especial form. More precisely, in order to determine the linear system $\mathcal{H}_{d-2}$ of adjoint curves of degree $(d-2)$ (see Step 2 of *Algorithm for Pencil Parametrization*), one considers the homogeneous polynomial $L(X)$ of degree $(d-2)$ with undetermined coefficients. For each singular point in $\mathbb{P}^2(\mathbb{K})$ of multiplicity r_i, one requires that L and all its partial derivatives up to order $(r_i - 1)$ vanish at the singular point. This generates a linear system of equations in the undetermined coefficients of L. In this system, the undetermined coefficient λ corresponding to x_3^{d-2} does not appear since the singular points are points at infinity. Hence, when we solve it, and we substitute in L, we get the defining homogeneous polynomial of $\mathcal{H}_{d-2}$ which we call again $L(X)$, that satisfies that $L(0,0,x_3) = \lambda x_3^{d-2}$. Note that if $d \leq 2$, we have that $L(0,0,x_3) = \lambda x_3$ (see statement 2 in Remark 1).

Under these conditions, if we have $(d-3)$ different simple points on $\mathcal{V}_1^*$ being of the form $(a_1 : a_2 : 0) \in \mathbb{P}^2(\mathbb{K}(t_1))$, we deduce that the linear subsystem $\mathcal{V}_2^*$ of $\mathcal{H}_{d-2}$, obtained by requiring that every simple point is a base point of multiplicity one, is defined by the polynomial $F_2(X,t_1) = G_2(X,t_1) - t_2 x_3^{\widetilde{d}} \in \mathbb{K}(t_1,t_2)[X]$ with $\deg(G_2) = \widetilde{d}$, where $\widetilde{d} = d-2$ for $d > 2$, and $\widetilde{d} = 1$ otherwise (see Steps 3 and 4 of *Algorithm for Pencil Parametrization*, and statement 2 in Remark 1).

However, note that the condition concerning that the simple points are $(a_1 : a_2 : 0) \in \mathbb{P}^2(\mathbb{K}(t_1))$ is equivalent to that the simple points are $(a_1 : a_2 : 0) \in \mathbb{P}^2(\mathbb{K}) \subset \mathbb{P}^2(\mathbb{K}(t_1))$ since $F_1(a_1,a_2,0) = G_1(a_1,a_2,0) = 0$, and $G_1(X) \in \mathbb{K}[X]$.

Thus, the adjoint pencil $\mathcal{V}_2^*$ is defined by the polynomial

$$F_2(X) = G_2(X) - t_2 x_3^{\widetilde{d}} \in \mathbb{K}(t_2)[X]$$

that is, F_2 does not depend on the variable t_1. Then, from Theorem 1, we deduce that $\mathcal{S}(x) = (g_1(x), g_2(x)) \in \mathbb{K}[x]^2$ is birational ($g_1(x) = G_1(x,1)$, $g_2(x,t_1) = G_2(x,1,t_1)$) and $\deg(\mathcal{S}) = d$ (see statement 2 in Corollary 2).

Hence, we get the following proposition which is equivalent to statement 2 in Corollary 2. This result, will be used to construct polynomial birational automorphisms of a given degree d (see Example 3).

Proposition 1 *Let $\mathcal{V}_1$ be a $\mathbb{K}(t)$-rational curve of degree d defined by an irreducible polynomial $f_1(x) = g_1(x) - t_1 \in \mathbb{K}(t_1)[x]$ such that if $d \geq 3$, $\mathcal{V}_1$ has at least $(d-3)$ different simple points in $\mathbb{P}^2(\mathbb{K})$ at infinity. Let $f_2(x) = g_2(x) - t_2 \in \mathbb{K}(t_2)[x]$ be an irreducible polynomial defining the adjoint pencil $\mathcal{V}_2$. Then, $\mathcal{S}(x) = (g_1(x), g_2(x)) \in \mathbb{K}[x]^2$ is birational, and $\deg(\mathcal{S}) = d$.*

Remark 5 *Note that if $d = 2$, we get that $\mathcal{V}_1$ always has at least 1 simple point in $\mathbb{P}^2(\mathbb{K})$ at infinity. Thus, no additional condition has to be imposed in Proposition 1.*

Proposition 1 shows that given a curve $\mathcal{V}_1$ of degree d in the conditions stated in Proposition 1, we may construct a birational polynomial automorphism $\mathcal{S}$ of degree d from the curves generating $\mathcal{V}_1$ and $\mathcal{V}_2$. For instance, let $\mathcal{C}$ be a plane algebraic curve of degree $d > 2$ defined by a polynomial $g_1(x)$, and having a singularity of maximum multiplicity, and $(d-3)$ different simple points at infinity. Then $\mathcal{S}(x) = (g_1(x), g_2(x))$ is birational, where $f_1(x) = g_1(x) - t_1 \in \mathbb{K}(t_1)[x]$ defines $\mathcal{V}_1$, and $f_2(x) = g_2(x) - t_2 \in \mathbb{K}(t_2)[x]$ defines $\mathcal{V}_2$. In the following example, we illustrate this statement, and we construct a birational polynomial automorphism of degree $d = 3$.

Example 4 *Let us compute a $\mathbb{C}(t)$-rational curve, $\mathcal{V}_1$, of degree $d = 3$, defined by an irreducible polynomial of the form $f_1(x) := g_1(x) - t_1$, $g_1 \in \mathbb{C}[x] \setminus \mathbb{C}$, with $d = \deg_x(f_1) = 3$. For this purpose, we determine $\mathcal{V}_1$ having $(1:0:0)$ as a point of multiplicity 2 ($\mathrm{genus}(\mathcal{V}_1) = 0$). In addition, since $d - 3 = 0$, we do not need to consider additional simple points (see Step 3 of Algorithm for Pencil Parametrization). We get that*

$$g_1(x) = 1 + 3x_2 + 4x_2^2 + 6x_2^3 - 5x_1 - 2x_1x_2 + 8x_1x_2^2 \in \mathbb{C}[x].$$

Now, let us compute $\mathcal{V}_2$. For this purpose, we determine a linear system of adjoint curves to $\mathcal{V}_1$ of degree $d - 2 = 1$. We remark that this linear system has $(1:0:0)$ as a point of multiplicity at least 1 (see Step 2 of Algorithm for Pencil Parametrization). In this case, we do not need to consider additional simple points (see Step 3 of Algorithm for Pencil Parametrization). We get that $\mathcal{V}_2$ is defined by $f_2(x) = g_2(x) - t_2$, where

$$g_2(x) = x_2 \in \mathbb{C}[x].$$

Thus, from Proposition 1, we conclude that

$$\mathcal{S}(x) = (g_1(x), g_2(x)) = \left(1 + 3x_2 + 4x_2^2 + 6x_2^3 - 5x_1 - 2x_1x_2 + 8x_1x_2^2,\ x_2\right) \in \mathbb{C}[x]^2$$

is a birational polynomial automorphism of degree $d = 3$.

In Theorem 3, we present a result that is equivalent to Theorem 2 but for the polynomial case. More precisely, we are given a birational polynomial automorphism of the plane, $\mathcal{P}(x) = (p_1(x), p_2(x)) \in \mathbb{K}[x]^2$, and we consider the associated birational automorphisms to $\mathcal{P}$ (see paragraph before Theorem 2) defined by

$$\mathcal{S}_1^{\mathcal{P}} = (p_1, g_2),\ \ g_2(x) = \frac{g_{2,1}(x, p_1(x))}{g_{2,2}(x, p_1(x))}, \qquad \mathcal{S}_2^{\mathcal{P}} = (g_1, p_2),\ \ g_1(x) = \frac{g_{1,1}(x, p_2(x))}{g_{1,2}(x, p_2(x))},$$

where $g_{j,1}(x, t_i) - t_j\, g_{j,2}(x, t_i)$ is an adjoint pencil of $\mathcal{Z}_i$ for $i, j \in \{1, 2\}$, $i \neq j$ (note that $\mathcal{S}_i^{\mathcal{P}}$, $i = 1, 2$, is birational and it does not parametrize a plane curve; see Theorem 1 and Remark 3).

Observe that if $\mathcal{P}$ is such that $\mathcal{Z}_i$, $i = 1, 2$ satisfy conditions in Proposition 1, then the associated birational automorphisms to $\mathcal{P}$ are defined as

$$\mathcal{S}_1^{\mathcal{P}} = (p_1, g_2) \in \mathbb{K}[x]^2, \quad \text{and} \quad \mathcal{S}_2^{\mathcal{P}} = (g_1, p_2) \in \mathbb{K}[x]^2,$$

where $g_j(x) - t_j$ is an adjoint pencil of $\mathcal{Z}_i$ for $i, j \in \{1, 2\}$, $i \neq j$ (apply Proposition 1 to $\mathcal{Z}_i$).

Under these conditions, we prove that, "up to composition with a polynomial De Jonquières transformation", the given polynomial birational automorphism $\mathcal{P}$ is *equivalent* to the associated birational automorphisms introduced above $\mathcal{S}_i^{\mathcal{P}}$, $i = 1, 2$ (that is, $\mathcal{P} = \mathcal{J}_i \circ \mathcal{S}_i^{\mathcal{P}}$, where $\mathcal{J}_i$ is a De Jonquières transformation). For this purpose, we first introduce the following lemma concerning specialization of resultants (see Lemma 4.3.1, pp. 96 in [37]).

Lemma 3 *Let $f, g \in \mathbb{K}[h_1, \ldots, h_n][t_1, \ldots, t_m]$, let $A = (a_1, \ldots, a_n) \in \mathbb{K}^n$ be such that $\deg_{t_1}(\varphi_A(f)) = \deg_{t_1}(f)$, and $\deg_{t_1}(\varphi_A(g)) = \deg_{t_1}(g) - k$, where φ_A denotes the natural evaluation homomorphism of $\mathbb{K}[h_1, \ldots, h_n][t_1, \ldots, t_m]$ into $\mathbb{K}[t_1, \ldots, t_m]$; that is,*

$$\begin{array}{rccc} \varphi_A: & \mathbb{K}[h_1, \ldots, h_n][t_1, \ldots, t_m] & \to & \mathbb{K}[t_1, \ldots, t_m] \\ & f(h_1, \ldots, h_n, t_1, \ldots, t_m) & \mapsto & f(a_1, \ldots, a_n, t_1, \ldots, t_m). \end{array}$$

Then,

$$\varphi_A(\mathrm{Res}_{t_1}(f, g)) = \varphi_A(\mathrm{lc}(f, t_1))^k \mathrm{Res}_{t_1}(\varphi_A(f), \varphi_A(g)),$$

where $\mathrm{lc}(f, t_1)$ denotes the leading coefficient of f w.r.t. t_1.

Theorem 3 *Let $\mathcal{P}(x) = (p_1(x), p_2(x)) \in \mathbb{K}[x]^2$ be a polynomial birational automorphism such that $\mathcal{Z}_i$, $i = 1, 2$ satisfy conditions in Proposition 1. The following statements holds:*

1. *$\mathcal{P}(x) = (x_1, B(x)) \circ \mathcal{S}_1^{\mathcal{P}}(x)$, where $B(x) = \alpha(x_1)x_2 + \beta(x_1) \in \mathbb{K}[x] \setminus \mathbb{K}[x_1]$.*
2. *$\mathcal{P}(x) = (A(x), x_2) \circ \mathcal{S}_2^{\mathcal{P}}(x)$, where $A(x) = m(x_2)x_1 + n(x_2) \in \mathbb{K}[x] \setminus \mathbb{K}[x_2]$.*

Proof. Since $\mathcal{P}$ is birational, we deduce that statements (1) and (2) in Theorem 2 hold. In the following, we prove that statement (1) in Theorem 2 implies statement (1) in Theorem 3. Similarly, one shows that statement (2) in Theorem 2 implies statement (2) in Theorem 3.

Thus, since statement (1) in Theorem 2 holds, we have that

$$\mathcal{P}(x) = (x_1, B(x)) \circ \mathcal{S}_1^{\mathcal{P}}(x), \ B(x) = \frac{\alpha(x_1)x_2 + \beta(x_1)}{\gamma(x_1)x_2 + \delta(x_1)},$$

$$\mathcal{S}_1^{\mathcal{P}}(x) = (p_1(x), g_2(x)) \in \mathbb{K}[x]^2,$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{K}[x_1]$, and $\gcd(\alpha x_2 + \beta, \gamma x_2 + \delta) = 1$. Thus, we have to prove that $\gamma(x_1)x_2 + \delta(x_1) \in \mathbb{K}$; that is, $\gamma = 0$, and $\delta \in \mathbb{K} \setminus \{0\}$. For this purpose, we first observe that from the equality $p_2 = B(\mathcal{S}_1^{\mathcal{P}}) = \frac{\alpha(p_1)g_2 + \beta(p_1)}{\gamma(p_1)g_2 + \delta(p_1)}$, we get that

$$h_n(x, p_1(x)) = p_2(x)\, h_d(x, p_1(x)) \qquad \text{(I)},$$

where

$$h_n(x, t_1) := \alpha(t_1)g_2(x) + \beta(t_1), \qquad h_d(x, t_1) := \gamma(t_1)g_2(x) + \delta(t_1).$$

Note that if $h_d(x, p_1) = c \in \mathbb{K}$, then $S_1^{\mathcal{P}}$ parametrizes the curve defined by $\gamma(x_1)x_2 + \delta(x_1) - c$ which implies that $\gamma(x_1)x_2 + \delta(x_1) = c$ (by Remark 3, we have that $S_1^{\mathcal{P}}$ does not parametrize a curve). Hence, $h_d(x, p_1) \in \mathbb{K}$ iff $\gamma(t_1)t_2 + \delta(t_1) \in \mathbb{K}$. Similarly, one has that $h_n(x, p_1) \in \mathbb{K}$ iff $\alpha(t_1)t_2 + \beta(t_1) \in \mathbb{K}$. Since $p_2 \notin \mathbb{K}$, from (I), we get that $h_n(x, p_1) \notin \mathbb{K}$.

Under these conditions, we assume that $h_d(x, p_1) \notin \mathbb{K}$, and we distinguish two different cases:

1. There exists an affine point $\mathcal{M} \in \mathbb{K}(t_1)^2$ such that $p_1(\mathcal{M}) - t_1 = h_d(\mathcal{M}, p_1(\mathcal{M})) = 0$. Thus, from (I), we get that $h_n(\mathcal{M}, p_1(\mathcal{M})) = h_n(\mathcal{M}, t_1) = 0$. Then,

$$\alpha(t_1)g_2(\mathcal{M}) + \beta(t_1) = \gamma(t_1)g_2(\mathcal{M}) + \delta(t_1) = 0,$$

   which is impossible since $\alpha\delta - \beta\gamma \neq 0$.

2. There does not exist any affine point $\mathcal{M} \in \mathbb{K}(t_1)^2$ such that $p_1(\mathcal{M}) - t_1 = h_d(\mathcal{M}, p_1(\mathcal{M})) = 0$. We again distinguish two different cases:

   (a) Let $\gamma(t_1) = 0$. Then, $h_d(x, t_1) := \delta(t_1) \notin \mathbb{K}$ and from (I), we deduce that

$$\alpha(p_1)g_2(x) + \beta(p_1) = (p_1(x) - r_0)N(x), \qquad N(x) \in \mathbb{K}[x],$$

   where $\delta(r_0) = 0$, $r_0 \in \mathbb{K}$ ($\delta(t_1) \notin \mathbb{K}$). Indeed: since $\alpha(p_1)g_2(x) + \beta(p_1) = p_2(x)\delta(p_1(x))$ (see (I)), the result follows taking into account that $\delta(t_1) = (t_1 - r_0)U(t_1)$.
   Now, using this equality, we get that for every $q \in \mathbb{K}^2$ such that $p_1(q) - r_0 = 0$ ($p_1 \notin \mathbb{K}$ since $\mathcal{P}$ does not parametrize a plane curve), it holds that $\alpha(r_0)g_2(q) + \beta(r_0) = \alpha(p_1(q))g_2(q) + \beta(p_1(q)) = 0$. Thus,

$$\alpha(r_0)g_2(x) + \beta(r_0) = (p_1(x) - r_0)L(x), \qquad L(x) \in \mathbb{K}[x] \qquad \text{(II)}.$$

   Note that if $\deg(p_1) := d$, we have that $\deg(g_2) = \widetilde{d}$, where $\widetilde{d} = d - 2$ for $d > 2$, and $\widetilde{d} = 1$ otherwise (see Step 4 of *Algorithm for Pencil Parametrization*, and statement 2 in Remark 1). Hence,

i. If $d \geq 2$, we get that $\deg(\alpha(r_0)g_2(x) + \beta(r_0)) < \deg(p_1(x) - r_0)$ which implies (using (II)) that $\alpha(r_0) = \beta(r_0) = L = 0$. Thus, $t_1 - r_0$ divides $\gcd(\alpha, \beta, \gamma, \delta) = 1$ which is impossible.

ii. If $d = 1$, then $\deg(\alpha(r_0)g_2(x) + \beta(r_0)) = \deg(p_1(x) - r_0) = 1$ which implies (using (II)) that

$$\alpha(r_0)g_2(x) + \beta(r_0) = \ell(p_1(x) - r_0), \qquad \ell \in \mathbb{K}.$$

Since $S_1^{\mathcal{P}} = (p_1, g_2)$ does not parametrize a plane curve, we get that $\alpha(r_0)x_2 + \beta(r_0) - \ell(x_1 - r_0) = 0$. This implies that $\alpha(r_0) = \beta(r_0) = \ell = 0$. Therefore, $t_1 - r_0$ divides $\gcd(\alpha, \beta, \gamma, \delta) = 1$ which is impossible.

(b) Let $\gamma(t_1) \neq 0$. We know that $\mathcal{V}_1^*$ is the $\mathbb{K}(t)$-rational pencil of curves of degree d defined by the polynomial $F_1(X) = P_1(X) - t_1 x_3^d$ (note that $\deg(P_1) = d$, and P_1 is the homogenization of p_1), and $\mathcal{V}_2^*$ is defined by the polynomial $F_2(X) = G_2(x) - t_2 x_3^{\widetilde{d}}$, where $\deg(G_2) = \widetilde{d}$, and G_2 is the homogenization of g_2 (see Proposition 1). Thus, taking into account Lemma 3 and that $\gamma(t_1) \neq 0$, we have that

$$\mathrm{Res}_{x_3}(F_1, F_2)(x, t_1, -\frac{\delta(t_1)}{\gamma(t_1)}) = C(x, t_1)^k \mathrm{Res}_{x_3}(F_1(X), G_2(x) + \frac{\delta(t_1)}{\gamma(t_1)}\, x_3^{\widetilde{d}}),$$

where $C(x, t_1) := \mathrm{lc}(F_1, x_3) \in \mathbb{K}(t_1)[x]$, and $k \in \mathbb{N}$. By construction, $\mathcal{V}_1^*$ and $\mathcal{V}_2^*$ intersect at $d\widetilde{d} - 1$ points at infinity (counted with multiplicity), and at one additional affine point $\mathcal{Q} \in \mathbb{K}(t_1, t_2)^2$ (see Step 5 of *Algorithm for Pencil Parametrization*). Then, from the above equality and using the results in [32] (see Chapter 2, Subsection 2.3), we deduce that $\mathcal{V}_1^*$ and the linear system of curves defined by $G_2(x) + \delta(t_1)/\gamma(t_1)\, x_3^{\widetilde{d}}$ intersect at $d\widetilde{d} - 1$ points at infinity (counted with multiplicity), and at the additional point $\mathcal{M}(t_1) := \mathcal{Q}(t_1, -\delta(t_1)/\gamma(t_1)) \in \mathbb{K}(t_1)^2$. Therefore, $p_1(\mathcal{M}) - t_1 = h_d(\mathcal{M}, p_1(\mathcal{M})) = 0$ which contradicts our assumption.

Since all the situations analyzed above are impossible, we deduce that $h_d(x, p_1) \in \mathbb{K}$ which implies that $\gamma(t_1)t_2 + \delta(t_1) \in \mathbb{K}$; that is, $\gamma = 0$, and $\delta \in \mathbb{K} \setminus \{0\}$. Finally, we observe that $\alpha \neq 0$ (that is, $B(x) \notin \mathbb{K}[x_1]$). Otherwise, $p_2 = \beta(p_1)$ which is impossible because $\mathcal{P}$ does not parametrize a plane curve. □

Remark 6 *The converse of Theorem 3 can be stated as follows:*

1. *Let $\mathcal{P}(x) = (x_1, B(x)) \circ \mathcal{S}(x)$, where $\mathcal{S}$ is a birational polynomial automorphism and B is a De Jonquières transformation defined as $B(x) = \alpha(x_1)x_2 + \beta(x_1) \in \mathbb{K}[x] \setminus \mathbb{K}[x_1]$. Then, $\mathcal{P}$ is a birational polynomial automorphism*

2. *Let $\mathcal{P}(x) = (A(x), x_2) \circ \mathcal{M}(x)$, where $\mathcal{M}$ is a birational polynomial automorphism and A is a De Jonquières transformation in $\mathbb{K}[x]$. Then, $\mathcal{P}$ is a birational automorphism.*

The above statements can be proved using Theorem 2.

4 Conclusions

In this paper, we prove that there exits a one-to-one correspondence between birational automorphisms of the plane and pairs of pencils $(\mathcal{V}_1, \mathcal{V}_2)$ intersecting in a unique point. As a consequence, we obtain a method that allows to construct birational automorphisms of the plane of a desired degree. The results are also stated for birational automorphism of the plane that are polynomial.

More precisely, we first construct a birational automorphism of the plane, $\mathcal{S}$, of a certain degree d (fixed in advance) from a pencil of curves $\mathcal{V}_1$ of degree d and a linear subsystem $\mathcal{V}_2$ of dimension 1 of the system of adjoint curves to $\mathcal{V}_1$. The birational automorphism $\mathcal{S}$ is obtained from the curves generating $\mathcal{V}_1$ and $\mathcal{V}_2$ (see Theorem 1 and statement 2 in Corollary 2). In Proposition 1, we show how these results can be stated for the important case of polynomial birational automorphisms. Additionally, we also are able to compute the inverse of $\mathcal{S}$ from $\mathcal{V}_1$ and $\mathcal{V}_2$, and we provide some formulae that allow to compute explicitly the degree of this inverse (see Corollary 1).

Reciprocally, we are given a birational automorphism $\mathcal{P}$, and we construct a certain $\mathcal{V}_1$ being $\mathbb{K}(t_1)$-rational, and its adjoint pencil $\mathcal{V}_2$. From $\mathcal{V}_1$ and $\mathcal{V}_2$, we obtain a birational automorphism $\mathcal{S}$ (constructed as in Theorem 1), and we show that "up to composition with a polynomial De Jonquières transformation", the given polynomial birational automorphism $\mathcal{P}$ is *equivalent* to the associated birational automorphism $\mathcal{S}$ (that is, $\mathcal{P} = \mathcal{J} \circ \mathcal{S}$, where $\mathcal{J}$ is a De Jonquières transformation; see Theorem 2). Furthermore, it is shown that the above results may be stated similarly for polynomial birational automorphisms. In this case, a polynomial De Jonquières transformation relates this correspondence (see Theorem 3).

The ideas presented in this paper open several important ways that may be used to provide significant results concerning Cremona transformations for dimensions $n \geq 3$. More precisely, given $\mathcal{V}_1$ defined by an irreducible polynomial $f_1(x) = g_{1,1}(x) - t_1 g_{1,2}(x) \in \mathbb{K}(t_1)[x]$, where $x = (x_1, x_2, x_3)$ is a generic element of $\mathbb{K}^3$, one would like to compute two polynomials

$$f_2(x, t_1) = g_{2,1}(x, t_1) - t_2\, g_{2,2}(x, t_1) \in \mathbb{K}(t_1, t_2)[x], \quad \text{and}$$
$$f_3(x, t_1, t_2) = g_{3,1}(x, t_1, t_2) - t_3\, g_{3,2}(x, t_1, t_2) \in \mathbb{K}(t_1, t_2, t_3)[x]$$

such that there exists exactly one intersection point in $\mathbb{K}(t_1, t_2, t_3)^3 \setminus \mathbb{K}^3$ of the polynomials f_1, f_2 and f_3. If this is the case, $\mathcal{S}(x) = (g_1(x), g_2(x), g_3(x))$, where

$$g_1(x) = \frac{g_{1,1}(x)}{g_{1,2}(x)}, \quad g_2(x) = \frac{g_{2,1}(x, g_1(x))}{g_{2,2}(x, g_1(x))}, \quad g_3(x) = \frac{g_{3,1}(x, g_1(x), g_2(x))}{g_{3,2}(x, g_1(x), g_2(x))}$$

is birational.

It is expected that, in order to solve this question, adjoints of high degree should be used (see [29]). In a future work, this problem will be developed in more detail.

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