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Asymptotes of space curves☆

Angel Blasco, Sonia Pérez-Díaz *

Dpto. de Física y Matemáticas, Universidad de Alcalá, E-28871 Madrid, Spain

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ABSTRACT

In Blasco and Pérez-Díaz (2014) (see [3]), a method for computing *generalized asymptotes* of a real algebraic plane curve implicitly defined is presented. Generalized asymptotes are curves that describe the status of a branch at points with sufficiently large coordinates and thus, it is an important tool to analyze the behavior at infinity of an algebraic curve. This motivates that in this paper, we analyze and compute the generalized asymptotes of a real algebraic space curve which could be parametrically or implicitly defined. We present an algorithm that is based on the computation of the *infinity branches* (this concept was already introduced for plane curves in Blasco and Pérez-Díaz (2014) [1]). In particular, we show that the computation of infinity branches in the space can be reduced to the computation of infinity branches in the plane and thus, the methods in Blasco and Pérez-Díaz (2014) (see [1]) can be applied.

1. Introduction

In [1], we introduce the notion of *infinity branches* and *approaching curves*. Some important properties are derived from these concepts which allow us to obtain an algorithm that compares the behavior of two implicitly defined algebraic plane curves at infinity. In particular, a characterization for the finiteness of the Hausdorff distance between two algebraic curves in the n-dimensional space can be obtained (see Section 5 in [1], and [2]). The characterization is related with the asymptotic behavior of the two curves and it can be easily checked.

Based on the notions and results presented in [1], in [3] we deal with the problem of computing the asymptotes of the infinity branches of a given plane curve $\mathcal{C}$ implicitly defined. The asymptotes of an infinity branch of $\mathcal{C}$ reflect the status of this branch at points with sufficiently large coordinates. It is well known that an asymptote of a curve is a line such that the distance between the curve and the line approaches zero as they tend to infinity. However, in [3], we show that an algebraic plane curve may have more general curves than lines describing the status of a branch at infinity. Thus, in [3], we develop an algorithm for the computation of *generalized asymptotes* (or *g-asymptotes*), and some important properties concerning this new concept are presented.

The applicability of the results presented in [1,3] is of central importance in the field of computer aided geometric design (CAGD) since these results provide new and important concepts as well as computational techniques that allow us to obtain information about the behavior of a plane curve at infinity. For instance, the infinity branches of an implicit plane curve $\mathcal{C}$ are essential for the study of the topology of $\mathcal{C}$ (see e.g. [4–6]) or for detecting its symmetries (see e.g. [7]). Also, the results obtained play an important role in the frame of approximate parametrization problems (see e.g. [8,9]) or in analyzing the Hausdorff distance between two curves (see [2]) which is specially interesting since the Hausdorff distance is an appropriate tool for measuring the closeness between two curves (see e.g. [10–13]).

☆ The author S. Pérez-Díaz is member of the Research Group ASYNACS (Ref. CCEE2011/R34).

* Corresponding author.
E-mail addresses: angel.blasco@uah.es (A. Blasco), sonia.perez@uah.es (S. Pérez-Díaz).

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The results obtained and the importance of the applications we mentioned above moved us to try to generalize the foundations and methods in [1,3] to the case of space curves.

For this purpose, in this paper, we consider an irreducible real algebraic space curve $\mathcal{C}$ over the field of complex numbers $\mathbb{C}$ implicitly defined by two irreducible real polynomials, and we deal with the problem of computing the asymptotes of the infinity branches of $\mathcal{C}$. For this purpose, we generalize the notions and some previous results presented in [1,3], and we develop an algorithm for computing the g-asymptotes of $\mathcal{C}$ (see Sections 2–4).

In addition, we also show how to compute the g-asymptotes if the given curve is defined by a rational real parametrization. This parametric approach can be easily generalized for parametric plane curves and in general, for a rational parametrization of a curve in the n-dimensional space (see Section 5). This new statement of the problem is specially interesting in some practical applications in CAGD, where objects are often given and manipulated parametrically.

Authors have not been able to find many references in the literature dealing with the analysis and computation of infinity branches and asymptotes of a given algebraic curve. Only in [14], linear asymptotes of space curves are briefly studied. In particular, it is proved how the tangents at the simple points at infinity of the curve (i.e. non-singular points at infinity) are related with the asymptotes. For the case of plane curves, some results concerning linear asymptotes can be found in [15,16].

CAGD is a natural environment for practical applications of algebraic curves and surfaces. In particular, the results and methods presented in this paper open new ways to study the behavior of algebraic space curves, with expected generalizations to higher dimension and the case of surfaces.

The applications expected of the results obtained in this paper can be included in the frame of those presented for the plane case. More precisely, the methods and techniques developed could be very useful to deal (for instance) with the following problems: the behavior at infinity of a space curve when approximate parametrization techniques are used (see e.g. [14] or [17]), the sketch of the graph or the computation of the topology of real algebraic space curves (see [17–19] or [20]), the detection of the symmetries of a given space curve (see [7]) or the computation of the Hausdorff distance between two curves (see [2,13]). The reader may find explanations of these and other problems in the vast literature on CAGD (see e.g. [20–25]).

The structure of the paper is as follows: in Section 2, we present the notation and we generalize some previous results developed in Sections 2, 3 and 4 in [1]. In particular, we introduce the notions of *infinity branch* and *convergent infinity branches*, and we characterize whether two implicit algebraic space curves approach each other at infinity. In Section 3, we show the relation between infinity branches of plane curves and infinity branches of space curves. More precisely, we obtain the infinity branches of a given space curve $\mathcal{C}$ from the infinity branches of a certain plane curve obtained by projecting $\mathcal{C}$ along some "valid projection direction". This approach allows us to use effective computational techniques existing in the plane case (see Section 3 in [1]) for the computation of the infinity branches of space curves. In Section 4, we introduce the notions of *perfect curve* and *generalized asymptote* or *g-asymptote*. These concepts are derived from the study of approaching curves and convergent branches in Section 3, and they generalize the notions introduced in [3] (see Section 3) for a given plane curve. Moreover, in Section 4, we also present an efficient algorithm that computes a g-asymptote for each infinity branch of a given space curve implicitly defined. We reach the expected situation, that is, the computation is similar to the case of implicit plane curves although the formalization and proofs of the results use approaches totally new since the computational techniques and tools in the space are necessarily different to those we have in the plane. Section 5 is devoted to the computation of g-asymptotes for a given parametric space curve. The method presented in this section is totally new. Moreover, it is easily applicable to parametric plane curves and in general, to rational parametrizations of curves in the n-dimensional space. We finish with a section of conclusions (see Section 6) where we summarize the results obtained, we emphasize the new contributions of this paper (compared with those presented in [1,3]), and we propose topics for further study.

2. Notation and terminology

In this section, we present some notions and terminology that will be used throughout the paper. In particular, we introduce some previous results concerning local parametrizations and Puiseux series (see Section 5.2 in [26], Section 2 in [1,27], Section 2.5 in [28,29] and Section 2 in Chapter 4 in [30]). In addition, we generalize the concept of *infinity branch* introduced in [1] (see Section 3) for a given algebraic plane curve. Important results and tools derived from this notion will be presented in the subsequent sections.

We denote by $\mathbb{C}[[t]]$ the domain of *formal power series* in the indeterminate t with coefficients in the field $\mathbb{C}$, i.e. the set of all sums of the form $\sum_{i=0}^{\infty} a_i t^i$, $a_i \in \mathbb{C}$. The quotient field of $\mathbb{C}[[t]]$ is called the field of *formal Laurent series*, and it is denoted by $\mathbb{C}((t))$. It is well known that every non-zero formal Laurent series $A \in \mathbb{C}((t))$ can be written in the form $A(t) = t^k \cdot (a_0 + a_1 t + a_2 t^2 + \cdots)$, where $a_0 \neq 0$ and $k \in \mathbb{Z}$. In addition, the field $\mathbb{C} \ll t \gg := \bigcup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$ is called the field of *formal Puiseux series*. Note that Puiseux series are power series of the form

$$\varphi(t) = m + a_1 t^{N_1/N} + a_2 t^{N_2/N} + a_3 t^{N_3/N} + \cdots \in \mathbb{C} \ll t \gg, \quad a_i \neq 0, \ \forall i \in \mathbb{N},$$

where $N, N_i \in \mathbb{N}$, $i \geq 1$, and $0 < N_1 < N_2 < \cdots$. The natural number N is known as *the ramification index* of the series. We denote it as $\nu(\varphi)$ (see [27]).

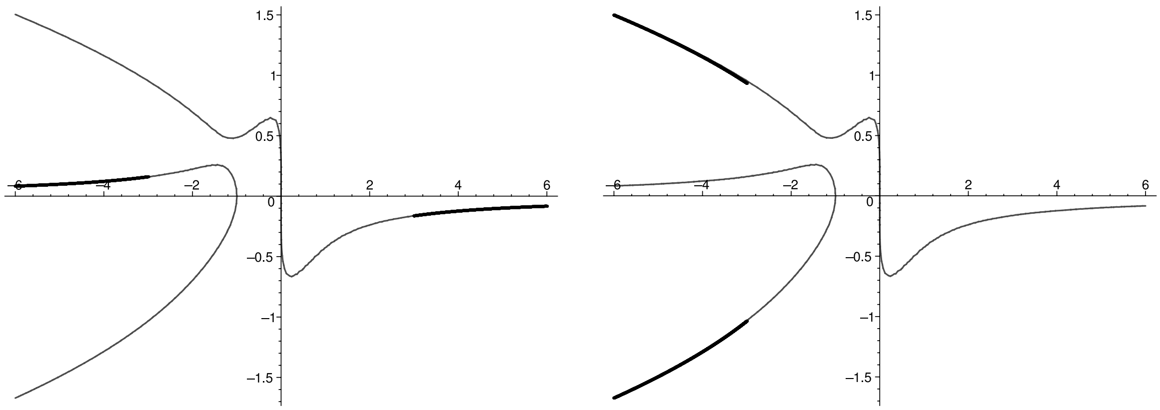

Fig. 1. Infinity branches B_1 (left) and B_2 (right).

The *order* of a non-zero (Puiseux or Laurent) series φ is the smallest exponent of a term with non-vanishing coefficient in φ. We denote it by $\mathrm{ord}(\varphi)$. We let the order of 0 be ∞.

The most important property of Puiseux series is given by Puiseux's theorem, which states that if $\mathbb{K}$ is an algebraically closed field, then the field $\mathbb{K} \ll x \gg$ is algebraically closed (see Theorems 2.77 and 2.78 in [28]). A proof of Puiseux's theorem can be given constructively by the Newton polygon method (see e.g. Section 2.5 in [28]).

In the following, we introduce the concept of infinity branch of a space curve, which is an essential tool in the development of the results presented in this paper. For this purpose, let $\mathcal{C} \subset \mathbb{C}^3$ be an irreducible space curve defined by two polynomials $f_i(x_1, x_2, x_3) \in \mathbb{R}[x_1, x_2, x_3]$, $i = 1, 2$. We assume that $\mathcal{C}$ is not planar (for planar space curves, one may apply the results in [1,3]). In general, an irreducible affine real (non-planar) space curve $\mathcal{C} \subset \mathbb{C}^3$ is defined as the zero set (over $\mathbb{C}$) of a finite set of real polynomials $\{f_1, \ldots, f_s\} \subset \mathbb{R}[x_1, x_2, x_3]$, $s \geq 2$. In this paper, we consider real algebraic space curves implicitly defined as the intersection of two surfaces.

Intuitively speaking, the infinity branches of a curve are the regions of the curve that spread out to infinity. They are associated to the infinity places of the corresponding projective curve (see e.g. Section 2 in [28]). In [1] (see Section 3), we define these branches, for the case of a given plane curve $\mathcal{C}$, as sets of the form $B = \{(z, r_j(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\} \subset \mathcal{C}$, where $r_j(z)$, $j = 1, \ldots, N$, are *conjugated Puiseux series* (see [27]) and $M \in \mathbb{R}^+$ (throughout the paper, $|\cdot|$ represents the module in $\mathbb{C}$). Thus, in particular, $f(z, r_j(z)) = 0$ for sufficient large values of z ($f(x_1, x_2)$ denotes the polynomial defining implicitly the plane curve $\mathcal{C}$). In Fig. 1, we plot a plane curve $\mathcal{C}$ and some points of the infinity branches B_1 and B_2 (see Example 3.5 in [1]).

In the following, we generalize the notion of infinity branch to the case of space curves and we provide a mathematical description of these entities. In addition, we show how to (theoretically) construct the infinity branches of a given space curve. Later, in Section 3, we will present an efficient computational method to obtain them.

We note that we work over $\mathbb{C}$, but we assume that the curve has infinitely many points in the affine plane over $\mathbb{R}$ and then, $\mathcal{C}$ has real defining polynomials (see Chapter 7 in [28]). We recall that the assumption of reality is included because of the nature of the problem, but the theory developed in this paper can be similarly developed for the case of complex non-real curves.

Let $\mathcal{C}^*$ be the corresponding projective curve defined by the homogeneous polynomials $F_i(x_1, x_2, x_3, x_4) \in \mathbb{R}[x_1, x_2, x_3, x_4]$, $i = 1, 2$. Furthermore, let $P = (1 : m_2 : m_3 : 0)$, $m_2, m_3 \in \mathbb{C}$, be a point at infinity of $\mathcal{C}^*$.

In addition, we consider the curve implicitly defined by the polynomials $g_i(x_2, x_3, x_4) := F_i(1, x_2, x_3, x_4) \in \mathbb{R}[x_2, x_3, x_4]$, for $i = 1, 2$. Observe that $g_i(p) = 0$, where $p = (m_2, m_3, 0)$. Let $I \in \mathbb{R}(x_4)[x_2, x_3]$ be the ideal generated by $g_i(x_2, x_3, x_4)$, $i = 1, 2$ in the ring $\mathbb{R}(x_4)[x_2, x_3]$. Since $\mathcal{C}$ is not contained in a hyperplane $x_4 = c$, $c \in \mathbb{C}$, we have that x_4 is not algebraic over $\mathbb{R}$. Under this assumption, the ideal I (i.e. the system of equations $g_1 = g_2 = 0$) has only finitely many solutions in the 3-dimensional affine space over the algebraic closure of $\mathbb{R}(x_4)$ (which is contained in $\mathbb{C} \ll x_4 \gg$). Then, there are finitely many pairs of Puiseux series $(\varphi_2(t), \varphi_3(t)) \in \mathbb{C} \ll t \gg^2$ such that $g_i(\varphi_2(t), \varphi_3(t), t) = 0$, $i = 1, 2$, and $\varphi_k(0) = m_k$, $k = 2, 3$. Each of the pairs $(\varphi_2(t), \varphi_3(t))$ is a solution of the system associated with the infinity point $(1 : m_2 : m_3 : 0)$, and $\varphi_2(t)$ and $\varphi_3(t)$ converge in a neighborhood of $t = 0$. Moreover, since $\varphi_k(0) = m_k$, $k = 2, 3$, these series do not have terms with negative exponents; in fact, they have the form

$$\varphi_k(t) = m_k + \sum_{i \geq 1} a_{i,k} t^{N_{i,k}/N_k}$$

where $N_k,\ N_{i,k} \in \mathbb{N},\ 0 < N_{1,k} < N_{2,k} < \cdots$.

It is important to remark that if $\varphi(t) := (\varphi_2(t), \varphi_3(t))$ is a solution of the system, then $\sigma_\epsilon(\varphi)(t) := (\sigma_\epsilon(\varphi_2)(t), \sigma_\epsilon(\varphi_3)(t))$ is another solution of the system, where

$$\sigma_\epsilon(\varphi_k)(t) = m_k + \sum_{i \geq 1} a_{i,k} \epsilon^{\lambda_{i,k}} t^{N_{i,k}/N_k}, \quad N_k,\ N_{i,k} \in \mathbb{N},\ 0 < N_{1,k} < N_{2,k} < \cdots,$$

$N := \mathrm{lcm}(N_2, N_3)$, $\lambda_{i,k} := N_{i,k}N/N_k \in \mathbb{N}$, and $\epsilon^N = 1$ (see Section 5.2 in [26]). We refer to these solutions as the *conjugates* of φ. The set of all (distinct) conjugates of φ is called the *conjugacy class* of φ, and the number of different conjugates of r is $N = \nu(\varphi)$.

Under these conditions and reasoning as in [1] (see Section 3), we get that there exists $M \in \mathbb{R}^+$ such that for $i \in \{1, 2\}$,

$$F_i(1 : \varphi_2(t) : \varphi_3(t) : t) = g_i(\varphi_2(t), \varphi_3(t), t) = 0, \quad \text{for } t \in \mathbb{C} \text{ and } |t| < M.$$

This implies that

$$F_i(t^{-1} : t^{-1}\varphi_2(t) : t^{-1}\varphi_3(t) : 1) = f_i(t^{-1}, t^{-1}\varphi_2(t), t^{-1}\varphi_3(t)) = 0,$$

for $t \in \mathbb{C}$ and $0 < |t| < M$.

Now, we set $t^{-1} = z$, and we obtain that for $i \in \{1, 2\}$,

$$f_i(z, r_2(z), r_3(z)) = 0, \quad z \in \mathbb{C} \text{ and } |z| > M^{-1}, \quad \text{where}$$
$$r_k(z) = z\varphi_k(z^{-1}) = m_k z + a_{1,k}z^{1-N_{1,k}/N_k} + a_{2,k}z^{1-N_{2,k}/N_k} + a_{3,k}z^{1-N_{3,k}/N_k} + \cdots,$$

$a_{j,k} \neq 0$, $N_k, N_{j,k} \in \mathbb{N}$, $j = 1, \ldots$, and $0 < N_{1,k} < N_{2,k} < \cdots$.

Since $\nu(\varphi) = N$, we get that there are N different series in its conjugacy class. Let $\varphi_{j,k}$, $j = 1, \ldots, N$ be these series, and $r_{j,k}(z) = z\varphi_{j,k}(z^{-1}) =$

$$m_k z + a_{1,k}c_j^{\lambda_{1,k}}z^{1-N_{1,k}/N_k} + a_{2,k}c_j^{\lambda_{2,k}}z^{1-N_{2,k}/N_k} + a_{3,k}c_j^{\lambda_{3,k}}z^{1-N_{3,k}/N_k} + \cdots \tag{1}$$

where $N := \mathrm{lcm}(N_2, N_3)$, $\lambda_{i,k} := N_{i,k}N/N_k \in \mathbb{N}$, and $c_1, \ldots, c_N$ are the N complex roots of $x^N = 1$. Now we are ready to introduce the notion of infinity branches. The following definitions and results generalize those presented in Sections 3 and 4 in [1] for algebraic plane curves.

Definition 1. An *infinity branch of a space curve* $\mathcal{C}$ associated to the point at infinity $P = (1 : m_2 : m_3 : 0)$, $m_2, m_3 \in \mathbb{C}$, is a set $B = \bigcup_{j=1}^N L_j$, where $L_j = \{(z, r_{j,2}(z), r_{j,3}(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M\}$, $M \in \mathbb{R}^+$, and the series $r_{j,2}$ and $r_{j,3}$ are given by (1). The subsets $L_1, \ldots, L_N$ are called the *leaves* of the infinity branch B.

Remark 1. Using Definition 1, we get that the points of the infinity branch have the form $(z, r_{j,2}(z), r_{j,3}(z))$, where $r_{j,k}(z)$, $j = 1, \ldots, N$ are conjugated Puiseux series (for $k = 2, 3$) and, by construction, they belong to the curve, so it holds that $f_1(z, r_{j,2}(z), r_{j,3}(z)) = f_2(z, r_{j,2}(z), r_{j,3}(z)) = 0$ for every $z \in \mathbb{C}$ with $|z| > M$. In addition, from (1) and taking into account that $1 - N_{i,k}/N_k < 1$, $i = 1, \ldots$, we get that $\lim_{z\to\infty} r_{j,k}(z)/z = m_k$ for $k = 2, 3$. That is, $\lim_{z\to\infty}(1 : r_{j,2}(z)/z : r_{j,3}(z)/z : 1/z) = (1 : m_2 : m_3 : 0)$.

Remark 2. An infinity branch is uniquely determined from one leaf, up to conjugation. That is, let B be an infinity branch and let us consider $L = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M\}$ one of its leaves, with

$$r_k(z) = z\varphi_k(z^{-1}) = m_k z + a_{1,k}z^{1-N_{1,k}/N_k} + a_{2,k}z^{1-N_{2,k}/N_k} + a_{3,k}z^{1-N_{3,k}/N_k} + \cdots$$

for $k = 2, 3$. Then, one has that any other leaf L_j of B has the form $L_j = \{(z, r_{j,2}(z), r_{j,3}(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M\}$ where $r_{j,k} = r_k$, $k = 2, 3$, up to conjugation; i.e.

$$r_{j,k}(z) = z\varphi_{j,k}(z^{-1}) = m_k z + a_{1,k}c_j^{\lambda_{1,k}}z^{1-N_{1,k}/N_k} + a_{2,k}c_j^{\lambda_{2,k}}z^{1-N_{2,k}/N_k} + a_{3,k}c_j^{\lambda_{3,k}}z^{1-N_{3,k}/N_k} + \cdots$$

$N, N_{i,k} \in \mathbb{N}$, $\lambda_{i,k} = N_{i,k}N/N_k \in \mathbb{N}$, $k = 2, 3$ and $c_j^N = 1$, $j = 1, \ldots, N$.

Remark 3. Observe that the above approach and Definition 1 is presented for points at infinity of the form $(1 : m_2 : m_3 : 0)$. For the points at infinity $(0 : m_2 : m_3 : 0)$, with $m_2 \neq 0$ or $m_3 \neq 0$, we reason similarly but we dehomogenize w.r.t x_2 (if $m_2 \neq 0$) or x_3 (if $m_3 \neq 0$). More precisely, we distinguish two different cases:

1. If $(0 : m_2 : m_3 : 0)$, $m_2 \neq 0$ is a point at infinity of the given space curve $\mathcal{C}$, we consider the curve defined by the polynomials $g_i(x_1, x_3, x_4) := F_i(x_1, 1, x_3, x_4) \in \mathbb{R}[x_1, x_3, x_4]$, $i = 1, 2$, and we reason as above. We get that an infinity branch of $\mathcal{C}$ associated to the point at infinity $P = (0 : m_2 : m_3 : 0)$, $m_2 \neq 0$, is a set $B = \bigcup_{j=1}^N L_j$, where $L_j = \{(r_{j,1}(z), z, r_{j,3}(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M\}$, $M \in \mathbb{R}^+$.
2. If $(0 : m_2 : m_3 : 0)$, $m_3 \neq 0$ is a point at infinity of the given space curve $\mathcal{C}$, we consider the curve defined by the polynomials $g_i(x_1, x_2, x_4) := F_i(x_1, x_2, 1, x_4) \in \mathbb{R}[x_1, x_2, x_4]$, $i = 1, 2$, and we reason as above. We get that an infinity branch of $\mathcal{C}$ associated to the point at infinity $P = (0 : m_2 : m_3 : 0)$, $m_3 \neq 0$, is a set $B = \bigcup_{j=1}^N L_j$, where $L_j = \{(r_{j,1}(z), r_{j,2}(z), z) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M\}$, $M \in \mathbb{R}^+$.

Additionally, instead of working with this type of branches, if the space curve $\mathcal{C}$ has points at infinity of the form $(0 : m_2 : m_3 : 0)$, one may consider a linear change of coordinates. Thus, in the following, we may assume w.l.o.g that the given algebraic space curve $\mathcal{C}$ only has points at infinity of the form $(1 : m_2 : m_3 : 0)$. More details on these branches for the planar case are given in [1] (see Definition 3.3 in Section 3).

In the following, we introduce the notions of convergent branches and approaching curves. Intuitively speaking, two infinity branches converge if they get closer as they tend to infinity. This concept will allow us to analyze whether two space curves approach each other and it generalizes the notion introduced for the plane case (see Section 4 in [1]).

Definition 2. Two infinity branches, B and $\overline{B}$, are convergent if there exist two leaves $L = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M\} \subset B$ and $\overline{L} = \{(z, \overline{r}_2(z), \overline{r}_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > \overline{M}\} \subset \overline{B}$ such that

$$\lim_{z\to\infty} d((r_2(z), r_3(z)), (\overline{r}_2(z), \overline{r}_3(z))) = 0.$$

In this case, we say that the leaves L and $\overline{L}$ converge.

Remark 4. 1. In our case, since we will be working over $\mathbb{C}$ or $\mathbb{R}$, d denotes the usual unitary or Euclidean distance (see Chapter 5 in [31]). Taking into account that all the distances are equivalent in $\mathbb{C}^2$, we easily get that $\lim_{z\to\infty} d((r_2(z), r_3(z)), (\overline{r}_2(z), \overline{r}_3(z))) = 0$ if and only if $\lim_{z\to\infty}(r_i(z) - \overline{r}_i(z)) = 0,\ i = 2, 3$.
2. Two convergent infinity branches are associated to the same point at infinity (see Remark 4.5 in [1]).

In the following lemma, we characterize the convergence of two given infinity branches. This result is obtained similarly as in the case of plane curves and thus, we omit the proof (see Lemma 4.2, and Proposition 4.6 in [1]).

Lemma 1. *The following statements hold:*

- *Two leaves $L = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M\}$ and $\overline{L} = \{(z, \overline{r}_2(z), \overline{r}_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > \overline{M}\}$ are convergent if and only if the terms with non negative exponent in the series $r_i(z)$ and $\overline{r}_i(z)$ are the same, for $i = 2, 3$.*
- *Two infinity branches B and $\overline{B}$ are convergent if and only if for each leaf $L \subset B$ there exists a leaf $\overline{L} \subset \overline{B}$ convergent with L, and reciprocally.*

In Definition 3, we introduce the notion of *approaching curves*, that is, *curves that approach each other*. For this purpose, we recall that given an algebraic space curve $\mathcal{C}$ over $\mathbb{C}$ and a point $p \in \mathbb{C}^3$, *the distance from p to $\mathcal{C}$* is defined as $d(p, \mathcal{C}) = \min\{d(p, q) : q \in \mathcal{C}\}$.

Definition 3. Let $\mathcal{C}$ be an algebraic space curve over $\mathbb{C}$ with an infinity branch B. We say that a curve $\overline{\mathcal{C}}$ *approaches* $\mathcal{C}$ at its infinity branch B if there exists one leaf $L = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\} \subset B$ such that $\lim_{z\to\infty} d((z, r_2(z), r_3(z)), \overline{\mathcal{C}}) = 0$.

In the following, we state some important results concerning two curves that approach each other. These results can be proved similarly as in the case of plane curves (see Lemma 3.6, Theorem 4.11, Remark 4.12 and Corollary 4.13 in [1]).

Theorem 1. *Let $\mathcal{C}$ be an algebraic space curve over $\mathbb{C}$ with an infinity branch B. An algebraic space curve $\overline{\mathcal{C}}$ approaches $\mathcal{C}$ at B if and only if $\overline{\mathcal{C}}$ has an infinity branch, $\overline{B}$, such that B and $\overline{B}$ are convergent.*

Remark 5. 1. Note that $\overline{\mathcal{C}}$ approaches $\mathcal{C}$ at some infinity branch B if and only if $\mathcal{C}$ approaches $\overline{\mathcal{C}}$ at some infinity branch $\overline{B}$. In the following, we say that $\mathcal{C}$ and $\overline{\mathcal{C}}$ approach each other or that they are *approaching curves*.
2. Two approaching curves have a common point at infinity.
3. $\overline{\mathcal{C}}$ *approaches* $\mathcal{C}$ at an infinity branch B if and only if for every leaf $L = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M\} \subset B$, it holds that $\lim_{z\to\infty} d((z, r_2(z), r_3(z)), \overline{\mathcal{C}}) = 0$.

Corollary 1. *Let $\mathcal{C}$ be an algebraic space curve with an infinity branch B. Let $\overline{\mathcal{C}}_1$ and $\overline{\mathcal{C}}_2$ be two different curves that approach $\mathcal{C}$ at B. Then:*

1. *$\overline{\mathcal{C}}_i$ has an infinity branch $\overline{B_i}$ that converges with B, for $i = 1, 2$.*
2. *$\overline{B_1}$ and $\overline{B_2}$ are convergent. Then, $\overline{\mathcal{C}}_1$ and $\overline{\mathcal{C}}_2$ approach each other.*

For the sake of simplicity, and taking into account that an infinity branch B is uniquely determined from one leaf, up to conjugation (see statement 1 in Remark 2), we identify an infinity branch by just one of its leaves. Hence, in the following

$$B = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M\}, \quad M \in \mathbb{R}^+$$

will stand for the infinity branch whose leaves are obtained by conjugation on

$$r_k(z) = m_k z + a_{1,k} z^{1-N_{1,k}/N_k} + a_{2,k} z^{1-N_{2,k}/N_k} + a_{3,k} z^{1-N_{3,k}/N_k} + \cdots,$$

$a_{i,k} \neq 0$, $\forall i \in \mathbb{N}$, $i \geq 1$, $N_k, N_{i,k} \in \mathbb{N}$, and $0 < N_{1,k} < N_{2,k} < \cdots$ for $k = 2, 3$. Observe that the results stated above hold for any leaf of B. In addition, we will also show that the results obtained in the following sections hold for any leaf (see statement 3 in Remark 8).

3. Computation of infinity branches

Let $\mathcal{C}$ be an irreducible algebraic space curve defined by the polynomials $f_i(x_1, x_2, x_3) \in \mathbb{R}[x_1, x_2, x_3]$ for $i = 1, 2$. In this section, we are interested in computing the infinity branches of $\mathcal{C}$. That is, points of the form $(z, r_{j,2}(z), r_{j,3}(z))$, where $r_{j,k}(z), j = 1, \dots, N$ are conjugated Puiseux series (for $k = 2, 3$), such that $f_1(z, r_{j,2}(z), r_{j,3}(z)) = f_2(z, r_{j,2}(z), r_{j,3}(z)) = 0$ for every $z \in \mathbb{C}$ with $|z| > M$ (see Definition 1 and Remark 1).

For this purpose, and taking into account the reasoning in Section 2, we get that we need to compute the finitely many pairs of Puiseux series $(\varphi_{j,2}(t), \varphi_{j,3}(t)) \in \mathbb{C} \ll t \gg^2$ such that $g_i(\varphi_{j,2}(t), \varphi_{j,3}(t), t) = 0$, $i = 1, 2$ (note that $r_{j,k}(z) = z\varphi_{j,k}(z^{-1})$).

In order to deal with this problem, some methods could be applied. For instance, in [26], an algorithm for computing local parametrizations of analytic branches of an implicitly defined curve in the n-dimensional space is presented. Furthermore, the algorithm can be used in space curve tracing near a singular point, as an alternative to symbolic computations based on resolutions of singularities. The method requires extensive recourses for solving systems of polynomial equations with finitely many solutions and it deals with arithmetics of algebraic numbers. Some interesting experiments are performed by means of CoCoA 1.5.3, but there is no current implementation of the algorithm. In [1] (see Section 3), it is shown how infinity branches for a given plane curve can be efficiently computed using well-known implemented algorithms (in particular, in order to compute the series expansions, we use the command puiseux included in the package algcurves of the computer algebra system Maple; see Example 3.5).

The development of this section is based on the idea of reducing the problem of computing infinity branches for space curves to the planar case. That is, we will try to compute the infinity branches of a given space curve $\mathcal{C}$ from the infinity branches of a birationally equivalent plane curve, say $\mathcal{C}^p$.

In order to get $\mathcal{C}^p$, we may apply the method presented in [32] (see Sections 2 and 3). The curve obtained using this approach is birationally equivalent to $\mathcal{C}$, that is, there exists a birational correspondence between the points of $\mathcal{C}^p$ and the points of $\mathcal{C}$. In [32], it is shown that $\mathcal{C}^p$ can always be obtained by projecting $\mathcal{C}$ along some *valid projection direction*.

Once we have $\mathcal{C}^p$, we may compute its infinity branches by applying the procedure developed in [1]. Finally, we use the birational correspondence mentioned above for obtaining the infinity branches of $\mathcal{C}$ from those of $\mathcal{C}^p$.

In the following we assume that the x_3-axis is a valid projection direction (otherwise, we apply a linear change of coordinates; see Section 2 in [32]). Let $\mathcal{C}^p$ be the projection of $\mathcal{C}$ along the x_3-axis, and let $f^p(x_1, x_2) \in \mathbb{R}[x_1, x_2]$ be the implicit polynomial defining $\mathcal{C}^p$. In [32] (see Section 3), it is shown how to construct a birational mapping $h(x_1, x_2) = h_1(x_1, x_2)/h_2(x_1, x_2)$ such that $(x_1, x_2, x_3) \in \mathcal{C}$ if and only if $(x_1, x_2) \in \mathcal{C}^p$ and $x_3 = h(x_1, x_2)$. For this purpose, one needs to compute a polynomial remainder sequence (PRS) along the projection direction. It can be computed in various different ways, see e.g [33], although in [32] the subresultant PRS scheme is chosen for its computational superiority (the subresultant PRS scheme can be computed using for instance the computer algebra system Maple; for more details see e.g. Section 5.1.2 in [34]).

We refer to $h(x_1, x_2)$ as the *lift function*, since we can obtain the points of the space curve $\mathcal{C}$ by applying h to the points of the plane projected curve $\mathcal{C}^p$. In addition, note that $x_3 = h(x_1, x_2)$ if and only if $h_1(x_1, x_2) - h_2(x_1, x_2)x_3 = 0$. Thus, $\mathcal{C}$ can be implicitly defined by the polynomials $f^p(x_1, x_2)$ and $f_3(x_1, x_2, x_3) = h_1(x_1, x_2) - h_2(x_1, x_2)x_3$.

In Theorem 2, we study the relation between the infinity branches of $\mathcal{C}$ and $\mathcal{C}^p$. The idea is to use the lift function h to obtain the infinity branches of the space curve $\mathcal{C}$ from the infinity branches of the plane curve $\mathcal{C}^p$. An efficient method to compute the infinity branches of a plane curve is presented in Section 3 in [1].

Theorem 2. *$B^p = \{(z, r_2(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M^p\}$ is an infinity branch of $\mathcal{C}^p$ for some $M^p \in \mathbb{R}^+$ iff there exists a series $r_3(z) = z\varphi_3(1/z)$, $\varphi_3(z) \in \mathbb{C} \ll z \gg$, such that $B = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\}$ is an infinity branch of $\mathcal{C}$ for some $M \in \mathbb{R}^+$.*

Proof. Clearly, if B is an infinity branch of $\mathcal{C}$, then B^p is an infinity branch of $\mathcal{C}^p$. Conversely, let $B^p = \{(z, r_2(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M^p\}$ be an infinity branch of $\mathcal{C}^p$, and we look for a series $r_3(z) = z\varphi_3(1/z), \varphi_3(z) \in \mathbb{C} \ll z \gg$, such that $B = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\}$ is an infinity branch of $\mathcal{C}$. Observe that, from the discussion above, we can get it as $r_3(z) = h(z, r_2(z))$ (note that since $(z, r_2(z)) \in \mathbb{C}^2$ for $|z| > M^p$, and $r_3(z) = h(z, r_2(z))$, we consider $(z, r_2(z), r_3(z)) \in \mathbb{C}^3$ for $|z| > M$, where $M = M^p$). However, we need to prove that $r_3(z) = z\varphi_3(1/z)$ for some Puiseux series $\varphi_3(z)$.

As we stated above, given $(a_1, a_2, a_3) \in \mathcal{C}$, it holds that $f_3(a_1, a_2, a_3) = h_1(a_1, a_2) - h_2(a_1, a_2)a_3 = 0$. Thus, in particular, $(z, r_2(z), r_3(z)) \in B \subset \mathcal{C}$ verifies that $f_3(z, r_2(z), r_3(z)) = 0$. Hence, $F_3(z, r_2(z), r_3(z), 1) = 0$, where $F_3(x_1, x_2, x_3, x_4)$ is the homogeneous polynomial of $f_3(x_1, x_2, x_3)$.

Taking into account the results in Section 3 in [1], we have that $r_2(z) = z\varphi_2(1/z)$, where $\varphi_2(z) \in \mathbb{C} \ll z \gg$. Now, we look for $\varphi_3(z) \in \mathbb{C} \ll z \gg$ such that $r_3(z) = z\varphi_3(1/z)$. This series must verify that (see statement above) that $F_3(z, z\varphi_2(1/z), z\varphi_3(1/z), 1) = 0$ for $|z| > M$. We set $z = t^{-1}$, and we get that $F_3(t^{-1}, t^{-1}\varphi_2(t), t^{-1}\varphi_3(t), 1) = 0$ or equivalently

$$F_3(1, \varphi_2(t), \varphi_3(t), t) = 0. \tag{I}$$

Note that equality (I) holds for $|t| < 1/M$. That is, equality (I) must be satisfied in a neighborhood of the point at infinity $(1, \varphi_2(0), \varphi_3(0), 0)$.

At this point, we observe that F_3 has the form

$$F_3(x_1, x_2, x_3, x_4) = x_4^{n_1} H_1(x_1, x_2, x_4) - x_4^{n_2} H_2(x_1, x_2, x_4) x_3$$

where $H_i(x_1, x_2, x_4)$ is the homogeneous polynomial of $h_i(x_1, x_2),\ i = 1, 2$, and $n_1, n_2 \in \mathbb{N}$. Then, we have that

$$F_3(1, \varphi_2(t), \varphi_3(t), t) = t^{n_1} H_1(1, \varphi_2(t), t) - t^{n_2} H_2(1, \varphi_2(t), t)\varphi_3(t)$$

and since (I) must hold, we obtain that

$$\varphi_3(t) = t^{n_1 - n_2} \frac{H_1(1, \varphi_2(t), t)}{H_2(1, \varphi_2(t), t)}.$$

Obviously, $\varphi_3(t)$ can be expressed as a Puiseux series since $\mathbb{C} \ll t \gg$ is a field. Therefore, we conclude that $B = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M\}$, where $r_3(z) = z\varphi_3(1/z)$, is an infinity branch of $\mathcal{C}$. $\square$

In the following, we illustrate the above theorem with an example.

Example 1. Let $\mathcal{C}$ be the irreducible space curve defined over $\mathbb{C}$ by

$$f_1(x_1, x_2, x_3) = -x_2^2 - 2x_1x_3 + 2x_2x_3 - x_1 + 3, \quad \text{and} \quad f_2(x_1, x_2, x_3) = x_3 + x_1x_2 - x_2^2.$$

The projection along the x_3-axis, $\mathcal{C}^p$ is given by the polynomial

$$f^p(x_1, x_2) = x_2^2 + x_1 - 3 - 2x_2x_1^2 + 4x_1x_2^2 - 2x_2^3$$

(f^p can be obtained by computing $\text{resultant}_{x_3}(f_1, f_2)$; see Section 2.3 in [28]).

By applying the method described in [1] (see Section 3) we compute the infinity branches of $\mathcal{C}_p$. For this purpose, we consider the algcurves package included in the computer algebra system Maple; in particular, the command puiseux is used. We obtain the branch $B_1^p = \{(z, r_{1,2}(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M_1^p\}$, where

$$r_{1,2}(z) = \frac{z^{-1}}{2} - \frac{3z^{-2}}{2} + \frac{z^{-3}}{2} - \frac{23z^{-4}}{8} + \frac{37z^{-5}}{8} - \frac{25z^{-6}}{4} + \cdots,$$

that is associated to the point at infinity $P_1 = (1 : 0 : 0)$, and the branch $B_2^p = \{(z, r_{2,2}(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M_2^p\}$, where

$$r_{2,2}(z) = z + \frac{\sqrt{2}z^{1/2}}{2} + \frac{1}{4} + \frac{9\sqrt{2}z^{-1/2}}{32} - \frac{z^{-1}}{4} - \frac{785\sqrt{2}z^{-3/2}}{1024} + \cdots,$$

that is associated to the point at infinity $P_2 = (1 : 1 : 0)$. Note that B_2^p has ramification index 2, so it has two leaves.

Once we have obtained the infinity branches of the projected curve $\mathcal{C}^p$, we compute the infinity branches of the space curve $\mathcal{C}$. For this purpose, we need to compute the lift function $h(x_1, x_2)$ (we apply Sections 2 and 3 in [32]) to get the third component of these branches. In this example, we only have to compute the remainder of f_1 divided by f_2 w.r.t. the variable x_3 (see e.g. Section 5.1.2 in [34]). We get that $\text{rem}_{x_3}(f_1, f_2) = -x_2^2 + 3 - x_1 + 2x_1^2x_2 - 4x_1x_2^2 + 2x_2^3$. Thus, the lift function $h(x_1, x_2)$ is obtained by solving the equation $f_2 = 0$ in the variable x_3. We get that $h(x_1, x_2) = -x_1x_2 + x_2^2$ and thus, the infinity branches of the space curve are $B_1 = \{(z, r_{1,2}(z), r_{1,3}(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M_1\}$, where

$$r_{1,3}(z) = h(z, r_{1,2}(z)) = -\frac{1}{2} - \frac{3z^{-1}}{2} - \frac{z^{-2}}{4} + \frac{11z^{-3}}{8} - \frac{15z^{-4}}{8} + \frac{15z^{-5}}{8} + \cdots$$

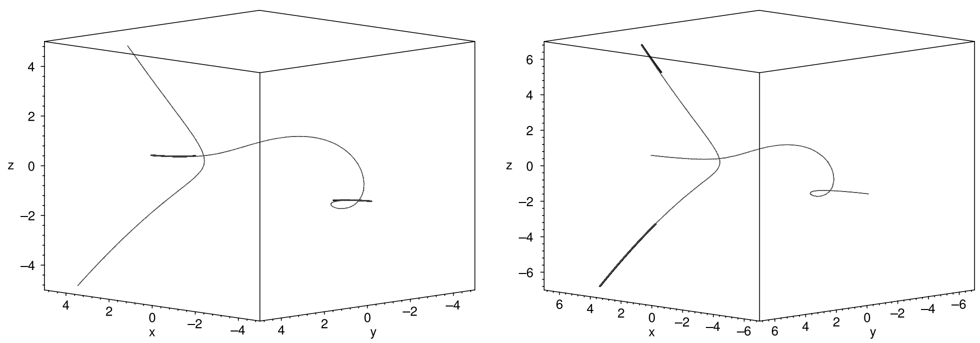

Fig. 2. Curve $\mathcal{C}$ and infinity branches B_1 (left) and B_2 (right).

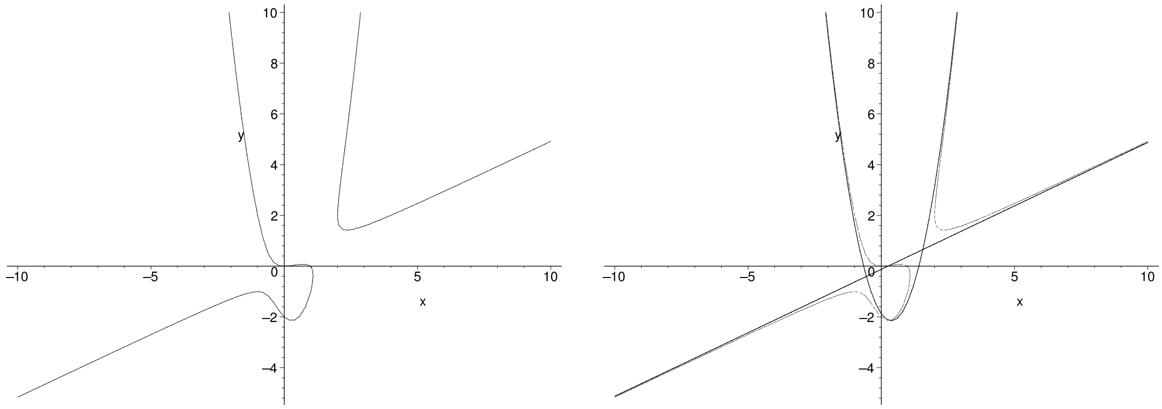

Fig. 3. Curve $\mathcal{C}$ (left) approached by a parabola and a line (right).

and $B_2 = \{(z, r_{2,2}(z), r_{2,3}(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M_2\}$, where

$$r_{2,3}(z) = h(z, r_{2,2}(z)) = \frac{\sqrt{2}z^{3/2}}{2} + \frac{3z}{4} + \frac{17\sqrt{2}z^{1/2}}{32} + \frac{3}{8} - \frac{897\sqrt{2}z^{-1/2}}{1024} + \cdots.$$

In Fig. 2, we plot the curve $\mathcal{C}$ and some points of the branches B_1 and B_2.

4. Computation of an asymptote of a given infinity branch

In [3] (see Section 3), we show how some algebraic plane curves can be approached at infinity by other curves of less degree. A well-known example is the case of hyperbolas that are curves of degree 2 approached at infinity by two lines (their asymptotes). Similar situations may also arise when we deal with curves of higher degree.

For instance, let $\mathcal{C}$ be the plane curve defined by the equation $-x_2x_1 - x_2^2 - x_1^3 + 2x_1^2x_2 + x_1^2 - 2x_2 = 0$. The curve $\mathcal{C}$ has degree 3 but it can be approached at infinity by the parabola $x_2 - 2x_1^2 + 3/2x_1 + 15/8 = 0$ (see Fig. 3). This example led us to introduce the notions of *perfect curve* and *g-asymptote* for plane curves (see Section 3 in [3]). Some properties on these concepts as well as some important applications for a given plane curve were presented in [3] (see Section 4).

Determining the asymptotes of an implicitly defined algebraic curve is an important topic considered in many text-books on analysis (see e.g. [35]). Some algorithms for computing the linear asymptotes of a real plane algebraic curve implicitly defined can be found in the literature (see e.g. [14,15] or [16]). However, as we show in [3], an algebraic plane curve may have more general curves than lines (that is the classical concept of asymptote) describing the status of a branch at the points with sufficiently large coordinates. The theory and practical methods concerning these special general curves, called *generalized asymptotes*, are presented in [3] (see Sections 3, 4 and 5) for the case of plane curves.

In this section, we intend to study and compute the *generalized asymptotes* for a given algebraic space curve. No results approaching this problem algorithmically and theoretically were known up to the moment. The results are new and open new ways in order to explore space curves as for instance different aspect concerning its topology (see [18]), the computation of the shapes in a family of space curves (see [19]) or even its symmetries (see [7]).

Some results cannot be seen as a straightforward generalization from the case of plane curves. Although the construction of asymptotes is similar (in the sense that in the space curve a new component has to be computed), the formalization of the results as well as the detailed proofs need to be considered from a different point of view (for instance, the computation of the degree of a space curve is totally different to the computation of the degree of a plane curve). So, Lemma 2 and Proposition 1 require new tools and different reasoning as in the plane case.

We start the section with some important definitions and previous results that will lead to the construction of asymptotes in Section 4.1.

Definition 4. A curve $\mathcal{C}$ of degree $d_{\mathcal{C}}$ is a *perfect curve* if it cannot be approached by any curve of degree less than $d_{\mathcal{C}}$.

A curve that is not perfect can be approached by other curves of less degree. If these curves are perfect, we call them *g-asymptotes*. More precisely, we have the following definition.

Definition 5. Let $\mathcal{C}$ be a curve with an infinity branch B. A *g-asymptote* (generalized asymptote) of $\mathcal{C}$ at B is a perfect curve that approaches $\mathcal{C}$ at B.

The notion of *g-asymptote* is similar to the classical concept of asymptote. The difference is that a g-asymptote does not have to be a line, but a perfect curve. Actually, it is a generalization, since every line is a perfect curve (this remark follows from Definition 4). Throughout the paper we refer to *g-asymptote* simply as *asymptote*.

Remark 6. The degree of an asymptote is less or equal than the degree of the curve it approaches. In fact, an asymptote of a curve $\mathcal{C}$ at a branch B has minimal degree among all the curves that approach $\mathcal{C}$ at B (see Remark 3 in [3]).

In the following, we prove that every infinity branch of a given algebraic space curve has, at least, one asymptote and we show how to obtain it (see Theorem 3). Most of the results introduced below for the case of space curves generalize the results presented in [3] for the plane case.

Let $\mathcal{C}$ be an irreducible space curve implicitly defined by the polynomials $f_i \in \mathbb{R}[x_1, x_2, x_3]$, $i = 1, 2$, and let $B = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M\}$ be an infinity branch of $\mathcal{C}$ associated to the point at infinity $P = (1 : m_2 : m_3 : 0)$. We know that r_2 and r_3 are given as

$$r_2(z) = m_2 z + a_{1,2} z^{-N_{1,2}/N_2+1} + a_{2,2} z^{-N_{2,2}/N_2+1} + a_{3,2} z^{-N_{3,2}/N_2+1} + \cdots$$
$$r_3(z) = m_3 z + a_{1,3} z^{-N_{1,3}/N_3+1} + a_{2,3} z^{-N_{2,3}/N_3+1} + a_{3,3} z^{-N_{3,3}/N_3+1} + \cdots$$

where $a_{i,2} \neq 0,\ N_2, N_{i,2} \in \mathbb{N},\ i \geq 1,\ 0 < N_{1,2} < N_{2,2} < \cdots$, and $a_{i,3} \neq 0,\ N_3, N_{i,3} \in \mathbb{N},\ i \geq 1$, and $0 < N_{1,3} < N_{2,3} < \cdots$. Let $N := \mathrm{lcm}(N_2, N_3)$, and note that $\nu(B) = N$.

Lemma 2. *It holds that* $\deg(\mathcal{C}) \geq N$.

Proof. In Section 2, we show that there exist $N := \mathrm{lcm}(N_2, N_3)$ conjugate tuples, $(\varphi_2(z), \varphi_3(z))$, which are solutions of the system $g_i(x_2, x_3, x_4) = 0$, $i = 1, 2$. Hence, the tuples $(z, r_{j,2}(z), r_{j,3}(z))$ with $r_{j,2}(z) = z\varphi_{j,2}(z^{-1})$ and $r_{j,3}(z) = z\varphi_{j,3}(z^{-1})$ for $j = 1, \ldots, N$, are solutions of the system $f_i(x_1, x_2, x_3) = 0$, $i = 1, 2$. That is, they are points of the curve $\mathcal{C}$.

Then, given z_0 such that $|z_0| > M$, we have N intersections between the curve $\mathcal{C}$ and the plane defined by the equation $x_1 - z_0 = 0$ (these points are $(z_0, r_{j,2}(z_0), r_{j,3}(z_0))$, $j = 1, \ldots, N$). Thus, by definition of degree of a space curve (see e.g. [23] or [36]), we get that $\deg(\mathcal{C}) \geq N$. □

In the following, let ℓ_k, $k = 1, 2$, be the first integer verifying that $N_{\ell_k,k} \leq N_k < N_{\ell_k+1,k}$. Then, we can write

$$r_2(z) = m_2 z + a_{1,2} z^{-\frac{N_{1,2}}{N_2}+1} + \cdots + a_{\ell_2,2} z^{-\frac{N_{\ell_2,2}}{N_2}+1} + a_{\ell_2+1,2} z^{-\frac{N_{\ell_2+1,2}}{N_2}+1} + \cdots$$
$$r_3(z) = m_3 z + a_{1,3} z^{-\frac{N_{1,3}}{N_3}+1} + \cdots + a_{\ell_3,3} z^{-\frac{N_{\ell_3,3}}{N_3}+1} + a_{\ell_3+1,3} z^{-\frac{N_{\ell_3+1,3}}{N_3}+1} + \cdots$$

where the exponents $-N_{j,k}/N_k + 1$ are non-negative for $j \leq \ell_k$ and negative for $j > \ell_k$.

Now, we simplify (if necessary) the non-negative exponents and rewrite the above expressions, and we get

$$\begin{aligned} r_2(z) &= m_2 z + a_{1,2} z^{-\frac{n_{1,2}}{n_2}+1} + \cdots + a_{\ell_2,2} z^{-\frac{n_{\ell_2,2}}{n_2}+1} + a_{\ell_2+1,2} z^{-\frac{N_{\ell_2+1,2}}{N_2}+1} + \cdots \\ r_3(z) &= m_3 z + a_{1,3} z^{-\frac{n_{1,3}}{n_3}+1} + \cdots + a_{\ell_3,3} z^{-\frac{n_{\ell_3,3}}{n_3}+1} + a_{\ell_3+1,3} z^{-\frac{N_{\ell_3+1,3}}{N_3}+1} + \cdots \end{aligned} \tag{2}$$

where $\gcd(n_k, n_{1,k}, \ldots, n_{\ell_k,k}) = 1$, $k = 1, 2$. Note that $0 < n_{1,k} < n_{2,k} < \cdots, n_{\ell_k,k} \leq n_k$.

Under these conditions, we introduce the definition of degree of a branch B as follows:

Definition 6. Let $B = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M\}$ defined by (2), be an infinity branch associated to $P = (1 : m_2 : m_3 : 0)$, $m_j \in \mathbb{C}$, $j = 1, 2$. We say that $n := \mathrm{lcm}(n_2, n_3)$ is the degree of B, and we denote it by $\deg(B)$.

Remark 7. Note that $n_i \leq N_i$, $i = 1, 2$. Thus, $n = \mathrm{lcm}(n_2, n_3) = \deg(B) \leq N = \mathrm{lcm}(N_2, N_3)$, and from Lemma 2 we get that $\deg(\mathcal{C}) \geq \deg(B)$.

Proposition 1. *Let $\overline{\mathcal{C}}$ be a curve that approaches $\mathcal{C}$ at its infinity branch B. It holds that* $\deg(\overline{\mathcal{C}}) \geq \deg(B)$.

Proof. From Theorem 1, we get that $\overline{\mathcal{C}}$ has an infinity branch $\overline{B} = \{(z, \overline{r}_2(z), \overline{r}_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > \overline{M}\}$ convergent with the branch $B = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M\}$. From Lemma 1, we deduce that the terms with non negative exponent in the series $r_i(z)$ and $\overline{r}_i(z)$, for $i = 2, 3$, are the same, and hence $\overline{B}$ is a branch of degree n of the form given in (2). Now, the result follows taking into account Remark 7. □

4.1. Construction of asymptotes

In this subsection, we present an algorithm that allows us to compute an asymptote for each of the infinity branches of a given implicit space curve. An example illustrating the algorithm is presented.

The algorithm is obtained from the results presented above and the construction developed throughout this subsection. In Theorem 3 we formally prove that, indeed, the construction presented leads to an asymptote of the given space curve. In addition, we show how the asymptote can be easily parametrized and in fact, we prove that this parametrization is proper. Although the results are equivalent to those presented for the plane case (see Section 3 in [3]), the proofs and detailed discussions have to be different since the tools used to deal with the space curve differ from those used in the plane case.

Let $\mathcal{C}$ be a space curve with an infinity branch $B = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\}$. Taking into account the results presented above, we have that any curve $\overline{\mathcal{C}}$ approaching $\mathcal{C}$ at B has an infinity branch $\overline{B} = \{(z, \overline{r}_2(z), \overline{r}_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > \overline{M}\}$ such that the terms with non negative exponent in $r_i(z)$ and $\overline{r}_i(z)$ (for $i = 2, 3$) are the same. We consider the series $\tilde{r}_2(z)$ and $\tilde{r}_3(z)$, obtained from $r_2(z)$ and $r_3(z)$ by removing the terms with negative exponent (see Eq. (2)). Then, we have that

$$\begin{aligned}\tilde{r}_2(z) &= m_2 z + a_{1,2} z^{-n_{1,2}/n_2+1} + \cdots + a_{\ell_2,2} z^{-n_{\ell_2,2}/n_2+1}\\ \tilde{r}_3(z) &= m_3 z + a_{1,3} z^{-n_{1,3}/n_3+1} + \cdots + a_{\ell_3,3} z^{-n_{\ell_3,3}/n_3+1}\end{aligned} \tag{3}$$

where $a_{j,k}, \ldots \in \mathbb{C} \setminus \{0\}$, $m_k \in \mathbb{C}$, $n_k, n_{j,k} \ldots \in \mathbb{N}$, $\gcd(n_k, n_{1,k}, \ldots, n_{\ell,k}) = 1$, and $0 < n_{1,k} < n_{2,k} < \cdots$. That is, $\tilde{r}_k$ has the same terms with non negative exponent as r_k, and $\tilde{r}_k$ does not have terms with negative exponent.

Let $\widetilde{\mathcal{C}}$ be the space curve containing the branch $\widetilde{B} = \{(z, \tilde{r}_2(z), \tilde{r}_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > \widetilde{M}\}$. Observe that

$$\begin{aligned}\widetilde{\mathcal{Q}}(t) = (t^n,\ & m_2 t^n + a_{1,2} t^{r_2(n_2-n_{1,2})} + \cdots + a_{\ell_2,2} t^{r_2(n_2-n_{\ell_2,2})},\\ & m_3 t^n + a_{1,3} t^{r_1(n_3-n_{1,3})} + \cdots + a_{\ell_3,3} t^{r_3(n_3-n_{\ell_3,3})}) \in \mathbb{C}[t]^3,\end{aligned} \tag{4}$$

where $n = \operatorname{lcm}(n_2, n_3)$, $r_k = n/n_k$, $n_k, n_{1,k}, \ldots, n_{\ell_k,k} \in \mathbb{N}$, $0 < n_{1,k} < n_{2,k} < \cdots n_{\ell_k,k}$ and $\gcd(n_k, n_{1,k}, \ldots, n_{\ell_k,k}) = 1$, $k = 2, 3$, is a polynomial parametrization of $\widetilde{\mathcal{C}}$. In addition, in Lemma 3, we prove that $\widetilde{\mathcal{Q}}$ is proper (i.e. invertible).

Lemma 3. *The parametrization $\widetilde{\mathcal{Q}}$ given in (4) is proper.*

Proof. Let us assume that $\widetilde{\mathcal{Q}}$ is not proper. Then, there exists $R(t) \in \mathbb{C}[t]$, with $\deg(R) = r > 1$, and $\mathcal{Q}(t) = (q_1(t), q_2(t), q_3(t)) \in \mathbb{C}[t]^3$, such that $\mathcal{Q}(R) = \widetilde{\mathcal{Q}}$ (see [37]). In particular, we get that $q_1(R(t)) = t^n$, which implies that $q_1(t) = (t-R(0))^k$, and $R(t) = t^r + R(0)$, where $rk = n$. Let us consider $R^\star(t) = R(t) - R(0) = t^r \in \mathbb{C}[t]$, and

$$\begin{aligned}\mathcal{Q}^\star(t) &= \mathcal{Q}(t + R(0)) = (t^k, q_2^\star(t), q_3^\star(t))\\ &= (t^k, c_0 + c_1 t + c_2 t^2 + \cdots + c_u t^u, d_0 + d_1 t + d_2 t^2 + \cdots + d_v t^v) \in \mathbb{C}[t]^3.\end{aligned}$$

Then, $\mathcal{Q}^\star(R^\star) = \mathcal{Q}(R) = \widetilde{\mathcal{Q}}$ and, in particular,

$$q_2^\star(R^\star) = q_2^\star(t^r) = m_2 t^n + a_{1,2} t^{r_2(n_2-n_{1,2})} + \cdots + a_{\ell_2,2} t^{r_2(n_2-n_{\ell_2,2})}.$$

That is,

$$c_0 + c_1 t^r + c_2 t^{2r} + \cdots + c_u t^{ur} = m_2 t^n + a_{1,2} t^{r_2(n_2-n_{1,2})} + \cdots + a_{\ell_2,2} t^{r_2(n_2-n_{\ell_2,2})}.$$

From this equality, and taking into account that $r_2 = n/n_2 = rk/n_2$, we deduce that $k/n_2(n_2-n_{i,2}) \in \mathbb{Z}$, and thus $kn_{i,2}/n_2 \in \mathbb{Z}$ for $i = 1, \ldots, \ell_2$. This implies that n_2 divides k since, otherwise, n_2 should divide $n_{i,2}$ for $i = 1, \ldots, \ell_2$, which contradicts the assumption that $\gcd(n_2, n_{1,2}, \ldots, n_{\ell_2,2}) = 1$ (see Eq. (4)).

On the other hand, reasoning similarly with the third component, we have that $q_3^\star(R^\star) = q_3^\star(t^r) = \tilde{q}_3(t)$ and we get that n_3 also divides k. Therefore, k is a common multiple of n_2 and n_3, which is impossible since $k < n$ (note that $rk = n$, $r > 1$) and $n = \operatorname{lcm}(n_2, n_3)$. □

From Lemma 3 and using the definition of degree for an implicitly defined space curve (see e.g. [23] or [36]), we obtain the following lemma.

Lemma 4. *Let $\widetilde{\mathcal{C}}$ be the space curve containing the infinity branch given in (3). It holds that* $\deg(\widetilde{\mathcal{C}}) = \deg(B)$.

Proof. The intersection of $\widetilde{\mathcal{C}}$ with a generic plane provides n points since $\widetilde{\mathcal{C}}$ is parametrized by the proper parametrization $\widetilde{\mathcal{Q}}$ that has degree n (see Lemma 3). In addition, we note that $n = \deg(B)$ (see Definition 6). □

In the following theorem, we prove that for any infinity branch B of a space curve $\mathcal{C}$, there always exists an asymptote that approaches $\mathcal{C}$ at B. Furthermore, we provide a method to obtain it (see algorithm Space Asymptotes Construction). The proof of this theorem is obtained from Lemmas 2 and 4, and Proposition 1. This proof is similar to the proof of Theorem 2 in [3], but for the sake of completeness, we include it.

Theorem 3. *The curve $\widetilde{\mathcal{C}}$ is an asymptote of $\mathcal{C}$ at B.*

Proof. From the construction of $\widetilde{\mathcal{C}}$, we have that $\widetilde{\mathcal{C}}$ approaches $\mathcal{C}$ at B. Thus, we need to show that $\widetilde{\mathcal{C}}$ cannot be approached by any curve with degree less than $\deg(\widetilde{\mathcal{C}})$ (that is, $\widetilde{\mathcal{C}}$ is perfect).

For this purpose, we first note that $\widetilde{\mathcal{C}}$ has a polynomial parametrization given by the form in (4). Hence, the unique infinity branch of $\widetilde{\mathcal{C}}$ is $\widetilde{B}$ (see [37]). In addition, we observe that by construction, $\widetilde{B}$ and B are convergent.

Under these conditions, we consider a space curve, $\overline{\mathcal{C}}$, that approaches $\widetilde{\mathcal{C}}$ at $\widetilde{B}$. Then, $\overline{\mathcal{C}}$ has an infinity branch $\overline{B}$ convergent with $\widetilde{B}$ (see Theorem 1). Since $\widetilde{B}$ and B are convergent, we deduce that $\overline{B}$ and B are convergent (see Corollary 1) which implies that $\overline{\mathcal{C}}$ approaches $\mathcal{C}$ at B. Finally, from Proposition 1 and Lemma 4, we deduce that $\deg(\overline{\mathcal{C}}) \geq \deg(\widetilde{\mathcal{C}})$ and thus, we conclude that $\widetilde{\mathcal{C}}$ is perfect. □

From these results, in the following we present an algorithm that computes an asymptote for each infinity branch of a given space curve.

We assume that we have prepared the input curve $\mathcal{C}$, by means of a suitable linear change of coordinates if necessary, such that $(0 : a : b : 0)$ ($a \neq 0$ or $b \neq 0$) is not a point at infinity of $\mathcal{C}$ (see Remark 3). In addition, we assume that there exists a birational correspondence between the points of $\mathcal{C}^p$ and the points of $\mathcal{C}$, where $\mathcal{C}^p$ is the plane curve obtained by projecting $\mathcal{C}$ along the x_3-axis (see Section 3).

Algorithm Space Asymptotes Construction.

Given an irreducible real algebraic space curve $\mathcal{C}$ implicitly defined by two polynomials $f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3) \in \mathbb{R}[x_1, x_2, x_3]$, the algorithm outputs an asymptote for each of its infinity branches.

1. Compute the projection of $\mathcal{C}$ along the x_3-axis. Let $\mathcal{C}_p$ be this projection and $f^p(x_1, x_2)$ the implicit polynomial defining $\mathcal{C}_p$.
2. Determine the lift function $h(x_1, x_2)$ (see Sections 2 and 3 in [5]).
3. Compute the infinity branches of $\mathcal{C}_p$ by applying the results in Section 3 in [8].
4. For each branch $B_i^p = \{(z, r_{i,2}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_{i,2}^p\}$, $i = 1, \ldots, s$, do:
   4.1. Compute the corresponding infinity branch of $\mathcal{C}$:
   $$B_i = \{(z, r_{i,2}(z), r_{i,3}(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M_i\}$$
   where $r_{i,3}(z) = h(z, r_{i,2}(z))$ is given as a Puiseux series.
   4.2. Consider the series $\tilde{r}_{i,2}(z)$ and $\tilde{r}_{i,3}(z)$ obtained by eliminating the terms with negative exponent in $r_{i,2}(z)$ and $r_{i,3}(z)$, respectively (see equation (3)).
   4.3. Return the asymptote $\widetilde{\mathcal{C}}_i$ defined by the proper parametrization (see Lemma 3), $\widetilde{Q}_i(t) = (t^{n_i}, \tilde{r}_{i,2}(t^{n_i}), \tilde{r}_{i,3}(t^{n_i})) \in \mathbb{C}[t]^3$, where $n_i = \deg(B_i)$ (see Definition 6).

Remark 8. 1. The implicit polynomial $f^p(x_1, x_2)$ defining $\mathcal{C}_p$ (see step 1) can be computed as $f^p(x_1, x_2) = \text{resultant}_{x_3}(f_1, f_2)$ (see Section 4.5 in [28]).
2. Since we have assumed that the given algebraic space curve $\mathcal{C}$ only has points at infinity of the form $(1 : m_2 : m_3 : 0)$ (see Remark 3), we have that $(0 : m : 0)$ is not a point at infinity of the plane curve $\mathcal{C}_p$. Thus, results in Section 3 in [1] can be applied.
3. Reasoning as in the correctness of the algorithm Asymptotes Construction in Section 3 in [3], one proves that the algorithm Space Asymptotes Construction outputs an asymptote $\widetilde{\mathcal{C}}$ that is independent of the leaf chosen to define the branch B (see Section 2).

In the following example, we illustrate algorithm Space Asymptotes Construction.

Example 2. Let $\mathcal{C}$ be the algebraic space curve over $\mathbb{C}$ introduced in Example 1. In Example 1, we show that $\mathcal{C}$ has two infinity branches $B_i = \{(z, r_{i,2}(z), r_{i3,}(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M_i\}$, $i = 1, 2$. These branches were obtained by applying steps 1, 2, 3, and 4.1 of Algorithm Space Asymptotes Construction. Now we apply step 4.2, and we compute the series $\tilde{r}_{i,j}(z)$ by removing the terms with negative exponent from the series $r_{i,j}(z)$, $i = 1, 2, j = 2, 3$. We get:

$$\tilde{r}_{1,2}(z) = 0, \qquad \tilde{r}_{2,2}(z) = z + \frac{\sqrt{2}z^{1/2}}{2} + \frac{1}{4},$$
$$\tilde{r}_{1,3}(z) = -\frac{1}{2}, \qquad \tilde{r}_{2,3}(z) = \frac{\sqrt{2}z^{3/2}}{2} + \frac{3z}{4} + \frac{17\sqrt{2}z^{1/2}}{32} + \frac{3}{8}.$$

Thus, in step 4.3, we obtain:

$$\widetilde{Q}_1(t) = (t, \tilde{r}_{1,2}(t), \tilde{r}_{1,3}(t)) = (t, 0, -1/2), \quad \text{and}$$
$$\widetilde{Q}_2(t) = (t^2, \tilde{r}_{2,2}(t^2), \tilde{r}_{2,3}(t^2)) = \left(t^2, t^2 + \frac{\sqrt{2}t}{2} + \frac{1}{4}, \frac{\sqrt{2}t^3}{2} + \frac{3t^2}{4} + \frac{17\sqrt{2}t}{32} + \frac{3}{8}\right).$$

$\widetilde{Q}_1$ and $\widetilde{Q}_2$ are proper parametrizations (see Lemma 3) of the asymptotes $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$, which approach $\mathcal{C}$ at its infinity branches B_1 and B_2, respectively.

In Fig. 4, we plot the curve $\mathcal{C}$ and its asymptotes $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$.

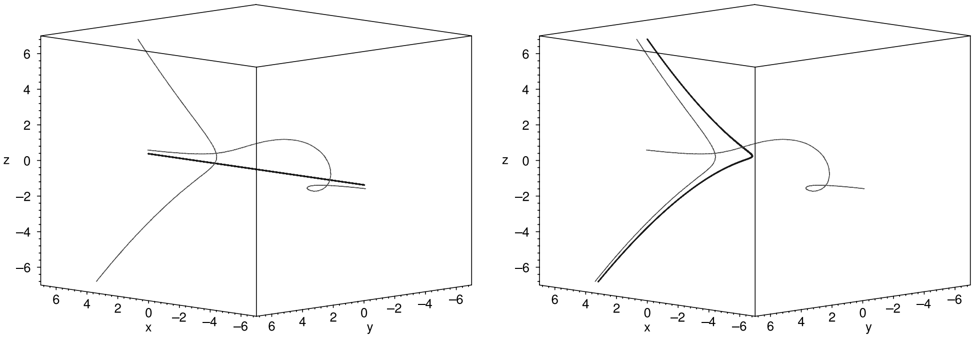

Fig. 4. Curve $\mathcal{C}$ approached by asymptotes $\widetilde{\mathcal{C}}_1$ (left) and $\widetilde{\mathcal{C}}_2$ (right).

5. Asymptotes of a parametric curve

Throughout this paper so far, we have dealt with real algebraic space curves implicitly defined by two polynomials. In this section, we present a method to compute infinity branches and asymptotes of rational curves from their parametric representation (without implicitizing).

The approach presented provides a new approach to construct generalized asymptotes by using only the parametrization. We have not been able to find a reference in the literature where this problem is solved (even for the plane case). Only some partial algorithms for parametric curves where the (linear) asymptotes are constructed can be found in classical text-books on analysis.

Thus, the contribution and advantage of the approach presented in this section is important not only because we avoid the computation of the implicit equation defining the curve but also for the important applications to the frame of the study of algebraic curves from its parametrization (see e.g. [7] or [17]).

In the following, we deal with real space curves defined parametrically. However, the method described can be trivially applied to the case of parametric real plane curves and in general, for a rational parametrization of a curve in the n-dimensional space. Similarly as in the previous sections, we work over $\mathbb{C}$, but we assume that the curve has infinitely many points in the affine plane over $\mathbb{R}$ and then, the curve has a real parametrization (see Chapter 7 in [28]).

Under these conditions, in the following, we consider a real space curve $\mathcal{C}$ defined by the parametrization

$$\mathcal{P}(s) = (p_1(s), p_2(s), p_3(s)) \in \mathbb{R}(s)^3 \setminus \mathbb{R}^3, \quad p_i(s) = p_{i1}(s)/p(s),\ i = 1, 2, 3.$$

We assume that we have prepared the input curve $\mathcal{C}$, by means of a suitable linear change of coordinates (if necessary) such that $(0 : a : b : 0)$ ($a \neq 0$ or $b \neq 0$) is not a point at infinity (see Remark 3). Note that this implies that $\deg(p_1) \geq 1$.

Observe that if $\mathcal{C}^*$ represents the projective curve associated to $\mathcal{C}$, we have that a parametrization of $\mathcal{C}^*$ is given by $\mathcal{P}^*(s) = (p_{11}(s) : p_{21}(s) : p_{31}(s) : p(s))$ or, equivalently,

$$\mathcal{P}^*(s) = \left(1 : \frac{p_{21}(s)}{p_{11}(s)} : \frac{p_{31}(s)}{p_{11}(s)} : \frac{p(s)}{p_{11}(s)}\right).$$

<u>A method to construct the asymptotes of $\mathcal{C}$.</u>

In order to compute the asymptotes of $\mathcal{C}$, first we need to determine the infinity branches of $\mathcal{C}$. That is, the sets

$$B = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C},\ |z| > M\}, \quad \text{where } r_j(z) = z\varphi_j(z^{-1}),\ j = 2, 3.$$

For this purpose, taking into account Remark 1 in Section 2, we have that

$$f_i(z : r_2(z) : r_3(z)) = F_i(1 : \varphi_2(z^{-1}) : \varphi_3(z^{-1}) : z^{-1}) = F_i(1 : \varphi_2(t) : \varphi_3(t) : t) = 0$$

around $t = 0$, where $t = z^{-1}$ and $F_i,\ i = 1, 2$ are the polynomials defining implicitly $\mathcal{C}^*$. Observe that in this section, we are given the parametrization $\mathcal{P}^*$ of $\mathcal{C}^*$ and then,

$$F_i(\mathcal{P}^*(s)) = F_i\left(1 : \frac{p_{21}(s)}{p_{11}(s)} : \frac{p_{31}(s)}{p_{11}(s)} : \frac{p(s)}{p_{11}(s)}\right) = 0.$$

Thus, intuitively speaking, in order to compute the infinity branches of $\mathcal{C}$, and in particular the series $\varphi_j,\ j = 2, 3$, one needs to rewrite the parametrization $\mathcal{P}^*(s) = \left(1 : \frac{p_{21}(s)}{p_{11}(s)} : \frac{p_{31}(s)}{p_{11}(s)} : \frac{p(s)}{p_{11}(s)}\right)$ in the form $(1 : \varphi_2(t) : \varphi_3(t) : t)$ around $t = 0$. For this purpose, the idea is to look for a value of the parameter s, say $\ell(t) \in \mathbb{C} \ll t \gg$, such that $\mathcal{P}^*(\ell(t)) = (1 : \varphi_2(t) : \varphi_3(t) : t)$ around $t = 0$.

Hence, from the above reasoning, we deduce that first, we have to consider the equation $p(s)/p_{11}(s) = t$ (or equivalently, $p(s) - tp_{11}(s) = 0$), and we solve it in the variable s around $t = 0$ (note that $\deg(p_1) \geq 1$). From Puiseux's theorem, there exist solutions $\ell_1(t), \ell_2(t), \ldots, \ell_k(t) \in \mathbb{C} \ll t \gg$ such that, $p(\ell_i(t)) - tp_{11}(\ell_i(t)) = 0,\ i = 1, \ldots, k$, in a neighborhood of $t = 0$.

Thus, for each $i = 1, \ldots, k$, there exists $M_i \in \mathbb{R}^+$ such that the points $(1 : \varphi_{i,2}(t) : \varphi_{i,3}(t) : t)$ or equivalently, the points $(t^{-1} : t^{-1}\varphi_{i,2}(t) : t^{-1}\varphi_{i,3}(t) : 1)$, where

$$\varphi_{i,j}(t) = \frac{p_{j,1}(\ell_i(t))}{p_{11}(\ell_i(t))}, \quad j = 2, 3, \tag{5}$$

are in $\mathcal{C}^*$ for $|t| < M_i$ (note that $\mathcal{P}^*(\ell(t)) \in \mathcal{C}^*$ since $\mathcal{P}^*$ is a parametrization of $\mathcal{C}^*$). Observe that $\varphi_{i,j}(t)$, $j = 2, 3$ are Puiseux series, since $p_{j,1}(\ell_i(t))$, $j = 2, 3$ and $p_{11}(\ell_i(t))$ can be written as Puiseux series and $\mathbb{C} \ll t \gg$ is a field.

Finally, we set $z = t^{-1}$. Then, we have that the points $(z, r_{i,2}(z), r_{i,3}(z))$, where $r_{i,j}(z) = z\varphi_{i,j}(z^{-1})$, $j = 2, 3$, are in $\mathcal{C}$ for $|z| > M_i^{-1}$. Hence, the infinity branches of $\mathcal{C}$ are the sets

$$B_i = \{(z, r_{i,2}(z), r_{i,3}(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, \ |z| > M_i^{-1}\}, \quad i = 1, \ldots, k.$$

Remark 9. Note that the series $\ell_i(t)$ satisfies that $p(\ell_i(t))/p_{11}(\ell_i(t)) = t$, for $i = 1, \ldots, k$. Then, from equality (5), we have that for $j = 2, 3$

$$\varphi_{i,j}(t) = \frac{p_{j,1}(\ell_i(t))}{p(\ell_i(t))} t = p_j(\ell_i(t))t, \quad \text{and} \quad r_{i,j}(z) = z\varphi_{i,j}(z^{-1}) = p_j(\ell_i(z^{-1})).$$

Once we have the infinity branches, we can compute an asymptote for each of them by simply removing the terms with negative exponent from $r_{i,2}$ and $r_{i,3}$ (see Section 4.1).

The following algorithm computes the infinity branches of a given parametric space curve and provides an asymptote for each of them. We remind that the input curve $\mathcal{C}$ is prepared such that $(0 : a : b : 0)$ ($a \neq 0$ or $b \neq 0$) is not a point at infinity of $\mathcal{C}^*$ (see Remark 3).

Algorithm Space Asymptotes Construction-Parametric Case.

Given a rational irreducible real algebraic space curve $\mathcal{C}$ defined by a parametrization $\mathcal{P}(s) = (p_1(s), p_2(s), p_3(s)) \in \mathbb{R}(s)^3$, $p_j(s) = p_{j1}(s)/p(s)$, $j = 1, 2, 3$, the algorithm outputs one asymptote for each of its infinity branches.

1. Compute the Puiseux solutions of $p(s) - tp_{11}(s) = 0$ around $s = 0$. Let them be $\ell_1(t), \ell_2(t), \ldots, \ell_k(t) \in \mathbb{C} \ll t \gg$.
2. For each $\ell_i(t) \in \mathbb{C} \ll t \gg$, $i = 1, \ldots, k$, do:
   2.1. Compute the corresponding infinity branch of $\mathcal{C}$:
   $$B_i = \{(z, r_{i,2}(z), r_{i,3}(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, \ |z| > M_i\}, \quad \text{where}$$
   $r_{i,j}(z) = p_j(\ell_i(z^{-1}))$, $j = 2, 3$ is given as Puiseux series (see Remark 9).
   2.2. Consider the series $\tilde{r}_{i,2}(z)$ and $\tilde{r}_{i,3}(z)$ obtained by eliminating the terms with negative exponent in $r_{i,2}(z)$ and $r_{i,3}(z)$, respectively (see equation (3) in Subsection 4.1).
   2.3. Return the asymptote $\widetilde{\mathcal{C}}_i$ defined by the proper parametrization (see Lemma 3), $\widetilde{Q}_i(t) = (t^{n_i}, \ \tilde{r}_{i,2}(t^{n_i}), \ \tilde{r}_{i,3}(t^{n_i})) \in \mathbb{C}[t]^3$, where $n_i = \deg(B_i)$ (see Definition 6).

Remark 10. We note that:

1. In step 1 of the algorithm, some of the solutions $\ell_1(t), \ell_2(t), \ldots, \ell_k(t) \in \mathbb{C} \ll t \gg$ might belong to the same conjugacy class. Thus, we only consider one solution for each of these classes.
2. Reasoning as in statement 3 in Remark 8, one also gets that the algorithm Space Asymptotes Construction-Parametric Case outputs an asymptote $\widetilde{\mathcal{C}}$ that is independent of the solutions $\ell_1(t), \ell_2(t), \ldots, \ell_k(t) \in \mathbb{C} \ll t \gg$ chosen in step 1 (see statement 1 above), and of the leaf chosen to define the branch B.

In the following example, we study a parametric space curve with only one infinity branch. We use algorithm Space Asymptotes Construction-Parametric Case to obtain the branch and compute an asymptote for it.

Example 3. Let $\mathcal{C}$ be the space curve defined by the parametrization

$$\mathcal{P}(s) = \left(\frac{-1+s^2}{s^3}, \frac{-1+s^2}{s^2}, \frac{1}{s}\right) \in \mathbb{R}(s)^3.$$

Step 1: We compute the solutions of the equation

$$p(s) - tp_{11}(s) = s^3 - t(-1+s^2) = s^3 - ts^2 + t = 0$$

around $t = 0$. For this purpose, we may use, for instance, the command puiseux included in the package algcurves of the computer algebra system Maple. There is only one solution that is given by the Puiseux series (see Proposition 2)

$$\ell(t) = (-t)^{1/3} + 1/3t + 1/9(-t)^{5/3} - 2/81(-t)^{7/3} + 2/729(-t)^{11/3} + \cdots$$

(note that $\ell(t)$ represents a conjugacy class composed of three conjugated series; one of them is real and the other two are complex).

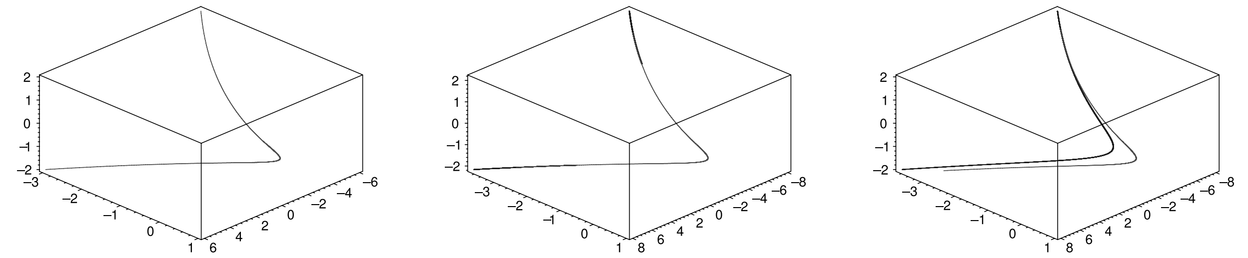

Fig. 5. Curve $\mathcal{C}$ (left), infinity branch B (center) and asymptote $\widetilde{\mathcal{C}}$ (right).

Step 2:

Step 2.1: We compute (see Proposition 2)

$$r_2(z) = p_2(\ell(z^{-1})) = -z^{2/3} + 1/3 - 1/9z^{-2/3} + 2/81z^{-4/3} - 2/729z^{-8/3} + \cdots$$
$$r_3(z) = p_3(\ell(z^{-1})) = -z^{1/3} - 1/3z^{-1/3} + 1/81z^{-5/3} - 1/243z^{-7/3} + \cdots$$

(we may use, for instance, the command series included in the computer algebra system Maple). The curve has only one infinity branch given by

$$B = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\}$$

for some $M \in \mathbb{R}^+$ (note that this branch has three leaves; one of them is real and the other two are complex).

Step 2.2: We obtain $\tilde{r}_2(z)$ and $\tilde{r}_3(z)$ by eliminating the terms with negative exponent in $r_2(z)$ and $r_3(z)$ respectively:

$$\tilde{r}_2(z) = -z^{2/3} + 1/3 \quad \text{and} \quad \tilde{r}_3(z) = -z^{1/3}.$$

Step 2.3: The input curve $\mathcal{C}$ has an asymptote $\widetilde{\mathcal{C}}$ at B that can be polynomially parametrized by:

$$\widetilde{Q}(t) = (t^3, \tilde{r}_2(t^3), \tilde{r}_3(t^3)) = (t^3, -t^2 + 1/3, -t).$$

In Fig. 5, we plot the curve $\mathcal{C}$, the infinity branch B, and the asymptote $\widetilde{\mathcal{C}}$.

Correctness.

The application of the algorithm Space Asymptotes Construction-Parametric Case presents some technical difficulties since infinite series are involved. In particular, when we compute the series ℓ_i in step 1, we cannot handle its infinite terms so it must be truncated, which may distort the computation of the series $r_{i,j}$ in step 2. However, this distortion may not affect to all the terms in $r_{i,j}$. In fact, the number of affected terms depends on the number of terms considered in ℓ_i. Nevertheless, note that we do not need to know the full expression of $r_{i,j}$ but only the terms with non negative exponent. Proposition 2 states that the terms with non negative exponent in $r_{i,j}$ can be obtained from a finite number of terms considered in ℓ_i. In fact, it provides a lower bound for the number of terms needed in ℓ_i.

Proposition 2. *Let $\ell(z) \in \mathbb{C} \ll z \gg$ be a solution obtained in step 1 of the algorithm Space Asymptotes Construction-Parametric Case. Let $B = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\}$, $r_j(z) = p_j(\ell(z^{-1}))$, $j = 2, 3$, be the infinity branch of $\mathcal{C}$ obtained in step 2.1 of the algorithm Space Asymptotes Construction-Parametric Case. It holds that the terms with non negative exponent in r_2 and r_3 can be obtained from the computation of $2\deg(p_1) + 1$ terms of ℓ.*

Proof. We prove the proposition for r_2 (similarly, one gets the result for r_3). For this purpose, we write $\ell(z)$ as

$$\ell(z) := b_0 + b_1 z^{-1/N} + \cdots + b_k z^{-k/N} + B(z), \quad B(z) = \sum_{j=1}^{\infty} a_j z^{j/N}, \ N \in \mathbb{N}^+,$$

$a_i, b_i \in \mathbb{C}$, and we consider $\ell^*(z) := \ell(z^N) = \nu/z^k$ where

$$\nu := b_0 z^k + b_1 z^{k-1} + \cdots + b_{k-1} z + b_k + z^k B(z^N), \quad B(z^N) = \sum_{j=1}^{\infty} a_j z^j.$$

Note that the terms with non negative exponent in $r_2(z)$ are the terms with non positive exponent in $r_2(1/z)$. In addition, these terms are the terms with non positive exponent in $r_2(1/z^N)$. On the other hand, $r_2(z) = p_2(\ell(z^{-1}))$ so $r_2(1/z^N) = p_2(\ell^*(z))$. Therefore, we need to determine the terms with non positive exponent in $p_2(\ell^*(z))$.

Now, we distinguish two different cases:

1. Let us assume that $\ell(z)$ has terms with negative exponent and thus, we assume w.l.o.g. that $b_k \neq 0,\ k > 0$. Thus,

$$p_2(\ell^*(z)) = \frac{p_{2,1}(\nu/z^k)}{p(\nu/z^k)} = \frac{\bar{p}_{2,1}(z)}{z^{k(m-n)}\bar{p}(z)}, \quad m := \deg(p_{2,1}),\ n := \deg(p),$$
$$\bar{p}_{2,1}(z) = c_m\nu^m + c_{m-1}z^k\nu^{m-1} + c_{m-2}z^{2k}\nu^{m-2} + \cdots + c_0 z^{km}, \quad c_m \neq 0$$
$$\bar{p}(z) = d_n\nu^n + d_{n-1}z^k\nu^{n-1} + d_{n-2}z^{2k}\nu^{n-2} + \cdots + d_0 z^{kn}, \quad d_n \neq 0.$$

Under these conditions, the generalized series expansion of $p_2(\ell^*(z))$ around $z = 0$ is given by $\frac{\bar{p}_{2,1}(z)}{z^{k(m-n)}} G(z)$, where $G(z)$ is the Taylor series of $1/\bar{p}(z)$ at $z = 0$. Observe that $G(z)$ exists since all the derivatives of $1/\bar{p}(z)$ at $z = 0$ exist (note that the denominator of all the derivatives is a power of the polynomial $\bar{p}(z)$, and $\bar{p}(0) = d_n\nu(0)^n = d_n b_k^n \neq 0$). In addition, taking into account that

$$\nu^{(j)}(0) = b_{k-j}, \quad 0 \le j \le k, \quad \text{and} \quad \nu^{(j)}(0) = a_{j-k}, \quad j \ge k+1,$$

and that $\frac{\partial^j(1/\bar{p}(z))}{\partial z^j}_{|z=0}$ is obtained from $\nu^{(i)}(0),\ 0 \le i \le j$, we get that

$$G(z) = \frac{1}{\bar{p}(0)} + z\frac{\partial(1/\bar{p}(z))}{\partial z}_{|z=0} + \cdots = h_0(b_k) + \cdots + z^k h_k(b_k,\ldots,b_0)$$
$$+ z^{k+1}h_{k+1}(b_k,\ldots,b_0,a_1) + \cdots + z^{k+u}h_{k+u}(b_k,\ldots,b_0,a_1,\ldots,a_u) + \cdots,$$

where $h_j(b_k,\ldots,b_0,a_1,\ldots,a_{j-k}),\ j \ge 0$, denotes a rational function depending on $b_k,\ldots,b_0,a_1,\ldots,a_{j-k}$.

As we stated above, we need to determine the terms with non positive exponent in

$$p_2(\ell^*(z)) = \frac{\bar{p}_{2,1}(z)}{z^{k(m-n)}} G(z).$$

In the following, we prove that they can be obtained by just computing $b_k,\ldots,b_0,a_1,\ldots,a_{km}$. Indeed:

1.1. Let $m = n$. Then, we need to compute the terms with non positive exponent in
$$\bar{p}_{2,1}(z)G(z) = (c_m\nu^m + c_{m-1}z^k\nu^{m-1} + c_{m-2}z^{2k}\nu^{m-2} + \cdots + c_0z^{km})$$
$$(h_0(b_k) + \cdots + z^k h_k(b_k,\ldots,b_0) + z^{k+1}h_{k+1}(b_k,\ldots,b_0,a_1) + \cdots).$$
Thus, we only need the independent term $c_m b_k^m h_0(b_k)$.

1.2. Let $m < n$. In this case, we need to determine the terms with non positive exponent in $z^{k(n-m)}\bar{p}_{2,1}(z)G(z)$. However, since $n - m > 0$, we conclude that there are no such terms.

1.3. Let $m > n$. Then, we need to compute the terms with non positive exponent in $\bar{p}_{2,1}G/z^{k(m-n)}$ which implies that we need to determine the terms having degree less or equal to $k(m-n)$ in the product $\bar{p}_{2,1}(z)G(z)$. Those terms are included in the product
$$(c_m(b_0z^k + b_1z^{k-1} + \cdots + b_{k-1}z + b_k)^m + c_{m-1}z^k(b_0z^k + b_1z^{k-1} + \cdots + b_{k-1}z + b_k)^{m-1} + \cdots + c_0z^{km})$$
$$\cdot (h_0(b_k) + \cdots + z^k h_k(b_k,\ldots,b_0) + z^{k+1}h_{k+1}(b_k,\ldots,b_0,a_1) + \cdots$$
$$+ z^{k(m-n)}h_{k(m-n)}(b_k,\ldots,b_0,a_1,\ldots,a_{k(m-n)}))$$
(we do not include the term $z^kB(z^N)$ in this product since after multiplying, it only provides terms of degree greater than km). Therefore, at most we have to compute $\ell(z)$ till the terms $b_k,\ldots,b_0,a_1,\ldots,a_{k(m-n)}$ appear. That is, $k+1+k(m-n)$ terms are needed.

Taking into account the cases 1.1–1.3, we deduce that at most we have to compute $k+1+k(m-n)$ terms in $\ell(z)$. Finally, we prove that $k+1+k(m-n) \le 2\deg(p_1)+1$. For this purpose, let $d(r_2)$ denote the maximum exponent of z in $r_2(z)$. We observe that $d(r_2) \le 1$; otherwise, since

$$F(z : r_2(z) : r_3(z) : 1) = F(z/r_2(z) : 1 : r_3(z)/r_2(z) : 1/r_2(z)) = 0$$

(for $|z| > M$) by continuity, we get

$$\lim_{z\to\infty} F(z/r_2(z) : 1 : r_3(z)/r_2(z) : 1/r_2(z)) = F(0 : 1 : C : 0) = 0$$

where $C := \lim_{z\to\infty} r_3(z)/r_2(z)$. If $C \in \mathbb{C}$, we get that $(0 : 1 : C : 0)$ is a point at infinity of the input curve which is impossible since we have assumed that the input curve does not have points at infinity of the form $(0 : a : b : 0)$. If $C = \infty$, we reason as above but we divide by $r_3(z)$. In this case, we get the point at infinity $(0 : 0 : 1 : 0)$ which is again impossible.

On the other hand, since $r_2(z) = p_2(\ell(z^{-1})) = \frac{p_{21}(\ell(z^{-1}))}{p(\ell(z^{-1}))}$, we get that $d(r_2) = (m-n)k/N$, where $m = \deg(p_{21})$ and $n = \deg(p)$ (see Chapter 4 in [30]). Hence, $(m-n)k/N \le 1$ which implies that $(m-n)k \le N$. In addition, since $N \le \deg_s(p(s) - tp_{11}(s)) = \deg(p_1)$ (see Remark 4 in [3]), we get that $k+1+k(m-n) \le 2k(m-n)+1 \le 2\deg(p_1)+1$.

2. Let us assume that $b_k = 0$ for $k > 0$. That is, there are no terms with negative exponent in $\ell(z)$. Then, we write $\ell(z) := b_0 + B(z)$, where

$$B(z) = \sum_{j=1}^{\infty} a_j z^{q_j/N}, \quad N \in \mathbb{N}^+,\ q_j \in \mathbb{N}^+,\ 0 < q_1 < q_2 < \cdots,\ a_j \in \mathbb{C}\setminus\{0\},$$

and

$$\ell^*(z) := \ell(z^N) = b_0 + B(z^N) = b_0 + z^{q_1}\left(a_1 + \sum_{j=2}^{\infty} a_j z^{q_j - q_1}\right), \quad B(z^N) = \sum_{j=1}^{\infty} a_j z^{q_j}.$$

In this case, we denote $\nu := b_0 + z^{q_1}(a_1 + \sum_{j=2}^{\infty} a_j z^{q_j - q_1})$. In addition, we write

$$p(t) = p^*(t)(t - b_0)^r, \qquad \gcd(p^*(t), t - b_0) = 1 \quad \text{for some } r \in \mathbb{N}.$$

Under these conditions, we get that $p_2(\ell^*(z)) =$

$$\frac{p_{2,1}(\nu)}{p(\nu)} = \frac{p_{2,1}(\nu)}{p^*(\nu)(\nu - b_0)^r} = \frac{p_{2,1}(\nu)}{z^{rq_1} p^*(\nu)\left(a_1 + \sum\limits_{j=2}^{\infty} a_j z^{q_j - q_1}\right)^r} := \frac{\bar{p}_{2,1}(z)}{z^{rq_1}\bar{p}(z)},$$

where

$$\bar{p}_{2,1}(z) = p_{2,1}(\nu) = c_m \nu^m + c_{m-1}\nu^{m-1} + \cdots + c_0,\ c_m \neq 0,\ m = \deg(p_{2,1})$$

and $\bar{p}(z) = p^*(\nu)(a_1 + \sum_{j=2}^{\infty} a_j z^{q_j - q_1})^r =$

$$(d_n \nu^n + d_{n-1}\nu^{n-1} + \cdots + d_0)\left(a_1 + \sum_{j=2}^{\infty} a_j z^{q_j - q_1}\right)^r, \quad d_n \neq 0,\ n := \deg(p^*).$$

The generalized series expansion of $p_2(\ell^*(z))$ around $z = 0$ is given by $\frac{\bar{p}_{2,1}(z)}{z^{rq_1}} G(z)$, where $G(z)$ is the Taylor series of $1/\bar{p}(z)$ at $z = 0$. Observe that $G(z)$ exists since all the derivatives of $1/\bar{p}(z)$ at $z = 0$ exist (note that the denominator of all the derivatives is a power of the polynomial $\bar{p}(z)$, and $\bar{p}(0) = p^*(\nu(0))a_1 = p^*(b_0)a_1 \neq 0$). Reasoning as in case 1, one may check that $G(z) = \frac{1}{\bar{p}(0)} + z\frac{\partial(1/\bar{p}(z))}{\partial z}_{|z=0} + \cdots =$

$$= h_0(b_0, a_1) + zh_1(b_0, a_1, a_2) + \cdots + z^k h_k(b_0, a_1, \ldots, a_{k+1}) + \cdots,$$

where $h_j(b_0, a_1, \ldots, a_{j+1}),\ j \geq 0$ is a rational function depending on $b_0, a_1, \ldots, a_{j+1}$.

Since we need to compute the terms with non positive exponent in

$$p_2(\ell^*(z)) = \frac{\bar{p}_{2,1}(z)}{z^{rq_1}} G(z),$$

we reason as in case 1.1 (if $r = 0$), or case 1.3 (if $r > 0$), and we conclude that at most, we have to determine $\ell(z)$ till the terms $b_0, a_1, \ldots, a_{rq_1+1}$ appear. That is, in this case, at most $rq_1 + 2$ terms are needed. Finally, we prove that $rq_1 + 2 \leq 2\deg(p_1) + 1$. For this purpose, we reason as above and since

$$r_2(z) = \frac{p_{21}(\ell(z^{-1}))}{p(\ell(z^{-1}))} = \frac{p_{21}(\ell(z^{-1}))}{\left(\sum\limits_{j=1}^{\infty} a_j z^{-q_j/N}\right)^r p^*(\ell(z^{-1}))},$$

and $\lim_{z\to\infty} p_{21}(\ell(z^{-1}))/p^*(\ell(z^{-1})) = p_{21}(b_0)/p^*(b_0) \in \mathbb{C}$ (and thus, $d(p_{21}(\ell(z^{-1}))) = d(p^*(\ell(z^{-1})))$, we get that $d(r_2) = rq_1/N$ (see Chapter 4 in [30])). Since $d(r_2) \leq 1$, we deduce that $rq_1 \leq N$. In addition, since $N \leq \deg(p_1)$ (see Remark 4 in [3]), we get that $rq_1 \leq \deg(p_1)$, and thus $rq_1 + 2 \leq \deg(p_1) + 2 \leq 2\deg(p_1) + 1$. □

6. Conclusion

In this paper, we present some important tools that will allow us to analyze the behavior at infinity of a real algebraic space curve implicitly or parametrically defined. In particular, we introduce the notions of *infinity branches* and *generalized asymptotes*, we study some properties, and we present algorithms where we show how to compute them. These notions were already introduced for an implicit real algebraic plane curve (see [1,3]) but the treatment in the space case has to be necessarily different. More precisely, in this paper, the following results and methods are obtained:

1. Some important previous notions and results are presented for a given algebraic space curve. In particular, the concepts of infinity branch and approaching curves are defined. These concepts are a straightforward generalization from the notions introduced in the case of plane curves (see Sections 3 and 4 in [1]).
2. A method for computing infinity branches in the space is presented. For this purpose, we reduce the problem from the space to the plane where effective and simply computations can be applied (see Sections 3 and 4 in [1]).
3. The computation of asymptotes of an implicitly defined space curve is presented. We reach the expected situation, that is, the computation is similar to the case of plane curves (see Section 3 in [3]). However, the construction and formalization of the results use approaches totally new since the computational techniques and tools in the space are necessarily different to those we have in the plane.

4. Finally, we show how to compute asymptotes for a given space curve parametrically defined. This approach developed for the parametric case is new and it can be easily applied to any algebraic curve in the n-dimensional space.

The results obtained in this paper open new ways to explore the algebraic space curves (and in particular, its behavior at infinity), with expected generalizations to surfaces. As a matter of future research, we plan to extend the results of this paper to surfaces.

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