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(Article begins on next page)

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Perturbation of polynomials and applications to the Hough transform *

M. Torrente, M.C. Beltrametti and J.R. Sendra

Abstract

Let f and g be complex polynomials of the same degree. We provide a new lower bound on the Euclidean distance of points belonging to their zero-loci in terms of Bombieri's norm. We also present a minimization of the Bombieri's norm of the difference $g - \lambda f$, for $\lambda \in \mathbb{C}^*$. In the real case, we apply the results above in the setting of the Hough transform, a standard technique to detect curves in images, suggesting a Bombieri's norm based recognition algorithm.

Introduction

Let's start briefly describing the applied idea from which the paper grew up. Image analysis nowadays can be performed by a validated method (see [2, 14]) based on a generalization of the Hough transform technique, a standard pattern recognition method to detect curves in images. According to the original definition [7], given a point $p = (x_p, y_p)$ in the image plane $\mathbb{A}^2_{(x,y)}(\mathbb{R})$ satisfying the equation of a straight line

$$y = ax + b, \tag{1}$$

with a, b independent real parameters, the Hough transform of p, with respect to the family of lines $\mathcal{F} = \{\ell_{a,b} : y = ax + b\}$, is the straight line

$$y_p = Ax_p + B, \tag{2}$$

in the parameter plane $\mathbb{A}^2_{(A,B)}(\mathbb{R})$. The usual point-line duality of projective plane implies that all points of the line (1) have as Hough transforms lines in the parameter plane that all intersect in the unique point (a, b) uniquely identifying the original straight line. This duality correspondence between the image and the parameters planes holds not only for the family of straight lines, but also for several families of algebraic curves (see equation (11)).

A key result (see [2, Lemma 2.3] and also [15, Section 2]) characterizes those families $\mathcal{F}$ for which the following general Hough-type correspondence between the image space and the parameter space holds true: each point p of a curve from the family in the image space is transformed into a curve $\Gamma_p(\mathcal{F})$ in the parameter space in such a way that all curves $\Gamma_p(\mathcal{F})$ meet in one and only one point uniquely identifying the original curve in the image space.

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Such a key result in [2] theoretically supports the formulation of a novel, rather general and effective pattern recognition algorithm. The basic ingredient of this algorithm is the construction of a catalogue of families of algebraic plane curves satisfying the above conditions. A pre-processing step of the algorithm consists of the application of a standard edge detection technique on the image. This step reduces the number of points of which one has to compute the Hough transform and, furthermore, eliminates the degree of freedom represented by the grey level in the image out of the game of the recognition task. Then a discretization of the parameter space is required, which possibly exploits bounds on the parameter values computed by utilizing either the cartesian or the parametric form of the curve in the image space. A last step constructs the accumulator function in the discretized parameter space, such that the value of the accumulator in a cell of such space corresponds to the number of the Hough transforms of the selected points crossing that cell. As a final outcome of the algorithm, the parameter values characterizing the curve, to be detected in the image space, corresponds to the parameter values identifying the cell where the accumulator function reaches its maximum.

Practical experimentation yields the following question, somehow underlying the whole recognition process.

> In the Hough transform technique, taking different values of the discretization step, say δ, may imply very different answers. Are there methods to estimate or bound the quantity δ with respect to a given context?

According to our knowledge, there are no general results in this direction, the choice of δ being done by ad hoc arguments depending on geometrical properties of the curves from the particular family one deals with. A novelty of the present paper is to address the above question from an algebraic geometric method based on perturbation results on the zero-locus of a polynomial and the behaviour of Bombieri's norm, ideally following results and methods as in [20] and [21].

The paper is organized as follows. Let $P = \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring over the complex field. In Section 2, for polynomials $f, g \in P$, with the same degree, we address the question to determine the value of $\lambda \in \mathbb{C} \setminus \{0\}$ which minimizes the Bombieri norm $\|g - \lambda f\|_{(\deg(g-\lambda f))}$ (see Theorem 2.1, a main result of the paper). This question naturally comes from the results of [21] and, specifically, allows us to refine the bound for the Euclidean distance $\|q - p\|_2$ where $p \in \mathbb{A}^n_{\mathbb{C}}$ is a real point of $f = 0$ and $q \in \mathbb{A}^n_{\mathbb{C}}$ is a point belonging to $g = 0$. In this way, we improve [21, Theorem 3.4] for $f, g \in P$, restating as well the result in the real case (see Theorem 2.2).

Section 3 is devoted to state perturbation results on the zero-locus of a polynomial, which can be considered as reverse type results of [21] (see, in particular, Theorem 4.1 loc. cit.) More precisely, we provide a new lower bound on the Euclidean distance of points belonging to the zero-loci of complex multivariate degree d polynomials f and g in terms of Bombieri's norm $\|g - f\|_{(d')}$, with $d' = \deg(g - f) \leq d$.

In Section 4 we recall the Hough transform technique elements we need to suggest an application of our Bombieri's norm based result to pattern recognition setting, with particular regard to the Hough-type duality correspondence between the image space and the parameter space.

In Section 5 we specialize [21, Lemma 3.3 and Theorem 3.4] (Theorem 1.4 in the paper) to the case when the polynomials f, g are the defining polynomials of the Hough transforms of two (general) points of the image space. This allows us to propose a heuristic suggestion to lower bound the discretization step δ, as described in Algorithm 5.12.

In Section 6 we describe a Bombieri's norm based recognition algorithm, providing examples which validate our results, also using the minimization results from Section 2 as well as the lower bounds for the discretization step suggested in Section 5.

We would like to thank the anonymous referee for helpful comments.

1 Background material

We follow standard notation in algebraic geometry, in particular, we keep those as in [19, 20, 21]. Let $P = \mathbb{C}[x_1, \dots, x_n]$ be the polynomial ring over the complex field. First, recall that, for polynomials f, g in P, the *inner product* $\langle f, g\rangle$ is the scalar product $\langle f, g\rangle = \mathbf{v}_f \cdot \mathbf{v}_g$, where $\mathbf{v}_f$, $\mathbf{v}_g$ are the coefficient vectors of f, g, respectively. We also use the 2-norm of a polynomial f, defined as $\|f\|_2 := \|\mathbf{v}_f\|_2$. Bombieri–Weyl's norm on the space of polynomials is defined as follows (see [1]).

Definition 1.1 Let $F = \sum_{|\boldsymbol{\alpha}|=d} c_{\boldsymbol{\alpha}} x_0^{\alpha_0} \dots x_n^{\alpha_n}$ and $G = \sum_{|\boldsymbol{\alpha}|=d} c'_{\boldsymbol{\alpha}} x_0^{\alpha_0} \dots x_n^{\alpha_n}$ be two homogeneous polynomials of $\overline{P}$ of degree d. Then the *Bombieri scalar product* of F and G is defined as

$$(F, G)_{(d)} = \sum_{|\boldsymbol{\alpha}|=d} \frac{\alpha_0! \dots \alpha_n!}{d!} c_{\boldsymbol{\alpha}} \overline{c'_{\boldsymbol{\alpha}}},$$

where "$\overline{\cdot}$" denotes the conjugate of complex numbers.

Such a scalar product induces an inner product on the linear space of all the degree d homogeneous polynomials of $\overline{P}$. We can then consider the canonically associated *Bombieri's norm*, defined as

$$\|F\|_{(d)} = \left(\sum_{|\boldsymbol{\alpha}|=d} \frac{\alpha_0! \dots \alpha_n!}{d!} |c_{\boldsymbol{\alpha}}|^2 \right)^{1/2}.$$

Moreover, for degree d polynomials f and g of P, the *Bombieri scalar product* of f and g is defined as Bombieri's scalar product of the homogenization of f and g, that is,

$$(f, g)_{(d)} = (f^{\text{hom}}, g^{\text{hom}})_{(d)}.$$

And the *Bombieri norm* of f is defined as the Bombieri norm of the homogenization of f, that is,

$$\|f\|_{(d)} = \|f^{\text{hom}}\|_{(d)}.$$

Let us stress the fact that in Bombieri's scalar product the polynomials must be of the same degree d (or considered so) and that the scalar product depends on d.

Let us recall the definition of some norms we need. For m, n positive integers, we let $\mathrm{Mat}_{m\times n}(\mathbb{R})$ be the set of $m \times n$ matrices with entries in $\mathbb{R}$. For any $M \in \mathrm{Mat}_{m\times n}(\mathbb{R})$, we will denote by M^T its transpose.

Definition 1.2 Let v be an element of $\mathrm{Mat}_{n\times 1}(\mathbb{R})$ and let $r \geq 1$ be a real number. Set $v^T := (v_1, \dots, v_n)$. The *r-norm*[1] of v is defined by

$$\|v\|_r := \left(\sum_{i=1}^{n} |v_i|^r\right)^{\frac{1}{r}}.$$

In particular, if $r = 1$, we get the expression $\|v\|_1 = \sum_{i=1}^{n} |v_i|$. If $r = 2$ we get the well-known *Euclidean norm* $\|v\|_2 = \left(\sum_{i=1}^{n} |v_i|^2\right)^{1/2}$. While, if $r \to \infty$, the r-norm approaches the *∞-norm* defined by $\|v\|_\infty := \max_{i=1,\dots,n} \{|v_i|\}$.

Definition 1.3 Let $M = (m_{ij})$ be a matrix in $\mathrm{Mat}_{m\times n}(\mathbb{R})$. The *$r$-matrix norm* is the norm on $\mathrm{Mat}_{m\times n}(\mathbb{R})$ induced by the r-norm on $\mathrm{Mat}_{n\times 1}(\mathbb{R})$, and defined by the formula

$$\|M\|_r := \max_{\|v\|_r = 1} \|Mv\|_r,$$

where $v \in \mathrm{Mat}_{n\times 1}(\mathbb{R})$. In particular, one has $\|M\|_1 = \max_{j=1,\dots,n} \left\{ \sum_{i=1,\dots,m} |m_{ij}| \right\}$ for $r = 1$. If $r = 2$, denoting by $\lambda_j(\cdot)$ the j-th eigenvalue, we have $\|M\|_2 = \left(\max_{j=1,\dots,n} \lambda_j(M^T M)\right)^{1/2}$. Moreover, for any vector $w \in \mathrm{Mat}_{1\times n}(\mathbb{R})$, one has

$$\|w\|_1 = \|w^T\|_\infty.$$

The following result is somehow the starting point of the paper (see [21, Lemma 3.3. and Theorem 3.4]).

Theorem 1.4 *Let f and g be polynomials of P of degree d and let $d' = \deg(g - f) \leq d$. Let $p \in \mathbb{A}^n_{\mathbb{C}}$ be a real point of $f = 0$ of multiplicity $s \geq 1$ such that $\frac{\partial^s g}{\partial x_i^s}(p) \neq 0$ for some index i. Then there exists a point $q \in \mathbb{A}^n_{\mathbb{C}}$ belonging to $g = 0$ such that:*

1. $$\|q - p\|_2 \leq \left(\frac{d!}{(d-s)!} \frac{(1 + \|p\|_2^2)^{d/2}}{\left\| \left(\frac{\partial^s g}{\partial x_1^s}(p), \dots, \frac{\partial^s g}{\partial x_n^s}(p)\right) \right\|_1} \right)^{1/s} \|g - f\|_{(d')}^{1/s}.$$

2. *If $\|g - f\|_{(d')}$ is small enough, namely* $\|g - f\|_{(d')} \leq \dfrac{(d-s)!}{2d!} \dfrac{\left\| \left(\frac{\partial^s f}{\partial x_1^s}(p), \dots, \frac{\partial^s f}{\partial x_n^s}(p)\right) \right\|_1}{(1 + \|p\|_2^2)^{\frac{d-s}{2}}}$, *then*

$$\|q - p\|_2 \leq \left(\frac{2d!}{(d-s)!} \frac{(1 + \|p\|_2^2)^{d/2}}{\left\| \left(\frac{\partial^s f}{\partial x_1^s}(p), \dots, \frac{\partial^s f}{\partial x_n^s}(p)\right) \right\|_1} \right)^{1/s} \|g - f\|_{(d')}^{1/s}.$$

We also use several times the following results.

I. *Bombieri's scalar product property*, [21, Lemma 2.5]. Let $f \in P$ be a polynomial of degree d and let $p = (p_1, \dots, p_n) \in \mathbb{A}^n_{\mathbb{C}}$. Then

$$f(p) = (f, (1 + \overline{p_1} x_1 + \dots + \overline{p_n} x_n)^d)_{(d)}. \tag{3}$$

To simplify the notation, for $p = (p_1, \dots, p_n) \in \mathbb{A}^n_{\mathbb{C}}$, we will denote by h_p the polynomial $(1 + \overline{p_1} x_1 + \dots + \overline{p_n} x_n)^d$. We recall that $\|h_p\|_{(d)} = (1 + \|p\|_2^2)^{\frac{d}{2}}$.

[1] We will only use matrix norms; however, let us mention that in the literature one also refers to this norm as the "r-norm of the vector $(v_1, \dots, v_n)$ in $\mathbb{R}^n$".

II. *Equivalence of norms*, [1]. Let $f \in P$ be a polynomial of degree d. Thus,

$$\left(\frac{1}{d!}\right)^{\frac{1}{2}} \|f\|_2 \leq \|f\|_{(d)} \leq \|f\|_2. \tag{4}$$

III. *Non-crossing area criterion*, [19, Section 3]. Let $g = g(\boldsymbol{x})$ be a real polynomial of P. Following the standard notation, we denote by $\mathrm{Jac}_g(\boldsymbol{x}) := \left(\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n}\right)$ the *Jacobian* (or *gradient*) of g, and by $H_g(\boldsymbol{x}) := \left(\frac{\partial^2 g}{\partial x_i \partial x_j}\right)_{i,j=1,\ldots,n}$ the $n \times n$ symmetric *Hessian matrix* of g, $\boldsymbol{x} = (x_1, \ldots, x_n)$.

Let $p \in \mathbb{A}^n_{\mathbb{C}}$ and fix a real number $0 < R \ll 1$. We then have that, whenever

$$|g(p)| > \|\mathrm{Jac}_g(p)\|_2 R + \frac{1}{2}\|H_g(p)\|_2 R^2,$$

and neglecting contributions of order $\mathbf{O}(R^3)$, the locus of equation $g = 0$ does not cross

$$\mathbf{C}(p) = \{\mathbf{x} \in \mathbb{R}^n \mid \|(\mathbf{x} - p)^T\|_2 \leq R\},$$

the $(\| \, \|_2, R)$-*unit ball of radius* R *centered at* p.

2 A minimization of Bombieri's norm

First, let us note that Bombieri's norm is defined for polynomials f but it may be thought to be used for hypersurfaces. This leads to a possible confusion, that is, if a point p satisfies $f = 0$ it also satisfies $kf = 0$ with $k \in \mathbb{C}^*$, but $||kf||_{\deg(f)} = |k|||f||_{\deg(f)}$. To this purpose, let's observe that whenever we consider hypersurfaces $f = 0$, we are indeed thinking to pairs $(f, V(f))$.

In this section we refine the bound for the quantity $\|q - p\|_2$ given by Theorem 1.4. For this purpose, let $\Phi \in P$ be a polynomial and p a point of multiplicity s such that not all partial derivatives $\frac{\partial^s \Phi}{\partial x_i^s}(p)$ are zero. Then, we introduce the quantity

$$\alpha(p, \Phi, s) = \left(\frac{\deg(\Phi)!}{(\deg(\Phi) - s)!} \frac{(1 + \|p\|_2^2)^{\deg(\Phi)/2}}{\left\|\left(\frac{\partial^s \Phi}{\partial x_1^s}(p), \ldots, \frac{\partial^s \Phi}{\partial x_n^s}(p)\right)\right\|_1}\right)^{1/s}.$$

We observe that, for $\lambda \in \mathbb{C} \setminus \{0\}$

$$\alpha(p, \lambda\Phi, s) = \frac{1}{\sqrt[s]{|\lambda|}} \alpha(p, \Phi, s). \tag{5}$$

In this situation, the claims in Theorem 1.4 can be rephrased as

1. $\|q - p\|_2 \leq \alpha(p, g, s) \, \|g - f\|_{(d')}^{1/s}$.

2. $\|q - p\|_2 \leq \sqrt[s]{2}\alpha(p, f, s) \, \|g - f\|_{(d')}^{1/s}$, under the assumption that

$$\|g - f\|_{(d')}^{1/s} \leq \frac{\sqrt{1 + \|p\|_2^2}}{\sqrt[s]{2}\alpha(p, f, s)}.$$

Now, since f and g are defining polynomials of a variety, we may consider for our purposes the product of them with any scalar. So, if in the previous inequalities we consider λg and μf instead of f and g, with $\lambda, \mu \in \mathbb{C} \setminus \{0\}$, we get

1. $\|q-p\|_2 \leq \alpha(p, \lambda g, s)\, \|\lambda g - \mu f\|_{(d')}^{1/s} = \alpha(p,g,s)\, \left\|g - \frac{\mu}{\lambda} f\right\|_{(d')}^{1/s}$.

2. $\|q-p\|_2 \leq \sqrt[s]{2}\alpha(p, \mu f, s)\, \|\lambda g - \mu f\|_{(d')}^{1/s} = \sqrt[s]{2}\alpha(p,f,s)\, \left\|\frac{\lambda}{\mu} g - f\right\|_{(d')}^{1/s}$, under the assumption that $\left\|\frac{\lambda}{\mu} g - f\right\|_{(d')}^{1/s} \leq \dfrac{\sqrt{1+\|p\|_2^2}}{\sqrt[s]{2}\alpha(p,f,s)}$.

Thus, the following natural question arises:

> For $\Phi,\ \Psi \in P$, with the same degree d, what is the value of $\lambda \in \mathbb{C} \setminus \{0\}$ minimizing $\|\Phi - \lambda\Psi\|_{(\deg(\Phi-\lambda\Psi))}$?

The following theorem answers the question.

Theorem 2.1 *Let $\Phi, \Psi \in P$ with the same degree d and let Φ_d, Ψ_d be the homogeneous forms of maximum degree of Φ and Ψ, respectively.*

1. *Let $\Phi_d/\Psi_d \notin \mathbb{C}$. Then*

$$\min_{\lambda\in\mathbb{C}^*} \left\{ \|\Phi - \lambda\Psi\|_{(d)} \right\} = \left\| \Phi - \frac{(\Phi,\Psi)_{(d)}}{\|\Psi\|_{(d)}^2} \Psi \right\|_{(d)}.$$

2. *Let $\Phi_d/\Psi_d \in \mathbb{C}$, $d^* = \deg\left(\Phi - \frac{(\Phi,\Psi)_{(d)}}{\|\Psi\|_{(d)}^2}\Psi\right)$ and $d^{**} = \deg\left(\Phi - \frac{\Phi_d}{\Psi_d}\Psi\right)$. Then*

$$\min_{\lambda\in\mathbb{C}^*} \left\{ \|\Phi - \lambda\Psi\|_{(\deg(\Phi-\lambda\Psi))} \right\} = \min \left\{ \left\| \Phi - \frac{(\Phi,\Psi)_{(d)}}{\|\Psi\|_{(d)}^2} \Psi \right\|_{(d^*)}, \left\| \Phi - \frac{\Phi_d}{\Psi_d} \Psi \right\|_{(d^{**})} \right\}.$$

Proof. Set $\Psi = \sum_{\mathbf{i} \in \mathbb{Z}^n_{\geq 0},\, |\mathbf{i}| \leq d} a_{\mathbf{i}} \boldsymbol{x}^{\mathbf{i}}$ and $\Phi = \sum_{\mathbf{i} \in \mathbb{Z}^n_{\geq 0},\, |\mathbf{i}| \leq d} b_{\mathbf{i}} \boldsymbol{x}^{\mathbf{i}}$, where $\mathbf{i} = (i_1, \ldots, i_n)$, $|\mathbf{i}| = \sum_{j=1}^n i_j$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, and $\boldsymbol{x}^{\mathbf{i}} = \prod_{j=1}^n x_j^{i_j}$. We observe that the degree of $\Phi - \lambda\Psi$ may drop if and only if the quotient of maximum homogeneous form of Φ and Ψ is constant. Then there exists $\lambda \in \mathbb{C}^*$ such that $\deg(\Phi - \lambda\Psi) < d$ if and only if $\Phi_d/\Psi_d \in \mathbb{C}$.

We consider the polynomial $h := \Phi - Z\Psi$, where Z is a new variable. We optimize $B(Z) := \|h\|_{(d)}^2$ with $Z \neq \Phi_d/\Psi_d$ whenever $\Phi_d/\Psi_d \in \mathbb{C}$. That is,

$$B(Z) = \sum_{\mathbf{i}\in\mathbb{Z}^n_{\geq 0}, |\mathbf{i}|\leq d} (d - |\mathbf{i}|)!\, \frac{\mathbf{i}!}{d!} \left| b_{\mathbf{i}} - a_{\mathbf{i}} Z \right|^2.$$

Set $Z = X + iY$, where X, Y are real variables. Then,

$$\begin{aligned} B(X,Y) &= \sum_{\mathbf{i}\in\mathbb{Z}^n_{\geq 0}, |\mathbf{i}|\leq d} (d - |\mathbf{i}|)!\, \frac{\mathbf{i}!}{d!} \left(|a_{\mathbf{i}}|^2 (X^2+Y^2) + |b_{\mathbf{i}}|^2 - 2\mathrm{Re}(a_{\mathbf{i}}\overline{b_{\mathbf{i}}})X + 2\mathrm{Im}(a_{\mathbf{i}}\overline{b_{\mathbf{i}}})Y \right) \\ &= \|\Psi\|_{(d)}^2 (X^2+Y^2) + \|\Phi\|_{(d)}^2 - 2X\mathrm{Re}((\Psi,\Phi)_{(d)}) + 2Y\mathrm{Im}((\Psi,\Phi)_{(d)}). \end{aligned}$$

Therefore,

$$\mathrm{Jac}_B(X,Y) = \left(2\|\Psi\|^2_{(d)}X - 2\mathrm{Re}((\Psi,\Phi)_{(d)}), 2\|\Psi\|^2_{(d)}Y + 2\mathrm{Im}((\Psi,\Phi)_{(d)})\right)$$

and the Hessian matrix of B is

$$H_B = \begin{pmatrix} 2\|\Psi\|^2_{(d)} & 0 \\ 0 & 2\|\Psi\|^2_{(d)} \end{pmatrix}.$$

Thus, $B(Z)$ reaches its minimum at

$$Z^* = \frac{\mathrm{Re}((\Psi,\Phi)_{(d)})}{\|\Psi\|^2_{(d)}} - \frac{\mathrm{Im}((\Psi,\Phi)_{(d)})}{\|\Psi\|^2_{(d)}} i = \frac{\overline{(\Psi,\Phi)_{(d)}}}{\|\Psi\|^2_{(d)}} = \frac{(\Phi,\Psi)_{(d)}}{\|\Psi\|^2_{(d)}},$$

where the last two equalities follow recalling that $(\Psi,\Phi)_{(d)} = \sum_{\mathbf{i}} \frac{(d-|\mathbf{i}|)!|\mathbf{i}|!}{d!} a_{\mathbf{i}}\overline{b_{\mathbf{i}}}$. Q.E.D.

Noting that for real polynomials one has $(g,f)_{(d)} = (f,g)_{(d)}$, Theorem 1.4 can be restated as follows.

Theorem 2.2 *Let f and g be real polynomials of P of degree d, let F_d and G_d be their homogeneous forms of maximum degree d and let $d' = \deg(g-f) \leq d$. Let $p \in \mathbb{A}^n_{\mathbb{C}}$ be a real point of $f=0$ of multiplicity $s \geq 1$ such that $\frac{\partial^s g}{\partial x_i^s}(p) \neq 0$ for some index i. Then there exists a point $q \in \mathbb{A}^n_{\mathbb{C}}$ belonging to $g=0$ such that the following holds true.*

1. *Let $G_d/F_d \notin \mathbb{R}$. Then*

 1.1. $\|q-p\|_2 \leq \alpha(p,g,s) \left\| g - \frac{(f,g)_{(d)}}{\|f\|^2_{(d)}} f \right\|^{\frac{1}{s}}_{(d)}.$

 1.2. *If* $\left\| \frac{(f,g)_{(d)}}{\|g\|^2_{(d)}} g - f \right\|^{\frac{1}{s}}_{(d)} \leq \frac{\sqrt{1+\|p\|_2^2}}{\sqrt[s]{2}\,\alpha(p,f,s)}$, *then*

$$\|q-p\|_2 \leq \sqrt[s]{2}\,\alpha(p,f,s) \left\| \frac{(f,g)_{(d)}}{\|g\|^2_{(d)}} g - f \right\|^{1/s}_{(d)}.$$

2. *Let $G_d/F_d \in \mathbb{R}$, $d^* = \deg\left(g - \frac{(f,g)_{(d)}}{\|f\|^2_{(d)}} f\right)$, and $d^{**} = \deg\left(g - \frac{G_d}{F_d} f\right)$. Then*

 2.1. $\|q-p\|_2 \leq \alpha(p,g,s) \min\left\{ \left\| g - \frac{(f,g)_{(d)}}{\|f\|^2_{(d)}} f \right\|^{\frac{1}{s}}_{(d^*)}, \left\| g - \frac{G_d}{F_d} f \right\|^{\frac{1}{s}}_{(d^{**})} \right\}.$

 2.2. *If* $\left\| \frac{(f,g)_{(d)}}{\|g\|^2_{(d)}} g - f \right\|^{\frac{1}{s}}_{(d^*)} \leq \frac{\sqrt{1+\|p\|_2^2}}{\sqrt[s]{2}\,\alpha(p,f,s)}$, *then*

$$\|q-p\|_2 \leq \sqrt[s]{2}\,\alpha(p,f,s)\, \min\left\{ \left\| \frac{(f,g)_{(d)}}{\|g\|^2_{(d)}} g - f \right\|^{\frac{1}{s}}_{(d^*)}, \left\| \frac{F_d}{G_d} g - f \right\|^{\frac{1}{s}}_{(d^{**})} \right\}.$$

A natural question would be to analyze the relationship between the bounds provided by Theorem 2.2 and Łojasiewicz inequality (see [12, Theorem 1.7] and [10]), or try to use them to get estimations on the constants appearing in Łojasiewicz inequality. Nevertheless, we have not investigate yet in this direction, leaving the topic for future work. We thank the referee for calling our attention to Łojasiewicz results.

The following examples highlight the improvement given by Theorem 2.2.

Example 2.3 Let $f = x + y + 1$ and $g = 1.1x + y + 0.9$. Then $G_1/F_1 \notin \mathbb{C}$, so that

$$\min_{\lambda\in\mathbb{C}^*}\left\{\|g-\lambda f\|_{(\deg(g-\lambda f))}\right\} = \left\|g - \left(\frac{(f,g)_{(1)}}{\|f\|^2_{(1)}}\right) f\right\|_{(1)} = \|g-f\|_{(1)} = 0.1414213.$$

Example 2.4 Let $g = x^2 + x + y + 3$ and $f = 2x^2 + 2x + 1$. $G_2/F_2 = 1/2 \in \mathbb{C}$. So,

$$\begin{aligned}\min_{\lambda\in\mathbb{C}^*}\left\{\|g-\lambda f\|_{(\deg(g-\lambda f))}\right\} &= \min\left\{\left\|g - \frac{(f,g)_{(2)}}{\|f\|^2_{(2)}} f\right\|_{(2)}, \left\|g - \frac{G_2}{F_2} f\right\|_{(1)}\right\}\\ &= \min\left\{\left\|g - \frac{6}{7} f\right\|_{(2)}, \left\|g - \frac{1}{2} f\right\|_{(1)}\right\} = \min\left\{\sqrt{\frac{29}{4}}, \sqrt{\frac{287}{49}}\right\}\\ &\approx \min\{2.69, 2.42\} = 2.42.\end{aligned}$$

This quantity is slightly better than the norm value $\|g-f\|_{(2)} = \sqrt{6} \approx 2.45$ as in Theorem 2.2(1).

Example 2.5 Let $g = x^2 + x + y + 1$ and $f = x^2 + 2x + 3y + 3$. Then, $G_2/F_2 = 1$. So,

$$\begin{aligned}\min_{\lambda\in\mathbb{C}^*}\left\{\|g-\lambda f\|_{(\deg(g-\lambda f))}\right\} &= \min\left\{\left\|g - \frac{(f,g)_{(2)}}{\|f\|^2_{(2)}} f\right\|_{(2)}, \left\|g - \frac{G_2}{F_2} f\right\|_{(1)}\right\}\\ &= \min\left\{\left\|g - \frac{13}{33} f\right\|_{(2)}, \|g-f\|_{(1)}\right\}\\ &= \min\left\{\frac{\sqrt{1914}}{66}, 3\right\} \approx \min\{0.66, 3\} = 0.66.\end{aligned}$$

This quantity is now definitely better than the norm value $\|g-f\|_{(1)} = 3$.

3 Lower bounds for distance of points

This section is devoted to state perturbation results on the zero-locus of a polynomial, which can be considered as reverse type results of Theorem 1.4 (see [21, Lemma 3.3 and Theorem 3.4]). In particular, we provide a new lower bound on the Euclidean distance of points belonging to the zero-loci of complex multivariate degree d polynomials f and g in terms of Bombieri's norm $\|g-f\|_{(d')}$, with $d' = \deg(g-f) \leq d$.

Let's summarize some notation we use throughout this section.

- f and g are real polynomials of P of degree $d \geq 2$.
  Note that we have excluded hyperplanes, and this is not a loss of generality. Indeed, let f, g be real polynomials of degree 1. Let p be a real point of $f = 0$. Then, the distance of p to $g = 0$ is
  $$\frac{|g(p)|}{\|\mathrm{Jac}_g(p)\|_2}. \tag{6}$$
  So, for every R smaller than the quantity above, each real point $q \in \mathbb{A}^n_{\mathbb{C}}$ belonging to $g = 0$ satisfies the condition $\|q - p\|_2 \geq R$.

- Set $d' := \deg(g - f)$.

- $p = (p_1, \ldots, p_n) \in \mathbb{A}^n_{\mathbb{C}}$ is a real smooth point of $f = 0$, and $g(p) \neq 0$.

- U_J, U_H are (positive) upper bounds of $\|\mathrm{Jac}_g(p)\|_2$ and $\|H_g(p)\|_2$, respectively.

- L_g is a (positive) lower bound of $|g(p)|$.

- R is real number such that $0 < R \ll 1$.

The following result makes use of arguments from [19, Section 3] and [21, Section 3], from where we take the notation.

Theorem 3.1 *Notation as above.*

1. *If $\|H_g(p)\|_2 \neq 0$, that is, $\frac{\partial^2 g}{\partial x_i \partial x_j}(p) \neq 0$ for some pair (i, j) of indices, and*
   $$R < \frac{-U_J + \sqrt{U_J^2 + 2U_H L_g}}{U_H}, \tag{7}$$
   then, up to an error of $\mathrm{O}(R^3)$,[2] each real point $q \in \mathbb{A}^n_{\mathbb{C}}$ belonging to $g = 0$ satisfies the condition $\|q - p\|_2 \geq R$.

2. *If $\|H_g(p)\|_2 = 0$, $\|\mathrm{Jac}_g(p)\|_2 \neq 0$, and*
   $$R < \frac{L_g}{U_J},$$
   then, up to an error of $\mathrm{O}(R^3)$, each real point $q \in \mathbb{A}^n_{\mathbb{C}}$ belonging to $g = 0$ satisfies the condition $\|q - p\|_2 \geq R$.

Proof. From (7), and using that $U_H > 0$, we get that

$$|g(p)| \geq L_g > \frac{1}{2} U_H R^2 + U_J R.$$

Therefore, using that U_J, U_H are upper bounds, we get that

$$|g(p)| > \frac{1}{2}\|H_g(p)\|_2 R^2 + \|\mathrm{Jac}_g(p)\|_2 R. \tag{8}$$

[2]We recall that, given a real value $\eta \ll 1$ and a real function $\omega : \mathbb{R}^n \to \mathbb{R}$, we write $\omega(\boldsymbol{x}) = \mathrm{O}(\eta^m)$, $m \in \mathbb{N}$, to mean that $\frac{\omega(\boldsymbol{x})}{\eta^m}$ is bounded near the origin.

Thus, from §1, property III, we get that the hypersurface of equation $g = 0$ does not cross $\mathbf{C}(p)$, the $(\| \, \|_2, R)$-unit ball of radius R centered at p, neglecting contributions of order $\mathbf{O}(R^3)$. This is just to say that each real point $q \in \mathbb{A}^n_{\mathbb{C}}$ belonging to $g = 0$ satisfies the condition $\|q - p\|_2 \geq R$, which proves statement 1). The same argument, by using equation (8) again, yields statement 2). Q.E.D.

Remark 3.2 We observe that if, instead of working with f, g, we take $g^* = \lambda g$ with $\lambda \in \mathbb{R} \setminus \{0\}$, then $|g^*(p)| = |\lambda| \, |g(p)|$, $\|\mathrm{Jac}_{g^*}(p)\|_2 = |\lambda| \, \|\mathrm{Jac}_g(p)\|_2$, $\|H_{g^*}(p)\|_2 = |\lambda| \, \|H_g(p)\|_2$. Therefore, if L_{g^*}, U_{J^*}, U_{H^*} denote the corresponding bounds associated to g^*, then they can be taken as $L_{g^*} = |\lambda| \; L_g$, $U_{J^*} = |\lambda| \; U_J$, $U_{H^*} = |\lambda| \; U_H$. Thus, the bound for R given in (7) will not depend on the defining polynomial g that we take. □

Theorem 3.1 provides a lower bound R for the distance between p and the (real) variety defined by $g = 0$. R depends on the upper bound in (7) that varies depending on the bounds L_g, U_J, U_H. Clearly, L_g, U_J, U_H can be taken as $L_g = |g(p)|, U_J = \|\mathrm{Jac}_g(p)\|_2, U_H = \|H_g(p)\|_2$. However, in some cases, g is given as a perturbation of f, and hence its exact coefficients are not known. In this case, the bounds L_g, U_J, U_H are required. Additionally, the question on what is the best election of these bounds appears. Because of the geometric interpretation of R, the best option for R appears when maximizing its upper bound in (7). This is studied in the next lemma. For this purpose, we introduce the following real function. Let $\Omega = [a_1, \, |g(p)| \,] \times [\, \|\mathrm{Jac}_g(p)\|_2, +\infty) \times [\, \|H_g(p)\|_2, \infty)$ and

$$\begin{array}{cccc} B: & \Omega & \longrightarrow & \mathbb{R} \\ & (x,y,z) & \longmapsto & \frac{-y+\sqrt{y^2+2zx}}{z}, \end{array}$$

where we assume that $a_1 > 0$ and $|g(p)| \, |\mathrm{Jac}_g(p)\|_2 \, \|H_g(p)\|_2 \neq 0$.

Lemma 3.3 *Notation as above. Let $\Sigma := [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2] \subset \Omega$. Then the absolute maximum of B in Σ is at (a_2, b_1, c_1).*

Proof. Since $\frac{\partial B}{\partial x} = 1/\sqrt{2xz + y^2}$ does not vanishes at Σ, the absolute maximum has to be on the faces of the cube Σ. Analyzing the faces one deduces that the only candidates to absolute extrema are the eight vertices of Σ. For $i, j, k \in \{1, 2\}$, let

$$\begin{array}{ll} B_{a,j,k}(t) = B(\ell_a(t), b_j, c_k) & \text{where } \ell_a(t) = ta_2 + (1-t)a_1 \text{ with } t \in [0,1] \\ B_{i,b,k}(t) = B(a_i, \ell_b(t), c_k) & \text{where } \ell_b(t) = tb_2 + (1-t)b_1 \text{ with } t \in [0,1] \end{array}$$

be the restrictions of B to the edges $[a_1, a_2] \times \{b_j\} \times \{c_k\}$ and $\{a_i\} \times [b_1, b_2] \times \{c_k\}$, respectively. For $t \in [0, 1]$ we get

$$\frac{\mathrm{d}B_{a,j,k}}{\mathrm{d}t} = \frac{\ell_a'(t)}{\sqrt{2c_k \ell_a(t) + b_j^2}} = \frac{a_2 - a_1}{\sqrt{2c_k \ell_a(t) + b_j^2}} > 0,$$

$$\frac{\mathrm{d}B_{i,b,k}}{\mathrm{d}t} = \frac{\ell_b'(t)(\ell_b(t) - \sqrt{2c_k a_i + \ell_b(t)^2})}{\sqrt{2c_k a_i + \ell_b(t)^2} c_k} = \frac{(b_2 - b_1)(\ell_b(t) - \sqrt{2c_k a_i + \ell_b(t)^2})}{\sqrt{2c_k a_i + \ell_b(t)^2} c_k} < 0.$$

Therefore, $B_{a,j,k}(t)$ is an increasing function and $B_{i,b,k}(t)$ is a decreasing function. Let us see that $B(a_2, b_1, c_1)$ is the maximum of B in Σ. Indeed,

$$\begin{array}{ll} B(a_2, b_2, c_k) & = B_{2,b,k}(1) < B_{2,b,k}(0) = B(a_2, b_1, c_k) \\ B(a_1, b_2, c_k) & = B_{1,b,k}(1) < B_{1,b,k}(0) = B(a_1, b_1, c_k) \\ B(a_1, b_1, c_k) & = B_{a,1,k}(0) < B_{a,1,k}(1) = B(a_2, b_1, c_k). \end{array}$$

So,

$$\max\{B(a_i,b_j,c_k)\}_{i,j,k\in\{1,2\}} = \max\{B(a_2,b_1,c_1), B(a_2,b_1,c_2)\}.$$

On the other hand, let us consider $C(z) = B(a_2, b_1, z)$. We observe that $\frac{\mathrm{d}C}{\mathrm{d}z}(z)$ is continuous in $\mathbb{R}^+$, it does not vanish at $\mathbb{R}^+$ and

$$\frac{\mathrm{d}C}{\mathrm{d}z}(b_1^2/a_2) = \frac{1}{3}\frac{(\sqrt{3}-2)\sqrt{3}a_2^2}{b_1^3} < 0.$$

Since $b_1^2/a_2 > 0$, then $C(z)$ is a decreasing function in $\mathbb{R}^+$. Thus,

$$\max\{B(a_2,b_1,c_1), B(a_2,b_1,c_2)\} = B(a_2,b_1,c_1).$$

Q.E.D.

Corollary 3.4 *The best choice in Theorem 3.1 is*

$$(L_g, U_J, U_H) = (|g(p)|, \|\mathrm{Jac}_g(p)\|_2, \|H_g(p)\|_2).$$

Proof. If $\|H_g(p)\|_2 \neq 0$, and $\|\mathrm{Jac}_g(p)\|_2 \neq 0$, the result follows from Lemma 3.3. If $\|H_g(p)\|_2 \neq 0$, and $\|\mathrm{Jac}_g(p)\|_2 = 0$, the upper bound is $\sqrt{2L_g/U_H}$, and if $\|H_g(p)\|_2 = 0$, and $\|\mathrm{Jac}_g(p)\|_2 \neq 0$, the upper bound is L_g/U_J. In both cases the result is obvious. Q.E.D.

The following examples illustrate Theorem 3.1.

Example 3.5 Let $f(x,y) = x^3 + y^3 - xy$, let $p = (\frac{1}{2}, \frac{1}{2})$ and let $g(x,y) = f(x,y) + \frac{1}{100}$. Obviously, we have $g - f = \frac{1}{100}$ and $d' = 0 < 3 = d$. Furthermore, $g(p) = \frac{1}{100}$. We apply Theorem 3.1, taking $L_g = |g(p)|$, $U_J = \|\mathrm{Jac}_g(p)\|_2$ and $U_H = \|H_g(p)\|_2$. Then, the bound provided by (7) is

$$\frac{-\|\mathrm{Jac}_g(p)\|_2 + \sqrt{\|\mathrm{Jac}_g(p)\|_2^2 + 2\|H_g(p)\|_2|g(p)|}}{\|H_g(p)\|_2} \approx 0.0248,$$

as computed below. So, we can take $R = 0.024$. Theorem 3.1 ensures that for each real point q belonging to $g = 0$ it holds that (up to an error of $\mathrm{O}(R^3)$)

$$\|q - p\|_2 > 0.024.$$

On the other hand, the real point of $g = 0$ closest to p is $p' \approx (0.478, 0.478)$, whose distance from p is approximately 0.031, that is, about the approximate lower bound found above. Indeed, direct computations give

$$\mathrm{Jac}_g(p) = \Big((3x^2 - y, 3y^2 - x)(p)\Big) = \left(\frac{1}{4}, \frac{1}{4}\right), \quad \text{so that} \quad \|\mathrm{Jac}_g(p)\|_2 = \frac{\sqrt{2}}{4}.$$

Moreover,

$$H_g(p) = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}, \quad \text{whence} \quad \|H_g(p)\|_2 = 4.$$

Thus, the right-hand side of inequality (7) becomes

$$\frac{-\frac{\sqrt{2}}{4} + \sqrt{\frac{1}{8} + 2\times 4 \times \frac{1}{100}}}{4} \approx 0.0248 \ll 1.$$

To compute the distance 0.031 from p to $g = 0$ we look at the real solution of the system (via Lagrange multipliers)

$$2\big(x - \frac{1}{2}\big) + \mu(3x^2 - y) = 2\big(y - \frac{1}{2}\big) + \mu(3y^2 - x) = x^3 + y^3 - xy + \frac{1}{100} = 0.$$

Next, consider a smaller perturbation, $g(x, y) = f(x, y) + \frac{1}{1000}$, so that now $g(p) = \frac{1}{1000}$, and the bound provided by Theorem 3.1 becomes

$$\frac{-\|\mathrm{Jac}_g(p)\|_2 + \sqrt{\|\mathrm{Jac}_g(p)\|_2^2 + 2\|H_g(p)\|_2 |g(p)|}}{\|H_g(p)\|_2} \approx 0.002784.$$

Therefore, we can take $R = 0.0027$. On the other hand, the minimum distance of p from $g = 0$ is approximately 0.002851, that is about the approximate bound 0.0027 found above.

Example 3.6 Let $f(x, y) = x^3 + y^3 - xy$, and $p = (\frac{1}{2}, \frac{1}{2})$. Let g be a perturbation of f of the form $g(x, y, \varepsilon) = f(x, y) + \varepsilon(x + y + 1)$. Then, we can take $L_g = |g(p)| = 2\varepsilon$, $U_J = \|\mathrm{Jac}_g(p)\|_2 = \frac{1}{4}\sqrt{2}(4\varepsilon + 1)$ and $U_H = \|H_g(p)\|_2 = 4$. Then, the bound (say $B(\varepsilon)$) provided in (7) is

$$B(\varepsilon) = -\frac{\sqrt{2}}{16}(4\varepsilon + 1) + \frac{1}{16}\sqrt{32\varepsilon^2 + 272\varepsilon + 2}.$$

Let us consider that ε varies as $0.01 \le \varepsilon \le 0.1$. Since $B(\varepsilon)$ is an increasing function for $\varepsilon \ge 0$, we have that

$$0.0442 \approx B(0.01) \le B(\varepsilon) \le B(0.1) \approx 0.2016.$$

Therefore, in the Euclidean ball centered at p and radius $R = 0.0442$ there is (up to an error of $\mathrm{O}(R^3)$) no real point of the curves $\{g(x, y, \varepsilon) = 0\}_{\varepsilon \in [0.01, 0.1]}$. In Figure 1 we plot the curve $f = 0$, the ball centered at p and the curves $\{g(x, y, \varepsilon) = 0 \,|\, \varepsilon = 0.01 + \frac{i \times 0.09}{40}, 0 \le i \le 40\}$.

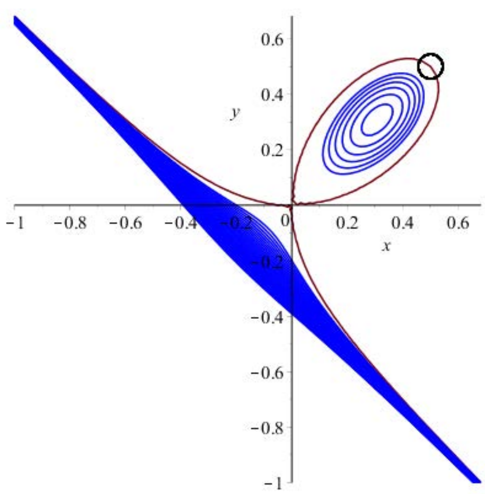

Figure 1: Curve $f(x, y) = 0$ (red), ball centered at p, and curves $\{g(x, y, \varepsilon) = 0 \,|\, \varepsilon = 0.01 + \frac{i \times 0.09}{40}, 0 \le i \le 40\}$ (blue)

4 The Hough transform setting

Most of the results in this section hold over an infinite integral ring K (see [8, Section 28]). However, let us restrict to the classical cases where either $K = \mathbb{R}$, or $K = \mathbb{C}$, the fields of real or complex numbers respectively.

For every t-tuples of independent parameters $\lambda := (\lambda_1, \ldots, \lambda_t)$, varying in an Euclidean open set $\mathcal{U} \subseteq K^t$, and indeterminates x, y, let

$$f_\lambda(x, y) = \sum_{i,j=0}^{\rho} x^i y^j g_{ij}(\lambda_1, \ldots, \lambda_t), \quad i + j \leq \rho, \tag{9}$$

be a family $\mathcal{G}$ of non-constant irreducible polynomials in $K[x, y]$, of a given degree ρ (not depending on λ), whose coefficients $g_{ij}(\lambda)$ are evaluations in the parameters $\lambda = (\lambda_1, \ldots, \lambda_t)$ of polynomials $g_{ij}(\Lambda) \in K[\Lambda]$ in new variables $\Lambda = (\Lambda_1, \ldots, \Lambda_t)$. Let $\mathcal{F}$ be the corresponding family of the zero-loci $\mathcal{C}_\lambda$, and let assume that $\mathcal{C}_\lambda$ is a curve in the affine plane $\mathbb{A}^2_{(x,y)}(K)$, for each λ (of course, this is always the case if $K = \mathbb{C}$). Clearly, if $K = \mathbb{C}$, the curves are irreducible, that is, they consist of a single component, since the polynomials of the family $\mathcal{G}$ are assumed to be irreducible in $K[x, y]$. If $K = \mathbb{R}$, the case of interest in the applications, we assume that $\mathcal{C}_\lambda$ is an *irreducible real curve*, that is, an irreducible curve over $\mathbb{C}$ with infinitely many points in the affine plane $\mathbb{A}^2_{(x,y)}(\mathbb{R})$ (see also [18, Chapter 7]). However, such a real curve may contain a finite set of isolated points; to this purpose, see also [15, Remark 1.4]. So, we want $\mathcal{F}$ to be a *family of real irreducible curves (with possibly a finite set of isolated points) which share the degree.*

Definition 4.1 Let $\mathcal{F}$ be a family of curves $\mathcal{C}_\lambda$ as above, and let $p = (x_p, y_p)$ be a point in the image space $\mathbb{A}^2_{(x,y)}(K)$. Let $\Gamma_p(\mathcal{F})$ be the locus defined in the affine t-dimensional parameter space $\mathbb{A}^t_{(\Lambda_1,\ldots,\Lambda_t)}(K)$ by the polynomial equation

$$f_p(\Lambda) := \sum_{i,j=0}^{\rho} x_p^i y_p^j \; g_{ij}(\Lambda_1, \ldots, \Lambda_t) = 0 \tag{10}$$

in the indeterminates $\Lambda = (\Lambda_1, \ldots, \Lambda_t)$. We say that $\Gamma_p(\mathcal{F})$ is the *Hough transform of the point p with respect to the family $\mathcal{F}$*, or, simply, that $\Gamma_p(\mathcal{F})$ is the *Hough transform of p.*

Summarizing, the polynomial family defined by (9) gives rise to a polynomial $F(x, y; \Lambda_1, \ldots, \Lambda_t) \in K[x, y; \Lambda_1, \ldots, \Lambda_t]$ such that, evaluating at each point $\lambda = (\lambda_1, \ldots, \lambda_t) \in K^t$ and at each point $p = (x_p, y_p) \in \mathbb{A}^2_{(x,y)}(K)$, we obtain the equations

$$\mathcal{C}_\lambda : f_\lambda(x, y) = F(x, y; \lambda_1, \ldots, \lambda_t) = 0 \quad \text{and} \quad \Gamma_p(\mathcal{F}) : f_p(\Lambda) = F(x_p, y_p; \Lambda_1, \ldots, \Lambda_t) = 0$$

of the curve $\mathcal{C}_\lambda$ and the Hough transform $\Gamma_p(\mathcal{F})$, respectively. And, clearly, the "duality condition"

$$p \in \mathcal{C}_\lambda \iff f_\lambda(x_p, y_p) = 0 \iff f_p(\lambda) = 0 \iff \lambda \in \Gamma_p(\mathcal{F}) \tag{11}$$

holds true.

It is convenient to introduce the notion of general points in the Hough sense, as follows. Rewrite $F(x, y; \Lambda_1, \ldots, \Lambda_t)$ as a polynomial in $K[x, y][\Lambda]$, in the variables Λ's, that is,

$$f_{(x,y)}(\Lambda) := \sum_{(i_1 \ldots i_t) \in \mathrm{I}} f_{i_1 \ldots i_t}(x, y) \; \Lambda_1^{i_1} \cdots \Lambda_t^{i_t}, \tag{12}$$

where $0 \leq i_1 + \cdots + i_t \leq \rho$, with $\rho := \deg_\Lambda(f_{(x,y)}(\Lambda))$. We will simplify the notation, letting $\mathbf{i} = (i_1, \ldots, i_t)$ and expressing $f_{(x,y)}(\Lambda)$ as

$$f_{(x,y)}(\Lambda) = \sum_{\mathbf{i} \in \mathrm{I}} f_{\mathbf{i}}(x,y)\Lambda^{\mathbf{i}}. \tag{13}$$

Of course, there exists a Zariski open set $\mathcal{U}_1 \subseteq \mathbb{A}^2_{(x,y)}(K)$, such that, for each point $p \in \mathcal{U}_1$, the Hough transform $\Gamma_p(\mathcal{F}) : f_p(\Lambda) = 0$ of p is a zero locus of a polynomial of degree ρ (not depending on p) in the parameter space. If $K = \mathbb{C}$, clearly such a locus $\Gamma_p(\mathcal{F})$ will be a hypersurface. If $K = \mathbb{R}$, then $\Gamma_p(\mathcal{F})$ is $(t-1)$-dimensional if and only if the polynomial $f_p = f_p(\Lambda) \in \mathbb{R}[\Lambda_1, \ldots, \Lambda_t]$ has a non-singular zero in $\lambda \in \mathbb{R}^t$, that is, the gradient $\left(\frac{\partial f_p}{\partial \Lambda_1}(\lambda), \ldots, \frac{\partial f_p}{\partial \Lambda_t}(\lambda)\right) \neq 0$ (see [4, Theorem 4.5.1] for details and equivalent conditions). Then, $\Gamma_p(\mathcal{F})$ is $(t-1)$-dimensional for p varying in an Euclidean open set $\mathcal{U}_2$ of $\mathbb{A}^2_{(x,y)}(K)$. (To this purpose, see also the more general result [16, Proposition 2.25]).

Based on the argumentation above, we introduce the following affine open set (depending on $\mathcal{F}$), that we will call *the invariance degree open set,*

$$\mathfrak{U}_1 = \begin{cases} \mathcal{U}_1 & \text{if } K = \mathbb{C} \\ \mathcal{U}_1 \cap \mathcal{U}_2 & \text{if } K = \mathbb{R}. \end{cases} \tag{14}$$

All the above justifies the following definition.

Definition 4.2 Notation as above. We say that a point p in the image space is *general* (in the Hough sense) if $p \in \mathfrak{U}_1$. That is, if *the Hough transform $\Gamma_p(\mathcal{F})$ of p is a hypersurface in the parameter space of a given degree ρ not depending on p.*

We refer to the first four sections of [15] for a complete, unified exposition on the Hough transform technique with respect to families of curves.

5 Lower bounds for the discretization step

As already pointed out, bounding the discretization step is quite critical point in the recognition algorithm based on the Hough transform technique. This section is devoted to suggest some ideas on that issue. We keep the notation as in Section 4, and we take $K = \mathbb{R}$. In particular, we refer to expression (13) for the Hough transform $f_{(x,y)}(\Lambda)$ of a point (x,y) varying in the image space $\mathbb{A}^2_{(x,y)}(\mathbb{R})$, and let $\mathfrak{U}_1$ be the invariance degree open set (see definition (14)) of points p whose Hough transforms $\Gamma_p(\mathcal{F})$ are zero loci of polynomials $f_p(\Lambda) \in \mathbb{R}[\Lambda]$ of degree ρ not depending on p.

It seems hard to express in a tractable way (compare with Proposition 5.4) the minimalized upper bound as given in Theorem 2.2 when applied in the parameter space to $f = f_p(\Lambda)$, $g = f_q(\Lambda)$. This is why, in this section, we then limit ourselves to use Theorem 1.4 instead of its improvement given by Theorem 2.2. Indeed the upper bounds as in Theorem 1.4, when applied to the parameter space, suggest lower bounds for the discretization step δ (see Proposition 5.6 and Algorithm 5.12). We aim to understand how much the Hough transform $\Gamma_p(\mathcal{F}) : f_p(\Lambda) = 0$ locally varies.

A first rather general result is the following proposition.

Proposition 5.1 *Let $\mathcal{F} = \{\mathcal{C}_\lambda\}$ be a family of real curves. Let p and q be points of $\mathfrak{U}_1$. Let $\Gamma_p(\mathcal{F}) : f_p(\Lambda) = 0$, $\Gamma_q(\mathcal{F}) : f_q(\Lambda) = 0$ be the Hough transforms of p, q, respectively. Let λ be a smooth point of $\Gamma_p(\mathcal{F})$ such that $\frac{\partial f_q(\Lambda)}{\partial \Lambda_i}(\lambda) \neq 0$ for some index $i \in \{1, \ldots, t\}$. Then there exists a point λ' belonging to $\Gamma_q(\mathcal{F})$ satisfying the following conditions.*

1. $$\|\lambda' - \lambda\|_2 \leq \frac{\rho\,(1 + \|\lambda\|_2^2)^{\rho/2}}{\left\|\mathrm{Jac}_{f_q(\Lambda)}(\lambda)\right\|_1} \|f_q(\Lambda) - f_p(\Lambda)\|_{(\deg(f_q(\Lambda) - f_p(\Lambda))}.$$

2. *If $\|f_q(\Lambda) - f_p(\Lambda)\|_{(\deg(f_q(\Lambda) - f_p(\Lambda))}$ is small enough, namely*
$$\|f_q(\Lambda) - f_p(\Lambda)\|_{(\deg(f_q(\Lambda) - f_p(\Lambda))} \leq \frac{\left\|\mathrm{Jac}_{f_p(\Lambda)}(\lambda)\right\|_1}{2\,\rho\,(1 + \|\lambda\|_2^2)^{\frac{\rho-1}{2}}},$$
then
$$\|\lambda' - \lambda\|_2 \leq \frac{2\,\rho\,(1 + \|\lambda\|_2^2)^{\rho/2}}{\left\|\mathrm{Jac}_{f_p(\Lambda)}(\lambda)\right\|_1} \|f_q(\Lambda) - f_p(\Lambda)\|_{(\deg(f_q(\Lambda) - f_p(\Lambda))}.$$

Proof. It follows from Theorem 1.4 applied to $f = f_p(\Lambda)$ and $g = f_q(\Lambda)$, respectively. Q.E.D.

Proposition 5.1 provides a local upper bound $\|\lambda' - \lambda\|_2$ in terms of Bombieri's norm $\|f_q(\Lambda) - f_p(\Lambda)\|_{(\deg(f_p(\Lambda) - f_q(\Lambda))}$. Now, we aim to relate this last quantity to the distance of the points p and q. To this end we need a new "ad hoc" notion of generality in the Hough sense. For this purpose, we introduce a polynomial associated to $f_{(x,y)}(\Lambda)$ where only the terms with non-constant coefficients are considered.

Definition 5.2 Let $f = \sum_{\mathbf{i} \in \mathrm{I}} f_{\mathbf{i}}(x,y)\Lambda^{\mathbf{i}}$ be expressed as in (13). Set $\mathrm{J} = \{\mathbf{i} \in \mathrm{I} \,|\, f_{\mathbf{i}}(x,y) \notin K\}$. We define the polynomial $f^*_{(x,y)}(\Lambda) := \sum_{\mathbf{i} \in \mathrm{J}} f_{\mathbf{i}}(x,y)\Lambda^{\mathbf{i}}$. We say that a collection of points $\mathcal{S}$ in the image space is *in relative general position* if for every $p, q \in \mathcal{S}$, $p \neq q$, it holds that
$$f_{\mathbf{i}}(p) \neq f_{\mathbf{i}}(q) \ \text{ for each } \mathbf{i} \in \mathrm{J}. \tag{15}$$

In particular, condition (15) implies that
$$\deg_\Lambda(f_p(\Lambda) - f_q(\Lambda)) = \deg_\Lambda(f^*_p(\Lambda) - f^*_q(\Lambda)) = \deg_\Lambda(f^*_{(x,y)}(\Lambda)).$$
We denote by $\rho^*(\leq \rho)$ the degree of $f^*_{(x,y)}(\Lambda)$, that is, $\rho^* = \deg_\Lambda(f^*_{(x,y)}(\Lambda))$.

Definition 5.3 Let $f_{(x,y)}(\Lambda)$ be expressed as in (13) and let J be as in Definition 5.2. For each point p in the image space, we define the polynomial
$$\mathfrak{J}_{f,p}(\Lambda) := \sum_{\mathbf{i} \in \mathrm{J}} \left\|\mathrm{Jac}_{f_{\mathbf{i}}}(p)\right\|_2 \Lambda^{\mathbf{i}}.$$

We observe that small perturbation of a set of points in relative general position will generate a set of points still in relative general position. Let us also emphasize the fact that a given set of points, not in relative general position, can be, via small perturbations, transformed in a set in relative general position.

In connection with Definition 5.3, let us consider the affine open set $\mathfrak{U}_2$ (depending on $\mathcal{F}$) of the image space defined by

$$\mathfrak{U}_2 = \mathbb{A}^2_{(x,y)}(\mathbb{R}) \setminus \bigcap_{\mathbf{i}\in \mathrm{J}, |\mathbf{i}|=\rho^*} \big(\{\mathrm{Jac}_{f_{\mathbf{i}}}(x,y)=0\}\big) = \{p \in \mathbb{A}^2_{(x,y)}(\mathbb{R}) \mid \deg_\Lambda(\mathfrak{J}_{f,p}(\Lambda)) = \rho^*\}. \quad (16)$$

We let

$$\mathfrak{U} := \mathfrak{U}_1 \cap \mathfrak{U}_2, \quad (17)$$

where $\mathfrak{U}_1$ is the invariance degree open set defined in (14).

In addition, from now on through this section, having in mind pixels in the image space, we consider perturbations of points in norm 1. □

Proposition 5.4 *Let $\mathcal{F} = \{\mathcal{C}_\lambda\}$ be a family of real curves. Let $p = (x_p, y_p)$, $q = (x_p + \varepsilon_1, y_p + \varepsilon_2)$ be points in the image space $\mathbb{A}^2_{(x,y)}(\mathbb{R})$, with q a perturbation of p under a threshold ε, that is, $0 < |\varepsilon_i| \le \varepsilon$, $i = 1, 2$. Assume that $p \in \mathfrak{U}$, $q \in \mathfrak{U}_1$ and that p and q are in relative general position. Then, up to a an error of $\mathrm{O}(\varepsilon^2)$, one has*

$$\|f_q(\Lambda) - f_p(\Lambda)\|_{(\rho^*)} \le \sqrt{2}\,\varepsilon\, \|\mathfrak{J}_{f,p}(\Lambda)\|_{(\rho^*)}.$$

Proof. According to Definition 1.3 and with clear meaning of the symbols, we have

$$||q^T - p^T||_\infty = \|q-p\|_1 = \max\{|\varepsilon_1|, |\varepsilon_2|\} \le \varepsilon.$$

The Hough transforms of the points p and q are defined by the equations

$$\Gamma_p(\mathcal{F}) : f_p(\Lambda) = \sum_{\mathbf{i}\in \mathrm{I}} f_{\mathbf{i}}(p)\ \Lambda^{\mathbf{i}}, \quad \text{and} \quad \Gamma_q(\mathcal{F}) : f_q(\Lambda) = \sum_{\mathbf{i}\in \mathrm{I}} f_{\mathbf{i}}(q)\ \Lambda^{\mathbf{i}}.$$

Letting J as in Definition 5.2 and denoting by Λ_0 the homogenizing variable in the parameter space, we find

$$f_p(\Lambda) - f_q(\Lambda) = \sum_{\mathbf{i}\in \mathrm{I}} \big(f_{\mathbf{i}}(p) - f_{\mathbf{i}}(q)\big)\Lambda^{\mathbf{i}} = \sum_{\mathbf{i}\in \mathrm{J}} \big(f_{\mathbf{i}}(p) - f_{\mathbf{i}}(q)\big)\Lambda^{\mathbf{i}},$$

and

$$\big(f_p(\Lambda) - f_q(\Lambda)\big)^{\mathrm{hom}} = \sum_{\mathbf{i}\in \mathrm{J}} \big(f_{\mathbf{i}}(p) - f_{\mathbf{i}}(q)\big)\Lambda_0^{i_0}\Lambda_1^{i_1}\cdots\Lambda_t^{i_t},$$

with $i_0 = \rho^* - (i_1 + \cdots + i_t)$. Thus,

$$\begin{aligned}
\big\|f_p(\Lambda) - f_q(\Lambda)\big\|^2_{(\rho^*)} &= \big\|\big(f_p(\Lambda) - f_q(\Lambda)\big)^{\mathrm{hom}}\big\|^2_{(\rho^*)} \\
&= \sum_{\mathbf{i}\in \mathrm{J}} \frac{i_0! i_1! \ldots i_t!}{\rho^*!}\big|f_{\mathbf{i}}(p) - f_{\mathbf{i}}(q)\big|^2. \quad (18)
\end{aligned}$$

On the other hand, for each t-tuple $\mathbf{i} \in \mathrm{J}$, we have

$$\big|f_{\mathbf{i}}(p) - f_{\mathbf{i}}(q)\big| \le |\mathrm{Jac}_{f_{\mathbf{i}}}(p)(q-p)^T| + \mathrm{O}(\|(q^T - p^T)\|_2^2).$$

Therefore, up to an error of $\mathrm{O}(\varepsilon^2)$, we get (by using 2-norm)

$$\begin{aligned}
\big|f_{\mathbf{i}}(p) - f_{\mathbf{i}}(q)\big|^2 &\le \|\mathrm{Jac}_{f_{\mathbf{i}}}(p)\|_2^2\ \|q^T - p^T\|_2^2 \\
&\le \|\mathrm{Jac}_{f_{\mathbf{i}}}(p)\|_2^2\big(\sqrt{2}\ \|q^T - p^T\|_\infty\big)^2 \\
&\le \|\mathrm{Jac}_{f_{\mathbf{i}}}(p)\|_2^2\ 2\varepsilon^2. \quad (19)
\end{aligned}$$

Now the result follows from Definition 5.3, by combining (18) and (19). Q.E.D.

Remark 5.5 In the previous result, the bound is given up to an error of $\mathrm{O}(\varepsilon^2)$. This can be avoided using the following argument. Since we are working with polynomials with real coefficients, and p, q are points with real coordinates, we can apply the mean value theorem in the proof of Proposition 5.4. More precisely, we get that, for $\mathbf{i} \in \mathrm{J}$, there exists a point $\xi_{\mathbf{i}}$ in the segment joining p, q such that $\left|f_{\mathbf{i}}(p) - f_{\mathbf{i}}(q)\right| = |\mathrm{Jac}_{f_{\mathbf{i}}}(\xi_{\mathbf{i}})(q-p)^T|$. So, $\left|f_{\mathbf{i}}(p) - f_{\mathbf{i}}(q)\right|^2 \leq \|\mathrm{Jac}_{f_{\mathbf{i}}}(\xi_{\mathbf{i}})\|_2^2 \; \|q^T - p^T\|_2^2$. Therefore, by using again equality (18), one concludes that

$$\|f_q(\Lambda) - f_p(\Lambda)\|_{(\rho^*)} \leq \sqrt{2}\,\varepsilon \max_{\mathbf{i}\in \mathrm{J}} \left\{ \max_{0<t<1} \left\{ \|\mathrm{Jac}_{f_{\mathbf{i}}}(q - t(\varepsilon_1, \varepsilon_2))\|_2 \right\} \right\} \left\| \sum_{\mathbf{i}\in \mathrm{J}} \Lambda^{\mathbf{i}} \right\|_{(\rho^*)}.$$

□

The following result provides local upper bounds in terms of $(q, f_q(\Lambda))$ and $(p, f_p(\Lambda))$.[3]

Proposition 5.6 *Let $\mathcal{F} = \{\mathcal{C}_\lambda\}$ be a family of real curves. Let $p = (x_p, y_p)$ be a point in the affine open set $\mathfrak{U}$ of the image space $\mathbb{A}^2_{(x,y)}(\mathbb{R})$, and let $q = (x_p + \varepsilon_1, y_p + \varepsilon_2)$ be a perturbation of p under a threshold ε, that is, $0 < |\varepsilon_i| \leq \varepsilon$, $i = 1, 2$, with $q \in \mathfrak{U}_1$. Further, assume that the points p and q are in relative general position. Let λ be a smooth point of $\Gamma_p(\mathcal{F})$. Thus, there exists a point λ' belonging to $\Gamma_q(\mathcal{F})$ such that, up to an error of $\mathrm{O}(\varepsilon^2)$ the following estimates hold true:*

1. $\|\lambda' - \lambda\|_2 \leq \dfrac{\rho \left(1 + \|\lambda\|_2^2\right)^{\frac{\rho}{2}}}{\left\|\mathrm{Jac}_{f_q(\Lambda)}(\lambda)\right\|_1} \sqrt{2}\,\varepsilon \, \|\mathfrak{J}_{f,p}(\Lambda)\|_{(\rho^*)} := \mathcal{R}_1.$

2. *If* $\varepsilon \leq \dfrac{\left\|\mathrm{Jac}_{f_p(\Lambda)}(\lambda)\right\|_1}{2\rho\,(1 + \|\lambda\|_2^2)^{\frac{\rho-1}{2}} \sqrt{2}\, \|\mathfrak{J}_{f,p}(\Lambda)\|_{(\rho^*)}}$, *then*

$$\|\lambda' - \lambda\|_2 \leq \frac{2\,\rho \left(1 + \|\lambda\|_2^2\right)^{\frac{\rho}{2}}}{\left\|\mathrm{Jac}_{f_p(\Lambda)}(\lambda)\right\|_1} \sqrt{2}\, \|\mathfrak{J}_{f,p}(\Lambda)\|_{(\rho^*)} \varepsilon := \mathcal{R}_2.$$

Proof. It follows from Propositions 5.1 and 5.4. Q.E.D.

Remark 5.7 The ideas in Proposition 5.6 can be used to develop global bounds. More precisely, let $M := \max_{\tau\in\mathcal{T}}\{\left(1 + \|\tau\|_2^2\right)\}$, where $\mathcal{T} \subset \mathbb{A}^t_{(\Lambda_1,\ldots,\Lambda_t)}(\mathbb{R})$ is the discretized region in the parameter space. Then,

$$\|\lambda' - \lambda\|_2 \leq \frac{\rho\,\sqrt{M^\rho}}{\min_{\tau\in\mathcal{T}} \left\{ \left\|\mathrm{Jac}_{f_q(\Lambda)}(\tau)\right\|_1 \right\}} \sqrt{2}\; \|\mathfrak{J}_{f,p}(\Lambda)\|_{(\rho^*)}\, \varepsilon := \mathcal{R}_3,$$

and, if

$$\varepsilon \leq \frac{\left\|\mathrm{Jac}_{f_p(\Lambda)}(\lambda)\right\|_1}{2\,\rho\,(1 + \|\lambda\|_2^2)^{\frac{\rho-1}{2}} \sqrt{2}\, \|\mathfrak{J}_{f,p}(\Lambda)\|_{(\rho^*)}},$$

then

$$\|\lambda' - \lambda\|_2 \leq \frac{2\,\rho\,\sqrt{M^\rho}}{\min_{\tau\in\mathcal{T}} \left\{ \left\|\mathrm{Jac}_{f_p(\Lambda)}(\tau)\right\|_1 \right\}} \sqrt{2}\, \|\mathfrak{J}_{f,p}(\Lambda)\|_{(\rho^*)} \varepsilon := \mathcal{R}_4.$$

[3] Note that, dividing by $1 + \|\lambda\|_2^2$, we get an approximate upper bound for the relative error $\frac{\|\lambda' - \lambda\|_2}{\|\lambda\|_2}$.

Remark 5.8 Proposition 5.6 can be formulated using in its proof Remark 5.5 instead of Proposition 5.4. Precisely, the bound $\mathcal{R}_5$ corresponding to $\mathcal{R}_1$ (similarly for the analog of $\mathcal{R}_2$) is given by

$$\|\lambda' - \lambda\|_2 \leq \frac{\rho\left(1 + \|\lambda\|_2^2\right)^{\frac{\rho}{2}}}{\left\|\mathrm{Jac}_{f_q(\Lambda)}(\lambda)\right\|_1} \sqrt{2}\,\varepsilon \max_{\mathbf{i}\in \mathrm{J}} \left\{ \max_{0<t<1} \left\{ \left\|\mathrm{Jac}_{f_{\mathbf{i}}}(q - t(\varepsilon_1, \varepsilon_2))\right\|_2 \right\} \right\} \left\| \sum_{\mathbf{i}\in \mathrm{J}} \Lambda^{\mathbf{i}} \right\|_{(\rho^*)} := \mathcal{R}_5.$$

□

Proposition 5.6 and Remark 5.7 can be at once interpreted as a criterion for the Hough transform $\Gamma_q(\mathcal{F})$ to cross the $(\| \ \|_2, \mathcal{R}_i)$-unit ball of radius $\mathcal{R}_i$ centered at λ, and therefore the $(\infty, \mathcal{R}_i)$-unit cell of radius $\mathcal{R}_i$ centered at λ,

$$\mathbf{C}(\lambda) = \{\tau \in \mathcal{T} \mid \|(\tau - \lambda)^T\|_2 \leq \mathcal{R}_i\}, \quad i = 1, 2, 3, 4, 5.$$

In the linear case, the above bounds specialize as follows.

Corollary 5.9 (The linear case) *Notation and assumptions as in Proposition 5.6. Furthermore, assume that $\rho = 1$ and let the equation of the curves $\mathcal{C}_\lambda$ from the family $\mathcal{F}$ be expressed as*

$$f_{(x,y)}(\lambda) = \sum_{k=1}^{t} f_k(x, y)\lambda_k + f_0(x, y).$$

Let λ be a smooth point of $\Gamma_p(\mathcal{F})$. Thus, there exists a point λ' belonging to $\Gamma_q(\mathcal{F})$ such that, up to an error of $\mathrm{O}(\varepsilon^2)$, the following estimates hold true:

1. $\|\lambda' - \lambda\|_2 \leq \frac{\left(1+\|\lambda\|_2^2\right)^{\frac{1}{2}}}{\max\left\{|f_1(q)|,\ldots,|f_t(q)|\right\}} \sqrt{2}\, \|\mathfrak{J}_{f,p}(\Lambda)\|_{(\rho^*)}\, \varepsilon := \mathcal{R}_1.$

2. *If* $\varepsilon \leq \frac{\max\left\{|f_1(p)|,\ldots,|f_t(p)|\right\}}{2(1+\|\lambda\|_2^2)\sqrt{2}\,\|\mathfrak{J}_{f,p}(\Lambda)\|_{(\rho^*)}}$, *then*

$$\|\lambda' - \lambda\|_2 \leq \frac{2\left(1 + \|\lambda\|_2^2\right)^{\frac{1}{2}}}{\max\left\{|f_1(p)|, \ldots, |f_t(p)|\right\}} \sqrt{2}\, \|\mathfrak{J}_{f,p}(\Lambda)\|_{(\rho^*)}\, \varepsilon := \mathcal{R}_2.$$

Proof. Since $\rho = 1$, then $\left\|\mathrm{Jac}_{f_p(\Lambda)}(\lambda)\right\|_1 = \max\left\{|f_1(p)|, \ldots, |f_t(p)|\right\}$, and similarly $\left\|\mathrm{Jac}_{f_q(\Lambda)}(\lambda)\right\|_1 = \max\left\{|f_1(q)|, \ldots, |f_t(q)|\right\}$. Now, the result follows from Proposition 5.6. Q.E.D.

As already noted, Proposition 5.6 may be used to suggest lower bounds for δ as discussed in the following examples.

Example 5.10 Let us consider the 3-parametrized family $\mathcal{F} = \{\mathcal{C}_\lambda\}$ of cubic curves of equation

$$\mathcal{C}_\lambda : x^2 = -\lambda_3 y^3 - \lambda_1 y + \lambda_2,$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ are real parameters and $\lambda_3 \neq 0$. For each point $p = (x_p, y_p)$ of the image space, the Hough transform is the plane, in the parameter space $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$, of equation

$$f_p(\Lambda) : y_p\Lambda_1 - \Lambda_2 + y_p^3\Lambda_3 + x_p^2 = 0.$$

Expressing $f_{(x,y)}(\Lambda)$ as in Corollary 5.9, we get that

$$f_0(x,y) = x^2, \quad f_1(x,y) = y, \quad f_2(x,y) = -1, \quad f_3(x,y) = y^3.$$

From Definition 5.3 it follows that $\mathfrak{J}_{f,p}(\Lambda) = 2|x_p| + \Lambda_1 + 3|y_p|^2\Lambda_3$, and therefore $\|\mathfrak{J}_{f,p}(\Lambda)\|_{(1)} = \sqrt{1+4x_p^2+9y_p^4}$. We observe that $\mathfrak{U}_1 = \mathbb{R}^2$, whence $\mathfrak{U} = \mathfrak{U}_2$ (see (14) and (17)), and $\mathfrak{U}_2 = \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 \,|\, y = 0\}$. Take $y_p \neq 0$ and let $q = (x_p + \varepsilon_1, y_p + \varepsilon_2)$, with $0 < |\varepsilon_i| \leq \varepsilon$, $i = 1, 2$. By taking $\varepsilon_1 \neq -2x_p$, conditions (15) are satisfied, so that the points p and q are in relative general position. Corollary 5.9(1) gives

$$\|\lambda' - \lambda\|_2 \leq \frac{\left(1 + \|\lambda\|_2^2\right)^{\frac{1}{2}}}{\max\left\{|y_p + \varepsilon_1|, 1, |(y_p + \varepsilon_2)^3|\right\}} \sqrt{1+4x_p^2+9y_p^4}\,\sqrt{2}\,\varepsilon := \mathcal{R}_1.$$

Or, applying Corollary 5.9(2), it holds that

$$\|\lambda' - \lambda\|_2 \leq \frac{2\left(1 + \|\lambda\|_2^2\right)^{\frac{1}{2}}}{\max\left\{|y_p|, 1, |y_p^3|\right\}} \sqrt{1+4x_p^2+9y_p^4}\,\sqrt{2}\,\varepsilon := \mathcal{R}_2.$$

Now, we consider the curve $\mathcal{C}_{(-\frac{1}{2},\frac{5}{2},1)} \in \mathcal{F}$, and the point $p = \left(\sqrt{\frac{21}{8}}, \frac{1}{2}\right) \in \mathcal{C}_{(-\frac{1}{2},\frac{5}{2},1)}$. We take the threshold $\varepsilon = 0.02$ and the point $q = (1.63, 0.51)$; note that $p \in \mathfrak{U}$, $q \in \mathfrak{U}_1$ and p and q are in relative general position. We have

$$\begin{aligned} f_p(\Lambda) &= \frac{1}{2}\Lambda_1 - \Lambda_2 + \frac{1}{8}\Lambda_3 + \frac{21}{8} \\ f_q(\Lambda) &= 0.51\Lambda_1 - \Lambda_2 + (0.51)^3\Lambda_3 + (1.63)^2. \end{aligned}$$

Therefore

$$\|\mathfrak{J}_{f,p}(\Lambda)\|_{(1)} = \sqrt{1 + 4 \times \frac{21}{8} + 9 \times \frac{1}{16}} = \frac{\sqrt{193}}{4}.$$

Then, letting $\lambda = \left(-\frac{1}{2}, \frac{5}{2}, 1\right)$, the local upper bounds given by Corollary 5.9 are (note that the assumption in Corollary 5.9(2) is satisfied since we get $\varepsilon \leq 0.035$) are

$$\mathcal{R}_1 = \frac{\sqrt{1 + \frac{1}{4} + \frac{25}{4} + 1}}{\max\left\{0.51, 1, 0.51^3\right\}} \times \tfrac{\sqrt{193}}{4} \times \sqrt{2} \times \tfrac{2}{100} \approx 0.286.$$

$$\mathcal{R}_2 = \frac{2 \times \sqrt{1 + \frac{1}{4} + \frac{25}{4} + 1}}{\max\left\{\frac{1}{2}, 1, \frac{1}{8}\right\}} \times \tfrac{\sqrt{193}}{4} \times \sqrt{2} \times \tfrac{2}{100} \approx 0.573 \approx 2\mathcal{R}_1.$$

While the effective distance between the point $\lambda = \left(-\frac{1}{2}, \frac{5}{2}, 1\right)$ and the plane $\Gamma_q(\mathcal{F})$ is

$$d = \frac{\left|f_q\left(-\frac{1}{2}, \frac{5}{2}, 1\right)\right|}{\sqrt{(0.51)^2 + 1 + (0.51)^6}} = 0.03056.$$

To determine $\mathcal{R}_5$ (see Remark 5.8) note that

$$\begin{aligned} &\max\nolimits_{\mathbf{i}\in J}\left\{\max\nolimits_{0<t<1}\left\{\|\mathrm{Jac}_{f_{\mathbf{i}}}(q - t(\varepsilon_1, \varepsilon_2))\|_2\right\}\right\} \\ &\leq \max\left\{\max\nolimits_{0\leq t\leq 1}\{2\,|1.63 - t\,0.02|\}, 1, \max\nolimits_{0\leq t\leq 1}\{3\,(0.51 - t\,0.02)^2\}\right\} \\ &= \max\{3.24, 1, 0.7803\} = 3.24. \end{aligned}$$

Moreover, $\left\|\sum_{\mathbf{i}\in\mathrm{J}}\Lambda^{\mathbf{i}}\right\|_{(1)} = \|1+\lambda_1+\lambda_3\|_{(1)} = \sqrt{3}$. Therefore

$$\mathcal{R}_5 = \frac{\sqrt{1+\frac{1}{4}+\frac{25}{4}+1}}{\max\{0.51, 1, 0.51^3\}} \times \sqrt{2} \times \frac{2}{100} \times 3.24 \times \sqrt{3} \approx 0.4628.$$

Coming to the global bound, let us consider again the same points p, q and the threshold $\varepsilon = 0.02$ as above. We fix the region $\mathcal{T} = [-1,0]\times[2,3]\times[0.75,1.25]$ (compare with [20, Example 6.4]). We have $\max_{\tau\in\mathcal{T}}\left(1+\|\tau\|_2\right)^{\frac{1}{2}} \leq (1+1+9+(1.25)^2)^{\frac{1}{2}} \approx 3.544$. Then our global upper bounds given by Remark 5.7 are

$$\mathcal{R}_3 = \tfrac{3.544}{1} \times \sqrt{2} \times \tfrac{\sqrt{193}}{4} \times \tfrac{2}{100} \approx 0.347,$$

$$\mathcal{R}_4 = 2\mathcal{R}_3 \approx 0.696.$$

To visualize the effective distance of any point $\tau \in \mathcal{T}$ belonging to (the plane) $\Gamma_p(\mathcal{F})$: $f_p(\Lambda) = 0$ to the plane $\Gamma_q(\mathcal{F}) : f_q(\Lambda) = 0$, we argue as follows. Let $\tau = \lambda$. By the above, $\tau \in \Gamma_p(\mathcal{F})$ rewrites as $f_p(\tau) = \frac{1}{2}\lambda_1 - \lambda_2 + \frac{1}{8}\lambda_3 + \frac{21}{8} = 0$, whence $\lambda_2 = \frac{1}{2}\lambda_1 + \frac{1}{8}\lambda_3 + \frac{21}{8}$. Therefore

$$\begin{aligned}
d(\tau,\Gamma_q(\mathcal{F})) &= \frac{|0.51\lambda_1 - \lambda_2 + (0.51)^3\lambda_3 + (1.63)^2|}{\sqrt{(0.51)^2+1+(0.51)^6}}\\
&= \frac{1}{\sqrt{(0.51)^2+1+(0.51)^6}}\left|0.51\lambda_1 + (0.51)^3\lambda_3 + (1.63)^2 - \frac{1}{2}\lambda_1 - \frac{1}{8}\lambda_3 - \frac{21}{8}\right|\\
&= \frac{1}{\sqrt{(0.51)^2+1+(0.51)^6}}\left|0.01\lambda_1 + \left((0.51)^3 - \frac{1}{8}\right)\lambda_3 + (1.63)^2 - \frac{21}{8}\right|\\
&=: d(\lambda_1,\lambda_3).
\end{aligned}$$

The graph of the function $d = d(\lambda_1,\lambda_3)$, when (λ_1,λ_3) varies in the region $[-1,0]\times[0.75,1.25]$, is a (portion of) a plane "almost" parallel to the $\langle\Lambda_1,\Lambda_3\rangle$ plane in the parameter space having distance $d \approx 0.016$ from the plane $\Gamma_q(\mathcal{F}) : f_q(\Lambda) = 0$.

Example 5.11 Let us consider the 2-parametrized family $\mathcal{F} = \{\mathcal{C}_\lambda\}$, where

$$\mathcal{C}_\lambda : (x-\lambda_1)^2 + y^2 = \lambda_2.$$

For each point $p = (x_p, y_p)$ of the image space, the Hough transform is the parabola, in the parameter space $\Lambda = (\Lambda_1,\Lambda_2)$, of equation

$$f_p(\Lambda) : \Lambda_1^2 - 2x_p\Lambda_1 - \Lambda_2 + x_p^2 + y_p^2 = 0.$$

From Definition 5.2 we get $\mathrm{J} = \{(0,0),(1,0)\}$, so that $\mathfrak{J}_{f,p}(\Lambda) = 2\sqrt{x_p^2+y_p^2} + 2\,\Lambda_1$. Further, note that $f^*_{(x,y)}(\Lambda) = (x^2+y^2) - 2x\Lambda_1$. Then $\rho^* = 1$ and $\|\mathfrak{J}_{f,p}(\Lambda)\|_{(1)} = 2\sqrt{1+x_p^2+y_p^2}$. We observe that $\mathfrak{U}_1 = \mathfrak{U}_2 = \mathbb{R}^2$ and (see (14), (16) and (17)). Let $q = (x_p+\varepsilon_1, y_p+\varepsilon_2)$, with $0 < |\varepsilon_i| \leq \varepsilon$, $i = 1,2$. We assume $\varepsilon_1(2x_p+\varepsilon_1) + \varepsilon_2(2y_p+\varepsilon_2) \neq 0$, so conditions (15) are satisfied, saying that the points p and q are in relative general position. Thus, using Proposition 5.6, and taking into account that $\rho = 2$, it holds that

$$\|\lambda' - \lambda\|_2 \leq \frac{4(1+\|\lambda\|_2^2)\,\sqrt{2}\,\sqrt{1+x_p^2+y_p^2}\;\varepsilon}{\max\{1, 2\,|\lambda_1 - x_q|\}} := \mathcal{R}_1.$$

Now, we consider the curve $\mathcal{C}_{(1,1)} \in \mathcal{F}$, and the point $p = (1,1) \in \mathcal{C}_{(1,1)}$. We take the point $q = p + \frac{9}{10}(\varepsilon, -\varepsilon)$ with $\varepsilon > 0$; note that p, q are in relative general position, and $\|q - p\|_1 < \varepsilon$. Then, the upper bound $\mathcal{R}_1$ is $\mathcal{R}_1(\varepsilon) = \dfrac{12\sqrt{6}\,\varepsilon}{\max\left\{1, \frac{9}{5}\varepsilon\right\}}$ Thus, under the assumption that $0 < \varepsilon < \frac{5}{9}$, we get

$$\mathcal{R}_1(\varepsilon) = 12\sqrt{6}\,\varepsilon.$$

Now, let ε vary in the set $\Omega = \left\{0.001 \times \frac{i}{60} \,|\, i \in \{0, \ldots, 60\}\right\} \subset [0, 0.001]$. It holds that

$$\max\{\mathcal{R}_1(\varepsilon) \,|\, \varepsilon \in \Omega\} = 0.029.$$

Therefore, for every $\varepsilon \in \Omega$, the perturbed parabolas of equation $f_q(\Lambda) = 0$ cross the Euclidean ball centered at p and radius 0.029 (see Figure 2).

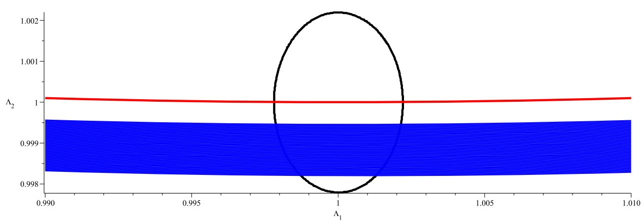

Figure 2: Curve $f_p(\Lambda) = 0$ (red), ball centered at $\lambda = (1,1)$ and radius 0.0022, and curves $\{f_q(\Lambda) = 0 \,|\, \varepsilon \in \Omega\}$ (blue).

The quantities $\mathcal{R}_i$ in Proposition 5.6 and Remark 5.7 may be taken, up to an error of $\mathrm{O}(\varepsilon^2)$, as lower bound candidates for the discretization step δ of the considered discretization; note that, for $\delta < \mathcal{R}_i$ we may not have crossing conditions, so we need $\delta \geq \mathcal{R}_i$. The following algorithmic procedure formalizes a *heuristic suggestion to lower bound* δ. It is inspired by Propositions 5.1 and 5.4, and it is expressed in terms of the local bound $\mathcal{R}_2$ as in Proposition 5.6. Such a bound depends only on p. The experimental quantities N and r below depend on the context, and, essentially, one needs practical experiments to get optimal choices for them; see Examples 6.3, 6.4.

Algorithm 5.12 (Lower bound discretization algorithm) Let $\mathcal{F} = \{\mathcal{C}_\lambda\}$ be a family of real curves as in Section 4. Let $\mathcal{P}$ be a profile of interest in the image plane $\mathbb{A}^2_{(x,y)}(\mathbb{R})$, pointed by a (finite) set $\mathcal{Q} = \{q_j\}_{j\in\mathcal{J}}$ of (exact) points q_j's belonging to the affine open set $\mathfrak{U}$ defined in (17). Perform the following instructions.

1. Compute $\varepsilon = \frac{1}{N}\min\{\|q_{j_1} - q_{j_2}\|_2 \mid j_1, j_2 \in \mathcal{J}, j_1 \neq j_2\}$ for a given positive number N.
2. Let $\mathbb{L}$ be an empty list and let r be a positive number. For $j \in \mathcal{J}$, do:
   (a) Take r smooth points $\lambda^{(i)}$ (depending on j) of $\Gamma_{q_j}(\mathcal{F})$, $i = 1, \ldots, r$.
   (b) If ε is small enough, namely, if
   $$\varepsilon \leq \frac{\max\left\{\left|\frac{\partial f_{q_j}(\Lambda)}{\partial \Lambda_1}\left(\lambda^{(i)}\right)\right|, \ldots, \left|\frac{\partial f_{q_j}(\Lambda)}{\partial \Lambda_t}\left(\lambda^{(i)}\right)\right|\right\}}{2\,\rho\left(1 + \|\lambda^{(i)}\|_2^2\right)^{\frac{\rho-1}{2}}\sqrt{2}\,\|\mathfrak{J}_{f,q_j}(\Lambda)\|_{(\rho^*)}},$$
   compute the bound $\mathcal{R}_2 = \mathcal{R}_2(i,j)$ and append it to the list $\mathbb{L}$. Otherwise go ahead with the next $\lambda^{(i)}$.

3. Compute the arithmetic mean value δ_{mean} of the entries of $\mathbb{L}$ and return it.

6 A Bombieri's norm based recognition algorithm

In this section we see how Theorem 2.2 can be applied to bound the number of Hough transforms crossing a cell of a given discretization of the parameter space $\mathcal{T}$. This voting procedure is the basic tool in the recognition algorithm on which the Hough transform technique is founded: we refer for this to [2, Sections 6, 7], [14, Section 4] and, for a more detailed version, to [15, Section 4]. This computation of the accumulator function and its maximization is the most time-consuming step of the algorithm. Further, it strongly depends on the number of parameters. So, in practice, the computational burden associated to the voting procedure and optimization leads to the need of restricting to families of curves depending on a small number of parameters. A novelty here is that our proposed algorithm does not depend of the number t of parameters in the game.

With the notations as in Section 4, let $K = \mathbb{R}$, and consider a discretization of a given bounded region $\mathcal{T}$ of the parameter space $\mathbb{A}^t_{(\Lambda_1,\dots,\Lambda_t)}(\mathbb{R})$, as follows. First, choose an *initializing point* $\lambda^* = (\lambda^*_1, \dots, \lambda^*_t)$ in $\mathcal{T}$ and let δ_k be the *sampling distance* with respect to the component λ_k. Then set

$$\lambda_{k,n_k} := \lambda^*_k + n_k\delta_k, \quad k = 1, \dots, t, \;\; n_k = 0, \dots, N_k - 1, \tag{20}$$

where $N_k \in \mathbb{N}$ denotes the number of considered samples for each component, and n_k the index of the sample. Then denote by

$$\mathbf{C}(\mathbf{n}) := \big\{\lambda = (\lambda_1, \dots, \lambda_t) \in \mathcal{T} \;\big|\; \lambda_k \in \big[\lambda_{k,n_k} - \frac{\delta_k}{2}, \lambda_{k,n_k} + \frac{\delta_k}{2}\big),\; k = 1, \dots, t\big\} \tag{21}$$

the cell with center in the *sampling point* $\lambda_{\mathbf{n}} := (\lambda_{1,n_1}, \dots, \lambda_{t,n_t})$ of the parameter space $\mathcal{T}$, where $\mathbf{n}$ denotes the multi-index $(n_1, \dots, n_t)$. We refer to $\mathbf{C}(\mathbf{n})$ as *the cell represented by the point* $\lambda_{\mathbf{n}}$. Then, the parameter space $\mathcal{T}$ is partitioned into a finite set of cells $\mathfrak{C} = \{\mathbf{C}(\mathbf{n})\}$. Now, we identify each cell with a parameter value λ, usually its center $\lambda_{\mathbf{n}}$, and we also write $\mathbf{C}(\lambda)$ to denote the cell containing a point $\lambda \in \mathcal{T}$.

Let us stress the fact that the discretization is defined by relation (20), that is, by the choice of the initializing point $\lambda^* \in \mathcal{T}$, the *discretization step* $\delta := (\delta_1, \dots, \delta_t)$ and the multi-index vector $\mathbf{n}$. As soon as the region $\mathcal{T}$ and the discretization step δ are given, then the multi-index vector $\mathbf{n}$ is determined. So, the discretization only depends on $\{\lambda^*, \delta\}$.

Let $\mathcal{P}$ be a profile of interest in the image plane $\mathbb{A}^2_{(x,y)}(\mathbb{R})$, pointed by the set $\mathcal{Q} = \{q_j\}_{j\in\mathcal{J}}$ of (exact) points q_j's. In the sequel, we assume that the discretization step δ is given by the results in Section 5, that is, we consider the output δ_{mean} of Algorithm 5.12 and the discretization $\{\lambda^*, \delta\}$, where the discretization step δ is defined by taking $\delta_k = \delta_{\text{mean}}$ for each component $k = 1, \dots, t$.

For each cell $\mathbf{C}(\lambda) \in \mathfrak{C}$, we aim to compute the number of Hough transforms of the points q_j, $j \in \mathcal{J}$, crossing it. That is, following the usual voting procedure as in the Hough transform techinque recognition algorithm, we want to estimate

$$\mu(\mathbf{C}(\lambda)) := \#\big\{j \in \mathcal{J} \,|\, \Gamma_{q_j}(\mathcal{F}) \cap \mathbf{C}(\lambda) \neq \emptyset\big\}.$$

In this situation, we then choose λ such that $\mu(\mathbf{C}(\lambda))$ is maximum, and we output as optimal approximation of the profile $\mathcal{P}$ the curve $\mathcal{C}_\lambda$ from the given family $\mathcal{F}$ of curves.

Let's briefly describe some preliminary input data for our recognition procedure.

I. The set $\mathcal{Q}$ of points in the image space highlighting the given profile $\mathcal{P}$, and belonging to the affine open set $\mathfrak{U}$ (see (17)).

II. The bounded region $\mathcal{T}$ (an hypercube) of the parameter space where we search our solution.

III. The point $\lambda^* \in \mathcal{T}$ and the discretization δ_{mean} defined as above. It may happen that the "safety discretization value" δ_{mean}, provided by Algorithm 5.12, is bigger than some of the sides of the "hypercube" defining the region $\mathcal{T}$. In that case we replace δ by $(r/2, \dots, r/2)$, where r is the lenght of the smallest side of the hypercube defining $\mathcal{T}$.

IV. A pair $\Upsilon = (w_1, w_2)$ of real numbers such that $0 < w_i \leq 100$. Precisely, Υ indicates that the algorithm will work with $w_1\%$ of the points in $\mathcal{Q}$, and $w_2\%$ of the cells in $\mathfrak{C}$; both cells and points are taken randomly.

Algorithm 6.1 Let $\mathcal{F} = \{\mathcal{C}_\lambda\}$ be a family of real curves as in Section 4. Let $\mathcal{Q} = \{q_j \,|\, j \in \mathcal{J}\}$ be a set of points in the image space belonging to the affine open set $\mathfrak{U}$ (defined in (17)) and let $\mathcal{T}$ be a hypercube in the parameter space. Let λ^* be a point in $\mathcal{T}$ and δ be the discretization step whose components are $\delta_k = \delta_{\text{mean}}$, with $k = 1, \dots t$, where δ_{mean} is the output of Algorithm 5.12. Let $\Upsilon = (w_1, w_2)$ be a pair of real numbers such that $0 < w_i \leq 100$. Perform the following steps.[4]

1. Consider the discretization $\{\lambda^*, \delta\}$ of the region $\mathcal{T}$.

2. For each cell $\mathbf{C}(\lambda) \in \{\lambda^*, \delta\}$, perform the following steps.

   (a) Let λ be the center of the cell; for each index $j \in \mathcal{J}$, verify that $\dfrac{\partial f_{q_j}}{\partial \Lambda_i}(\lambda) \neq 0$, for some $i \in \{1, \dots, t\}$. If this would not be the case, slightly perturb λ so that the above condition is satisfied.

   (b) Pick a point $p \in \mathfrak{U}_1$ in the image space such that $\lambda \in \Gamma_p(\mathcal{F})$: by the duality condition (11), this is equivalent to ask $\mathcal{C}_\lambda \ni p$. For each $j \in \mathcal{J}$ verify that $\dfrac{(f_{q_j}(\lambda))_\rho}{(f_p(\lambda))_\rho} \notin \mathbb{R}$, where $\rho = \deg(f_{q_j}(\lambda)) = \deg(f_p(\lambda))$ and $(f_{q_j}(\lambda))_\rho, (f_p(\lambda))_\rho$ are the homogeneous components of degree ρ of $f_{q_j}(\lambda), f_p(\lambda)$.

   (c) For each $j \in \mathcal{J}$, compute the bound expressed by Theorem 2.2(1),

   $$\mathcal{B}_{\lambda,q_j} := \alpha(\lambda, f_{q_j}, 1) \left\| f_{q_j} - \frac{(f_{q_j}, f_p)_{(\rho)}}{\|f_p\|^2_{(\rho)}} f_p \right\|_{(\rho)},$$

   so that the Euclidean distance $d(\lambda, \Gamma_{q_j}(\mathcal{F}))$ is bounded above by $\mathcal{B}_{\lambda,q_j}$.

   (d) Taking into account that $\|\delta\|_\infty > \mathcal{B}_{\lambda,q_j}$ implies that $\Gamma_{q_j}(\mathcal{F}) \cap \mathbf{C}(\lambda) \neq \emptyset$, compute the quantity $\nu_\lambda := \#\{j \in \mathcal{J} \mid \mathcal{B}_{\lambda,q_j} < \|\delta\|_\infty\}$.

3. Compute the quantity $\nu_{\mathfrak{C}} := \max_{\mathbf{C}(\lambda) \in \mathfrak{C}}\{\nu_\lambda\}$, which is called the *crossing number* of $\mathfrak{C}$, and define the set $\mathfrak{C}_{\text{good}} := \{\mathbf{C}(\lambda) \mid \nu_\lambda = \nu_{\mathfrak{C}}\}$.

[4] Let us point out that, once we have the value δ_{mean} from the results of Section 5, we only use Theorem 2.2(1) in the algorithm's steps. Thus, we only need here the assumption $p \in \mathfrak{U}_1$ for the point $p \in \mathcal{C}_\lambda$ in the game.

4. If $\nu_{\mathfrak{C}} = 0$, increase the discretization step δ (the same increase for each component) and go to step 1. Otherwise continue with step 5.

5. While $0 < \nu_{\mathfrak{C}}$ and $\nu_{\mathfrak{C}} \neq 1$ do:

   (a) For each $\mathbf{C}(\lambda) \in \mathfrak{C}_{\text{good}}$ let $\theta_\lambda := \max_{j \in \mathcal{J}}\{\mathcal{B}_{\lambda,q_j} \mid \mathcal{B}_{\lambda,q_j} < \|\delta\|_\infty\}$.

   (b) Choose $\mathbf{C}(\lambda_{\text{best}})$ with λ_{best} defined by the condition $\theta_{\lambda_{\text{best}}} := \min_{\mathbf{C}(\lambda) \in \mathfrak{C}_{\text{good}}}\{\theta_\lambda\}$.

   Note: this is done to find the potentially best answer; in fact, in general, $\mathfrak{C}_{\text{good}}$ has more than one element and, theoretically, any of them would be a suitable answer.

   (c) Set $\delta = \frac{1}{2}\delta$ and perform the instructions of steps 2 and 3.

6. Return λ_{best}.

Remark 6.2 We make a few observation regarding the choice of the point p in step 2(b) of Algorithm 6.1.

- There exist infinitely many points p that can be taken in step 2(b); indeed, all (say real) points in the curve $\mathcal{C}_\lambda$ defined by the polynomial $f_\lambda(x, y) = F(x, y; \lambda)$. To choose the point p, observe that Theorem 2.2(1) reads the bound

$$\begin{aligned}\mathcal{B}_{\lambda,q_j} &= \alpha(\lambda, f_{q_j}, 1) \ \left\| f_{q_j} - \frac{\left(f_{q_j}, f_p\right)_{(\rho)}}{\|f_p\|^2_{(\rho)}} f_p \right\|_{(\rho)} \\ &\leq \alpha(\lambda, f_{q_j}, 1) \ \left\| f_{q_j} - f_p \right\|_{(\rho)} \\ &\leq \alpha(\lambda, f_{q_j}, 1) \ \left(\left\| f_{q_j} \right\|_{(\rho)} + \|f_p\|_{(\rho)} \right).\end{aligned}$$

  Note that all quantities in the above upper bound of $\mathcal{B}_{\lambda,q_j}$ are fixed, but the norm $\|f_p\|_{(\rho)}$. So, take different random points $p \in \mathcal{C}_\lambda$ and choose one such that $\|f_p\|_{(\rho)}$ is the smallest.

- If the curve $\mathcal{C}_\lambda$ defined by $f_\lambda(x, y) = 0$ (see equation (9)) over the complex field $\mathbb{C}$ has genus less than or equal to six, we can take a radical parametrization to generate p (see [17]). If not, one may use an approximation of p. If we use a parametrization $\chi(t) = (x(t), y(t))$ (either rational or radical) of $\mathcal{C}_\lambda$, we will find a parameter value t_0 such that $\|f_\lambda(\chi(t_0)\|_{(\rho)}$ is minimum (see previous item). If no parametrization is provided, different random choices of the approximation of p are taken to finally select the one such that $\|f_p\|_{(\rho)}$ is the smallest.

We finish this section illustrating these ideas with some examples; the examples are taken from [20].

Example 6.3 We apply Algorithm 6.1 to the input given in Example 6.3. (part 1) in [20]. We consider a family of affine curves $\mathcal{F} = \{\mathcal{C}_\lambda\}$, $\lambda = (\lambda_1, \lambda_2)$, defined by the polynomial

$$f_\lambda(x, y) = (x^2 + y^2)^3 - (\lambda_1(x^2 + y^2) - \lambda_2(x^3 - 3xy^2))^2.$$

The curves in $\mathcal{F}$ are known as the sextic *curves with* 3 *convexities*. In this example, the data set $\mathcal{Q}$ of points pointing the profile of interest in the image consists in 320 points.

We apply our algorithm in the parameter region $\mathcal{T} = [0.35, 0.9] \times [0.175, 0.5]$ (as in [20] loc. cit.).

For the initializing step, we follow Algorithm 5.12 with $r = N = 10$, and we take

$$\delta^1 = \delta := \delta_{\text{mean}} = (0.28, 0.28).$$

As percentage vector we take $\Upsilon = (100, 100)$. In the first execution, Algoritm 6.1 found

$$\mathfrak{C}_{\text{good}} = \{\mathbf{C}((0.632, 0.175))\}$$

with $\theta_{(0.632,0.175)} = 0.273$ and $\nu_{(0.632,0.175)} = 10$ (see Figure 3, left panel). So, since the crossing number of $\mathfrak{C}$ is 10, a second iteration is performed. We take $\delta^2 = \frac{1}{2}\delta^1$, and the algorithm generates as $\mathfrak{C}_{\text{good}}$ the set

$$\mathfrak{C}_{\text{good}} = \{(0.632, .457), (0.773, 0.316), (0.773, 0.457), (0.773, 0.175), (0.491, 0.175), (0.491, 0.316)\}.$$

In Table 1, we show the $\lambda's$ found by the algorithm, as well as the numbers ν_λ and θ_λ associated to each $\mathbf{C}(\lambda) \in \mathfrak{C}_{\text{good}}$. We observe that the number of crossings of the new

$(\lambda, \nu_\lambda, \theta_\lambda)$	$(\lambda, \nu_\lambda, \theta_\lambda)$
((0.632, 0.457), 1, 0.07869809181)	((0.773, 0.316), 1, 0.07881713181)
((0.773, 0.457) 1, 0.0804252671)	((0.773, 0.175), 1, 0.09459597765)
((0.491, 0.175), 1, 0.1243367035)	((0.491, 0.316), 1, 0.1285879526)

Table 1: List of values $(\lambda, \nu_\lambda, \theta_\lambda)$ generated by the last iteration of Algorithm 6.1 applied to Example 6.3.

partition is 1. Then the algorithm chooses as output the λ with minimum θ_λ; in this case, it outputs $\lambda_{\text{best}} = (0.632, 0.457)$ (see Figure 3, right panel).

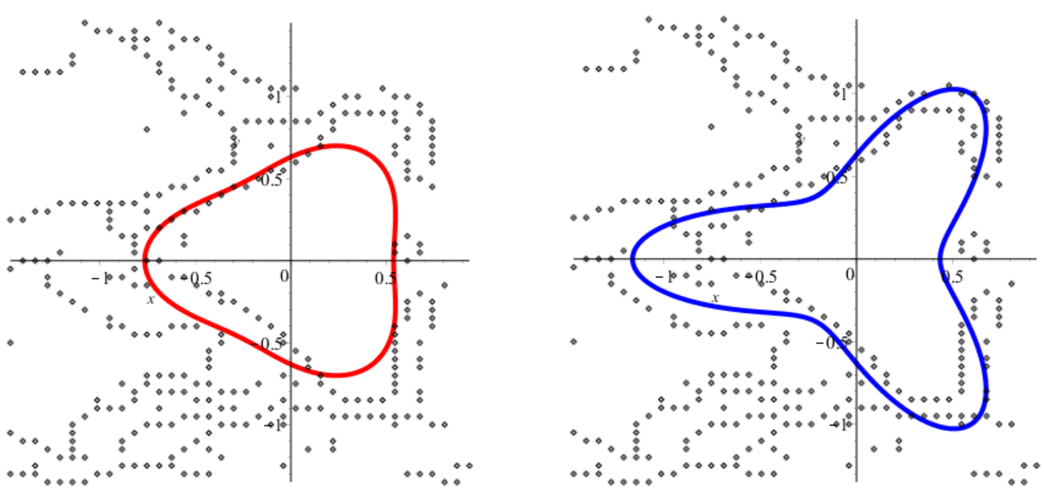

Figure 3: Plot of $\mathcal{Q}$ and $\mathcal{C}_{\lambda_{\text{best}}}$ (Left: after the first iteration; Right: after the second iteration) in Example 6.3.

Example 6.4 We apply our procedure to the input given in Example 6.3. (part 2) in [20]. We consider again a family $\mathcal{F} = \{\mathcal{C}_\lambda\}$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, of curves with 3 convexities defined now by the polynomial

$$f_\lambda(x, y) = (\lambda_3 x^2 + y^2)^3 - (\lambda_1(\lambda_3 x^2 + y^2) - \lambda_2(x^3 - 3xy^2))^2.$$

In this example, the data set $\mathcal{Q}$ of points pointing the profile of interest in the image consists in 132 points. We take the parameter region $\mathcal{T} = [0.7, 1] \times [0, 0.18] \times [0.9, 1.1]$ (see [20] loc. cit.).

For the initializing step, we follow Algorithm 5.12 with $r = N = 10$, and we take

$$\delta^1 = \delta := \delta_{\text{mean}} = (1.24, 1.24, 1.24).$$

Since $\|\delta\|_\infty$ is bigger than the length of the sides of $\mathcal{T}$, we proceed as explained in the input data (II)-(III) of the procedure description and we replace δ by $\delta = (0.09, 0.09, 0.09)$; note the 0.09 is half of the smallest length sides defining $\mathcal{T}$. As percentage vector we take $\Upsilon = (100, 100)$. In the first execution, the algorithm found

$$\begin{aligned}\mathfrak{C}_{\text{good}} =& \{(0.79, 0, 0.9), (0.88, 0, 0.9), (0.970, 0, 0.9), (0.79, 0, 0.99), (0.88, 0, 0.99), \\ & (0.97, 0, 0.99), (0.790, 0.09, 0.9), (0.88, 0.09, 0.9)\}.\end{aligned}$$

In Table 2, we show the $\lambda' s$ found by the algorithm, as well as the numbers ν_λ and θ_λ associated to each $\mathbf{C}(\lambda) \in \mathfrak{C}_{\text{good}}$. We observe that the crossings number of $\mathfrak{C}$ is 1. So,

$(\lambda, \nu_\lambda, \theta_\lambda)$	$(\lambda, \nu_\lambda, \theta_\lambda)$
((0.790, 0, 0.9), 1, 0.7217843317)	((0.880, 0, 0.9), 1, 0.7304567414)
((0.970, 0, 0.9), 1, 0.7508195658)	((0.790, 0, 0.990), 1, 0.7553552111)
((0.880, 0, 0.990), 1, 0.7586344023)	((0.970, 0, 0.990), 1, 0.7739644826)
((0.790, 009, 0.9), 1, 0.7805628033)	((0.880, 0.09, 0.9), 1, 0.7836837743)

Table 2: List of values $(\lambda, \nu_\lambda, \theta_\lambda)$ generated by the last iteration of Algorithm 6.1 applied to Example 6.4.

the algorithm ends choosing as output the λ with minimum θ_λ; in this case, it outputs $\lambda_{\text{best}} = (0.790, 0, 0.9)$ (see Figure 4).

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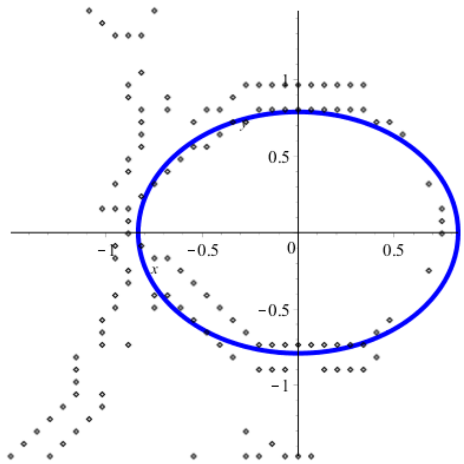

Figure 4: Plot of $\mathcal{Q}$ and $\mathcal{C}_{\lambda_{\text{best}}}$ in Example 6.4.

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M. Torrente, Istituto di Matematica Applicata e Tecnologie Informatiche "E. Magenes" CNR, Via de Marini, 6, I-16149 Genova, Italy. e-mail `torrente@dima.unige.it`

M.C. Beltrametti, Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, I-16146 Genova, Italy. e-mail `beltrame@dima.unige.it`

J. R. Sendra, Research Group ASYNACS. Dpto. de Física y Matemáticas, Universidad de Alcalá, 28871 Alcalá de Henares (Madrid), Spain. e-mail `Rafael.sendra@uah.es`