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(Article begins on next page)

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Rational General Solutions of Systems of First-Order Algebraic Partial Differential Equations

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Abstract

We study the rational solutions of systems of first-order algebraic partial differential equations and relate them to those of an associated autonomous system. We also describe how rational general solutions of these systems are related, and provide an algorithm in some particular case concerning the dimension of the associated algebraic variety. Our results can be considered as a generalization of the approach by L. X. C. Ngô and F. Winkler on algebraic ordinary differential equations of order one, adapted to systems of first-order algebraic partial differential equations.

Keywords: Algebraic partial differential equation, rational general solution, exact computation

1. Introduction

The study of solutions of systems of partial differential equations in the complex domain goes along a parallel study of that of a single equation. This is a classical approach in mathematics, which has proved successful in many circumstances. There are widespread approximation techniques of the solutions by different methods. For a historical review on the analytic approach we refer to [5]. In this paper, we study exact symbolic solutions of systems of algebraic partial differential equations. Our work is based on differential algebra techniques, put forward in [20, 25].

More precisely, our main aim is to investigate the exact rational solutions of systems of algebraic partial differential equations (APDEs, for short) from the point of view of an algebro-geometric treatment. We deal with systems of first-order partial differential equations. To such a system we associate an algebraic variety. If this associated variety admits a rational parametrization, we derive information on the rational solvability from such a parametrization.

An algebraic-geometric method for solving differential equations was proposed in [15] via Gröbner bases; see also [26]. Several advances have been made concerning not only the treatment of equations of higher order, but also the extension of results for autonomous equations to more general ones. Algebraic ordinary differential equations (AODEs, for short) were considered in [6, 7], where the authors develop an algorithm for deciding whether an autonomous first-order AODE admits a rational solution, and, in the affirmative case, for computing it. These investigations have been extended to radical solutions in [10, 12], to the non-autonomous case in [22, 24, 23], and to higher order AODEs in [14].

In the case of systems of ordinary differential equations, a first step is taken in [21] when studying systems of AODEs of algebro-geometric dimension one. A similar direction is explored in [13].

The multivariate setting of the problem deals with partial differential equations. The development concerning APDEs has progressed in a manner analogous to AODEs. In [11] we describe a solution method

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for a single autonomous first-order APDE. The latter method is not restricted to finding rational solutions. It also provides in some cases an implicit description of the solution even if the method fails. The present work deals with a next step in the theory, namely, the study of solutions of systems of first-order (not necessarily autonomous) APDEs. The method heavily relies on the construction of an auxiliary system of partial differential equations which is associated to the initial one, in the spirit of the univariate case studied in [22]. The so-called associated system is linear and autonomous, which might be easier to handle, as suggested by our examples. Thus, the present paper can be seen as a generalization to APDEs of the results developed for AODEs in [22].

Given a system of first-order APDEs, $\mathrm{S}_{\mathrm{diff}}$, we construct its associated system, $\mathrm{S}_{\mathrm{ass}}^{\mathcal{P}}$, via a proper parametrization $\mathcal{P}$ of the associated algebraic variety. $\mathcal{P}$-covered rational solutions of $\mathrm{S}_{\mathrm{diff}}$ generate a certain type of rational solutions of $\mathrm{S}_{\mathrm{ass}}^{\mathcal{P}}$; and rational solutions of $\mathrm{S}_{\mathrm{ass}}^{\mathcal{P}}$ under the natural assumption of being $\mathcal{P}$-suitable (see Definition 23) yield rational solutions of $\mathrm{S}_{\mathrm{diff}}$. As a matter of fact, this yields a one-to-one relation. The study of the behavior of general solutions of $\mathrm{S}_{\mathrm{diff}}$ and $\mathrm{S}_{\mathrm{ass}}^{\mathcal{P}}$ is much more involved.

Under the assumption of the differential primality of the ideal associated to $\mathrm{S}_{\mathrm{diff}}$, we show that a general solution of $\mathrm{S}_{\mathrm{diff}}$ has the expected number of free parameters and corresponds to a solution of $\mathrm{S}_{\mathrm{ass}}^{\mathcal{P}}$ having also the expected number of free parameters; and vice-versa, see Theorem 28 and Theorem 29. If we drop the hypothesis of primality, we still have this correspondence between the number of free parameters but we cannot ensure that the solutions are general in the classical sense of Ritt (see [25]).

The paper is structured as follows. In the last paragraphs of the current section, a summary of the notations considered in the work is stated for the sake of clarity of the interested readers. Section 2 states the hypotheses for the problem and several assumptions required in the subsequent sections. In Sections 2.1 and 2.2 we recall basic notions with respect to rational solutions of systems of APDEs and characteristic sets, respectively. In Section 3, we give details on a particular case of the problem, and describe an algorithm providing rational solutions of the systems under study. In Section 4, we describe the field extension containing all the coefficients of a general solution of $\mathrm{S}_{\mathrm{diff}}$ (see Theorem 16). Essentially, the number of indeterminate functions in the system is also the number of transcendental elements required in the extension of the ground field (see Corollary 17). In Section 5, we construct the associated system $\mathrm{S}_{\mathrm{ass}}^{\mathcal{P}}$ related to the initial problem $\mathrm{S}_{\mathrm{diff}}$, via a proper parametrization $\mathcal{P}$ of the associated algebraic variety. This is a system of first-order autonomous APDEs, which is easier to handle. A one-to-one correspondence of the so called $\mathcal{P}$-covered rational solutions of $\mathrm{S}_{\mathrm{diff}}$ (see Definition 6) and $\mathcal{P}$-suitable rational solutions of $\mathrm{S}_{\mathrm{ass}}^{\mathcal{P}}$ (see Definition 23) is stated in Theorem 18 and Theorem 21. Under the assumption of differential primality, Section 6 is devoted to describing the transformation from $\mathcal{P}$-covered rational general solutions of $\mathrm{S}_{\mathrm{diff}}$ to $\mathcal{P}$-suitable rational rank-general solutions of $\mathrm{S}_{\mathrm{ass}}^{\mathcal{P}}$ (see Theorem 28), and vice-versa (see Theorem 29). In the last section, we drop the requirement of differential primality and we see how rank-general solutions of both systems relate (see Section 7). We end with some conclusions and possible future lines of research in Section 8.

Notation. Throughout this paper we use the following notation:

1. $\mathbb{K}$ stands for an algebraically closed field of characteristic zero, and we consider the usual partial derivative $\frac{\partial}{\partial x_i}$ where $\frac{\partial x_j}{\partial x_i} = \delta_{ij}$ and the elements x_i are trascendental over $\mathbb{K}$.
2. We fix positive integers k, ℓ such that k stands for the number of independent variables and ℓ for the number of dependent variables.
3. By bold face letters we denote the tuples involved. More precisely, in the differential frame, we write
   (a) $\mathbf{x} = (x_1, \ldots, x_k)$ for the tuple of independent variables,
   (b) $\mathfrak{u}(\mathbf{x}) = (\mathfrak{u}_1(\mathbf{x}), \ldots, \mathfrak{u}_\ell(\mathbf{x}))$ for the tuple of indeterminate functions on $\mathbf{x}$, and
   (c) $\mathfrak{u}'(\mathbf{x}) = \left(\frac{\partial \mathfrak{u}_1}{\partial x_1}(\mathbf{x}), \ldots, \frac{\partial \mathfrak{u}_1}{\partial x_k}(\mathbf{x}), \ldots, \frac{\partial \mathfrak{u}_\ell}{\partial x_1}(\mathbf{x}), \ldots, \frac{\partial \mathfrak{u}_\ell}{\partial x_k}(\mathbf{x})\right)$ for the tuple of all first-order derivatives.
   (d) Analogously, if $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), \ldots, u_\ell(\mathbf{x}))$ is a tuple of differentiable functions depending on $\mathbf{x}$ we write $\mathbf{u}(\mathbf{x})' = \left(\frac{\partial u_1}{\partial x_1}(\mathbf{x}), \ldots, \frac{\partial u_1}{\partial x_k}(\mathbf{x}), \ldots, \frac{\partial u_\ell}{\partial x_1}(\mathbf{x}), \ldots, \frac{\partial u_\ell}{\partial x_k}(\mathbf{x})\right)$.
   (e) We deal with two different differential systems, both depending on $\mathbf{x}$. The main system, $\mathrm{S}_{\mathrm{diff}}$, involves $\mathbf{x}, \mathfrak{u}(\mathbf{x}), \mathfrak{u}'(\mathbf{x})$. For the second system, $\mathrm{S}_{\mathrm{ass}}^{\mathcal{P}}$, we introduce a tuple $\mathfrak{s}$ of $\beta := k + \ell$ new

indeterminate functions on $\mathbf{x}$, $\mathfrak{s}(\mathbf{x}) = (\mathfrak{s}_1(\mathbf{x}), \dots, \mathfrak{s}_\beta(\mathbf{x}))$, and similarly we denote by $\mathfrak{s}'$ the tuple of first-order derivatives.

In the next section we associate an algebraic variety to the differential system. When working in this algebraic frame, we also use the following tuples of undetermined variables

(a) $\mathbf{y} = (y_1, \dots, y_\ell)$, will replace $\mathfrak{u}(\mathbf{x})$,
(b) $\mathbf{z} = (z_{11}, \dots, z_{1k}, \dots, z_{\ell 1}, \dots, z_{\ell k})$, will replace $\mathfrak{u}'(\mathbf{x})$,
(c) $\mathbf{w} = (w_1, \dots, w_\beta)$, will replace $\mathfrak{s}(\mathbf{x})$.
(d) The associated algebraic variety in the next section is assumed to have a rational component of dimension β, i.e. the length of the tuple $\mathfrak{s}$. When expressing the variety parametrically, we use $\mathbf{t} = (t_1, \dots, t_\beta)$ for the tuple of parameters.

4. Finally, let $(f_1, \dots, f_m)$ be a tuple of functions in the variables $\mathbf{z} = (z_1, \dots, z_j)$. We denote by $\mathrm{Jac}_{\mathbf{z}}(f_1, \dots, f_m)$ the Jacobian (matrix) of the tuple of functions with respect to $\mathbf{z}$. If the set of variables is clear from the context we might also write $\mathrm{Jac}(f_1, \dots, f_m)$.

2. Preliminaries

We consider a (potentially) non-autonomous system of first-order algebraic partial differential equations

$$\mathrm{S}_{\mathrm{diff}} = \{F(\mathbf{x}, \mathfrak{u}, \mathfrak{u}') = 0\}_{F \in \mathcal{F}_{\mathrm{a}}} , \tag{1}$$

where $\mathcal{F}_{\mathrm{a}} \subset \mathbb{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ is a nonempty finite set of polynomials. Our goal is to analyze the existence of rational solutions of $\mathrm{S}_{\mathrm{diff}}$. For this purpose, we associate to $\mathrm{S}_{\mathrm{diff}}$ the system of algebraic equations

$$\mathrm{S}_{\mathrm{alg}} = \{F(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0\}_{F \in \mathcal{F}_{\mathrm{a}}} . \tag{2}$$

Let $\mathbb{V}(\mathcal{F}_{\mathrm{a}}) \subset \mathbb{K}^{k+\ell+k\ell}$ be the variety over $\mathbb{K}$ defined by $\mathcal{F}_{\mathrm{a}}$. We call it the *variety associated to* $\mathrm{S}_{\mathrm{alg}}$ and we denote it by $\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}$. We denote by $\mathcal{F}_{\mathrm{d}}$ the set of differential polynomials $\{F(\mathbf{x}, \mathfrak{u}, \mathfrak{u}')\}_{F \in \mathcal{F}_{\mathrm{a}}}$. Let $\mathcal{J}_{\mathrm{d}} = [\mathcal{F}_{\mathrm{d}}]$ be the differential ideal generated by $\mathcal{F}_{\mathrm{d}}$ in $\mathbb{K}(\mathbf{x})\{\mathfrak{u}\}$, and let $\mathcal{J}_{\mathrm{a}} = \langle \mathcal{F}_{\mathrm{a}} \rangle$ be the ideal in $\mathbb{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ generated by $\mathcal{F}_{\mathrm{a}}$. Additionally, we make the following

Assumptions:

(I) The differential ideal $\mathcal{J}_{\mathrm{d}}$, is prime;
(II) $\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}$ has a rational component of maximal dimension;
(III) $\dim(\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}) = \beta := k + \ell$;
(IV) We fix an orderly ranking (see e.g. page 75 in [20]) and let $\mathcal{A}$ be a characteristic set of $\mathcal{J}_{\mathrm{d}}$. We assume that every polynomial in $\mathcal{A}$ is of order 1 and that every first-order partial derivative of every component of $\mathfrak{u}$ is the leader of some polynomial in $\mathcal{A}$ (this entails that no equivalent system to $\mathrm{S}_{\mathrm{diff}}$ can contain algebraic equations);
(IVa) $\mathcal{J}_{\mathrm{a}} \cap \mathbb{K}[\mathbf{x}, \mathbf{y}] = \{0\}$.

Whether a differential ideal is prime, as stipulated in (I), is a difficult question. For algebraic ideals in multivariate polynomial rings over a field, we can decompose a radical ideal into prime ideals; see for instance [18]. In [2, Sec. 5], we read "*To our knowledge, there does not exist any algorithm which decides if a differential ideal given by a basis (...) is prime*". Hubert [16] revisits this problem area, but without giving an algorithm for testing primality of differential ideals. Golubitsky [9] raises the question in the conclusion, when he asks "*Can one efficiently construct a universal regular (Boulier et al., 1995)/characterizable (Hubert, 2000) decomposition of a radical differential ideal?*". To our knowledge, the question of whether a differential ideal is prime, is still not known to be decidable.

Also the question whether an algebraic set in arbitrary dimension is rational is algorithmically still open. Observe that Assumption (IV) implies Assumption (IVa), namely $\mathcal{J}_{\mathrm{a}} \cap \mathbb{K}[\mathbf{x}, \mathbf{y}] = \{0\}$. Indeed, polynomials of order less than one cannot be reduced to 0.

The next lemma ensures that under our hypotheses, the rank of the Jacobian of any proper parametrization of $\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}$ is reached by a principal minor. This property plays an important role in the process of generating the associated system in Section 5.

Lemma 1. *Under the assumptions (II), (III), and (IVa), let $\mathcal{P}$ be a proper rational parametrization of a component of maximal dimension of $\mathbb{V}_{\mathrm{S_{alg}}}$. Then, the $(k+\ell)\times(k+\ell)$ principal minor of the Jacobian of $\mathcal{P}$ is non-zero.*

Proof. Let W be such a component of maximal dimension. Let us assume that the $(k+\ell)\times(k+\ell)$ principal minor is zero. Let $\mathcal{P}_{k+\ell}$ be the tuple consisting of the first $(k+\ell)$ components of $\mathcal{P}$. By construction, $\mathcal{P}_{k+\ell}$ parametrizes the Zariski closure of the projection of W over the first $(k+\ell)$ coordinates. Therefore, the dimension of this projection equals the rank of the Jacobian of $\mathcal{P}_{k+\ell}$, and hence it is smaller than $k+\ell$. Therefore, $\mathcal{J}_{\mathrm{a}} \cap \mathbb{K}[\mathbf{x}, \mathbf{y}] \neq \{0\}$, which is a contradiction. □

The results in this paper do not all depend on the whole set of assumptions (I)–(IVa). In the following table we specify which assumptions are required in which sections.

Section	2.1	2.2	3	4	5	6	7
Assumptions	(II)	(I), (IV)	(II)	(I), (IV)	(II), (III), (IVa)	All	All but (I)

Table 1: Assumptions and sections

2.1. Rational solutions

In this subsection we assume hypothesis (II). We recall the basic notions of solution, and we introduce the concept of solution variety and covered solution. Let us consider the partial differential field $(\mathbb{K}(\mathbf{x}), \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k})$ and let $(\mathbb{F}, \delta_1, \ldots, \delta_k)$ be a differential field extension.

Definition 2. *A* solution *of* $\mathrm{S_{diff}}$ *is a* $\mathbf{u}(\mathbf{x}) = (u_1, \ldots, u_\ell) \in \mathbb{F}^\ell$ *such that*

$$F(\mathbf{x}, \mathbf{u}(\mathbf{x}), \delta_1(\mathbf{u}(\mathbf{x})), \ldots, \delta_k(\mathbf{u}(\mathbf{x}))) = 0\,,$$

for all $F \in \mathcal{F}_{\mathrm{a}}$*, and where* $\delta_i(\mathbf{u}(\mathbf{x})) = (\delta_i(u_1), \ldots, \delta_i(u_\ell))$.

Definition 3. *Let* $\mathbf{u}(\mathbf{x}) \in \mathbb{K}(\mathbf{x})^\ell$ *be a rational solution of* $\mathrm{S_{diff}}$*, and let* $\Omega_{\mathbf{u}} \subset \mathbb{K}^k$ *be the Zariski open subset where the evaluations of* $\mathbf{u}$ *are well-defined. The Zariski closure of*

$$\{(\mathbf{x}_0, \mathbf{u}(\mathbf{x}_0), \mathbf{u}'(\mathbf{x}_0)) : \mathbf{x}_0 \in \Omega_{\mathbf{u}}\}$$

is called the solution variety *of* $\mathbf{u}$. *We denote it by* $\mathbb{V}_{\mathbf{u}}$.

We observe that

$$\mathbb{V}_{\mathbf{u}} \subset \mathbb{V}_{\mathrm{S_{alg}}}.$$

In the next lemma we prove further properties of $\mathbb{V}_{\mathbf{u}}$.

Lemma 4. *Let* $\mathbf{u}(\mathbf{x})$ *be a rational solution of* $\mathrm{S_{diff}}$. *Then*

(i) $\mathbb{V}_{\mathbf{u}}$ *is rational. Moreover*

$$\mathbf{U}(\mathbf{x}) = (\mathbf{x}, \mathbf{u}(\mathbf{x}), \mathbf{u}(\mathbf{x})') \tag{3}$$

properly parametrizes $\mathbb{V}_{\mathbf{u}}$. *We call* $\mathbf{U}(\mathbf{x})$ *the* parametrization derived from the solution $\mathbf{u}$.

(ii) $\dim(\mathbb{V}_{\mathbf{u}}) = k$.

Proof. It is clear that $\mathbf{U}(\mathbf{x})$ parametrizes $\mathbb{V}_{\mathbf{u}}$. Moreover, since $\mathbf{x}$ appears as components of the parametrization $\mathbf{U}$, it holds that $\mathbb{K}(\mathbf{U}(\mathbf{x}))) = \mathbb{K}(\mathbf{x})$. So, the parametrization is proper. In addition, since the Jacobian of $\mathbf{U}$ has rank k, it holds that $\dim(\mathbb{V}_{\mathbf{u}}) = k$. □

Remark 5. *Lemma 4 (ii) implies that* $\dim(\mathbb{V}_{\mathrm{S_{alg}}}) \geq k$ *and obviously* $\beta < k + \ell + k\ell$. *The particular case* $\dim(\mathbb{V}_{\mathrm{S_{alg}}}) = k$ *is studied in Section 3.*

The previous lemma gives a hint on where the rational solutions of $\mathrm{S}_{\mathrm{diff}}$ come from. They are parametrizations of rational subvarieties of $\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}$ of dimension k. In order to analyze these subvarieties, we introduce the next definition.

Definition 6. *Let $\mathbf{u}(\mathbf{x})$ be a rational solution of $\mathrm{S}_{\mathrm{diff}}$, and let $\mathcal{P}$ be a proper rational parametrization of a maximal component of $\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}$. We say that $\mathbf{u}(\mathbf{x})$ is $\mathcal{P}$-*covered *if*

- *$\mathcal{P}^{-1}$ is well-defined at $\mathbf{U}(\mathbf{x})$.*
- *$\mathcal{P}$ is well-defined at $\mathcal{P}^{-1}(\mathbf{U}(\mathbf{x}))$.*

Remark 7. *The previous definition essentially requires that*

$$\mathbb{V}_{\mathbf{u}} \cap \mathrm{dom}(\mathcal{P}^{-1}) \cap \mathrm{Im}(\mathcal{P})$$

is a Zariski dense subset of $\mathbb{V}_{\mathbf{u}}$.

2.2. Basic facts on characteristic sets

In this section we assume hypotheses (I), (IV). We fix an orderly ranking. Let $\mathcal{A}$ be a characteristic set (see e.g. page 82 in [20]) of $\mathcal{J}_{\mathrm{d}}$, and let H be the product of all initials and separants of the polynomials in $\mathcal{A}$. Then, we introduce the saturation of the ideal $[\mathcal{A}]$

$$[\mathcal{A}] : H^{\infty} = \{p \in \mathbb{K}(\mathbf{x})\{\mathfrak{u}\} \,|\, \exists n \in \mathbb{N} \cup \{0\} \text{ such that } H^n \cdot p \in [\mathcal{A}]\}\,.$$

Since $\mathcal{J}_{\mathrm{d}}$ is prime and $\mathcal{A}$ is a characteristic set of $\mathcal{J}_{\mathrm{d}}$, we have (see [20, Lem. 2, p. 167])

$$\mathcal{J}_{\mathrm{d}} = [\mathcal{A}] : H^{\infty}\,.$$

Characteristic sets can be computed using the characteristic set method [25, 34], Rosenfeld-Groebner algorithm [2, 3, 17] or differential Thomas decomposition [31, 32]. The Rosenfeld-Groebner algorithm is included in Maple in the DifferentialAlgebra package developed by F. Boulier and E. S. Cheb-Terrab. There are also implementations for Thomas decomposition (see for instance [1]). Thus, Assumption (IV) can be easily checked. In the following remark, we collect some facts from differential algebra that are used throughout the paper.

Remark 8. *Under the assumptions made for this section, the following statements hold:*

(a) *The order of H is at most 1.*

Proof. By (IV), each polynomial in $\mathcal{A}$ is of order 1. We have $\mathrm{ord}(\mathrm{sep}(P)) \leq \mathrm{ord}(P)$, and for every $P \in \mathcal{A}$, the order of its initial is at most 1. □

(b) *Let G be of order 1, and G^* its pseudo-remainder w.r.t. $\mathcal{A}$. Then,*

$$\left(\prod_{A\in\mathcal{A}} I_A^{m_A} S_A^{n_A}\right) G = G^* + \sum_{A\in\mathcal{A}} T_A A\,,$$

where $m_A, n_A \in \mathbb{N} \cup \{0\}, T_A \in \mathbb{K}(\mathbf{x})\{\mathfrak{u}\}$, and where I_A and S_A denote the initial and the separant of $A \in \mathcal{A}$.

Proof. By Proposition 1, p. 79, [20], we have

$$\left(\prod_{A\in\mathcal{A}} I_A^{m_A} S_A^{n_A}\right) G = G^{\star} + \sum_{A\in\mathcal{B}} T_A A,$$

where

$$\mathcal{B} = \{\theta A : A \in \mathcal{A}, \theta(\mathrm{ld}(A)) \leq \mathrm{ld}(G), \theta \in \Theta\}$$

(see page 59 in [20] for the notation for the set of derivative operators). By Assumption (IV), every polynomial in $\mathcal{A}$ has a leader of order 1, hence any proper derivative of the leader is of order 2 and hence greater than the leader of G due to the orderly ranking. We conclude $\mathcal{B} \subseteq \mathcal{A}$. □

(c) *Neither any initial nor any separant of the polynomials in $\mathcal{A}$ belong to $\mathcal{J}_{\mathrm{d}}$.*

Proof. Since $\mathcal{J}_{\mathrm{d}}$ is prime, the result follows from the second to the last paragraphs in [26], page 252. □

(d) *If $G(\mathbf{x}, \mathfrak{u}, \mathfrak{u}')$ is an initial or a separant of a differential polynomial in $\mathcal{A}$, then $G(\mathbf{x}, \mathbf{y}, \mathbf{z}) \notin \mathcal{J}_{\mathrm{a}}$.*

Proof. Let $G(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathcal{J}_{\mathrm{a}}$, then $G(\mathbf{x}, \mathfrak{u}, \mathfrak{u}') \in \mathcal{J}_{\mathrm{d}}$. The conclusion follows from (IV). □

3. Case $\dim(\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}) = k$

In this section, assuming (II), we analyze the special case where $\dim(\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}) = k$. Let $\mathbf{u}(\mathbf{x})$ be a rational solution of $\mathrm{S}_{\mathrm{diff}}$. Then, $\mathbb{V}_{\mathbf{u}} \subset \mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}$, $\dim(\mathbb{V}_{\mathbf{u}}) = \dim(\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}})$. Thus, $\mathbb{V}_{\mathbf{u}}$ is an irreducible component of $\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}$. We have the next theorem.

Theorem 9. *Let $\dim(\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}) = k$ and let W be a k-dimensional rational irreducible component of $\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}$. The following statements are equivalent*

(a) *There exists a proper rational parametrization $\mathcal{P}(\mathbf{t}) = (\chi_1(\mathbf{t}), \ldots, \chi_{k+\ell+k\ell}(\mathbf{t}))$ of W such that*
   (i) *$\mathcal{P}_1 := (\chi_1(\mathbf{t}), \ldots, \chi_k(\mathbf{t}))$ defines a birational map from $\mathbb{K}^k$ onto $\mathbb{K}^k$.*
   (ii) *There exists $\xi(\mathbf{t}) \in \mathbb{K}(\mathbf{t})^\ell$ such that $\mathcal{P}(\mathcal{P}_1^{-1}(\mathbf{t})) = (\mathbf{t}, \xi(\mathbf{t}), \xi(\mathbf{t})')$.*

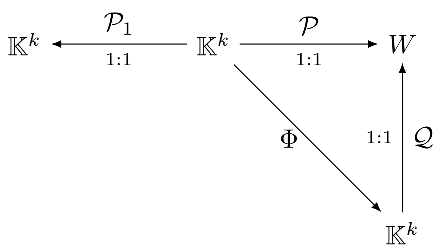
(b) *Every proper rational parametrization of W satisfies conditions (a)(i) and (a)(ii).*

Rational solutions of $\mathrm{S}_{\mathrm{diff}}$ come from the parametrizations of irreducible components of $\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}$ satisfying the above property.

Proof. First we prove the equivalence of (a) and (b). Let $\mathcal{P}(\mathbf{t}), \mathcal{P}_1(\mathbf{t}), \xi(\mathbf{t})$ satisfy (a), and let $\mathcal{Q}(\mathbf{t})$ be a proper rational parametrization of W. Since both parametrizations are proper, there exists $\Phi(\mathbf{t}) \in \mathbb{K}(\mathbf{t})^k$ such that $\mathcal{Q}(\Phi(\mathbf{t})) = \mathcal{P}(\mathbf{t})$. Therefore,

$$\mathcal{Q}(\Phi(\mathcal{P}_1^{-1}(\mathbf{t}))) = \mathcal{P}(\mathcal{P}_1^{-1}(\mathbf{t})) = (\mathbf{t}, \xi(\mathbf{t}), \xi(\mathbf{t})'),$$

and clearly $\Phi \circ \mathcal{P}_1^{-1}$ is a birational map from $\mathbb{K}^k$ on $\mathbb{K}^k$ (see Diagram (4)). So, (b) holds.



(4)

(b) clearly implies (a).

Let $\mathbf{u}(\mathbf{x})$ be a rational solution of $\mathrm{S}_{\mathrm{diff}}$. Then, by Lemma 4, $\mathbf{U}(\mathbf{x})$ is a proper rational parametrization of a k-dimensional component of $\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}$ satisfying the conditions in (a). On the other hand, let $\mathcal{P}(\mathbf{t})$ and $\xi(\mathbf{t})$ be as in (a). Then, since $\mathcal{P}(\mathcal{P}_1^{-1}(\mathbf{t}))$ parametrizes a k-dimensional component of $\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}$, we get that $\xi(\mathbf{t})$ is a rational solution of $\mathrm{S}_{\mathrm{diff}}$. □

Note that Theorem 9 implies that the number of k-dimensional rational components of $\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}$ is an upper bound for the number of rational solutions of $\mathrm{S}_{\mathrm{diff}}$.

The following algorithm is derived from the previous theorem.

Algorithm 1 Rational Solutions when $\dim(\mathbb{V}_{\mathrm{S_{alg}}}) = k$ (General assumptions as above).

The algorithm takes as input a system of APDEs $\mathrm{S_{diff}}$ such that $\dim(\mathbb{V}_{\mathrm{S_{alg}}}) = k$, and it returns the set L of all rational solutions.

1: Determine the k-dimensional rational components $\{W_1, \ldots, W_r\}$ of $\mathbb{V}_{\mathrm{S_{alg}}}$.
2: Set $L := \emptyset$.
3: **for** i from 1 to r **do**
4: Compute a proper rational parametrization $\mathcal{P}(\mathbf{t}) = (\chi_1(\mathbf{t}), \ldots, \chi_{k+\ell+k\ell}(\mathbf{t}))$ of W_i.
5: Check whether $\mathcal{P}_1 := (\chi_1, \ldots, \chi_k)$ is a birational map from $\mathbb{K}^k$ to $\mathbb{K}^k$.
$\triangleright$ One may decide this using e.g. Gröbner bases. If $k = 1$, $\mathcal{P}_1$ is birational iff $\mathcal{P}_1$ is a linear rational function.
6: **if** $\mathcal{P}_1$ is birational **then**
7: compute $\mathcal{P}(\mathcal{P}_1^{-1})$ $\triangleright$ Say that $\mathcal{P}(\mathcal{P}_1^{-1}) = (\mathbf{t}, \psi_{k+1}(\mathbf{t}), \ldots, \psi_{k+\ell+k\ell}(\mathbf{t}))$
8: **if** $(\psi_{k+1}, \ldots, \psi_{k+\ell})' = (\psi_{k+\ell+1}, \ldots, \psi_{k+\ell+k\ell})$ **then** $L := L \cup \{(\psi_{k+1}, \ldots, \psi_{k+\ell})\}$
9: **end if**
10: **end if**
11: **end for**
12: **return** L.

In the following we briefly comment on how to perform the steps of Algorithm 1. In Step 1 one needs to compute the irreducible components of $\mathbb{V}_{\mathrm{S_{alg}}}$. This can be done, for instance, by means of Gröbner bases (see e.g. [8], [33]). Furthermore, one can also compute the dimension with similar techniques (see e.g. Chapter 9 in [4]). For checking the rationality of the components of $\mathbb{V}_{\mathrm{S_{alg}}}$, one may use the methods described in [29] for dimension 1, and in [28] for dimension 2; if the dimension is higher than 2 one may find answers for particular cases but we do not know any general method and the problem is open. The algorithms described in [29] and [28] provide proper parametrizations when the dimension is 1 or 2, respectively; Again for dimension higher than 2 the problem is open. The invertibility of $\mathcal{P}_1$ (see Step 5), and the actual computation of the inverse (see Step 7), can be performed by using the ideas in [27] or in [29].

Example 10. *We consider the system of ODEs*

$$\mathrm{S_{diff}} = \begin{cases} \mathfrak{u}_1' + \mathfrak{u}_1^2 = 0 \\ x^3 \mathfrak{u}_1' \mathfrak{u}_1 + 1 = 0 \end{cases}$$

In this example we have $k = 1 = \ell$. We work in $\mathbb{C}^3$ using coordinates x, y, z. We have $\mathcal{F}_{\mathrm{a}} = \{z + y^2, x^3 yz + 1\}$, which decomposes into prime ideals $J_1 = \langle z + y^2, -1 + zx^2 - xy \rangle$, and $J_2 = \langle -1 + xy, z + y^2, xz + y \rangle$. Both ideals are of dimension 1, which we proceed to study. On the one hand, J_1 determines a curve in space which does not provide any solutions of the problem under study, while J_2 provides the solution $u(x) = 1/x$.

Example 11. *We consider the system of ODEs*

$$\mathrm{S_{diff}} = \begin{cases} -x^3 \mathfrak{u}_1' + x^3 - x\mathfrak{u}_1' + 2\mathfrak{u}_1 = 0 \\ x^4 \mathfrak{u}_1' - x^4 + 2x^2 \mathfrak{u}_1' - 3x^2 + \mathfrak{u}_1' = 0 \end{cases}$$

So, $k = 1 = \ell$. We work in $\mathbb{C}^3$ with coordinates x, y, z. Furthermore, $\mathcal{F}_{\mathrm{a}} = \{x^4 z - x^4 + 2x^2 z - 3x^2 + z, -x^3 z + x^3 - xz + 2y\}$. Therefore, $\mathbb{V}_{\mathrm{S_{alg}}}$ is rational of dimension 1. Hence $\dim(\mathbb{V}_{\mathrm{S_{alg}}}) = k$. Moreover, a proper rational parametrization is

$$\mathcal{P}(t) = \left(\frac{t+1}{t}, \frac{(t+1)^3}{t\,(2\,t^2 + 2\,t + 1)}, \frac{4\,t^4 + 10\,t^3 + 9\,t^2 + 4\,t + 1}{4\,t^4 + 8\,t^3 + 8\,t^2 + 4\,t + 1} \right).$$

We observe that $\mathcal{P}_1$ is invertible and $\mathcal{P}_1^{-1} = \frac{1}{t-1}$. We compute

$$\mathcal{P}(\mathcal{P}_1^{-1}) = \left(t, \frac{t^3}{t^2 + 1}, \frac{t^2\,(t^2 + 3)}{t^4 + 2\,t^2 + 1} \right).$$

Moreover,

$$\left(\frac{t^3}{t^2+1}\right)' = \frac{t^2\left(t^2+3\right)}{t^4+2\,t^2+1}\,.$$

So, we conclude that $u(x) = \frac{x^3}{x^2+1}$ *is a rational solution of* $\mathrm{S_{diff}}$*.*

Example 12. *We consider the system of PDEs*

$$\mathrm{S_{diff}} = \begin{cases} -\dfrac{\partial \mathfrak{u}_1}{\partial x_2}(x_1,x_2) + x_1 = 0 \\ \dfrac{\partial \mathfrak{u}_1}{\partial x_2}(x_1,x_2)\,\dfrac{\partial \mathfrak{u}_1}{\partial x_1}(x_1,x_2) - \left(\dfrac{\partial \mathfrak{u}_1}{\partial x_2}(x_1,x_2)\right)^2 - \mathfrak{u}_1(x_1,x_2) = 0 \\ 2\,\dfrac{\partial \mathfrak{u}_1}{\partial x_2}(x_1,x_2) - \dfrac{\partial \mathfrak{u}_1}{\partial x_1}(x_1,x_2) + x_2 = 0 \end{cases}$$

So, $k = 2, \ell = 1$*. We work in* $\mathbb{C}^5$ *with variables* x_1, x_2, y_1, z_1, z_2*. Furthermore,* $\mathcal{F}_{\mathrm{a}} = \{-z_2 + x_1, 2z_2 - z_1 + x_2, z_1 z_2 - z_2^2 - y_1\}$*. Therefore,* $\mathbb{V}_{\mathrm{S_{alg}}}$ *is rational of dimension 2. Hence* $\dim(\mathbb{V}_{\mathrm{S_{alg}}}) = k$*. Moreover, a proper rational parametrization is*

$$\mathcal{P}(t) = \left(t_1, -t_1^2 + t_2, t_1^2 + (-t_1^2 + t_2)t_1, -t_1^2 + 2t_1 + t_2, t_1\right).$$

We observe that $\mathcal{P}_1$ *is invertible and* $\mathcal{P}_1^{-1} = (t_1, t_1^2 + t_2)$*. We compute*

$$\mathcal{P}(\mathcal{P}_1^{-1}) = \left(t_1, t_2, t_1^2 + t_1 t_2, 2t_1 + t_2, t_1\right)\,.$$

Moreover,

$$\frac{\partial}{\partial t_1}(t_1^2 + t_1 t_2) = 2t_1 + t_2,\quad \frac{\partial}{\partial t_2}(t_1^2 + t_1 t_2) = t_1\,.$$

So, we conclude that $u(x_1, x_2) = x_1^2 + x_1 x_2$ *is a rational solution of* $\mathrm{S_{diff}}$*.*

4. Rational General Solutions of First-Order Systems of Algebraic PDEs

In this section, we assume hypotheses (I) and (IV). We analyze properties of a rational general solution of $\mathrm{S_{diff}}$. As in Subsection 2.2, we consider an orderly ranking in $\mathbb{K}(\mathbf{x})\{\mathfrak{u}\}$ and we assume that $\mathcal{A}$ is a characteristic set of $\mathcal{J}_{\mathrm{d}}$.

Definition 13. *Let* $\mathbb{F}$ *be an extension field of* $\mathbb{K}$ *and let* $\mathbf{u}(\mathbf{x}) \in \mathbb{F}(\mathbf{x})^\ell$ *be a rational solution of* $\mathrm{S_{diff}}$*. We say that* $\mathbf{u}(\mathbf{x})$ *is a* rational general solution *of* $\mathrm{S_{diff}}$ *if* $\mathbf{u}(\mathbf{x})$ *is a generic zero of* $\mathcal{J}_{\mathrm{d}}$*. In other words, if for* $G \in \mathbb{K}(\mathbf{x})\{\mathfrak{u}\}$*, it holds that* $G \in \mathcal{J}_{\mathrm{d}}$ *if and only if* $G(\mathbf{u}(\mathbf{x})) = 0$*.*

The following lemma characterizes the notion of general solution. For this purpose, we need to introduce first some additional terminology. By hypothesis (IV), we know that all polynomials in $\mathcal{A}$ have order 1, and all partial derivatives appear in the heads of the polynomials in $\mathcal{A}$. Let $\mathcal{R}_{\mathrm{d}}$ be the set of all possible pseudo-remainders w.r.t. $\mathcal{A}$, that is (ld denotes the leader)

$$\mathcal{R}_{\mathrm{d}} = \{G \in \mathbb{K}(\mathbf{x})\{\mathfrak{u}\} \mid \mathrm{ord}(G) \leq 1, \deg_{\mathrm{ld}(A)}(G) < \deg_{\mathrm{ld}(A)}(A) \text{ for all } A \in \mathcal{A}\}, \tag{5}$$

and let $\mathcal{R}_{\mathrm{a}}$ be the set of all algebraic polynomial associated to the differential polynomials in $\mathcal{R}_{\mathrm{d}}$. Note that $\mathcal{R}_{\mathrm{a}} \subset \mathbb{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$.

Lemma 14. *Let* $\mathbf{u}(\mathbf{x})$ *be a rational solution of* $\mathrm{S_{diff}}$*. Then, the following statements are equivalent*

(i) $\mathbf{u}(\mathbf{x})$ *is a* rational general solution *of* $\mathrm{S_{diff}}$*.*
(ii) *For every* $G(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathcal{R}_{\mathrm{a}}$ *it holds that* $G(\mathbf{x}, \mathbf{u}(\mathbf{x}), \mathbf{u}(\mathbf{x})') = 0 \Longleftrightarrow G(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0$*.*

Proof. We first observe that, by construction, $\mathrm{prem}(G(\mathbf{x},\mathfrak{u},\mathfrak{u}'),\mathcal{A}) = G(\mathbf{x},\mathfrak{u},\mathfrak{u}')$. Therefore, if $\mathbf{u}(\mathbf{x})$ is a general solution, using that $\mathcal{J}_{\mathrm{d}}$ is prime (see pp. 31 in [30]), we have that $G(\mathbf{x},\mathbf{u}(\mathbf{x}),\mathbf{u}(\mathbf{x})') = 0$ iff $G(\mathbf{x},\mathfrak{u},\mathfrak{u}') \in \mathcal{J}_{\mathrm{d}}$ iff $\mathrm{prem}(G(\mathbf{x},\mathfrak{u},\mathfrak{u}'),\mathcal{A}) = 0$ iff $G(\mathbf{x},\mathfrak{u},\mathfrak{u}') = 0$ identically . So, (i) implies (ii).

Conversely, let us see that (ii) implies (i). Let $G(\mathbf{x},\mathfrak{u}) \in \mathbb{K}(\mathbf{x})\{\mathfrak{u}\}$ and $G^*(\mathbf{x},\mathfrak{u}) = \mathrm{prem}(G,\mathcal{A})$. Then, $G^* \in \mathcal{R}_{\mathrm{d}}$ and hence $G^*(\mathbf{x},\mathbf{y},\mathbf{z}) \in \mathcal{R}_{\mathrm{a}}$. Thus, $G \in \mathcal{J}_{\mathrm{d}}$ iff $G^*(\mathbf{x},\mathfrak{u},\mathfrak{u}') = 0$ iff $G^*(\mathbf{x},\mathbf{y},\mathbf{z}) = 0$ iff $G^*(\mathbf{x},\mathbf{u}(\mathbf{x}),\mathbf{u}(\mathbf{x})') = 0$. So, it only remains to prove that $G(\mathbf{x},\mathbf{u}(\mathbf{x}),\mathbf{u}(\mathbf{x})') = 0$ iff $G^*(\mathbf{x},\mathbf{u}(\mathbf{x}),\mathbf{u}(\mathbf{x})') = 0$. Since G^* is the pseudo-remainder of G w.r.t. $\mathcal{A}$, by Prop. 1 page 79 in [20], $G^* - (\prod_{A\in\mathcal{A}} I_A^{m_A} S_A^{n_A})G \in [\mathcal{A}]$, where $m_A, n_A \in \mathbb{N}\cup\{0\}$, and where I_A and S_A are the initial and the separant of A, respectively. So,

$$G^*(\mathbf{x},\mathbf{u}(\mathbf{x}),\mathbf{u}(\mathbf{x})') \equiv \left(\prod_{A\in\mathcal{A}} I_A^{m_A}(\mathbf{x},\mathbf{u}(\mathbf{x}),\mathbf{u}(\mathbf{x})')S_A^{n_A}(\mathbf{x},\mathbf{u}(\mathbf{x}),\mathbf{u}(\mathbf{x})')\right) G(\mathbf{x},\mathbf{u}(\mathbf{x}),\mathbf{u}(\mathbf{x})') \pmod{[\mathcal{A}]}.$$

Observe that $I_A, S_A \in \mathcal{R}_{\mathrm{d}}$ and are not zero. So, by our assumptions in (ii) they cannot vanish at $(\mathbf{x},\mathbf{u}(\mathbf{x}),\mathbf{u}(\mathbf{x})')$. This implies the result. □

Lemma 15. *Let $\mathbf{u}_{\mathrm{g}}(\mathbf{x})$ be a rational general solution of $\mathrm{S}_{\mathrm{diff}}$. Let $\mathbb{L}$ be the smallest field extension of $\mathbb{K}$ containing all coefficients of $\mathbf{u}_{\mathrm{g}}(\mathbf{x})$. Then, $\mathbb{L}$ is transcendental over $\mathbb{K}$.*

Proof. Let $\mathbf{u}_{\mathrm{g}}(\mathbf{x}) = (u_1(\mathbf{x})\ldots,u_\ell(\mathbf{x}))$ with $u_j(\mathbf{x}) = \frac{A_j(\mathbf{x})}{B_j(\mathbf{x})}$, where $A_j, B_j \in \mathbb{L}[\mathbf{x}]$ and $\gcd(A_j,B_j) = 1$. We consider the polynomials

$$T_j(\mathbf{x},\mathbf{y}) = y_jB_j(\mathbf{x}) - A_j(\mathbf{x}) \in \mathbb{L}[\mathbf{x},\mathbf{y}].$$

Let $\mathcal{T} = \{T_1,\ldots,T_\ell\}$. If $\mathbb{L} = \mathbb{K}$ then, $\mathcal{T} \subset \mathbb{L}[\mathbf{x},\mathbf{y}] = \mathbb{K}[\mathbf{x},\mathbf{y}] \subset \mathcal{R}_{\mathrm{a}}$. Moreover, let $V = \mathbb{V}(\mathcal{T})$ be the variety defined by $\mathcal{T}$ over $\mathbb{K}^{k+\ell}$. We observe that $T_j(\mathbf{x},\mathbf{u}_{\mathrm{g}}(\mathbf{x})) = 0$. Therefore, V contains an irreducible component W parametrized by $(\mathbf{x},\mathbf{u}_{\mathrm{g}}(\mathbf{x}))$. Thus,

$$\dim(V) \geq \dim(W) = \mathrm{rank}(\mathrm{Jac}_{\mathbf{x}}(\mathbf{x},\mathbf{u}_{\mathrm{g}}(\mathbf{x}))) = k.$$

Now, since $\mathcal{T} \subset \mathcal{R}_{\mathrm{a}}$ and all polynomials in $\mathcal{T}$ vanish at $\mathbf{u}_{\mathrm{g}}(\mathbf{x})$, and since $\mathbf{u}_{\mathrm{g}}(\mathbf{x})$ is a rational general solution, by Lemma 14 we have that $\mathcal{T} = \{0\}$. Thus $V = \mathbb{K}^{k+\ell}$. So, $V = W$ and $k+\ell = \dim(V) = \dim(W) = k$ which is a contradiction. So $\mathbb{K} \subsetneq \mathbb{L}$, and, since $\mathbb{K}$ is algebraically closed we get that $\mathbb{L}$ is transcendental over $\mathbb{K}$. □

Theorem 16. *Let $\mathbf{u}_{\mathrm{g}}(\mathbf{x})$ be a rational general solution of $\mathrm{S}_{\mathrm{diff}}$. Let $\mathbb{L}$ be the smallest field extension of $\mathbb{K}$ containing all coefficients of $\mathbf{u}_{\mathrm{g}}(\mathbf{x})$. There exists a tuple $\mathbf{c}$ of transcendental constants over $\mathbb{K}$, such that $\mathbb{L} = \mathbb{K}(\mathbf{c})$, and $\mathrm{rank}(\mathrm{Jac}_{\mathbf{c}}(\mathbf{u}_{\mathrm{g}}(\mathbf{x})) = \ell$.*

Proof. By Lemma 15, we know that $\mathbb{L}$ is transcendental over $\mathbb{K}$. $\mathbb{L}$ is obtained by adjoining all coefficients of $\mathbf{u}_{\mathrm{g}}(\mathbf{x})$ to $\mathbb{K}$. Since $\mathbb{K}$ is algebraically closed, then $\mathbb{L}$ can be expressed as $\mathbb{L} = \mathbb{K}(\mathbf{c})$, with $\mathbf{c} = (c_1,\ldots,c_\rho)$, for some $\rho \in \mathbb{N}$, and all c_i transcendental over $\mathbb{K}$. The coefficients of $\mathbf{u}_{\mathrm{g}}(\mathbf{x})$ depend on $\mathbf{c}$. Let us emphasize it by writing $\mathbf{u}_{\mathrm{g}}(\mathbf{x},\mathbf{c})$. Let $u_j = \frac{A_j(\mathbf{x},\mathbf{c})}{B_j(\mathbf{x},\mathbf{c})}$ with $A_j, B_j \in \mathbb{K}[\mathbf{x},\mathbf{c}]$ and $\gcd(A_j,B_j) = 1$. We consider the polynomials

$$T_j(\mathbf{x},\mathbf{y},\mathbf{c}) = y_jB_j(\mathbf{x},\mathbf{c}) - A_j(\mathbf{x},\mathbf{c}) \in \mathbb{K}[\mathbf{x},\mathbf{y},\mathbf{c}].$$

Let V be the variety defined by $\{T_1,\ldots,T_\ell\}$ in $\mathbb{K}^{k+\ell+\rho}$. We also consider the ideal, over $\mathbb{K}[\mathbf{x},\mathbf{y},\mathbf{c}]$, $I = \langle T_1,\ldots,T_\ell\rangle$. We consider a Gröbner basis G of $I \cap \mathbb{K}[\mathbf{x},\mathbf{y}]$. Note that $G \subset \mathbb{K}[\mathbf{x},\mathbf{y}] \subset \mathcal{R}_{\mathrm{a}}$. Let $\pi : \mathbb{K}^{k+\ell+\rho} \to \mathbb{K}^{k+\ell}; (\mathbf{x},\mathbf{y},\mathbf{c}) \mapsto (\mathbf{x},\mathbf{y})$. Let W be the Zariski closure of $\pi(V)$. In this situation, we observe that

- By construction, $T_j(\mathbf{x},\mathbf{u}_{\mathrm{g}}(\mathbf{x},\mathbf{c}),\mathbf{c}) = 0$. Thus, for all $g \in G$ it holds that $g(\mathbf{x},\mathbf{u}_{\mathrm{g}}(\mathbf{x},\mathbf{c})) = 0$. This implies that $(\mathbf{x},\mathbf{u}_{\mathrm{g}}(\mathbf{x},\mathbf{c}))$ parametrizes an irreducible component W^* of W. Moreover, $\dim(W^*) = \mathrm{rank}(\mathrm{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{x},\mathbf{u}_{\mathrm{g}}(\mathbf{x},\mathbf{c})))$
- By Lemma 14, since $\mathbf{u}_{\mathrm{g}}(\mathbf{x})$ is general and $G \subset \mathcal{R}_{\mathrm{a}}$, we have that $G = \{0\}$. Thus, $W = \mathbb{K}^{k+\ell}$.

Since W is irreducible, we have that $W^* = W = \mathbb{K}^{k+\ell}$. This implies that $k + \ell = \operatorname{rank}(\operatorname{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{x}, \mathbf{u}_{\mathrm{g}}(\mathbf{x}, \mathbf{c})))$. We note that

$$\operatorname{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{x}, \mathbf{u}_{\mathrm{g}}(\mathbf{x}, \mathbf{c})) = \begin{pmatrix} \mathrm{Id}_{k\times k} & \mathrm{O}_{k\times\rho} \\ \operatorname{Jac}_{\mathbf{x}}(\mathbf{u}_{\mathrm{g}}(\mathbf{x}, \mathbf{c})) & \operatorname{Jac}_{\mathbf{c}}(\mathbf{u}_{\mathrm{g}}(\mathbf{x}, \mathbf{c})) \end{pmatrix},$$

where $\mathrm{Id}_{k\times k}$ is the $k \times k$ identity matrix and $\mathrm{O}_{k\times\rho}$ is the $k \times \rho$ zero matrix. Therefore, we get that $\operatorname{rank}(\operatorname{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{x}, \mathbf{u}_{\mathrm{g}}(\mathbf{x}, \mathbf{c}))) = k + \operatorname{rank}(\operatorname{Jac}_{\mathbf{c}}(\mathbf{u}_{\mathrm{g}}(\mathbf{x}, \mathbf{c})))$.

In conclusion, this implies the statement that $\operatorname{rank}(\operatorname{Jac}_{\mathbf{c}}(\mathbf{x}, \mathbf{u}_{\mathrm{g}}(\mathbf{x}, \mathbf{c}))) = \ell$. □

Observe that in the previous theorem we are only requiring the elements of $\mathbf{c}$ to be transcendental over $\mathbb{K}$. There might be algebraic relations between the components of $\mathbf{c}$.

Corollary 17. *Let $\mathbf{u}_{\mathrm{g}}(\mathbf{x}) \in \mathbb{K}(\mathbf{c})(\mathbf{x})^\ell$ be a rational general solution of $\mathrm{S}_{\mathrm{diff}}$, where $\mathbf{c} = (c_1, \ldots, c_\rho)$ are transcendental constants. If $\rho > \ell$, there exists a subset $\{c_{i_1}, \ldots, c_{i_{\rho-\ell}}\} \subset \{c_1, \ldots, c_\rho\}$ and there exists $\{a_{i_1}, \ldots, a_{i_{\rho-\ell}}\} \subset \mathbb{K}(\mathbf{x})$ such that the specialization $\mathbf{u}_{\mathrm{g}}(\mathbf{x})^*$ of $\mathbf{u}_{\mathrm{g}}(\mathbf{x})$ at $(c_{i_1}, \ldots, c_{i_{\rho-\ell}}) = (a_{i_1}, \ldots, a_{i_{\rho-\ell}})$ is a rational solution and $\operatorname{rank}(\operatorname{Jac}_{\mathbf{c}}(\mathbf{u}_{\mathrm{g}}(\mathbf{x})^*)) = \ell$.*

Proof. Let us assume w.l.o.g. that the principal $\ell \times \ell$-minor of $\operatorname{Jac}_{\mathbf{c}}(\mathbf{u}_{\mathrm{g}}(\mathbf{x}))$ is non-zero; we denote it by $M(\mathbf{x}, \mathbf{c})$. Let $\mathbf{c}_\ell = (c_1, \ldots, c_\ell)$ and $\mathbf{a} = (a_1, \ldots, a_{\rho-\ell}) \in \mathbb{K}(\mathbf{x})^{\rho-\ell}$. We consider the evaluation homomorphism

$$\varphi_{\mathbf{a}} : \mathbb{K}(\mathbf{x})[\mathbf{c}] \to \mathbb{K}(\mathbf{x})[\mathbf{c}_\ell];\ F(\mathbf{x}, \mathbf{c}) \mapsto F(\mathbf{x}, \mathbf{c}_\ell, \mathbf{a}).$$

By abuse of notation we also denote by $\varphi_{\mathbf{a}}$ the extension of $\varphi_{\mathbf{a}}$ to the subset of $\mathbb{K}(\mathbf{x})(\mathbf{c})$ consisting in those rational functions whose denominators do not vanishes at $\mathbf{a}$. In this situation, let $\Omega \subset \mathbb{K}(\mathbf{x})[\mathbf{c}]$ be the set of all denominators appearing in $\mathbf{u}_{\mathrm{g}}(\mathbf{x}, \mathbf{c})$ and the numerator of the $M(\mathbf{x}, \mathbf{c})$. We take $\mathbf{a}^0 \in \mathbb{K}(\mathbf{x})^{\rho-\ell}$ such that no $\omega \in \Omega$ vanishes on $\mathbf{a}^0$. Let

$$\mathbf{u}_{\mathrm{g}}^0(\mathbf{x}, \mathbf{c}_\ell) = (u_1(\mathbf{x}, \mathbf{c}_\ell, \mathbf{a}^0), \ldots, u_\ell(\mathbf{x}, \mathbf{c}_\ell, \mathbf{a}^0)).$$

Note that $\mathbf{u}_{\mathrm{g}}^0$ is well-defined. In addition $\varphi_{\mathbf{a}^0}(M)$ is the principal $\ell \times \ell$-minor of $\operatorname{Jac}_{\mathbf{c}}(\mathbf{u}_{\mathrm{g}}^0)$. So, we have that $\mathbf{u}_{\mathrm{g}}^0$ is a rational solution of $\mathrm{S}_{\mathrm{diff}}$, and $\operatorname{rank}(\operatorname{Jac}_{\mathbf{c}}(\mathbf{u}_{\mathrm{g}}^0)) = \ell$. □

5. The Associated System $\mathbf{S}_{\mathbf{ass}}^{\mathcal{P}}$ corresponding to $\mathbf{S}_{\mathbf{diff}}$

In this section we assume hypotheses (II), (III) and (IVa), and we see how to associate an autonomous first-order system of algebraic PDEs to $\mathrm{S}_{\mathrm{diff}}$. Later, in Section 6 we study how the solutions of $\mathrm{S}_{\mathrm{diff}}$ and its associated system are related. In the following, for a given parametrization $\mathcal{P}(\mathbf{t}) \in \mathbb{K}(\mathbf{t})^{k+\ell+k\ell}$ of a rational component of maximal dimension W of $\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}} \subset \mathbb{K}^{k+\ell+k\ell}$ we distinguish three parts as follows

$$\mathcal{P}(\mathbf{t}) = (\overbrace{\chi_1(\mathbf{t}), \ldots, \chi_k(\mathbf{t})}^{\mathcal{P}_1(\mathbf{t})}, \overbrace{\chi_{k+1}(\mathbf{t}), \ldots, \chi_{k+\ell}(\mathbf{t})}^{\mathcal{P}_2(\mathbf{t})}, \overbrace{\chi_{k+\ell+1}(\mathbf{t}), \ldots, \chi_{k+\ell+k\ell}(\mathbf{t})}^{\mathcal{P}_3(\mathbf{t})}). \tag{6}$$

Note that $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ correspond, respectively, to the independent variables, the indeterminate functions and the first-order derivatives. We start with the following theorem.

Theorem 18. *Let $\mathcal{P}(\mathbf{t}) = (\mathcal{P}_1(\mathbf{t}), \mathcal{P}_2(\mathbf{t}), \mathcal{P}_3(\mathbf{t}))$ be a proper parametrization of W. Then, the following statements are equivalent*

(a) *$\mathrm{S}_{\mathrm{diff}}$ has a $\mathcal{P}$-covered rational solution.*

(b) *There exists $\mathbf{s} \in \mathbb{K}(\mathbf{x})^{k+\ell}$ such that $\mathcal{P}(\mathbf{s}(\mathbf{x}))$ and $\mathcal{P}^{-1}(\mathcal{P}(\mathbf{s}(\mathbf{x})))$ are well-defined and*

$$\begin{cases} \mathcal{P}_1(\mathbf{s}(\mathbf{x})) = \mathbf{x}, \\ (\mathcal{P}_2(\mathbf{s}(\mathbf{x})))' = \mathcal{P}_3(\mathbf{s}(\mathbf{x})). \end{cases} \tag{7}$$

In that case, $\mathcal{P}_2(\mathbf{s}(\mathbf{x}))$ is a $\mathcal{P}$-covered rational solution of $\mathrm{S}_{\mathrm{diff}}$

Proof. Let $\mathbf{u}(\mathbf{x}) \in \mathbb{K}(\mathbf{x})^\ell$ be a $\mathcal{P}$-covered rational solution of $\mathrm{S}_{\mathrm{diff}}$, and let $\mathbf{U}$ be the parametrization derived from $\mathbf{u}$ (see Lemma 4). Since $\mathbf{u}(\mathbf{x})$ is $\mathcal{P}$-covered, then $\mathcal{P}^{-1} \circ \mathbf{U}$ is well defined. Let

$$\mathbf{s}(\mathbf{x}) := \mathcal{P}^{-1}(\mathbf{U}(\mathbf{x})) \in \mathbb{K}(\mathbf{x})^{k+\ell}.$$

Since $\mathbf{u}(\mathbf{x})$ is $\mathcal{P}$-covered, the formal substitution $\mathcal{P}(\mathbf{s}(\mathbf{x}))$ is well-defined. Then, $\mathbf{U}(\mathbf{x}) = \mathcal{P}(\mathbf{s}(\mathbf{x}))$. Taking into account how $\mathbf{U}$ is defined (see (3)), one deduces that $\mathbf{s}(\mathbf{x})$ satisfies the equalities in (7). Since $\mathcal{P}_2(\mathbf{s}(\mathbf{x})) = \mathbf{u}(\mathbf{x})$, it is clear that it is a $\mathcal{P}$-covered rational solution.

For the other implication, let $\mathbf{u}(\mathbf{x}) = \mathcal{P}_2(\mathbf{s}(\mathbf{x}))$. Since $\mathcal{P}$ parametrizes $\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}$, one has that $\mathbf{u}$ is a rational solution of $\mathrm{S}_{\mathrm{diff}}$. Finally, note that $\mathbf{U} = \mathcal{P}(\mathbf{s}(\mathbf{x}))$ and hence the hypotheses imply that $\mathbf{u}$ is $\mathcal{P}$-covered. □

By Theorem 18 we know that $\mathcal{P}$-covered rational solutions of $\mathrm{S}_{\mathrm{diff}}$ are related with the rational solutions of the system (7). In the following we study (7) in more detail. So, let $\mathbf{s}(\mathbf{x})$ and $\mathcal{P}$ be as in Theorem 18. Taking derivatives in the first equality in (7) we get

$$\mathrm{Jac}(\mathcal{P}_1)(\mathbf{s}(\mathbf{x})) \cdot \mathrm{Jac}(\mathbf{s})(\mathbf{x}) = \mathrm{Id}_{k\times k} \tag{8}$$

where $\mathrm{Id}_{k\times k}$ is the $k \times k$ identity matrix. That, combined with (7), yields the system

$$\begin{cases} \mathrm{Jac}(\mathcal{P}_1)(\mathbf{s}(\mathbf{x})) \cdot \mathrm{Jac}(\mathbf{s})(\mathbf{x}) = \mathrm{Id}_{k\times k}\,, \\ (\mathcal{P}_2(\mathbf{s}(\mathbf{x})))' = \mathcal{P}_3(\mathbf{s}(\mathbf{x}))\,, \end{cases} \tag{9}$$

that is

$$\begin{cases} \mathrm{Jac}(\mathcal{P}_1)(\mathbf{s}(\mathbf{x})) \cdot \mathrm{Jac}(\mathbf{s})(\mathbf{x}) = \mathrm{Id}_{k\times k}\,, \\ \mathrm{Jac}(\mathcal{P}_2)(\mathbf{s}(\mathbf{x})) \cdot \mathrm{Jac}(\mathbf{s})(\mathbf{x}) = \mathcal{P}_3(\mathbf{s}(\mathbf{x}))\,. \end{cases} \tag{10}$$

Observe, that the system in (10) consists of linear partial differential equations in the indeterminates $\{\partial s_j/\partial x_h\}_{1\le j\le k+\ell, 1\le h\le k}$; that is, in the undetermined entries of $\mathrm{Jac}(\mathbf{s})(\mathbf{x})$. Let $s_{j,h}$ denote $\partial s_j/\partial x_h(\mathbf{x})$, and let the coordinates of $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ be expressed as in (6), i.e. its entries are $\chi_j(\mathbf{x})$. Then, (9) can be expressed as

$$\mathrm{Jac}(\mathcal{P}_1, \mathcal{P}_2)(\mathbf{s}(\mathbf{x})) \cdot \begin{pmatrix} s_{1,1} & \cdots & s_{1,k} \\ \vdots & \ddots & \vdots \\ s_{k+\ell,1} & \cdots & s_{k+\ell,k} \end{pmatrix} = \Upsilon(\mathbf{s}(\mathbf{x}))\,, \tag{11}$$

where

$$\Upsilon := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \chi_{k+\ell+1} & \chi_{k+\ell+2} & \cdots & \chi_{2k+\ell} \\ \chi_{2k+\ell+1} & \chi_{2k+\ell+2} & \cdots & \chi_{3k+\ell} \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{\ell k+\ell+1} & \chi_{\ell k+\ell+2} & \cdots & \chi_{(\ell+1)k+\ell} \end{pmatrix}.$$

By Lemma 1, $\mathrm{Jac}(\mathcal{P}_1, \mathcal{P}_2)(\mathbf{t})$ is regular. Let us assume that $\det(\mathrm{Jac}(\mathcal{P}_1, \mathcal{P}_2)(\mathbf{s}(\mathbf{x}))) \neq 0$. Then, one can rewrite (11) in the form

$$\begin{pmatrix} s_{1,1} & \cdots & s_{1,k} \\ \vdots & \ddots & \vdots \\ s_{k+\ell,1} & \cdots & s_{k+\ell,k} \end{pmatrix} = (\mathrm{Jac}(\mathcal{P}_1, \mathcal{P}_2)(\mathbf{s}(\mathbf{x})))^{-1}\, \Upsilon(\mathbf{s}(\mathbf{x}))\,. \tag{12}$$

Thus $\mathbf{s}(\mathbf{x})$ is a solution of the system of APDEs

$$\left(\frac{\partial \mathfrak{s}_i}{\partial x_j}\right)_{\substack{1\le i\le k+\ell\\ 1\le j\le k}} = \left(\mathrm{Jac}(\mathcal{P}_1,\mathcal{P}_2)(\mathfrak{s})\right)^{-1}\Upsilon(\mathfrak{s})\,, \tag{13}$$

where $\mathfrak{s}=(\mathfrak{s}_1,\ldots,\mathfrak{s}_{k+\ell})$ is a $(k+\ell)$-tuple of undetermined functions in $\mathbf{x}$.

Remark 19. *Observe that in the case* $\dim(\mathbb{V}_{\mathrm{S_{alg}}})=k$ *the matrix* Υ *turns out to be the identity matrix.*

Definition 20. *The autonomous system (13) is called the* associated system *of* $\mathrm{S_{diff}}$ *with respect to* $\mathcal{P}(\mathbf{s})$*. We denote it by* $\mathrm{S}^{\mathcal{P}}_{\mathrm{ass}}$

In this situation, we get the following theorem.

Theorem 21. *Assume* $\mathcal{P}(\mathbf{t})=(\mathcal{P}_1(\mathbf{t}),\mathcal{P}_2(\mathbf{t}),\mathcal{P}_3(\mathbf{t}))$ *to be proper (see (6)).*

(i) *Let* $\mathbf{u}(\mathbf{x})$ *be a* $\mathcal{P}$*-covered rational solution of* $\mathrm{S_{diff}}$ *and* $\mathbf{s}(\mathbf{x}):=\mathcal{P}^{-1}(\mathbf{U}(\mathbf{x}))$*. If* $\det(\mathrm{Jac}(\mathcal{P}_1,\mathcal{P}_2)(\mathbf{s}(\mathbf{x})))\neq 0$*, then* $\mathbf{s}(\mathbf{x})$ *is a rational solution of* $\mathrm{S}^{\mathcal{P}}_{\mathrm{ass}}$ *such that* $\mathcal{P}_1(\mathbf{s}(\mathbf{x}))=\mathbf{x}$*.*

(ii) *Let* $\mathbf{s}(\mathbf{x})$ *be a rational solution of* $\mathrm{S}^{\mathcal{P}}_{\mathrm{ass}}$ *such that* $\det(\mathrm{Jac}(\mathcal{P}_1,\mathcal{P}_2)(\mathbf{s}(\mathbf{x})))\neq 0$*. If*

$$\mathcal{P}_1(\mathbf{s}(\mathbf{x}))=\mathbf{x}, \tag{14}$$

then $\mathcal{P}_2(\mathbf{s}(\mathbf{x}))$ *is a* $\mathcal{P}$*-covered rational solution of* $\mathrm{S_{diff}}$*, and* $\mathbf{U}=\mathcal{P}(\mathbf{s}(\mathbf{x}))$*.*

Observe that the first statement in the previous theorem follows directly from the construction of $\mathrm{S}^{\mathcal{P}}_{\mathrm{ass}}$ in terms of $\mathrm{S_{diff}}$. On the other hand, the condition (14) is required in order to derive a solution of $\mathrm{S_{diff}}$, from one of $\mathrm{S}^{\mathcal{P}}_{\mathrm{ass}}$. If we remove this condition, we derive

$$\mathcal{P}_1(\mathbf{s}(\mathbf{x}))=\mathbf{x}+\mathbf{c}$$

for a tuple of transcendental constants $\mathbf{c}$. This does not provide a solution unless these constants are chosen to be zero, as we point out in the following example, studied in [19, I.415].

Example 22. *Consider* $\mathrm{S_{diff}}=\{F(x,\mathfrak{u},\mathfrak{u}')=0\}$*, with* $F(x,\mathfrak{u},\mathfrak{u}')=\mathfrak{u}'^2x+\mathfrak{u}'\mathfrak{u}-\mathfrak{u}^4$*. We have* $k=\ell=1$ *and* $\dim(\mathbb{V}_{\mathrm{S_{alg}}})=2$*.* $\mathbb{V}_{\mathrm{S_{alg}}}$ *has only one component. A proper rational parametrization of* $\mathbb{V}_{\mathrm{S_{alg}}}$ *is given by*

$$\mathcal{P}=(\chi_1(t_1,t_2),\chi_2(t_1,t_2),\chi_3(t_1,t_2))=\left(t_1,\frac{t_2}{t_2^2-t_1},-\frac{t_2^3}{t_1(t_1-t_2^2)^2}\right).$$

Then, $\mathrm{S}^{\mathcal{P}}_{\mathrm{ass}}$ *is*

$$\begin{pmatrix}\mathfrak{s}_1'\\ \mathfrak{s}_2'\end{pmatrix}=\begin{pmatrix}1\\ \mathfrak{s}_2/\mathfrak{s}_1\end{pmatrix}.$$

This system has the rational solution $s_1(x)=x+c_1$ *and* $s_2(x)=(x+c_1)c_2$*, for any choice of the constants* c_1,c_2*. If* $c_2=0$*, then* $u(x)=0$ *is a solution of* $\mathrm{S_{diff}}$ *which is not* $\mathcal{P}$*-covered because* $\mathcal{P}^{-1}(x_1,x_2,x_3)=\left(x,-\frac{xz}{y^2}\right)$ *is not well defined for* $\mathfrak{u}=(x,0,0)$ *(see Definition 6). Let* $c_2\neq 0$*. Then we have* $u(x)=\chi_2(s_1(x),s_2(x))=\frac{c_2}{(x+c_1)c_2^2-1}$*, which is not a solution of* $\mathrm{S_{diff}}$*, unless* c_1 *vanishes. Condition (14) states that*

$$x=\chi_1(s_1(x),s_2(x))=x+c_1\,,$$

so we choose $c_1=0$*.*

The additional assumption (14) made when constructing a rational solution of $\mathrm{S_{diff}}$ from another rational solution of $\mathrm{S}^{\mathcal{P}}_{\mathrm{ass}}$ motivates the next definition.

Definition 23. *Let* $\mathbf{s}(\mathbf{x})$ *be a rational solution of* $\mathrm{S}^{\mathcal{P}}_{\mathrm{ass}}$*. We say it is* $\mathcal{P}$-suitable *if* $\mathcal{P}_1(\mathbf{s}(\mathbf{x}))=\mathbf{x}$*.*

Given a proper parametrization of the associated variety, we introduce the following sets

$$\mathrm{SolCov}(\mathrm{S}_{\mathrm{diff}}) = \left\{ \begin{array}{c} \mathbf{u}(\mathbf{x}) \mid \mathbf{u}(\mathbf{x}) \text{ is a } \mathcal{P}\text{-covered rational solution of } \mathrm{S}_{\mathrm{diff}} \\ \text{such that } \det(\mathrm{Jac}(\mathcal{P}_1,\mathcal{P}_2)(\mathcal{P}^{-1}(\mathbf{U}(\mathbf{x}))) \neq 0 \end{array} \right\}$$

$$\mathrm{SolLift}(\mathrm{S}^{\mathcal{P}}_{\mathrm{ass}}) = \left\{ \begin{array}{c} \mathbf{s} \mid \mathbf{s} \text{ is a } \mathcal{P}\text{-suitable rational solution of } \mathrm{S}^{\mathcal{P}}_{\mathrm{ass}} \\ \text{such that } \det(\mathrm{Jac}(\mathcal{P}_1,\mathcal{P}_2)(\mathbf{s}(\mathbf{x})) \neq 0 \end{array} \right\}$$

Theorem 21 establishes a one-to-one relation between $\mathrm{SolCov}(\mathrm{S}_{\mathrm{diff}})$ and $\mathrm{SolLift}(\mathrm{S}^{\mathcal{P}}_{\mathrm{ass}})$. Observe that in Lemma 25 we prove that the requirement on the determinant not being 0 in $\mathrm{SolCov}(\mathrm{S}_{\mathrm{diff}})$ follows from the notion of $\mathcal{P}$-covered.

6. General Solutions of $\mathrm{S}_{\mathrm{diff}}$ versus $\mathrm{S}^{\mathcal{P}}_{\mathrm{ass}}$

In this section we assume all the hypotheses in Section 2, see Table 1. Note that hypothesis (I) implies that $\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}$ is irreducible. So, taking into account hypothesis (II), one gets that $\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}$ is rational of dimension $k+\ell$; this is indeed used in the proof of Theorem 29. In addition, throughout the whole section, $\mathrm{S}_{\mathrm{diff}}, \mathcal{P}(\mathbf{t}) = (\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3) = (\chi_1,\ldots,\chi_{k+\ell+k\ell})$ (see (6)) and $\mathrm{S}^{\mathcal{P}}_{\mathrm{ass}}$ are fixed; moreover, $\mathcal{P}$ is assumed to be proper.

Definition 24. *Let* $\mathbf{c}$ *be a tuple of* ℓ *independent transcendental constants over* $\mathbb{K}$. *Let* $\mathbb{F} = \overline{\mathbb{K}(\mathbf{c})}$ *be the algebraic closure of* $\mathbb{K}(\mathbf{c})$, *and let* $\mathbf{s} \in \mathbb{F}(\mathbf{x})^{k+\ell}$ *be a rational solution of* $\mathrm{S}^{\mathcal{P}}_{\mathrm{ass}}$. *Then* $\mathbf{s}$ *is* rank-general *if and only if* $\mathrm{rank}(\mathrm{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{s}(\mathbf{x}))) = k+\ell$.

We start with a few technical lemmas.

Lemma 25. *Let* $\mathbf{u}_{\mathrm{g}}(\mathbf{x})$ *be a* $\mathcal{P}$*-covered rational solution of* $\mathrm{S}_{\mathrm{diff}}$. *Then*

$$\det(\mathrm{Jac}(\mathcal{P}_1,\mathcal{P}_2)(\mathcal{P}^{-1}(\mathbf{U}_{\mathrm{g}}))) \neq 0\,.$$

Proof. We know that $\mathcal{P}\circ\mathcal{P}^{-1} = \mathrm{id}$. Hence,

$$\mathrm{Id}_{k+\ell+k\ell} = \mathrm{Jac}(\mathcal{P}\circ\mathcal{P}^{-1})(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathrm{Jac}(\mathcal{P})(\mathcal{P}^{-1}(\mathbf{x},\mathbf{y},\mathbf{z}))\cdot\mathrm{Jac}(\mathcal{P}^{-1})(\mathbf{x},\mathbf{y},\mathbf{z}),$$

where $\mathrm{Jac}(\mathcal{P})$ is a $(k+\ell+k\ell)\times(k+\ell)$-matrix and $\mathrm{Jac}(\mathcal{P}^{-1})$ is a $(k+\ell)\times(k+\ell+k\ell)$-matrix and $\mathrm{Id}_{k+\ell+k\ell}$ is the $(k+\ell+k\ell)\times(k+\ell+k\ell)$ identity matrix. In fact we are interested in the first $(k+\ell)$ rows of $\mathrm{Jac}(\mathcal{P})$. Observe that $\mathbf{u}_{\mathrm{g}}(\mathbf{x})$ is $\mathcal{P}$-covered, and thus the matrices in the following line are well-defined. We have that

$$\mathrm{Jac}(\mathcal{P}_1,\mathcal{P}_2)(\mathcal{P}^{-1}(\mathbf{U}_{\mathrm{g}}))\cdot\mathrm{Jac}(\mathcal{P}^{-1})(\mathbf{U}_{\mathrm{g}}) = \mathrm{Id}^{k+\ell}_{k+\ell+k\ell},$$

where $\mathrm{Id}^{k+\ell}_{k+\ell+k\ell}$ is the matrix consisting of the first $k+\ell$ rows of $\mathrm{Id}_{k+\ell+k\ell}$. Hence,

$$\begin{aligned} k+\ell = \mathrm{rank}(\mathrm{Id}^{k+\ell}_{k+\ell+k\ell}) &= \mathrm{rank}(\mathrm{Jac}(\mathcal{P}_1,\mathcal{P}_2)(\mathcal{P}^{-1}(\mathbf{U}_{\mathrm{g}}))\cdot\mathrm{Jac}(\mathcal{P}^{-1})(\mathbf{U}_{\mathrm{g}})) \\ &\leq \min\{\mathrm{rank}(\mathrm{Jac}(\mathcal{P}_1,\mathcal{P}_2)(\mathcal{P}^{-1}(\mathbf{U}_{\mathrm{g}}))), \mathrm{rank}(\mathrm{Jac}(\mathcal{P}^{-1})(\mathbf{U}_{\mathrm{g}}))\}. \end{aligned}$$

Therefore, $\mathrm{rank}(\mathrm{Jac}(\mathcal{P}_1,\mathcal{P}_2)(\mathcal{P}^{-1}(\mathbf{U}_{\mathrm{g}}))) = k+\ell$ and thus we have shown the claim. □

Lemma 26. *Let* $\mathbf{s}_{\mathrm{g}}(\mathbf{x})$ *be a rational rank-general solution of* $\mathrm{S}^{\mathcal{P}}_{\mathrm{ass}}$. *Then*

$$\det(\mathrm{Jac}(\mathcal{P}_1,\mathcal{P}_2)(\mathbf{s}_{\mathrm{g}}(\mathbf{x}))) \neq 0\,.$$

Proof. Let $L(\mathbf{w})$ be the numerator of $\det(\mathrm{Jac}(\mathcal{P}_1,\mathcal{P}_2)(\mathbf{w}))) \in \mathbb{K}[\mathbf{w}] \subset \mathbb{K}(\mathbf{x})[\mathbf{w}]$. By Lemma 1, $L(\mathbf{w}) \neq 0$. To emphasize that $\mathbf{s}_{\mathrm{g}}$ does depend on $\mathbf{x}$ and $\mathbf{c}$ we write $\mathbf{s}_{\mathrm{g}}(\mathbf{c},\mathbf{x})$. Let us now assume that $L(\mathbf{s}_{\mathrm{g}}(\mathbf{c},\mathbf{x}))$ is identically zero. This means that $\mathbf{s}_{\mathrm{g}}(\mathbf{c},\mathbf{x})$ is a parametrization of a component W^* of the hypersurface W, in $\mathbb{K}^{k+\ell}$, defined by $L(\mathbf{w})$. So, since $\mathbf{u}_{\mathrm{g}}(\mathbf{x})$ is rank-general, then $\mathrm{rank}(\mathrm{Jac}_{\mathbf{c},\mathbf{x}}(\mathbf{s}_{\mathrm{g}})) = k+\ell$, and this implies that $\dim(W^*) = k+\ell$. So, $W^* = \mathbb{K}^{k+\ell} \subset W \subset \mathbb{K}^{k+\ell}$. Thus, $W = \mathbb{K}^{k+\ell}$, and hence $L = 0$ which is a contradiction. □

The next lemma is a weaker version of Lemma 14 for the case of rank-general solutions.

Lemma 27. *Let $\mathbf{s}_{\mathrm{g}}$ be a rational rank-general solution of $\mathrm{S}_{\mathrm{ass}}^{\mathcal{P}}$. Then, for every $G(\mathbf{w}) \in \mathbb{K}[\mathbf{w}]$ we have: $G(\mathbf{s}_{\mathrm{g}}) = 0 \Longleftrightarrow G(\mathbf{w}) = 0$.*

Proof. Clearly, if $G(\mathbf{w}) = 0$ then $G(\mathbf{s}_{\mathrm{g}}) = 0$. Let us assume that $G(\mathbf{s}_{\mathrm{g}}) = 0$, and let W be the variety generated by $G(\mathbf{w})$ in $\mathbb{K}^{k+\ell}$. So, $\mathbf{s}_{\mathrm{g}}$ is a rational parametrization of an irreducible component W^* of W. Then

$$k + \ell = \mathrm{rank}(\mathrm{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{s}_{\mathrm{g}})) = \dim(W^*) \leq \dim(W) \leq k + \ell.$$

So, $\dim(W) = k + \ell$, and hence $W = \mathbb{K}^{k+\ell}$ and therefore $G(\mathbf{w}) = 0$. □

Theorem 28 (From $\mathrm{S}_{\mathrm{diff}}$ to $\mathrm{S}_{\mathrm{ass}}^{\mathcal{P}}$)**.** *Let $\mathbf{u}_{\mathrm{g}}(\mathbf{x})$ be a $\mathcal{P}$-covered rational general solution of $\mathrm{S}_{\mathrm{diff}}$. Then, $\mathbf{s}_{\mathrm{g}}(\mathbf{x}) := \mathcal{P}^{-1}(\mathbf{U}_{\mathrm{g}}(\mathbf{x}))$ is a $\mathcal{P}$-suitable rational rank-general solution of $\mathrm{S}_{\mathrm{ass}}^{\mathcal{P}}$.*

Proof. By Lemma 25 and by Theorem 21 (i), we have that $\mathbf{s}_{\mathrm{g}}(\mathbf{x})$ is a rational $\mathcal{P}$-suitable solution of $\mathrm{S}_{\mathrm{ass}}^{\mathcal{P}}$. By Corollary 17, we know that $\mathbf{u}_{\mathrm{g}}(\mathbf{x})$ can be taken depending exactly on ℓ transcendental constants $\mathbf{c}$ and such that $\mathrm{rank}(\mathrm{Jac}_{\mathbf{c}}(\mathbf{u}_{\mathrm{g}}(\mathbf{x}))) = \ell$. Now, we show that $\mathrm{rank}(\mathrm{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{s}_{\mathrm{g}}(\mathbf{x}))) = k + \ell$. We consider $\mathbf{U}_{\mathrm{g}} = (\mathbf{x}, \mathbf{u}_{\mathrm{g}}(\mathbf{x}), \mathbf{u}_{\mathrm{g}}(\mathbf{x})')$. We observe that

$$\mathrm{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{U}_{\mathrm{g}}) = \begin{pmatrix} \mathrm{Id}_{k\times k} & \mathrm{O}_{k\times \ell} \\ \mathrm{Jac}_{\mathbf{x}}(\mathbf{u}_{\mathrm{g}}(\mathbf{x})) & \mathrm{Jac}_{\mathbf{c}}(\mathbf{u}_{\mathrm{g}}(\mathbf{x})) \\ \mathrm{Jac}_{\mathbf{x}}(\mathbf{u}_{\mathrm{g}}(\mathbf{x})') & \mathrm{Jac}_{\mathbf{c}}(\mathbf{u}_{\mathrm{g}}(\mathbf{x})') \end{pmatrix}.$$

So, $\mathrm{rank}(\mathrm{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{U}_{\mathrm{g}})) = k + \ell$. Since, $\mathcal{P} \circ \mathbf{s}_{\mathrm{g}} = \mathbf{U}_{\mathrm{g}}$, taking Jacobians, we get

$$\mathrm{Jac}_{\mathbf{x},\mathbf{c}}(\mathcal{P})(\mathbf{s}_{\mathrm{g}}) \cdot \mathrm{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{s}_{\mathrm{g}}) = \mathrm{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{U}_{\mathrm{g}}).$$

Thus,

$$\begin{aligned} k + \ell &= \mathrm{rank}(\mathrm{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{U}_{\mathrm{g}}) \\ &\leq \min\{\mathrm{rank}(\mathrm{Jac}_{\mathbf{x},\mathbf{c}}(\mathcal{P})(\mathbf{s}_{\mathrm{g}})), \mathrm{rank}(\mathrm{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{s}_{\mathrm{g}}))\} \\ &\leq \mathrm{rank}(\mathrm{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{s}_{\mathrm{g}})) \\ &\leq k + \ell \end{aligned}$$

Therefore, $\mathrm{rank}(\mathrm{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{s}_{\mathrm{g}})) = k + \ell$. □

Theorem 29 (From $\mathrm{S}_{\mathrm{ass}}^{\mathcal{P}}$ to $\mathrm{S}_{\mathrm{diff}}$)**.** *Let $\mathbf{s}_{\mathrm{g}}(\mathbf{x})$ be a $\mathcal{P}$-suitable rational rank-general solution of $\mathrm{S}_{\mathrm{ass}}^{\mathcal{P}}$. Then $\mathcal{P}_2(\mathbf{s}_{\mathrm{g}}(\mathbf{x}))$ is a $\mathcal{P}$-covered rational general solution of $\mathrm{S}_{\mathrm{diff}}$.*

Proof. Let $\mathbf{u}_{\mathrm{g}}(\mathbf{x}) := \mathcal{P}_2(\mathbf{s}_{\mathrm{g}}(\mathbf{x}))$. By Lemma 26 and Theorem 21 (ii) we have that $\mathbf{u}_{\mathrm{g}}(\mathbf{x})$ is a $\mathcal{P}$-covered rational solution of $\mathrm{S}_{\mathrm{diff}}$. It only remains to check that $\mathbf{u}_{\mathrm{g}}(\mathbf{x})$ is a general solution of $\mathrm{S}_{\mathrm{diff}}$. In view of Definition 13, we have to prove that $G \in \mathcal{J}_{\mathrm{d}}$ iff $G(\mathbf{u}_{\mathrm{g}}(\mathbf{x})) = 0$.

Since $\mathcal{J}_{\mathrm{d}}$ is radical, by the Nullstellensatz (see e.g. [26, p. 105]), it is equal to the ideal of the differential polynomials vanishing on all the solutions of $\mathrm{S}_{\mathrm{diff}}$. Therefore, since $G \in \mathcal{J}_{\mathrm{d}}$ and $\mathbf{u}_{\mathrm{g}}(\mathbf{x})$ is a solution of $\mathrm{S}_{\mathrm{diff}}$ we get that $G(\mathbf{u}_{\mathrm{g}}(\mathbf{x})) = 0$.

We now prove the converse statement. For this purpose, and within this part of the proof, we find it useful to utilize the following notation: when a polynomial is differential (resp. algebraic) we emphasize this fact by writing the subindex d (resp. a). Let $G_{\mathrm{d}} \in \mathbb{K}(\mathbf{x})\{\mathrm{u}\}$ such that $G_{\mathrm{d}}(\mathbf{u}_{\mathrm{g}}(\mathbf{x})) = 0$. We need to prove that $G_{\mathrm{d}} \in \mathcal{J}_{\mathrm{d}}$ which is equivalent to prove that the pseudo-remainder G_{d}^* of G_{d} w.r.t. $\mathcal{A}$ is 0. Because of Hypothesis (IV), $\mathrm{ord}(G_{\mathrm{d}}^*) \leq 1$. Indeed, $G_{\mathrm{d}}^* \in \mathcal{R}_{\mathrm{d}}$ (see (5)). So we consider the algebraic polynomial $G_{\mathrm{a}}^*(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathcal{R}_{\mathrm{a}} \subset \mathbb{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$. Now, let $M_{\mathrm{a}}(\mathbf{w}) = G_{\mathrm{a}}^*(\mathcal{P}(\mathbf{w}))$. We observe that

$$M_{\mathrm{a}}(\mathbf{s}_{\mathrm{g}}(\mathbf{x})) = G_{\mathrm{a}}^*(\mathcal{P}(\mathbf{s}_{\mathrm{g}}(\mathbf{x}))) = G_{\mathrm{a}}^*(\mathbf{x}, \mathbf{u}_{\mathrm{g}}(\mathbf{x}), \mathbf{u}_{\mathrm{g}}(\mathbf{x})') = G_{\mathrm{d}}^*(\mathbf{u}_{\mathrm{g}}(\mathbf{x}))$$

By Remark 8 (b)

$$\left(\prod_{A\in\mathcal{A}} I_A^{m_A} S_A^{n_A}\right) G_{\rm d} = G_{\rm d}^{\star} + \sum_{A\in\mathcal{B}} T_A A,$$

Since $G_{\rm d}(\mathbf{u}_{\rm g}(\mathbf{x})) = 0$ by hypothesis and $A(\mathbf{u}_{\rm g}(\mathbf{x})) = 0$ because $A \in \mathcal{J}_{\rm d}$, we get that $G_{\rm d}^*(\mathbf{u}_{\rm g}(\mathbf{x})) = 0$. Therefore, $M_{\rm a}(\mathbf{s}_{\rm g}(\mathbf{x})) = 0$. Thus, by Lemma 27, the numerator of $M_{\rm a}(\mathbf{w})$ is zero, and hence $M_{\rm a}(\mathbf{w})$ is zero. In this situation, we consider the Zariski open subset

$$\Omega = \mathbb{V}_{\mathrm{S_{alg}}} \cap \mathrm{dom}(\mathcal{P}^{-1}).$$

Let $P := (\mathbf{a},\mathbf{b},\mathbf{c}) \in \Omega$. There exists $\mathbf{t}^0 \in \mathbb{K}^{k+\ell}$ such that $\mathcal{P}(\mathbf{t}^0) = P$. Then, $G_{\rm a}^*(P) = G_{\rm a}^*(\mathcal{P}(\mathbf{t}^0)) = M_{\rm a}(\mathbf{t}^0) = 0$. Thus, $G_{\rm a}^*(\mathbf{w})$ vanishes on Ω. Since $\mathbb{V}_{\mathrm{S_{alg}}}$ is irreducible, then Ω is dense in $\mathbb{V}_{\mathrm{S_{alg}}}$, and therefore $G_{\rm a}^*(\mathbf{w})$ vanishes on $\mathbb{V}_{\mathrm{S_{alg}}}$. This implies that $G_{\rm a}^*(\mathbf{w}) \in \mathcal{J}_{\rm a}$. So, $G_{\rm d}^*(\mathfrak{u}) \in \mathcal{J}_{\rm d}$. However, $G_{\rm d}^*(\mathfrak{u})$ is a pseudo-remainder, and thus $G_{\rm d}^* = 0$.

□

We illustrate the previous ideas by some examples.

Example 30. *We consider the system of first-order algebraic partial differential equations*

$$\mathrm{S_{diff}} = \begin{cases} x_2^2 \dfrac{\partial \mathfrak{u}}{\partial x_1} = 1\,, \\ 4x_1^2 \left(\dfrac{\partial \mathfrak{u}}{\partial x_1}\right)^3 - \left(\dfrac{\partial \mathfrak{u}}{\partial x_2}\right)^2 = 0\,, \\ 2x_1 \dfrac{\partial \mathfrak{u}}{\partial x_1} + x_2 \dfrac{\partial \mathfrak{u}}{\partial x_2} = 0\,, \\ 2x_1x_2 \left(\dfrac{\partial \mathfrak{u}}{\partial x_1}\right)^2 + \dfrac{\partial \mathfrak{u}}{\partial x_2} = 0\,. \end{cases}$$

Therefore, $k = 2, \ell = 1$. *Then* $\mathcal{A} = \left\{ x_2^2 \dfrac{\partial \mathfrak{u}}{\partial x_1} - 1,\ \dfrac{\partial \mathfrak{u}}{\partial x_2} x_2^3 + 2x_1 \right\}$ *is a characteristic set. It fulfills Assumption (IV). The set*

$$\mathcal{F}_{\rm a} = \{x_2^2 z_1 - 1, 4x_1^2 z_1^3 - z_2^2, 2x_1 z_1 + x_2 z_2, 2x_1 x_2 z_1^2 + z_2\}$$

defines an irreducible variety $\mathbb{V}_{\mathrm{S_{alg}}}$ *in* $\mathbb{C}^5$ *of dimension* $3 = k + \ell$. *Furthermore, the variety is rational and can be properly parametrized by*

$$\mathcal{P}(\mathbf{t}) = \left(t_1, t_2, \frac{t_2^2 t_3 + t_1}{t_2^2}, \frac{1}{t_2^2}, -2\frac{t_1}{t_2^3}\right).$$

Indeed,

$$\mathcal{P}^{-1}(\mathbf{x},\mathbf{y},\mathbf{z}) = \left(x_1, x_2, \frac{1}{2} x_2 z_2 + y\right).$$

So,

$$\mathcal{P}_1(\mathbf{t}) = (t_1, t_2),\ \mathcal{P}_2(\mathbf{t}) = \frac{t_2^2 t_3 + t_1}{t_2^2},\ \textit{and}\ \mathcal{P}_3(\mathbf{t}) = \left(\frac{1}{t_2^2}, -2\frac{t_1}{t_2^3}\right).$$

In addition, we observe that $\mathcal{J}_{\rm a} \cap \mathbb{K}[\mathbf{x}, y] = \{0\}$. *Now, the associated system* $\mathrm{S}_{\rm ass}^{\mathcal{P}}$ *in* $\mathfrak{s}_1(\mathbf{x}), \mathfrak{s}_2(\mathbf{x})$ *is given by*

$$\mathrm{S}_{\rm ass}^{\mathcal{P}} = \begin{cases} \dfrac{\partial \mathfrak{s}_1}{\partial x_1} = 1\,, & \dfrac{\partial \mathfrak{s}_1}{\partial x_2} = 0\,, \\ \dfrac{\partial \mathfrak{s}_2}{\partial x_1} = 0\,, & \dfrac{\partial \mathfrak{s}_2}{\partial x_2} = 1\,, \\ \dfrac{\partial \mathfrak{s}_3}{\partial x_1} = 0\,, & \dfrac{\partial \mathfrak{s}_3}{\partial x_2} = 0\,, \end{cases}$$

with solution $\mathbf{s}_1(x_1,x_2)=x_1+c_1, \mathbf{s}_2(x_1,x_2)=x_2+c_2, \mathbf{s}_3(x_1,x_2)=c_3$*, for arbitrary constants* c_1,c_2,c_3*. So,* $\mathbf{s}_{\mathrm{g}}(\mathbf{x})=(x_1,x_2,c)$ *is a* $\mathcal{P}$*-suitable rational rank-general solution. Therefore,* $\mathcal{P}_2(\mathbf{s}_{\mathrm{g}}(\mathbf{x}))$ *is a rational general solution of* $\mathrm{S}_{\mathrm{diff}}$*, namely,*

$$\mathbf{u}_{\mathrm{g}}(\mathbf{x}) = c + \frac{x_1}{x_2^2}.$$

Example 31. *We consider the system of first-order algebraic partial differential equations*

$$\mathrm{S}_{\mathrm{diff}} = \begin{cases} \frac{\partial \mathrm{u}_1}{\partial x_1}\frac{\partial \mathrm{u}_2}{\partial x_2} - \frac{\partial \mathrm{u}_1}{\partial x_2}\frac{\partial \mathrm{u}_2}{\partial x_1} = 0\,, \\ 4\mathrm{u}_2^2 - \frac{\partial \mathrm{u}_2}{\partial x_1}\left(\frac{\partial \mathrm{u}_2}{\partial x_2}\right)^2 = 0\,, \\ \mathrm{u}_1\frac{\partial \mathrm{u}_2}{\partial x_2} - \mathrm{u}_2\frac{\partial \mathrm{u}_1}{\partial x_2} = 0\,, \\ \mathrm{u}_1\frac{\partial \mathrm{u}_2}{\partial x_1} - \mathrm{u}_2\frac{\partial \mathrm{u}_1}{\partial x_1} = 0\,, \\ 4\mathrm{u}_1\mathrm{u}_2 - \frac{\partial \mathrm{u}_1}{\partial x_2}\frac{\partial \mathrm{u}_2}{\partial x_1}\frac{\partial \mathrm{u}_2}{\partial x_2} = 0\,, \\ 4\mathrm{u}_1^2 - \left(\frac{\partial \mathrm{u}_1}{\partial x_2}\right)^2\frac{\partial \mathrm{u}_2}{\partial x_1} = 0\,, \\ -8x_1\mathrm{u}_2\frac{\partial \mathrm{u}_1}{\partial x_2} + 4x_1^2\frac{\partial \mathrm{u}_1}{\partial x_2}\frac{\partial \mathrm{u}_2}{\partial x_1} + \frac{\partial \mathrm{u}_1}{\partial x_2}\left(\frac{\partial \mathrm{u}_2}{\partial x_2}\right)^2 - 4\frac{\partial \mathrm{u}_2}{\partial x_1}\frac{\partial \mathrm{u}_2}{\partial x_2} = 0\,, \\ -8x_1\mathrm{u}_1 + 4x_1^2\frac{\partial \mathrm{u}_1}{\partial x_1} - 4\frac{\partial \mathrm{u}_2}{\partial x_1} + \frac{\partial \mathrm{u}_1}{\partial x_2}\frac{\partial \mathrm{u}_2}{\partial x_2} = 0\,. \end{cases}$$

So, $k=2$*,* $\ell=2$*. Then*

$$\mathcal{A} = \begin{cases} u_2\frac{\partial \mathrm{u}_1}{\partial x_2} - u_1\frac{\partial \mathrm{u}_2}{\partial x_2}\,, \\ 4\frac{\partial \mathrm{u}_2}{\partial x_1}u_1x_1^2 - 4\frac{\partial \mathrm{u}_2}{\partial x_1}u_2 + \left(\frac{\partial \mathrm{u}_2}{\partial x_2}\right)^2 u_1 - 8u_1u_2x_1\,, \\ 4\frac{\partial \mathrm{u}_1}{\partial x_1}u_1u_2x_1^2 - 4\frac{\partial \mathrm{u}_1}{\partial x_1}u_2^2 + \left(\frac{\partial \mathrm{u}_2}{\partial x_2}\right)^2 u_1^2 - 8u_1^2u_2x_1\,, \\ \left(\frac{\partial \mathrm{u}_2}{\partial x_2}\right)^4 u_1 - 8\left(\frac{\partial \mathrm{u}_2}{\partial x_2}\right)^2 u_1u_2x_1 + 16u_1u_2^2x_1^2 - 16u_2^3\,. \end{cases}$$

is a characteristic set which fulfills Assumption (IV). The set

$$\begin{aligned}\mathcal{F}_{\mathrm{a}} = \{&z_{11}z_{22}-z_{12}z_{21}, 4y_2^2 - z_{21}z_{22}^2, y_1z_{22}-y_2z_{12}, y_1z_{21}-y_2z_{11}, 4y_1y_2 - z_{12}z_{21}z_{22},\\ &4y_1^2 - z_{12}^2z_{21}, -8y_2x_1z_{12} + 4x_1^2z_{12}z_{21} + z_{12}z_{22}^2 - 4z_{21}z_{22},\\ &-8y_1x_1 + 4x_1^2z_{11} - 4z_{21} + z_{12}z_{22}\}\end{aligned}$$

defines an irreducible variety $\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}$ *in* $\mathbb{C}^8$*.* $\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}$ *is rational and a proper rational parametrization is*

$$\mathcal{P}(t) = \left(t_1+t_3, t_2t_4, \frac{t_1t_2^2}{t_3^2}, t_1t_2^2, \frac{t_2^2}{t_3^2}, 2\,\frac{t_1t_2}{t_3^2}, t_2^2, 2\,t_1t_2\right).$$

The inverse of $\mathcal{P}$ *is given by*

$$\mathcal{P}^{-1}(\mathbf{x},\mathbf{y},\mathbf{z}) = \left(\frac{1}{4}\frac{z_{22}^2}{y_2}, \frac{2y_2}{z_{22}}, \frac{1}{4}\frac{4x_1y_2 - z_{22}^2}{y_2}, \frac{1}{2}\frac{x_2z_{22}}{y_2}\right).$$

The associated system $\mathrm{S}_{\mathrm{ass}}^{\mathcal{P}}$ *in* $\mathfrak{s}_1(\mathbf{x}), \mathfrak{s}_2(\mathbf{x}), \mathfrak{s}_3(\mathbf{x}), \mathfrak{s}_4(\mathbf{x})$ *is given by*

$$\mathrm{S}_{\mathrm{ass}}^{\mathcal{P}} = \begin{cases} \dfrac{\partial \mathfrak{s}_1}{\partial x_1}(\mathbf{x}) = 1\,, & \dfrac{\partial \mathfrak{s}_1}{\partial x_2}(\mathbf{x}) = 0\,, \\ \dfrac{\partial \mathfrak{s}_2}{\partial x_1}(\mathbf{x}) = 0\,, & \dfrac{\partial \mathfrak{s}_2}{\partial x_2}(\mathbf{x}) = 1\,, \\ \dfrac{\partial \mathfrak{s}_3}{\partial x_1}(\mathbf{x}) = 0\,, & \dfrac{\partial \mathfrak{s}_3}{\partial x_2}(\mathbf{x}) = 0\,, \\ \dfrac{\partial \mathfrak{s}_4}{\partial x_1}(\mathbf{x}) = 0\,, & \dfrac{\partial \mathfrak{s}_4}{\partial x_2}(\mathbf{x})\,\mathfrak{s}_2(\mathbf{x}) + \mathfrak{s}_4(\mathbf{x}) = 1\,. \end{cases}$$

with solution

$$\mathfrak{s}_1(\mathbf{x}) = x_1 + c_1\,, \mathfrak{s}_2(\mathbf{x}) = x_2 + c_3\,, \mathfrak{s}_3(\mathbf{x}) = c_2\,, \mathfrak{s}_4(\mathbf{x}) = \frac{x_2 + c_4}{x_2 + c_3},$$

for arbitrary constants c_1, c_2, c_3, c_4*. We have* $\det(\mathrm{Jac}(\mathcal{P}_1, \mathcal{P}_2)(\mathbf{s}(\mathbf{x}))) \neq 0$*. We consider* $c_1 = -c_2, c_4 = 0$ *in order to apply Theorem 21 (ii), and deduce that* $\mathcal{P}_2(\mathbf{s}(\mathbf{x}))$ *is a* $\mathcal{P}-$*covered rational solution of* $\mathrm{S}_{\mathrm{diff}}$*, and* $\mathfrak{u} = \mathcal{P}(\mathbf{s}(\mathbf{x}))$*. In view of Theorem 29, we obtain that* $\mathcal{P}_2(\mathbf{s}_{\mathrm{g}}(\mathbf{x}))$ *turns out to be a* $\mathcal{P}-$*covered rational general solution of* $\mathrm{S}_{\mathrm{diff}}$*. More precisely,*

$$\mathbf{u}_{\mathrm{g}}(\mathbf{x}) = \mathcal{P}_2(\mathbf{s}_{\mathrm{g}}(\mathbf{x})) = \left(-\frac{(-x_1 + c_2)(x_2 + c_3)^2}{{c_2}^2}, -(-x_1 + c_2)(x_2 + c_3)^2 \right).$$

7. Extension to non-prime differential ideals

Let W be a rational component of maximum dimension $(k + \ell)$ of $\mathbb{V}_{\mathrm{S}_{\mathrm{alg}}}$ and $\mathcal{P}$ a proper rational parametrization of W. We associate to W a differential system $\mathcal{S}(W)_{\mathrm{diff}}$ in the following way. Consider a set of generators $\{g_1, \ldots, g_r\} \subset \mathbb{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ of the ideal of W, which is algebraically prime. Then $\mathcal{S}(W)_{\mathrm{diff}}$ is the system $\{g_i(\mathbf{x}, \mathfrak{u}, \mathfrak{u}') = 0\}_{i=1,\ldots,r}$. Using the ideas in Section 5 we associate to $\mathcal{S}(W)_{\mathrm{diff}}$ an autonomous first order system $\mathcal{S}(W)_{\mathrm{ass}}^{\mathcal{P}}$. Let $u(\mathbf{x}, \mathbf{c})$ be a rational solution $\mathcal{S}(W)_{\mathrm{diff}}$, where $\mathbf{c}$ are transcendental constants. We say that $u(\mathbf{x}, \mathbf{c})$ is rank-general if and only if

$$\mathrm{rank}(\mathrm{Jac}_{\mathbf{c}}(u(\mathbf{x}, \mathbf{c}))) = \ell\,.$$

Observe that in Theorem 28 we have proved that if u is a rational general solution in the classical Ritt sense, then it is rank-general. For the associated system we consider the notion of rank-general given in Definition 24.

Theorem 32. *Rational* $\mathcal{P}$*-covered rank-general solutions of* $\mathcal{S}(W)_{\mathrm{diff}}$ *are in 1:1 correspondence with rational* $\mathcal{P}$*-suitable rank-general solutions of* $\mathcal{S}(W)_{\mathrm{ass}}^{\mathcal{P}}$*.*

Proof. Because of Lemmas 25 and 26 and Theorem 21, there is a 1:1 correspondence between $\mathcal{P}$-covered solutions of $\mathcal{S}(W)_{\mathrm{diff}}$ and $\mathcal{P}$-suitable solutions of $\mathcal{S}(W)_{\mathrm{ass}}^{\mathcal{P}}$. Let us see that rank-general solutions also correspond. Reasoning as in proof of Theorem 28, we get a $\mathcal{P}$-covered rational rank-general solution of $\mathcal{S}(W)_{\mathrm{diff}}$ provides a rank-general solution of $\mathcal{S}(W)_{\mathrm{ass}}^{\mathcal{P}}$. Conversely, let $\mathbf{s}(\mathbf{x}, \mathbf{c})$ be a rational $\mathcal{P}$-suitable rank-general solution of $\mathcal{S}(W)_{\mathrm{ass}}^{\mathcal{P}}$. This implies that $\mathrm{rank}(\mathrm{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{s})) = k + \ell$. Since $\mathrm{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{s})$ is of dimension $(k + \ell) \times (k + \ell)$, it must be a regular matrix. By Lemma 26, we get that

$$\det(\mathrm{Jac}_{\mathbf{t}}(\mathcal{P}_1, \mathcal{P}_2))(\mathbf{s})) \neq 0. \tag{15}$$

Now, we consider $(\mathbf{x}, \mathbf{u}(\mathbf{x})) = (\mathcal{P}_1, \mathcal{P}_2)(\mathbf{s})$. Since $\mathbf{s}$ is $\mathcal{P}$-suitable, using (15) and Theorem 21, we see that $(\mathbf{x}, \mathbf{u}(\mathbf{x}))$ is well-defined. Therefore, by the chain rule we get

$$\begin{pmatrix} \mathrm{Id}_{k\times k} & \mathrm{O}_{k\times \ell} \\ \mathrm{Jac}_{\mathbf{x}}(\mathbf{u}(\mathbf{x})) & \mathrm{Jac}_{\mathbf{c}}(\mathbf{u}(\mathbf{x})) \end{pmatrix} = \mathrm{Jac}_{\mathbf{x},\mathbf{c}}((\mathcal{P}_1, \mathcal{P}_2)(\mathbf{s})) = \mathrm{Jac}_{\mathbf{t}}((\mathcal{P}_1, \mathcal{P}_2)(\mathbf{s})) \cdot \mathrm{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{s})\,.$$

Taking into account that $\mathrm{Jac}_{\mathbf{x},\mathbf{c}}(\mathbf{s})$ is regular, this implies

$$k + \mathrm{rank}(\mathrm{Jac}_{\mathbf{c}}(\mathbf{u}(\mathbf{x}))) = \mathrm{rank}(\mathrm{Jac}_{\mathbf{t}}((\mathcal{P}_1, \mathcal{P}_2)(\mathbf{s}))) \,. \tag{16}$$

Furthermore, by Lemma 1 we have

$$\mathrm{rank}\left(\mathrm{Jac}_{\mathbf{t}}(\mathcal{P}_1, \mathcal{P}_2)\right) = k + \ell. \tag{17}$$

Using (15), (16) and (17) we get that $\mathrm{rank}(\mathrm{Jac}_{\mathbf{c}}(\mathbf{u}(\mathbf{x}))) = \ell$ and hence $\mathbf{u}(\mathbf{x})$ is rank-general. □

8. Conclusion

The present study deals with the problem of solving systems of first-order partial differential equations. This is a theoretical approach to the problem. The rational solutions of such systems are related in a one-to-one correspondence with the rational solutions of an associated autonomous system of first-order partial differential equations. Such an autonomous system might be solved more easily. Moreover, we describe how rational general solutions are transmitted in both directions, and we provide a concrete algorithm in some particular case concerning the dimension of the associated algebraic variety.

This paper can be seen as a generalization of the results obtained in [22] for systems of first-order algebraic partial differential equations. In this sense, some open questions remain unsolved and will be studied in future research. In particular, the study of systems of APDEs of order higher than one is completely open. Further research is also necessary for relaxing the restrictions on the dimensions of the varieties involved, which we have been required to assume in our treatment of the problem.

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