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A solution method for autonomous first-order algebraic partial differential equations

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In this paper we present a procedure for solving first-order autonomous algebraic partial differential equations in an arbitrary number of variables. The method uses rational parametrizations of algebraic (hyper)surfaces and generalizes a similar procedure for first-order autonomous ordinary differential equations. In particular we are interested in rational solutions and present certain classes of equations having rational solutions. However, the method can also be used for finding non-rational solutions.

1 Introduction

The problem of finding exact solutions to partial differential equations has been deeply studied in the literature. However, there is not a general method to be followed when handling a specific equation but different methods and techniques which might be applied to solving equations of certain form or under some assumptions on the elements involved. We provide [24, Sec. II.B] as a reference in this direction.

The main aim of the present work is to provide an alternative novel exact method for solving partial differential equations. More precisely, we study algebraic partial differential equations (APDEs) which are autonomous and of first-order (see Section 2 for a precise definition).

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1.1 Algebro-geometric treatment of AODEs — State of the art

Recently algebraic-geometric solution methods for algebraic ordinary differential equations (AODEs) were investigated. First results on solving first order AODEs can be found in [15] where Gröbner bases are used and [5] where a degree bound is computed which might be used for making an ansatz. The starting point for algebraic-geometric methods, such as the one described in this paper, was an algorithm by Feng and Gao [7, 8] which decides whether or not an autonomous AODE, $F(y, y') = 0$ has a rational solution and in the affirmative case computes a rational general solution. This result was then generalized by Ngô and Winkler [19, 21, 20] to the non-autonomous case $F(x, y, y') = 0$. First results on higher order AODEs can be found in [12, 13, 14]. Ngô, Sendra and Winkler [17] also classified AODEs in terms of rational solvability by considering affine linear transformations. A generalization to birational transformations can be found in [18]. In [9, 11] a solution method for autonomous AODEs is presented which generalizes the method of Feng and Gao to finding radical and also non-radical solutions. A generalization of the procedure to APDEs in two variables can be found in [10]. In this paper we present a further generalization to the case of an arbitrary number of variables.

1.2 The novel solution method proposed

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0, and let $F \in \mathbb{K}(x_1, \ldots, x_n)\{u\}$ be an element of the ring of differential polynomials in u (derivatives are w.r.t. $x_1, \ldots, x_n$), which is also a polynomial in $x_1, \ldots, x_n$. For our purposes we assume:

a) F only depends on the first order derivatives of u w.r.t. $x_1, \ldots, x_n$, say $u_{x_1}, \ldots, u_{x_n}$.

b) F does not depend on $x_1, \ldots, x_n$.

Under the previous assumptions we may write the first-order autonomous APDE associated to F as

$$F(u, u_{x_1}, \ldots, u_{x_n}) = 0. \tag{1}$$

Our method aims to solve such equations.

The method departs from a given proper rational parametrization of the hypersurface $F(z, p_1, \ldots, p_n) = 0$, say

$$\mathcal{Q}(s_1, \ldots, s_n) = (q_0(s_1, \ldots, s_n), q_1(s_1, \ldots, s_n), \ldots, q_n(s_1, \ldots, s_n)) \, .$$

We assume that the parametrization can be expressed in the form

$$\mathcal{Q}(s_1, \ldots, s_n) = \mathcal{L}(g(s_1, \ldots, s_n)) \, ,$$

where g is an invertible map. If we are able to compute $h = g^{-1}$, then

$$\mathcal{Q}(g^{-1}(s_1, \ldots, s_n)) = \mathcal{L}(s_1, \ldots, s_n) \, ,$$

provides a solution $q_1(h_1, \ldots, h_n)$ of (1).

The solution method, algorithmically described in Procedure 1, makes use of the method of characteristics applied to an auxiliary system of quasilinear equations (see (8)). If it returns a function, it is a solution of the initial problem (see Theorem 3.6).

It is worth remarking a distinguished difference with respect to the case of ordinary differencial equations. Whilst a non-constant solution of an autonomous AODE always provides a proper parametrization of the associated curve, this is no longer valid when working with APDEs. The method provides, if it successfully arrives to a rational solution of the APDE under study, a proper solution of suitable dimension (see Definition 2.2 and Definition 2.3 for the definitions of proper solution and solution of suitable dimension respectively, and Theorem 4.2 for this statement).

Another important feature under discussion in the work is completeness of the solution (see Definition 2.2) which is also attained. From the knowledge of a complete solution one can construct any other solution of the problem (see [4]), so we only focus our results on those.

Our method provides a tool for systematically solving various well-known equations (see Table 2) . Some of them are enumerated throughout the text, such as those studied in the examples in Section 4. Moreover, the algorithm may be applied to find other solutions rather than just rational ones (see Section 5).

1.3 Main contributions of the paper

The main contribution of the paper is the development of a novel method to find exact solutions for autonomous first-order algebraic partial differential equations. The value of this approach is motivated by the following facts:

- In our method, if it returns a rational function, then this function provides a solution to the autonomous first-order ADPE under study. Moreover, this solution turns out to be proper, complete, and of suitable dimension.
- Our method is not restricted to obtain rational solutions of APDEs (see Section 5).
- Our method generalizes known results obtained in the framework of ODEs. It gives solutions to some well-known partial differential equations studied in the literature.
- Even if the method fails, it often leads to an implicit description of the solution.

1.4 Structure of the paper

In Section 2 we recall and introduce the necessary definitions and concepts. The procedure presented in this paper is a generalization of the case for two variables [10]. We do not go into details of this case but show first an extension to three variables in Section 3. Then we present the general procedure for solving APDEs in arbitrary many variables. In Section 4 we consider the case of rational solutions. The section is divided into two parts. The first part proves some properties of rational solutions which can be found by the procedure. The second part presents APDEs which have rational solutions. Finally, in Section 5 we show that the procedure is not restricted to finding rational solutions.

2 Preliminaries

We consider the field of rational functions $\mathbb{K}(x_1, \ldots, x_n)$ for some algebraically closed field $\mathbb{K}$ of characteristic 0; in practice, one may think of $\mathbb{K}$ as the field $\mathbb{C}$ of complex numbers. We denote the usual derivative w.r.t. x_i by $\frac{\partial}{\partial x_i}$. Sometimes we might use the abbreviations $u_{x_i} = \frac{\partial u}{\partial x_i}$. In case $n = 2$ we also write x for x_1 and y for x_2. The ring of differential polynomials is denoted as $\mathbb{K}(x_1, \ldots, x_n)\{u\}$. It consists of all polynomials in u and its derivatives, i.e.

$$\mathbb{K}(x_1, \ldots, x_n)\{u\} = \mathbb{K}(x_1, \ldots, x_n)[u, u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_1}, \ldots, u_{x_n x_n}, \ldots] \,.$$

An algebraic partial differential equation (APDE) is defined by a differential polynomial $F \in \mathbb{K}(x_1, \ldots, x_n)\{u\}$ which is also a polynomial in $x_1, \ldots, x_n$. We write

$$F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_1}, \ldots, u_{x_n x_n}, \ldots) = 0$$

for the corresponding APDE. In this paper we restrict our attention to the first-order autonomous case, i.e.

$$F(u, u_{x_1}, \ldots, u_{x_n}) = 0 \,.$$

An algebraic hypersurface $\mathcal{S}$ is an algebraic variety of codimension 1, i.e. the zero set of a squarefree non-constant polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$,

$$\mathcal{S} = \{(a_1, \ldots, a_n) \in \mathbb{A}^n \mid f(a_1, \ldots, a_n) = 0\} \,,$$

where $\mathbb{A}^n$ is the n-dimensional affine space over $\mathbb{K}$. We call the polynomial f the defining polynomial. An important aspect of algebraic hypersurfaces is their rational parametrizability. We consider an algebraic hypersurface defined by an irreducible polynomial f. We write $\bar{s} = (s_1, \ldots, s_{n-1})$. A tuple of rational functions $\mathcal{P}(s_1, \ldots, s_{n-1}) = (p_1(\bar{s}), \ldots, p_n(\bar{s}))$ is called a rational parametrization of the hypersurface if $f(\mathcal{P}(\bar{s})) = 0$ for all $\bar{s}$ and the jacobian of $\mathcal{P}$ has generic rank $n - 1$. We observe that this condition is fundamental since, otherwise, we are parametrizing a lower dimensional subvariety on the hypersurface. A parametrization can be considered as a dominant map $\mathcal{P}(\bar{s}) : \mathbb{A}^{n-1} \to \mathcal{S}$. By abuse of notation we also call this map a parametrization. We call a parametrization $\mathcal{P}(\bar{s})$ proper if it is a birational map or, in other words, if for almost every point $a = (a_1, \ldots, a_n)$ on the hypersurface we find exactly one tuple $(s_1, \ldots, s_{n-1})$ such that $\mathcal{P}(\bar{s}) = a$, i.e. only a finite number of lower dimensional subvarieties might not be attained. Equivalently $\mathcal{P}$ is proper iff $\mathbb{K}(\mathcal{P}(\bar{s})) = \mathbb{K}(\bar{s})$.

Remark 2.1. *The jacobian of a proper parametrization $\mathcal{P}(s_1, \ldots, s_{n-1})$ of a hypersurface in $\mathbb{A}^n$ has generic rank $n - 1$. Since $\mathcal{P}$ is proper we know that $\mathbb{K}(s_1, \ldots, s_{n-1}) = \mathbb{K}(\mathcal{P}(\bar{s}))$. Hence, there is a rational function $R(a_1, \ldots, a_n) = (R_1(\bar{a}), \ldots, R_n(\bar{a})) \in \mathbb{K}(\bar{a})^n$ such that $R(\mathcal{P}(\bar{s})) = (s_1, \ldots, s_{n-1})$. Thus, $\mathcal{J}_{\mathrm{id}} = \mathcal{J}_{R \circ \mathcal{P}} = \mathcal{J}_R(\mathcal{P}) \cdot \mathcal{J}_{\mathcal{P}}$. Taking into account, that the rank of a product of two matrices is smaller equal the minimal rank of the two matrices, we get that* $\mathrm{rank}(\mathcal{J}_{\mathcal{P}}) = n - 1$.

Above we have considered rational parametrizations of a hypersurface. However, we might want to deal with more general parametrizations. If so, we will say that a tuple of differentiable functions $\mathcal{Q}(\bar{s}) = (q_1(\bar{s}), \ldots, q_n(\bar{s}))$ is a parametrization of the hypersurface if $f(\mathcal{Q}(\bar{s}))$ is identically zero and the jacobian of $\mathcal{Q}(\bar{s})$ has generic rank $n-1$.

Let $F(u, u_{x_1}, \ldots, u_{x_n}) = 0$ be an autonomous APDE. We consider the corresponding algebraic hypersurface by replacing the derivatives by independent transcendental variables, $F(z, p_1, \ldots, p_n) = 0$. Given any differentiable function $u(x_1, \ldots, x_n)$ which satisfies $F(u, u_{x_1}, \ldots, u_{x_n}) = 0$, then

$$\mathcal{L}(s_1, \ldots, s_n) = (u(s_1, \ldots, s_n), u_{x_1}(s_1, \ldots, s_n), \ldots, u_{x_n}(s_1, \ldots, s_n))$$

is a parametrization. We call this parametrization the *corresponding parametrization of the solution*. We observe that the corresponding parametrization of a solution is not necessarily a parametrization of the associated hypersurface, since the condition on the rank of the Jacobian may fail. For instance, let us consider the APDE $u_x = 0$ with $n = 2$. A solution would be of the form $u(x, y) = g(y)$, with g differentiable. However, this solution generates $(g(s_2), 0, g'(s_2))$ that is a curve in the surface; namely the plane $p = 0$. Now, consider the APDE $u_x = \lambda$, with λ a nonzero constant. Hence, the solutions are of the form $u(x, y) = \lambda x + g(y)$. Then, $u(x, y) = \lambda x + y$ generates the line $(\lambda s_1 + s_2, \lambda, 1)$ while $u(x, y) = \lambda x + y^2$ generates the parametrization $(\lambda s_1 + s_2^2, \lambda, 2s_2)$ of the associated plane $p = \lambda$. These examples motivate the following definition. Clearly a solution of an APDE is a function $u(x_1, \ldots, x_n)$ such that $F(u, u_{x_1}, \ldots, u_{x_n}) = 0$.

Definition 2.2. *A solution of an APDE is* rational *iff $u(x_1, \ldots, x_n)$ is a rational function over $\mathbb{K}$.*
A rational solution of an APDE is proper *iff the corresponding parametrization is proper.*

In the case of autonomous ordinary differential equations, every non-constant solution induces a proper parametrization of the associated curve (see [7]). However, this is not true in general for autonomous APDEs. For instance, the solution $x + y^3$ of $u_x = 1$, induces the parametrization $(s_1 + s_2^3, 1, 3s_2^2)$ which is, although its jacobian has rank 2, not proper.

In addition, we observe that it can happen that none of the rational solutions of an APDE is proper. This is the case for instance, of $u_x = 0$, since all rational solutions are of the form $u = R(y)$, with R a rational function and $\mathbb{K}(R(s_1), 0, R'(s_1)) \subsetneq \mathbb{K}(s_1, s_2)$. Furthermore, we see that none of the solutions of this APDE generates a parametrization of the associated hypersurface, since the Jacobian has rank 1.

Every solution of the problem under consideration in this work can be attained by the knowledge of a set of complete solutions. A general solution can be obtained from a complete solution by envelope computations (see [4] for the details).

For this reason, we focus on finding families of complete solutions. This notion of a complete solution is due to Lagrange and can also be found in [16].

Definition 2.3. *Let $F(u, u_{x_1}, \ldots, u_{x_n}) = 0$ be an autonomous APDE. Let u be a rational solution depending on n arbitrary constants $c_1, \ldots, c_n$. Let $\mathcal{L} = (p_0, p_1, \ldots, p_n)$ be the*

parametrization induced by the solution, i.e. $p_0 = u$ *and* $p_i = u_{x_i}$ *for* $i \geq 1$. *We call the solution* complete *if the jacobian* $\mathcal{J}_{\mathcal{L}}^{c_1,\ldots,c_n}$ *of* $\mathcal{L}$ *with respect to* $c_1, \ldots, c_n$ *has generic rank* n.
We call the solution complete of suitable dimension *if it is complete and the jacobian* $\mathcal{J}_{\mathcal{L}}^{s_1,\ldots,s_n}$ *of* $\mathcal{L}$ *with respect to* $s_1, \ldots, s_n$ *has generic rank* n.

Intuitively speaking, the notion of complete solution is requiring that the corresponding parametrization of the solution parametrizes an algebraic set on the hypersurface, independently of the constants $c_1, \ldots, c_n$. On the other hand, the notion of suitable dimension ensures that the corresponding parametrization really parametrizes the associated hypersurface and not a lower dimensional subvariety.

In the following example we will see complete and non-complete solutions of APDEs.

Example 2.4. *We consider the APDE* $u_x = 0$, $F(z, p, q) = p$, *as well as the solution* $u(x, y) = y + c_1 + c_2$. *The corresponding parametrization is* $\mathcal{L} = (s_2 + c_1 + c_2, 0, 1)$. *Then*

$$\mathcal{J}_{\mathcal{L}}^{c_1,c_2} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} ,$$

and hence $u(x, y)$ *is not complete. However, if we take* $u(x, y) = c_1 y + c_2$, *the jacobian with respect to* c_1, c_2 *has generic rank 2, and u is complete but not of suitable dimension, since the jacobian of* $\mathcal{L}$ *with respect to* s_1, s_2 *has rank 1.*

Now, if we take the APDE, $u_x = 1$. *In Table 1 we see solutions and their properties. Note that the solution* $x + c_1 + y^2 + c_2$ *is not complete and hence, not complete of suitable dimension. However, the other property of suitable dimension is fulfilled.*

solution	complete	suitable dim	proper	rank($\mathcal{J}_{\mathcal{L}}^{s_1,s_2}$)
$x + c_1$	F	F	F	1
$x + y + c_1 + c_2$	F	F	F	1
$x + c_1 + c_2 y$	T	F	F	1
$x + c_1 + y^2 + c_2$	F	F	T	2
$x + c_1 + c_2 y^2$	T	T	T	2
$x + c_1 + (y + c_2)^2$	T	T	T	2
$x + c_1 + (y + c_2)^3$	T	T	F	2

Table 1: Properties of some solutions of $u_x = 1$ where T means true, F false

3 A method for solving first-order autonomous APDEs

Let $F(u, u_{x_1}, \ldots, u_{x_n}) = 0$ be an algebraic partial differential equation, where F is an irreducible non-constant polynomial. We consider the hypersurface $F(z, p_1, \ldots, p_n) = 0$ and assume it admits a proper (rational) hypersurface parametrization

$$\mathcal{Q}(s_1, \ldots, s_n) = (q_0(s_1, \ldots, s_n), q_1(s_1, \ldots, s_n), \ldots, q_n(s_1, \ldots, s_n)) .$$

An algorithm for computing a proper rational parametrization of a three-dimensional surface can be found for instance in [22]. For higher-dimensional hypersurfaces there is no general algorithm for computing rational parametrizations. Here, we will stick to rational parametrizations, but the procedure which we present will work as well with other kinds of parametrizations, for instance radical ones. First results on radical parametrizations of three-dimensional surfaces can be found in [23]. Assume that $\mathcal{L}(s_1, \ldots, s_n) = (v_0, \ldots, v_n)$ corresponds to a solution of the APDE. Furthermore we assume that the parametrization $\mathcal{Q}$ can be expressed as

$$\mathcal{Q}(s_1, \ldots, s_n) = \mathcal{L}(g(s_1, \ldots, s_n))$$

for some invertible map $g(s_1, \ldots, s_n) = (g_1(s_1, \ldots, s_n), \ldots, g_n(s_1, \ldots, s_n))$. This assumption is motivated by the fact that in case of rational algebraic curves every non-constant rational solution of an AODE yields a proper rational parametrization of the associated algebraic curve and each proper rational parametrization can be obtained from any other proper one by a rational transformation. In the case of APDEs, however, not all rational solutions provide a proper parametrization, as mentioned in the remark after Definition 2.2. Talking about hypersurface parametrizations, we still know that any proper rational parametrization can be obtained from any other proper one by a rational transformation. At this point, if we can compute g^{-1} we have a solution $\mathcal{Q}(g^{-1}(s_1, \ldots, s_n))$.

Let $\mathcal{J}$ be the jacobian matrix. The solution of our problem comes from the solution of

$$\mathcal{J}_{\mathcal{Q}}(s_1, \ldots, s_n) = \mathcal{J}_{\mathcal{L}}(g(s_1, \ldots, s_n)) \cdot \mathcal{J}_g(s_1, \ldots, s_n) \,.$$

Taking a look at the rows we get that

$$\left.\begin{aligned} \frac{\partial q_0}{\partial s_1} &= \sum_{i=1}^{n} \frac{\partial v_0}{\partial s_i}(g) \frac{\partial g_i}{\partial s_1} = \sum_{i=1}^{n} q_i(s_1, \ldots, s_n) \frac{\partial g_i}{\partial s_1} \,, \\ &\vdots \\ \frac{\partial q_0}{\partial s_n} &= \sum_{i=1}^{n} \frac{\partial v_0}{\partial s_i}(g) \frac{\partial g_i}{\partial s_n} = \sum_{i=1}^{n} q_i(s_1, \ldots, s_n) \frac{\partial g_i}{\partial s_n} \,. \end{aligned}\right\} \tag{2}$$

This is a system of quasilinear equations in the unknown functions g_1 to g_n. In case q_i is zero for some i the problem reduces to lower order. Since $\mathcal{Q}$ is a proper parametrization of a hypersurface, at most one of its components can be zero. So, we can ensure that there exists a non-zero q_i with $i > 0$. Let us assume that $q_1 \neq 0$. If this is not the case, we can always change the role of x_1 and x_i with $i > 1$. First we divide by q_1:

$$\left.\begin{aligned} a_1 &= \frac{\partial g_1}{\partial s_1} + \sum_{i=2}^{n} b_i \frac{\partial g_i}{\partial s_1} \,, \\ &\vdots \\ a_n &= \frac{\partial g_1}{\partial s_n} + \sum_{i=2}^{n} b_i \frac{\partial g_i}{\partial s_n} \,. \end{aligned}\right\} \tag{3}$$

with $a_i = \frac{\frac{\partial q_0}{\partial s_i}}{q_1}$ and $b_i = \frac{q_i}{q_1}$. ¿From this system we will get by differentiation the following system (where for each $j \in \{1, \ldots, n\}$ we take derivatives of the j-th equation in (3) w.r.t. the variables s_k for $j \neq k$).

$$\frac{\partial a_j}{\partial s_k} = \frac{\partial^2 g_1}{\partial s_k \partial s_j} + \sum_{i=2}^{n} \frac{\partial b_i}{\partial s_k}\frac{\partial g_i}{\partial s_j} + b_i \frac{\partial^2 g_i}{\partial s_k \partial s_j} \qquad \text{for } j \neq k\,. \tag{4}$$

Now we take the difference of two equations each and get the following equations where the second derivatives vanished.

$$a_{j,k} = \sum_{i=2}^{n} b_{i,k}\frac{\partial g_i}{\partial s_j} - b_{i,j}\frac{\partial g_i}{\partial s_k} \qquad \text{for } j < k\,, \tag{5}$$

where $a_{j,k} = \frac{\partial a_j}{\partial s_k} - \frac{\partial a_k}{\partial s_j}$ and $b_{i,k} = \frac{\partial b_i}{\partial s_k}$.

The aim now will be to take suitable linear combinations of the equations from (5) such that all derivatives of g_i vanish except for $i = n$, i.e. we are left with a quasilinear PDE in g_n. In [10] this was shown for $n = 2$ and in the Section 3.1 we will do so for $n = 3$. Later in Section 3.2 we will prove the general case. Finally in Section 3.3 we will give a step by step description of the procedure for solving APDEs in arbitrary many variables.

3.1 The case $n = 3$

In the case of three variables the system (5) reads as

$$\left.\begin{aligned}
a_{1,2} &= b_{2,2}\frac{\partial g_2}{\partial s_1} - b_{2,1}\frac{\partial g_2}{\partial s_2} + b_{3,2}\frac{\partial g_3}{\partial s_1} - b_{3,1}\frac{\partial g_3}{\partial s_2}\,,\\
a_{1,3} &= b_{2,3}\frac{\partial g_2}{\partial s_1} - b_{2,1}\frac{\partial g_2}{\partial s_3} + b_{3,3}\frac{\partial g_3}{\partial s_1} - b_{3,1}\frac{\partial g_3}{\partial s_3}\,,\\
a_{2,3} &= b_{2,3}\frac{\partial g_2}{\partial s_2} - b_{2,2}\frac{\partial g_2}{\partial s_3} + b_{3,3}\frac{\partial g_3}{\partial s_2} - b_{3,2}\frac{\partial g_3}{\partial s_3}\,.
\end{aligned}\right\} \tag{6}$$

By a linear combination we get

$$\begin{aligned}
&b_{2,3}a_{1,2} + b_{2,1}a_{2,3} - b_{2,2}a_{1,3}\\
&\quad = (b_{2,3}b_{3,2} - b_{2,2}b_{3,3})\frac{\partial g_3}{\partial s_1} + (b_{2,1}b_{3,3} - b_{2,3}b_{3,1})\frac{\partial g_3}{\partial s_2} + (b_{2,2}b_{3,1} - b_{2,1}b_{3,2})\frac{\partial g_3}{\partial s_3}\,.
\end{aligned}$$

This is a quasilinear PDE in g_3. Hence, it can be solved by the method of characteristics. Once we have g_3 we get a quasilinear PDE in g_2 adding the two first equations of (6):

$$\begin{aligned}
a_{1,2} + a_{1,3} &- \left((b_{3,2} + b_{3,3})\frac{\partial g_3}{\partial s_1} - b_{3,1}\frac{\partial g_3}{\partial s_2} - b_{3,1}\frac{\partial g_3}{\partial s_3}\right)\\
&= (b_{2,2} + b_{2,3})\frac{\partial g_2}{\partial s_1} - b_{2,1}\frac{\partial g_2}{\partial s_2} - b_{2,1}\frac{\partial g_2}{\partial s_3}\,.
\end{aligned}$$

Again, this can be solved by the well known method of characteristics. Finding g_1 is finally computing an integral from (3).

Note, here we have shown a recursive way. However, some computations can also be done in parallel. Indeed, we may consider this second quasilinear PDE in g_2

$$\begin{aligned}&b_{3,3}a_{1,2}+b_{3,1}a_{2,3}-b_{3,2}a_{1,3}\\&\quad=(b_{2,2}b_{3,3}-b_{2,3}b_{3,2})\frac{\partial g_2}{\partial s_1}+(b_{2,3}b_{3,1}-b_{2,1}b_{3,3})\frac{\partial g_2}{\partial s_2}+(b_{2,1}b_{3,2}-b_{2,2}b_{3,1})\frac{\partial g_2}{\partial s_3}\,.\end{aligned}$$

Indeed, the two quasilinear PDEs can be expressed as

$$\frac{1}{q_1^2}\det\begin{pmatrix}\frac{\partial q_0}{\partial s_1}&\frac{\partial q_0}{\partial s_2}&\frac{\partial q_0}{\partial s_3}\\\frac{\partial q_1}{\partial s_1}&\frac{\partial q_1}{\partial s_2}&\frac{\partial q_1}{\partial s_3}\\b_{2,1}&b_{2,2}&b_{2,3}\end{pmatrix}=\det\begin{pmatrix}\frac{\partial g_3}{\partial s_1}&\frac{\partial g_3}{\partial s_2}&\frac{\partial g_3}{\partial s_3}\\b_{2,1}&b_{2,2}&b_{2,3}\\b_{3,1}&b_{3,2}&b_{3,3}\end{pmatrix}$$

$$-\frac{1}{q_1^2}\det\begin{pmatrix}\frac{\partial q_0}{\partial s_1}&\frac{\partial q_0}{\partial s_2}&\frac{\partial q_0}{\partial s_3}\\\frac{\partial q_1}{\partial s_1}&\frac{\partial q_1}{\partial s_2}&\frac{\partial q_1}{\partial s_3}\\b_{3,1}&b_{3,2}&b_{3,3}\end{pmatrix}=\det\begin{pmatrix}\frac{\partial g_2}{\partial s_1}&\frac{\partial g_2}{\partial s_2}&\frac{\partial g_2}{\partial s_3}\\b_{2,1}&b_{2,2}&b_{2,3}\\b_{3,1}&b_{3,2}&b_{3,3}\end{pmatrix}$$

In both cases there is no reason for the choice of the roles of g_i (compare Remark 3.2).

3.2 The general case

Theorem 3.1. *Let $n \geq 2$ be the number of independent variables. Let further $M = (b_{k,\ell})_{2\leq k\leq n,1\leq \ell\leq n}$, where $b_{i,j}$ are as in (5). Then system (3) yields a quasilinear PDE in g_n of the following form*

$$\sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}} a_{i,j}(-1)^{i+j+n}\det(M_{\{n\},\{i,j\}})=\sum_{i=1}^{n}\frac{\partial g_n}{\partial s_i}(-1)^i\det(M_{\emptyset,\{i\}})\,, \tag{7}$$

where $M_{R,S}$ denotes the matrix which is obtained from M by deleting all rows with index in R and all columns with index in S.

Proof. We will start with rearranging the left hand side. Some technical details we will outsource to lemmata which are shown later. Using equation (5) to replace the $a_{i,j}$ the left hand side of (7) reads as

$$\begin{aligned}&\sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}}\left(\sum_{k=2}^{n} b_{k,j}\frac{\partial g_k}{\partial s_i}-b_{k,i}\frac{\partial g_k}{\partial s_j}\right)(-1)^{i+j+n}\det(M_{\{n\},\{i,j\}})\\&=\sum_{k=2}^{n}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(b_{k,j}\frac{\partial g_k}{\partial s_i}-b_{k,i}\frac{\partial g_k}{\partial s_j}\right)(-1)^{i+j+n}\det(M_{\{n\},\{i,j\}})\\&=\sum_{k=2}^{n}\left(\sum_{i=1}^{n}\sum_{j=i+1}^{n} b_{k,j}\frac{\partial g_k}{\partial s_i}(-1)^{i+j+n}\det(M_{\{n\},\{i,j\}})\right.\end{aligned}$$

$$
\left. - \sum_{i=1}^{n} \sum_{j=i+1}^{n} b_{k,i} \frac{\partial g_k}{\partial s_j} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \right)
$$

$$
= \sum_{k=2}^{n} \left(\sum_{i=1}^{n} \sum_{j=i+1}^{n} b_{k,j} \frac{\partial g_k}{\partial s_i} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \right.
$$

$$
\left. - \sum_{i=2}^{n} \sum_{j=1}^{i-1} b_{k,j} \frac{\partial g_k}{\partial s_i} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \right)
$$

$$
= \sum_{k=2}^{n} \left(\sum_{j=2}^{n} b_{k,j} \frac{\partial g_k}{\partial s_1} (-1)^{1+j+n} \det(M_{\{n\},\{i,j\}}) \right.
$$

$$
+ \sum_{i=2}^{n} \frac{\partial g_k}{\partial s_i} \left(\sum_{j=i+1}^{n} b_{k,j} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \right.
$$

$$
\left.\left. - \sum_{j=1}^{i-1} b_{k,j} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \right) \right)
$$

$$
= \sum_{k=2}^{n} \left(\sum_{i=1}^{n} \frac{\partial g_k}{\partial s_i} \left(\sum_{j=i+1}^{n} b_{k,j} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \right. \right.
$$

$$
\left.\left. - \sum_{j=1}^{i-1} b_{k,j} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \right) \right)
$$

$$
= \sum_{i=1}^{n} \frac{\partial g_n}{\partial s_i} \left(\sum_{j=i+1}^{n} b_{n,j} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) - \sum_{j=1}^{i-1} b_{n,j} (-1)^{i+j+n} \det(M_{\{n\},\{i,j\}}) \right)
$$

$$
= \sum_{i=1}^{n} \frac{\partial g_n}{\partial s_i} (-1)^i \det(M_{\emptyset,\{i\}}) \,.
$$

In the last two steps we used backward Laplace expansion and got a matrix with an additional line. This line does already appear in the matrix except for $k = n$. □

There is no reason for the special role of g_n. Hence, we can give a similar quasilinear equation for each g_ν for $\nu > 1$ and solve them in parallel.

Remark 3.2. *The equations we have to solve are*

$$
\left\{ \sum_{\substack{i,j \in \{1,\dots,n\} \\ i<j}} a_{i,j} (-1)^{i+j+\nu} \det(M_{\{\nu\},\{i,j\}}) = \sum_{i=1}^{n} \frac{\partial g_\nu}{\partial s_i} (-1)^i \det(M_{\emptyset,\{i\}}) \right\}_{\nu \in \{2,\dots,n\}} . \tag{8}
$$

Finally, g_1 has to be computed by using the system (3). The system of quasilinear PDEs in (8) can be expressed as (compare to the case of 3 variables)

$$\left\{ \frac{(-1)^\nu}{q_1^2} \det \begin{pmatrix} \nabla q_0 \\ \nabla q_1 \\ M_{\{\nu\},\emptyset} \end{pmatrix} = \det \begin{pmatrix} \nabla g_\nu \\ M \end{pmatrix} \right\}_{\nu\in\{2,\dots,n\}} .$$

This is a consequence of using backward Laplace expansion by the first row, of the right hand side determinant, and generalized Laplace's expansion by the two first rows of the left hand side determinant.

The system of quasilinear PDEs depends on the choice of parametrization. This might influence computational complexity. However, investigation of a suitable choice is subject to further research.

Note, that the determinants on the right hand side of (8) do not depend on ν. In the following we will see some cases where the the determinants on the right hand side have special properties. Mainly, we are asking some or all of them to be zero.

Remark 3.3. *If* $\det(M_{\emptyset,\{i\}}) = 0$ *for every* $i \in \{1,\dots,n\}$ *but one index, say* ℓ*, then the equations* (8) *reduce to* $n-1$ *ODEs with solution*

$$g_\nu = \int \frac{\sum_{\substack{i,j\in\{1\dots n\}\\ i<j}} a_{i,j}(-1)^{i+j}\det(M_{\{\nu\},\{i,j\}})}{(-1)^{\ell-\nu}\det(M_{\emptyset,\{\ell\}})} ds_\ell + K(s_1,\dots,s_{\ell-1},s_{\ell+1},\dots,s_n)\,.$$

In the following remark and theorem we will see what happens if the right hand side of (8) is zero. Two possible cases appear: Either the left hand side is zero as well, or it is not.

Remark 3.4. *If* $\det(M_{\emptyset,\{i\}}) = 0$ *for every* $i \in \{1,\dots,n\}$ *and*

$$\sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}} a_{i,j}(-1)^{i+j+\nu}\det(M_{\{\nu\},\{i,j\}}) \neq 0$$

for some $\nu \in \{1,\dots,n\}$*, then we get a contradiction, and hence, the assumption* $\mathcal{Q} = \mathcal{L}(g)$ *was wrong. This, however, means that there is no proper rational solution (compare the remarks on parametrization in the beginning of Section 3). Nevertheless, there might be a non-proper rational solution, which we cannot find with the procedure presented here.*

We will now show that the left hand side cannot be zero according to our assumptions. Note, that the proof can also be applied in the case when $\mathcal{Q}$ is not rational.

Theorem 3.5. *If* $\det(M_{\emptyset,\{i\}}) = 0$ *for every* $i \in \{1,\dots,n\}$*, and*

$$\sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}} a_{i,j}(-1)^{i+j+\nu}\det(M_{\{\nu\},\{i,j\}}) = 0$$

for every $\nu \in \{1,\dots,n\}$*, then* $\mathcal{Q}$ *turns out to be a parametrization of a variety of dimension strictly less than* n*.*

Proof. In order to prove this statement, we take the matrix $M = (b_{k,\ell})_{\substack{2\leq k\leq n\\ 1\leq \ell\leq n}}$. From the fact that the determinant $\det(M_{\emptyset,\{i\}}) = 0$ for every $i \in \{1,\dots,n\}$, the rank of M is, at most, $n-2$. By definition of the $b_{k,\ell}$ we know

$$b_{k,\ell} = \frac{\partial b_k}{\partial s_\ell} = \frac{\partial}{\partial s_\ell}\left(\frac{q_k}{q_1}\right) = q_1^{-2}\left(\frac{\partial q_k}{\partial s_\ell}q_1 - \frac{\partial q_1}{\partial s_\ell}q_k\right)$$

for every $k \in \{2,\dots,n\}$ and $\ell \in \{1,\dots,n\}$. Let $M^\star = (\frac{\partial q_k}{\partial s_\ell})_{\substack{2\leq k\leq n\\ 1\leq \ell\leq n}}$. Then each row in M is obtained from a linear combination of the corresponding row in $M^\star$ and the vector $(\frac{\partial q_1}{\partial s_\ell})_{1\leq \ell\leq n}$. More precisely, one has that the ν-th row in M is given by

$$\nabla(q_{\nu+1})\frac{1}{q_1} - \nabla(q_1)\frac{q_{\nu+1}}{q_1^2}$$

for every $\nu \in \{1,\dots,n-1\}$, and where $\nabla(q_j) = (\frac{\partial q_j}{\partial s_1},\dots,\frac{\partial q_j}{\partial s_n})$. So the rank of $\begin{pmatrix}\nabla q_1\\ M^\star\end{pmatrix}$ is upper bounded by $n-1$. It remains to prove that this rank is preserved when the vectors $(\frac{\partial q_0}{\partial s_j})_{1\leq j\leq n}$ and $(\frac{\partial q_1}{\partial s_j})_{1\leq j\leq n}$ are incorporated to $M^\star$ as new rows. If this occurs, then the matrix $(\frac{\partial q_j}{\partial s_k})_{\substack{0\leq j\leq n\\ 1\leq \ell\leq n}}$ would have rank strictly lower than n, and the parametrization does not correspond to a variety of dimension n.

From their definition,

$$a_{i,j} = \frac{\partial a_i}{\partial s_j} - \frac{\partial a_j}{\partial s_i} = \frac{\partial}{\partial s_j}\left(\frac{\frac{\partial q_0}{\partial s_i}}{q_1}\right) - \frac{\partial}{\partial s_i}\left(\frac{\frac{\partial q_0}{\partial s_j}}{q_1}\right) = \frac{1}{q_1^2}\left(\frac{\partial q_0}{\partial s_j}\frac{\partial q_1}{\partial s_i} - \frac{\partial q_0}{\partial s_i}\frac{\partial q_1}{\partial s_j}\right).$$

The hypotheses held in the statement of the theorem, that the left hand side of equation (8) vanishes for every $\nu \in \{2,\dots,n\}$ yields

$$\sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}} \left(\frac{\partial q_0}{\partial s_j}\frac{\partial q_1}{\partial s_i} - \frac{\partial q_0}{\partial s_i}\frac{\partial q_1}{\partial s_j}\right)(-1)^{i+j}\det(M_{\{\nu\},\{i,j\}}) = 0 \tag{9}$$

for $\nu \in \{2,\dots,n\}$. Regarding the generalized Laplace expansion (see for instance [6]), the left hand side of (9) is the determinant of a single $n\times n$-matrix and we get

$$\det\begin{pmatrix}\nabla q_0\\ \nabla q_1\\ M_{\{\nu\},\emptyset}\end{pmatrix} = 0\,.$$

Hence, all such $n\times n$ matrices have rank $n-1$. We still need to show, that the rank of $\begin{pmatrix}\nabla q_0\\ M^\star\end{pmatrix}$ is at most $n-1$. Assume to the contrary, that the rank is n. Then $(\nabla q_2,\dots,\nabla q_n)$ are linearly independent. Since the rank of $\begin{pmatrix}\nabla q_1\\ M^\star\end{pmatrix}$ is at most $n-1$, we know that $(\nabla q_1, \ldots, \nabla q_n)$ are linearly dependent. Hence, ∇q_1 can be written as a linear combination of $\nabla q_1 = \sum_{j=2}^{n} \lambda_j \nabla q_j$. We take k such that $\lambda_k \neq 0$. Then $\nabla q_k = \frac{1}{\lambda_k}\left(\nabla q_1 - \sum_{\substack{j=2 \\ j\neq k}}^{n} \lambda_j \nabla q_j\right)$. Hence, the rank of $\begin{pmatrix} \nabla q_0 \\ M^\star \end{pmatrix}$ equals the rank of $\begin{pmatrix} \nabla q_0 \\ \nabla q_1 \\ M_{\{k\},\emptyset} \end{pmatrix}$ which we have shown to be at most $n-1$ so we have a contradiction.

From this we conclude that the rank of $(\frac{\partial q_j}{\partial s_k})_{\substack{0\le j\le n \\ 1\le k\le n}}$ is, at most, $n-1$, and the parametrization does not correspond to a variety of dimension n. □

For the rest of the paper we will assume that the quasilinear equations (8) are non-trivial, i.e. we are not in one of the special cases described above.

Method of characteristics. The quasilinear equations (8) can be solved by using the method of characteristics (see for instance [24]). Doing so we need to solve the following system of ordinary differential equations.

$$\left.\begin{aligned} \frac{ds_i}{dt} &= (-1)^i \det(M_{\emptyset,\{i\}}) \qquad\qquad \text{for } 1 \le i \le n\,, \\ \frac{dv}{dt} &= \sum_{\substack{i,j\in\{1,\ldots,n\} \\ i<j}} a_{i,j}(-1)^{i+j+\nu} \det(M_{\{\nu\},\{i,j\}})\,. \end{aligned}\right\} \tag{10}$$

In case $n = 2$ this can be transformed to a decoupled system which can be solved by methods presented in [19, 20, 21]. Compare [10] for this case. For $n \ge 3$ system (10) is no longer uncoupled in general. The first n equations will form a possibly coupled system, whereas (as in the case $n = 2$) the last one can then be solved by integration. Hence, an arbitrary constant is involved. We will show later that the introduction of these constants can be postponed.

First we solve the first n equations in (10). Observe that it is a first-order system of autonomous ordinary differential equations which provides solutions depending on $n-1$ arbitrary constants, say $k_2, \ldots, k_n$. We write $s_i(t) = \chi_i(t, k_2, \ldots, k_n)$, $1 \le i \le n$, to describe the dependence on that constants. In view of this information, we solve the last equation in (10) by integration. The dependence on the arbitrary constants is inherited by v so that we may write $v(t) = v(t, k_2, \ldots, k_n) = \bar{v}(t, k_2, \ldots, k_n) + \omega(k_2, \ldots, k_n)$ for some $\bar{v}$ and an arbitrary function ω.

In order to resolve the constants appearing, we search for explicit functions ξ_k satisfying $s_i(t) = \chi_i(\xi_1, \ldots, \xi_n)$ for all i.

We remark it is not always possible to obtain these functions explicitly. In the negative case, the procedure will fail to find a solution of the APDE. In this situation, we will not know whether a solution exists or not, whereas in the positive case, we get $g_\nu(s_1, \ldots, s_n) = \bar{v}(\xi_1, \ldots, \xi_n) + \omega$, where ω depends on a constant c. For the sake of simplicity, we fix $\omega = c$ as a special case in the procedure. The question of how to choose ω is a matter of further research.

Note, that the first n equations of (10) do not depend on ν since the right hand side of (8) did not either. This means we can solve this part of the system of ODEs once for each APDE. What remains is to solve the last equation of (10). This needs to be done for every $\nu > 1$, but can be done in parallel.

3.3 Solution procedure

Finally, using the results from the previous sections we give a procedure for solving APDEs in n variables is as follows

Procedure 1. *Given an autonomous first-order APDE, $F(u, u_{x_1}, \ldots, u_{x_n}) = 0$, where F is an irreducible and non-constant polynomial, and a proper rational parametrization $\mathcal{Q}(s_1, \ldots, s_n) = (q_0, \ldots, q_n)$ of F.*

1. *Compute the coefficients $a_i = \frac{\frac{\partial q_0}{\partial s_i}}{q_1}$, and $b_i = \frac{q_i}{q_1}$.*
   Compute further $a_{j,k} = \frac{\partial a_j}{\partial s_k} - \frac{\partial a_k}{\partial s_j}$ and $b_{i,\ell} = \frac{\partial b_i}{\partial s_\ell}$.
2. *Compute $\det(M_{\emptyset,\{i\}})$ for all i. If only one of them is non-zero, solve the ODEs by integration as described in Remark 3.3 and continue with step 4.*
   If all determinants are zero, compute $\sum_{\substack{i,j\in\{1,\ldots,n\}\\ i<j}} a_{i,j}(-1)^{i+j+\nu}\det(M_{\{\nu\},\{i,j\}})$. If this is non-zero, there is no proper rational solution. The procedure stops. If this is zero, then $\mathcal{Q}$ does not fulfill the requirements.
3. *Solve (in parallel) the quasilinear PDEs (8) for g_ν, $n \geq \nu > 1$, respectively. Using the method of characteristics proceed as follows.*
   a) *Solve the system of ODEs, $\frac{ds_i}{dt} = (-1)^i \det(M_{\emptyset,\{i\}})$, for all $1 \leq i \leq n$ and get solutions $s_i(t) = \chi_i(t, k_2, \ldots, k_n)$.*
   b) *Solve the ODE, $\frac{dv}{dt} = \sum_{\substack{i,j\in\{1,\ldots,n\}\\ i<j}} a_{i,j}(-1)^{i+j+\nu}\det(M_{\{\nu\},\{i,j\}})$, by integration.*
   c) *Compute ξ_k such that $s_i = \chi_i(\xi_1, \ldots, \xi_n)$ for all i.*
   d) *Compute $g_\nu(s_1, \ldots, s_n) = \bar{v}(\xi_1, \ldots, \xi_n) + c$.*
4. *Use (3) to compute g_1.*
5. *Compute $h_1, \ldots, h_n$ such that*
   $g(h_1(s_1, \ldots, s_n), \ldots, h_n(s_1, \ldots, s_n)) = (s_1, \ldots, s_n)$.
6. *Compute the solution $q_0(h_1, \ldots, h_n)$.*

Theorem 3.6. *Let $F(u, u_{x_1}, \ldots, u_{x_n}) = 0$ be an autonomous APDE. If Procedure 1 returns a function $v(x_1, \ldots, x_n)$ for input F, then v is a solution of $F = 0$.*

Proof. By the last step of the procedure we know that

$$v(x_1, \ldots, x_n) = q_0(h_1(x_1, \ldots, x_n), \ldots, h_n(x_1, \ldots, x_n)).$$

with h_i such that $g(h_1(s_1, \ldots, s_n), \ldots, h_n(s_1, \ldots, s_n)) = (s_1, \ldots, s_n)$. The function g fulfills the assumption that $u(g_1, \ldots, g_n) = q_0$ for a solution u since it is a solution of the system (5). Hence, v is a solution. We have seen a more detailed description at the beginning of this section. □

Now, we will show that the result does not change if we postpone the introduction of the constants $c_1, \dots, c_n$ to the end of the procedure. It is easy to show that if $u(x_1, \dots, x_n)$ is a solution of an autonomous APDE then so is $u(x_1 + c_1, \dots, x_n + c_n)$ for any constants c_i, $1 \leq i \leq n$. ¿From the procedure we get that $g_i = \bar{g}_i + c_i$ for $i \geq 2$ and $\bar{g}_i$ not depending on c_j for all j. Furthermore, we see that in the computation of g_1 we use the derivatives of g_i only (and hence the c_i disappear). Therefore, we have that $g_1 = \bar{g}_1 + c_1$. Let $g = (g_1, \dots, g_n)$ and $\bar{g} = (\bar{g}_1, \dots, \bar{g}_n)$. In step 5 we are looking for a function h such that $g \circ h = \text{id}$. Now $g \circ h = \bar{g} \circ h + (c_1, \dots, c_n)$. Take $\bar{h}$ such that $\bar{g} \circ \bar{h} = \text{id}$. Then $g \circ \bar{h}(s_1 - c_1, \dots, s_n - c_n)) = \text{id}$. Hence, we can introduce the constants at the end.

In the next section, we provide additional information on the solution obtained by the preceding method, in case it yields a rational solution. Namely, we study whether the output is a proper complete solution of suitable dimension.

In case the original APDE is in fact an AODE, the ODE in (10) turns out to be trivial and the integral in step 4 is exactly the one which appears in the procedure for AODEs [9, 11]. Of course then g is univariate and so is its inverse. In this sense, this new procedure generalizes the procedure in [9, 11]. We do not specify Procedure 1 to handle this case. Furthermore, if $n = 2$ this procedure is exactly the one which can be found in [10].

Remark 3.7. *Procedure 1 might fail in several steps. First of all, we avoided to talk about parametrizability by assuming there is a parametrization of the corresponding hypersurface. In case such a parametrization does not exist in a certain class there cannot exist a solution in this class either. Further we use the method of characteristics which might not give an explicit solution (compare [24]). Later we compute g_1 by integration where a solution might only be found in a field extension, i.e. we might get out of the class of functions we are looking for. Nevertheless, if we find an integral in a field extension and the subsequent steps are successful as well, we might still get a solution. See for instance the examples in Section 5. We have made the initial assumption that the solution can be written in the form $\mathcal{Q} \circ g$, for some invertible function g. In step 5 one may approach the actual computation of the inverse of g by means of elimination techniques, such as Gröbner bases. Nevertheless, it might happen that there is no explicit solution for h_i while inverting g.*
In all the above mentioned cases, we say that the procedure fails and then we do not know anything about solvability of the input APDE. In the latter case, however, we might state the solution implicitly.

4 Rational Solutions

For first-order autonomous AODEs the algorithm of Feng and Gao [7] gives an answer on whether or not a rational solution exists. As Procedure 1 is a generalization of the procedure for ODEs in [9, 11], it also generalizes this algorithm. However, as in [9, 11], any final result of the procedure is a solution of the differential equation, but the procedure might fail and then it does not tell us whether a solution might exist. In the following we describe properties of rational solutions found by Procedure 1 and we give a class of APDEs that has a rational solution which can be found by the procedure.

4.1 Properties of Rational Solutions

In the following we will discuss the properties of rational solutions computed by our procedure. We will show that these solutions are proper and complete of suitable dimension.

Lemma 4.1. *If Procedure 1 yields a rational solution, then the solution is proper.*

Proof. Let $\mathcal{L}$ be the corresponding parametrization of the output solution. In the procedure we start with a proper parametrization $\mathcal{Q}$ of the associated surface. When the procedure is successful we know that $\mathcal{L}(g) = \mathcal{Q}$ and the inverse h of g exists. Hence, $\mathcal{L} = \mathcal{Q}(h)$ is proper as well. □

Recall Remark 2.1 which proves that the jacobian of the corresponding parametrization of a proper solution computed by the procedure has generic rank n.

Theorem 4.2. *Assume Procedure 1 yields a rational solution $u(x_1, \ldots, x_n)$. Then the solution u is complete of suitable dimension.*

Proof. From the investigation below Theorem 3.6 we know that the output of the procdure is $u(x_1, \ldots, x_n) = u^*(x_1 + c_1, \ldots, x_n + c_n)$ for some u^*. As usual let $\mathcal{L}$ be the corresponding parametrization of u. For the case of two variables we see that

$$\mathcal{J}_{\mathcal{L}}^{c_1,c_2} = \begin{pmatrix} u_x(x + c_1, y + c_2) & u_y(x + c_1, y + c_2) \\ u_{xx}(x + c_1, y + c_2) & u_{xy}(x + c_1, y + c_2) \\ u_{yx}(x + c_1, y + c_2) & u_{yy}(x + c_1, y + c_2) \end{pmatrix} = \mathcal{J}_{\mathcal{L}}^{x,y} = \mathcal{J}_{\mathcal{L}} \, .$$

The equation $\mathcal{J}_{\mathcal{L}}^{c_1,\ldots,c_n} = \mathcal{J}_{\mathcal{L}}^{x_1,\ldots,x_n}$ also holds in general. From Lemma 4.1 we know that $\mathcal{L}$ is proper and from Remark 2.1 we know that a proper solution has a jacobian of rank n. □

4.2 APDEs with Rational Solutions

Examples with two variables can be found in [10]. Here we will therefore focus on an example with more than two variables.

Example 4.3 (Example 7.11 of Kamke [16])**.** *We consider the autonomous APDE,*

$$F(u, u_{x_1}, u_{x_2}, u_{x_3}) = d_1 u_{x_1}^2 + d_2 u_{x_2}^2 + d_3 u_{x_3}^2 - u = 0 \, ,$$

where d_1, d_2 and d_3 are non-zero constants. A possible parametrization is

$$\mathcal{Q} = (s_1, s_2, \frac{-s_1 + d_1 s_2^2 + d_3 s_3^2}{2 d_2 s_3} \sqrt{-\frac{d_2}{d_3}}, \frac{s_1 - d_1 s_2^2 + d_3 s_3^2}{2 d_3 s_3}) \, .$$

The coefficients as computed in the procedure are

$$a_1 = \frac{1}{s_2}, \qquad a_2 = 0, \qquad a_3 = 0\,,$$
$$b_2 = \frac{-s_1 + d_1 s_2^2 + d_3 s_3^2}{2 d_2 s_2 s_3}\sqrt{-\frac{d_2}{d_3}}, \qquad b_3 = \frac{s_1 - d_1 s_2^2 + d_3 s_3^2}{2 d_3 s_2 s_3}\,.$$

Then we have to solve the following quasilinear equations

$$\frac{-s_1 + d_1 s_2^2 + d_3 s_3^2}{2 d_3 s_2^3 s_3^2} = -\frac{\sqrt{-\frac{d_2}{d_3}}\left(s_3 \frac{\partial g_2}{\partial s_3} + s_2 \frac{\partial g_2}{\partial s_2} + 2 s_1 \frac{\partial g_2}{\partial s_1}\right)}{2 d_2 s_2^3 s_3}\,,$$
$$-\frac{\sqrt{-\frac{d_2}{d_3}}\left(s_1 - d_1 s_2^2 + d_3 s_3^2\right)}{2 d_2 s_2^3 s_3^2} = -\frac{\sqrt{-\frac{d_2}{d_3}}\left(s_3 \frac{\partial g_3}{\partial s_3} + s_2 \frac{\partial g_3}{\partial s_2} + 2 s_1 \frac{\partial g_3}{\partial s_1}\right)}{2 d_2 s_2^3 s_3}\,.$$

Since $q_0 = s_1$ *we have that* $a_i = 0$ *for* $i \geq 2$*, and hence only one pair of indeces in the sum on the left hand side of* (8) *yields a non-zero contribution. Simplifying these equations and using the ideas of the method of characteristics, we have to solve the following system of ODEs.*

$$s_1' = \frac{2 s_1 s_3}{\sqrt{-\frac{d_2}{d_3}} d_3}\,,$$
$$s_2' = -\frac{\sqrt{-\frac{d_2}{d_3}} s_2 s_3}{d_2}\,,$$
$$s_3' = -\frac{\sqrt{-\frac{d_2}{d_3}} s_3^2}{d_2}\,,$$
$$v' = \frac{d_1 s_2^2 + d_3 s_3^2 - s_1}{d_3}\,, \qquad \textit{resp.} \qquad v = -\frac{\sqrt{-\frac{d_2}{d_3}}\left(-d_1 s_2^2 + d_3 s_3^2 + s_1\right)}{d_2}\,.$$

The first three equations are independent on the last one. They yield solutions

$$s_1 = \frac{c_2}{\left(c_1 \sqrt{-\frac{d_2}{d_3}} d_3 + t\right)^2}\,, \qquad s_2 = \frac{c_3}{c_1 d_2 - \sqrt{-\frac{d_2}{d_3}} t}\,, \qquad s_3 = -\frac{d_2}{c_1 d_2 - \sqrt{-\frac{d_2}{d_3}} t}\,,$$

for some arbitrary constants c_1, c_2, c_3*. Resolving* t *and the constants we get*

$$t = -\frac{\sqrt{-\frac{d_2}{d_3}} d_3}{s_3}\,, \qquad c_2 = -\frac{d_2 d_3 s_1}{s_3^2}\,, \qquad c_3 = -\frac{d_2 s_2}{s_3}\,. \tag{11}$$

Solving the last equation of the system of ODEs by integration we get

$$v = \frac{\frac{c_3^2 d_1}{d_2} + \frac{c_2}{d_3} + d_2 d_3}{t}\,, \qquad \textit{resp.} \qquad v = \frac{\sqrt{-\frac{d_2}{d_3}}\left(c_3^2 d_1 d_3 + c_2 d_2 - d_2^2 d_3^2\right)}{d_2^2 t}$$

Using (11) get the solutions

$$g_2 = \frac{\sqrt{-\frac{d_2}{d_3}}\left(-s_1 + d_1 s_2^2 + d_3 s_3^2\right)}{s_3}\,, \qquad g_3 = \frac{s_1 - d_1 s_2^2 + d_3 s_3^2}{s_3}\,.$$

Now, we need to compute g_1. We do so be taking the first equation of (3). As a solution we get

$$g_1 = m_1(s_2, s_3)\,,$$

where m_1 is an arbitrary function. Using the second equation of (3) we compute m_1 and get

$$m_1 = 2d_1 s_2 + m_2(s_3)\,.$$

Finally, we compute m_2 using the last equation in (3) and get $m_2 = c_1$, which we choose to be 0. Hence,

$$g_1 = 2a_1 s_2\,.$$

Solving the system $g_i(h) = s_i$, we get

$$h_1 = \frac{1}{4}\left(\frac{s_1^2}{d_1} + \frac{s_2^2}{d_2} + \frac{s_3^2}{d_3}\right)\,, \qquad h_2 = \frac{s_1}{2d_1}\,, \qquad h_3 = \frac{\frac{s_2}{\sqrt{-\frac{d_2}{d_3}}} + s_3}{2d_3}\,.$$

Hence,

$$q_0(h(x_1, x_2, x_3)) = h_1(x_1, x_2, x_3) = \frac{1}{4}\left(\frac{s_1^2}{d_1} + \frac{s_2^2}{d_2} + \frac{s_3^2}{d_3}\right)$$

is a solution of the APDE and $q_0(h(x_1 + c_1, x_2 + c_2, x_3 + c_3))$ is a complete one.

5 Other Solutions

We will first show some properties of arbitrary solutions found by the procedure. Similarly to Lemma 4.1 we get the following.

Lemma 5.1. *If Procedure 1 yields a solution, then the corresponding parametrization is injective almost everywhere.*

Proof. Let $\mathcal{L}$ be the corresponding parametrization of the output solution. In the procedure we start with a proper parametrization $\mathcal{Q}$ of the associated surface. When the procedure is successful we know that $\mathcal{L}(g) = \mathcal{Q}$ and the inverse h of g exists. Hence, $\mathcal{L} = \mathcal{Q}(h)$ is injective almost everywhere. $\square$

A parametrization which is injective almost everywhere is also called almost injective. Note, that jacobian of an almost injective parametrization $\mathcal{P}(s_1, \ldots, s_n)$ has generic rank n. Indeed, since $\mathcal{P}$ is almost injective, there exists a map R such that $\mathrm{id} = R \circ \mathcal{P}$ generically. Thus $\mathcal{J}_{\mathrm{id}} = \mathcal{J}_{R\circ\mathcal{P}} = \mathcal{J}_R(\mathcal{P}) \cdot \mathcal{J}_{\mathcal{P}}$. Taking into account, that the rank of a product of two matrices is smaller equal the minimal rank of the two matrices, we getthat $\mathrm{rank}(\mathcal{J}_{\mathcal{P}}) = n$.

Theorem 5.2. *Assume Procedure 1 yields a solution $u(x_1, \ldots, x_n)$. Then the solution u is complete of suitable dimension.*

Proof. As usual let $\mathcal{L}$ be the corresponding parametrization of u. Then the equation $\mathcal{J}_{\mathcal{L}}^{c_1,\ldots,c_n} = \mathcal{J}_{\mathcal{L}}^{x_1,\ldots,x_n}$ holds in general. From Lemma 5.1 we know that $\mathcal{L}$ is almost injective and from the notes above we know that an almost injective solution has a jacobian of expected rank. □

The following examples show that the method is not restricted to finding rational solutions. It might happen that the steps in Procedure 1 can be done working in some extension field. In this case we can of course continue in the procedure and might get a non-rational solution.

Table 2 presents a list of some well known equations in two variables and the solutions found by the procedure. For the sake of readability we negelct the arbitrary constants and present only specific solutions. Details can be found in [10].

Name	APDE	Parametrization	Solution
Burgers (inviscid) [24]	$uu_x + u_y$	$(-\frac{t}{s}, s, t)$	$\frac{x}{y}$
Traffic [3]	$u_y - u_x\left(\frac{2uv_m}{r_m} - v_m\right)$	$(\frac{r_m(t+sv_m)}{2sv_m}, s, t)$	$\frac{r_m(-x+yv_m)}{2v_m y}$
Eikonal [2]	$u_x^2 + u_y^2 - 1$	$(s, \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$	$\pm\sqrt{x^2+y^2}$
Convection-Reaction [1]	$u_x + cu_y - du$	$(\frac{s+ct}{d}, s, t)$	$\frac{\mathrm{e}^{dx} + c\mathrm{e}^{\frac{dy}{c}}}{d}$
Generalized Burgers (special case) [24]	$u_y + uu_x + \alpha u + \beta u^2$	(sB, tB, B) $B = -\frac{(1+s\alpha)}{st+s^2\beta}$	$\frac{\mathrm{e}^{-x\beta}(1-\mathrm{e}^{x\beta})\alpha}{(1+\mathrm{e}^{\alpha y})\beta}$

Table 2: Well known PDEs and their solutions found by the method in [10], which is a special case of the method presented here.

The procedure might as well find non-rational solutions to APDEs in more than two variables as we will see in the following examples. In both examples a parametrization with $q_0 = s_1$ is chosen which simplifies the left hand side of the quasilinear PDEs (8). Nevertheless, all important aspects of the procedure can be seen.

Example 5.3 (Eikonal equation with 5 variables)**.** *We consider the APDE,*

$$F(u, u_{x_1}, \ldots, u_{x_5}) = \left(\sum_{i=1}^{5} u_{x_i}^2\right) - 1 = 0\,.$$

A possible rational parametrization of the corresponding surface is

$$\mathcal{Q} = (s_1, \frac{s_2^2 + s_3^2 + s_4^2 + s_5^2 - 1}{D}, \frac{2s_2}{D}, \frac{2s_3}{D}, \frac{2s_4}{D}, \frac{2s_5}{D})\,,$$

where $D = s_2^2 + s_3^2 + s_4^2 + s_5^2 + 1$*. The parametrization is proper. Indeed, the inverse is given by*

$$s_1 = z\,, \qquad s_2 = -\frac{p_2}{p_1 - 1}\,, \qquad s_3 = \frac{p_3(p_1 + 1)}{p_2^2 + p_3^2 + p_4^2 + p_5^2}\,,$$
$$s_4 = \frac{p_4(p_1 + 1)}{p_2^2 + p_3^2 + p_4^2 + p_5^2}\,, \qquad s_5 = \frac{p_5(p_1 + 1)}{p_2^2 + p_3^2 + p_4^2 + p_5^2}\,.$$

The coefficients appearing in the procedure are

$$a_1 = \frac{s_2^2 + s_3^2 + s_4^2 + s_5^2 + 1}{s_2^2 + s_3^2 + s_4^2 + s_5^2 - 1}\,, \qquad a_i = 0\,, \qquad \text{for } i \geq 2\,,$$
$$b_i = \frac{2s_i}{s_2^2 + s_3^2 + s_4^2 + s_5^2 - 1}\,.$$

Then we get the following quasilinear equations for $2 \leq i \leq 5$*.*

$$\frac{32s_i}{\left(s_2^2 + s_3^2 + s_4^2 + s_5^2 - 1\right)^5} = \frac{16\left(s_2^2 + s_3^2 + s_4^2 + s_5^2 + 1\right)\frac{\partial g_i}{\partial s_1}}{\left(s_2^2 + s_3^2 + s_4^2 + s_5^2 - 1\right)^5}\,.$$

Here we are in the case of Remark 3.3 and hence, we get by integration

$$g_i = \frac{2s_1 s_i}{D} \qquad \text{for } i \geq 2\,.$$

Note, that for simplicity we chose the arbitrary functions which occur in the solutions to be 0*. Now we need to compute* g_1*. We do so by taking the first equation of (3). As a solution we get* $g_1 = \frac{s_1\left(s_2^2+s_3^2+s_4^2+s_5^2-1\right)}{D} + m_1(s_2, s_3, s_4, s_5)$*, where* m_1 *is an arbitrary function. Step by step we will compute* m_1 *now by using the other equations of (3). Using the second equation we have an ODE in* m_1*. We get* $m_1 = m_2(s_3, s_4, s_5)$*. Continuing like this we finally get* $m_1 = c_1$ *for an arbitrary constant. Since, we can deal with the constants at the end of the procedure, we will take it to be zero for the moment. Now we have to solve the system* $g_i(h) = s_i$*. A solution of this system is*

$$h_1 = \frac{\sqrt{s_2^2\left(s_1^2 + s_2^2 + s_3^2 + s_4^2 + s_5^2\right)}}{s_2}\,,$$
$$h_i = \frac{s_1 s_2 s_i - s_i\sqrt{s_2^2\left(s_1^2 + s_2^2 + s_3^2 + s_4^2 + s_5^2\right)}}{s_2\left(s_2^2 + s_3^2 + s_4^2 + s_5^2\right)} \qquad \text{for } i \geq 2\,.$$

Hence we conclude that,

$$q_0(h(x)) = h_1(x) = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2}$$

is a solution of the APDE.

Example 5.4. *We consider the APDE,* $F(u, u_{x_1}, u_{x_2}, u_{x_3}) = (u_{x_1}+d_1)u_{x_2}-(u+d_2)u_{x_3} = 0$. *A possible proper parametrization is* $\mathcal{Q} = (s_1, s_2, s_3, \frac{(s_2+d_1)s_3}{s_1+d_2})$. *The coefficients are*

$$a_1 = \frac{1}{s_2}\,, \qquad a_2 = 0\,, \qquad a_3 = 0\,,$$

$$b_2 = \frac{s_3}{s_2}\,, \qquad b_3 = \frac{(s_2+d_1)s_3}{(s_1+d_2)s_2}\,.$$

Then we have to solve the following quasilinear equations

$$\frac{d_1+s_2}{(d_2+s_1)s_2^3} = \frac{(d_1+s_2)s_3}{(d_2+s_1)^2 s_2^3}\left(s_3\frac{\partial g_2}{\partial s_3} + s_2\frac{\partial g_2}{\partial s_2}\right) + \frac{s_3}{(d_2+s_1)s_2^2}\frac{\partial g_2}{\partial s_1}\,,$$

$$-\frac{1}{s_2^3} = \frac{(d_1+s_2)s_3}{(d_2+s_1)^2 s_2^3}\left(s_3\frac{\partial g_3}{\partial s_3} + s_2\frac{\partial g_3}{\partial s_2}\right) + \frac{s_3}{(d_2+s_1)s_2^2}\frac{\partial g_3}{\partial s_1}\,.$$

Omiting the details and intermediate steps we get the solutions

$$g_2 = -\frac{(d_2+s_1)(d_1-\log(s_2)s_2)}{(d_1+s_2)s_3}\,, \qquad g_3 = \frac{(d_2+s_1)^2(d_1-\log(s_2)s_2)}{(d_1+s_2)^2 s_3}\,.$$

Now, we need to compute g_1. *We do so by taking the first equation of* (3). *As a solution we get*

$$g_1 = \frac{(1+\log(-s_2))s_1}{d_1+s_2} + m_1(s_2, s_3)\,,$$

where m_1 *is an arbitrary function. Using the second equation of* (3) *we compute* m_1 *and get*

$$m_1 = \frac{d_2(1+\log(-s_2))}{d_1+s_2} + m_2(s_3)$$

Finally, we compute m_3 *using the last equation in* (3) *and get* $m_2 = c_1$, *which we choose to be* 0. *Hence,*

$$g_1 = \frac{(1+\log(-s_2))(d_2+s_1)}{d_1+s_2}\,.$$

Solving the system $g_i(h) = s_i$, *we get*

$$h_1 = -\frac{d_2 s_2 + d_1 s_3 - \mathrm{e}^{-1-\frac{s_1 s_2}{s_3}} s_3}{s_2}\,,$$

$$h_2 = -\mathrm{e}^{-1-\frac{s_1 s_2}{s_3}}\,,$$

$$h_3 = \frac{\mathrm{e}^{-1-\frac{s_1 s_2}{s_3}}\left(-s_1 s_2 + \left(-1 + d_1 \mathrm{e}^{1+\frac{s_1 s_2}{s_3}}\right)s_3\right)}{s_2^2}\,.$$

Hence,

$$q_0(h(x)) = h_1(x_1, x_2, x_3) = -\frac{d_2x_2 + d_1x_3 - \mathrm{e}^{-1-\frac{x_1x_2}{x_3}}x_3}{x_2}$$

is a solution of the APDE.

6 Conclusion

We have presented an exact procedure for solving first-order algebraic differential equations in an arbitrary number of independent variables. In case the procedure yields a result, it is proven to be a complete solution of suitable dimension. Even if the method fails, it often leads to an implicit description of the solution. The method is a generalization of several methods which were already known, in particular also for ordinary differential equations.

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