Lastra, A. & Malek, S. 2020, "On parametric Gevrey asymptotics for some nonlinear initial value problems in symmetric complex time variables", Asymptotic Analysis, vol. 118, no. 1-2, pp. 49-79

Available at https://doi.org/10.3233/ASY-191568

On parametric Gevrey asymptotics for some nonlinear initial value problems in symmetric complex time variables

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Abstract

The asymptotic behavior of a family of singularly perturbed PDEs in two time variables in the complex domain is studied. The appearance of a multilevel Gevrey asymptotics phenomenon in the perturbation parameter is observed. We construct a family of analytic sectorial solutions in ϵ which share a common asymptotic expansion at the origin, in different Gevrey levels. Such orders are produced by the action of the two independent time variables.

Key words: asymptotic expansion, Borel-Laplace transform, Fourier transform, initial value problem, formal power series, nonlinear integro-differential equation, nonlinear partial differential equation, singular perturbation. 2010 MSC: 35C10, 35C20.

1 Introduction

This work is devoted to the study of a family of nonlinear initial value problems of the form

$$Q(\partial_z)\partial_{t_1}\partial_{t_2}u(t_1,t_2,z,\epsilon) = (P_1(\partial_z,\epsilon)u(t_1,t_2,z,\epsilon))(P_2(\partial_z,\epsilon)u(t_1,t_2,z,\epsilon)) + P(t_1,t_2,\partial_{t_1},\partial_{t_2},\partial_z,\epsilon)u(t_1,t_2,z,\epsilon) + f(t_1,t_2,z,\epsilon), \quad (1)$$

with initial null data $u(0,t_2,z,\epsilon) \equiv u(t_1,0,z,\epsilon) \equiv 0$.

The terms Q, P, P_1, P_2 are polynomials in all their variables except from the perturbation parameter ϵ in which they turn out to be holomorphic in a neighborhood of the origin. Moreover, we assume that the polynomial

$$\mathcal{P}(t_1,t_2,\partial_{t_1},\partial_{t_2},\partial_z,\epsilon) := Q(\partial_z)\partial_{t_1}\partial_{t_2} - L_1(t_1,t_2,\partial_{t_1},\partial_{t_2},\partial_z,\epsilon) \quad (2)$$

[*]The author is partially supported by the project MTM2016-77642-C2-1-P of Ministerio de Economía y Competitividad, Spain
[†]The author is partially supported by the project MTM2016-77642-C2-1-P of Ministerio de Economía y Competitividad, Spain.

where L_1 involves leading terms from the differential operator P which can be factorized in such a way that each of the factors only depend on one of the times variables, i.e.

$$\mathcal{P}(t_1,t_2,\partial_{t_1},\partial_{t_2},\partial_z,\epsilon)=\mathcal{P}_1(t_1,\partial_{t_1},\partial_z,\epsilon)\mathcal{P}_2(t_2,\partial_{t_2},\partial_z,\epsilon), \tag{3}$$

where the factors will be concreted later in the introduction.

The present work is a natural continuation of that in [6].

In the previous study [6], we focused on families of equations of the form

$$Q(\partial_z)\partial_t y(t,z,\epsilon)=H(t,\epsilon,\partial_t,\partial_z)y(t,z,\epsilon)+Q_1(\partial_z)y(t,z,\epsilon)Q_2(\partial_z)y(t,z,\epsilon)+f(t,z,\epsilon) \tag{4}$$

for given vanishing initial data $y(0,z,\epsilon)\equiv 0$, where Q_1,Q_2,H are polynomials in their corresponding variables, and the forcing term $f(t,z,\epsilon)$ turns out to be an analytic function near the origin for $(t,\epsilon)\in\mathbb{C}^2$, holomorphic with respect to z on a horizontal strip of the form

$$H_\beta:=\{z\in\mathbb{C}:|\mathrm{Im}(z)|<\beta\},$$

for some $\beta>0$. We constructed a set of genuine bounded holomorphic solutions $y_p(t,z,\epsilon)$, for $0\le p\le\varsigma-1$, defined on domains $\mathcal{T}\times H_\beta\times\mathcal{E}_p$, for well chosen bounded sectors $\mathcal{T}$ with vertex at the origin, and $(\mathcal{E}_p)_{0\le p\le\varsigma-1}$ represents a set of bounded sectors whose union contains a full neighborhood of the origin in $\mathbb{C}^\star$, which is called a good covering in $\mathbb{C}^\star$. Such functions were built as Laplace transform of order k and inverse Fourier transform

$$y_p(t,z,\epsilon)=\frac{k}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\int_{L_{\gamma_p}}\omega_p(u,m,\epsilon)\exp\left(-\left(\frac{u}{\epsilon t}\right)^k\right)e^{izm}\frac{du}{u}dm,$$

along halflines $L_{\gamma_p}=\mathbb{R}_+e^{i\gamma_p}\subseteq S_{d_p}\cup\{0\}$, where S_{d_p} is an unbounded sector with bisecting direction $d_p\in\mathbb{R}$, and ω_p represents a holomorphic function with exponential growth w.r.t u on $S_{d_p}\cup D(0,r)$, continuous w.r.t $m\in\mathbb{R}$ with exponential decay and holomorphic w.r.t ϵ on $D(0,\epsilon_0)\setminus\{0\}$, for some $\epsilon_0>0$. The highest order term of H in (4) is of irregular type

$$L(t,\epsilon,\partial_t,\partial_z)=\epsilon^{(\delta_D-1)k}t^{(\delta_D-1)(k+1)}\partial_t^{\delta_D}R_D(\partial_z),$$

for some positive integer k, some integer $\delta_D\ge 2$ and a polynomial $R_D(X)$. We proved that the functions y_p share a common asymptotic expansion w.r.t ϵ, say $\hat{y}(t,z,\epsilon)=\sum_{n\ge0}y_n(t,z)\epsilon^n$, a formal series with bounded holomorphic coefficients $y_n(t,z)$ on $\mathcal{T}\times H_\beta$. This asymptotic expansion is (at most) of Gevrey order $1/k$, meaning that

$$\sup_{t\in\mathcal{T},z\in H_\beta}\left|y_p(t,z,\epsilon)-\sum_{l=0}^{n-1}y_l(t,z)\epsilon^l\right|\le CM^n\Gamma\left(1+\frac{n}{k}\right)|\epsilon|^n,$$

for all $n\ge1$, all $\epsilon\in\mathcal{E}_p$. In case the aperture of $\mathcal{E}_p$ can be taken slightly larger than π/k, the function y_p is the $k-$sum of $\hat{y}$ on $\mathcal{E}_p$.

In view of the shape of equation (4), we assume that the factors in (3) are presented as a product of factors of the form

$$\mathcal{P}_1=Q_1(\partial_z)\partial_{t_1}-\epsilon^{(\delta_{D_1}-1)k_1}t_1^{(\delta_{D_1}-1)(k_1+1)}\partial_{t_1}^{\delta_{D_1}}R_{D_1}(\partial_z),$$

and

$$\mathcal{P}_2=Q_2(\partial_z)\partial_{t_2}-\epsilon^{(\tilde{\delta}_{D_2}-1)k_2}t_2^{(\tilde{\delta}_{D_2}-1)(k_2+1)}\partial_{t_2}^{\tilde{\delta}_{D_2}}R_{D_2}(\partial_z).$$

The importance of the present work with respect to that previous one is mainly due to the appearance of a multilevel Gevrey asymptotics phenomenon in the perturbation parameter, when dealing with a multivariable approach in time.

A recent overview on summability and multisummability techniques under different points of view is displayed in [10].

In recent years, an increasing interest on complex singularly perturbed PDEs has been observed in the area. Parametric Borel summability has been described in semilinear systems of PDEs of Fuchsian type by H. Yamazawa and M. Yoshino in [14]

$$\eta\sum_{j=1}^{n}\lambda_j x_j\frac{\partial}{\partial x_j}u(x)=f(x,u),$$

where $x=(x_1,\dots,x_n)\in\mathbb{C}^n$ and $f(x,u)=(f_1(x,u),\dots,f_N(x,u))$, for $n,N\geq 1$, and $\lambda_j\in\mathbb{C}$. η is a small complex perturbation parameter and f stands for a holomorphic vector function in a neighborhood of the origin in $\mathbb{C}^n\times\mathbb{C}^N$. Also, in partial differential equations of irregular singular type by M. Yoshino in [15]:

$$\eta\sum_{j=1}^{n}\lambda_j x_j^{s_j}\frac{\partial}{\partial x_j}u(x)=g(x,u,\eta),$$

where $s_j\geq 2$ for $1\leq j\leq m<n$, and $g(x,u,\eta)$ is a holomorphic vector function in some neighborhood of the origin in $\mathbb{C}^n\times\mathbb{C}^N\times\mathbb{C}$.

Recently, S.A. Carrillo and J. Mozo-Fernández have studied properties on monomial summability and the extension of Borel-Laplace methods to this theory in [3, 4]. In the last section of the second work, a further development on multisummability with respect to several monomials is proposed. Novel Gevrey asymptotic expansions and summability with respect to an analytic germ are described in [11] and applied to different families of ODEs and PDEs such as

$$\left(x_2\frac{\partial P}{\partial x_2}+\alpha P^{k+1}+PA\right)x_1\frac{\partial f}{\partial x_1}-\left(x_1\frac{\partial P}{\partial x_1}+\beta P^{k+1}+PB\right)x_2\frac{\partial f}{\partial x_2}=h,$$

where P is an homogeneous polynomial, $k\in\mathbb{N}^\star$ and h,A,B are convergent power series, and α,β satisfy certain conditions.

We now give some words on the main results obtained in our present study.

In the first principal outcome (Theorem 1), we construct a double indexed family of bounded holomorphic solutions to our main problem (1), presented as a double Laplace transform and Fourier integral

$$u_{p_1,p_2}(\boldsymbol{t},z,\epsilon)=\frac{k_1k_2}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_{\gamma_{p_1}}}\int_{L_{\gamma_{p_2}}}\omega_{\boldsymbol{k}}(u_1,u_2,m,\epsilon)e^{-(\frac{u_1}{\epsilon t_1})^{k_1}-(\frac{u_2}{\epsilon t_2})^{k_2}}e^{izm}\frac{du_2}{u_2}\frac{du_1}{u_1}dm \tag{5}$$

defined on $(\mathcal{T}_1\cap D(0,h))\times(\mathcal{T}_2\cap D(0,h))\times H_\beta\times\mathcal{E}_{p_1,p_2}$, for some bounded sectors $\mathcal{T}_1,\mathcal{T}_2$ with vertex at 0, some $h>0$, and where H_β is a horizontal strip. Here, the set $(\mathcal{E}_{p_1,p_2})_{\substack{0\leq p_1\leq\varsigma_1-1\\0\leq p_2\leq\varsigma_2-1}}$ is a good covering of $\mathbb{C}^\star$ of finite sectors with vertex at the origin (see Definition 4).

In the second main result, Theorem 2, we show that all the functions $\epsilon\mapsto u_{p_1,p_2}(t,z,\epsilon)$ share a common asymptotic expansion $\hat{u}(\boldsymbol{t},z,\epsilon)=\sum_{n\geq 0}H_n(\boldsymbol{t},z)\epsilon^n$, with bounded holomorphic coefficients H_m on $\mathcal{T}_1\times\mathcal{T}_2\times H_\beta$. This formal power series can be split as a sum of two formal power series, and each of the holomorphic solutions is decomposed accordingly in such a way that different Gevrey asymptotic behavior can be observed in each term of the sum. This phenomenon is the key point to multisummability, as described in [1] and also Section 7.5 in [10].

We have labelled these equations as *symmetric* for the following reason: the function $\omega_{\boldsymbol{k}}(\boldsymbol{\tau}, m, \epsilon)$, lying in the Borel plane, is defined on a domain which has the same shape with respect to τ_1 and τ_2, namely a product of unions of discs and unbounded sectors. In this respect, it can be seen as a first step in the generalization of our previous work [6]. In comparison to this work, the result obtained in [9] is a step further in which we study a particular case that we call *asymmetric* since it deals with a situation where the Borel map presents an asymmetry in its domain of holomorphy. These two geometric configurations give rise to different parametric Gevrey asymptotic behavior of the true solutions.

Throughout the whole paper, we denote $\boldsymbol{t} := (t_1, t_2)$, $\boldsymbol{T} = (T_1, T_2)$, $\boldsymbol{\tau} = (\tau_1, \tau_2)$.

The structure of the paper is as follows.
Section 2 states and gives details on the main problem under study (9), and the elements involved in it. It is also explained how to reduce the study of the main problem to that of two auxiliary equations, when searching for solutions under the form of a double Laplace and Fourier transform. In Section 3 we state the definition and some properties of the Banach spaces involved in the solution of the two auxiliary problems. Subsequently, a fixed point argument gives rise to the solutions of these auxiliary problems which lead to an analytic solution of the main problem under study in Section 4, Theorem 1. Also, upper estimates on the difference of two consecutive solutions are described in order to apply a novel version of the multilevel Ramis-Sibuya theorem and provide the existence of a formal solution of the main problem in Theorem 2. The analytic and formal solutions are related by means of a multilevel Gevrey asymptotic representation.

2 Description of the main problem

Let $k_1, k_2 \geq 1$ and $D_1, D_2 \geq 2$ be integers. For $j = 1, 2$ and $1 \leq l_j \leq D_j$, let $d_{l_1}, \delta_{l_1}, \Delta_{l_1,l_2}, \tilde{d}_{l_2}, \tilde{\delta}_{l_2}$ be non negative integers. We assume that

$$1 = \delta_1 = \tilde{\delta}_1 \quad , \quad \delta_{l_1} < \delta_{l_1+1} \quad , \quad \tilde{\delta}_{l_2} < \tilde{\delta}_{l_2+1} \tag{6}$$

for all $1 \leq l_1 \leq D_1 - 1$ and $1 \leq l_2 \leq D_2 - 1$.

Let $Q_1(X), Q_2(X), R_{D_j}(X), R_{l_1,l_2}(X) \in \mathbb{C}[X]$, and for $1 \leq l_j \leq D_j$, $j = 1, 2$.

Let P_1, P_2 be polynomials belonging to $\mathcal{O}_b(D(0,\epsilon_0))[X]$, with coefficients in the space of bounded holomorphic functions on a disc $D(0,\epsilon_0)$, for some $\epsilon_0 > 0$. We assume

$$\deg(Q_j) \geq \deg(R_{D_j}), \quad j = 1, 2, \quad \text{and} \tag{7}$$

$$\begin{aligned} &\deg(Q_j) \geq \deg(R_{D_j}) \quad , \quad \deg(R_{D_1,D_2}) \geq \deg(R_{l_1,l_2}) \\ &\deg(R_{D_1,D_2}) \geq \deg(P_1) \quad , \quad \deg(R_{D_1,D_2}) \geq \deg(P_2) \quad , \quad Q_j(im) \neq 0 \quad , \quad R_{D_1,D_2}(im) \neq 0 \end{aligned} \tag{8}$$

for all $m \in \mathbb{R}$, all $j=1,2$ and $0 \leq l_j \leq D_j - 1$.

$$
\begin{aligned}
(9) \quad & \Big(Q_1(\partial_z)\partial_{t_1} - \epsilon^{(\delta_{D_1}-1)k_1} t_1^{(\delta_{D_1}-1)(k_1+1)} \partial_{t_1}^{\delta_{D_1}} R_{D_1}(\partial_z) \Big) \\
& \times \Big(Q_2(\partial_z)\partial_{t_2} - \epsilon^{(\tilde{\delta}_{D_2}-1)k_2} t_2^{(\tilde{\delta}_{D_2}-1)(k_2+1)} \partial_{t_2}^{\tilde{\delta}_{D_2}} R_{D_2}(\partial_z) \Big) u(\boldsymbol{t},z,\epsilon) \\
& = P_1(\partial_z,\epsilon)u(\boldsymbol{t},z,\epsilon)P_2(\partial_z,\epsilon)u(\boldsymbol{t},z,\epsilon) \\
& + \sum_{1 \leq l_1 \leq D_1-1, 1 \leq l_2 \leq D_2-1} \epsilon^{\Delta_{l_1,l_2}} t_1^{d_{l_1}} t_2^{\tilde{d}_{l_2}} \partial_{t_1}^{\delta_{l_1}} \partial_{t_2}^{\tilde{\delta}_{l_2}} R_{\ell_1,\ell_2}(\partial_z) u(\boldsymbol{t},z,\epsilon) + f(\boldsymbol{t},z,\epsilon).
\end{aligned}
$$

We first introduce the following Banach space of functions.

Definition 1 *Let $\beta,\mu \in \mathbb{R}$. We denote by $E_{(\beta,\mu)}$ the vector space of continuous functions $h : \mathbb{R} \rightarrow \mathbb{C}$ such that*

$$||h(m)||_{(\beta,\mu)} = \sup_{m \in \mathbb{R}} (1+|m|)^{\mu} \exp(\beta|m|)|h(m)|$$

is finite. The space $E_{(\beta,\mu)}$ equipped with the norm $||.||_{(\beta,\mu)}$ is a Banach space.

The forcing term is constructed as follows. For every $n_1,n_2 \geq 1$, let $m \mapsto F_{n_1,n_2}(m,\epsilon)$, belonging to the Banach space $E_{(\beta,\mu)}$ for some $\beta > 0$ and $\mu > \max\{\deg(P_1)+1, \deg(P_2)+1\}$, depending holomorphically on $\epsilon \in D(0,\epsilon_0)$, for some positive ϵ_0. We assume the existence of $K_0, T_0 > 0$ such that

$$||F_{n_1,n_2}(m,\epsilon)||_{(\beta,\mu)} \leq K_0 (\frac{1}{T_0})^{n_1+n_2} \tag{10}$$

for all $n_1,n_2 \geq 1$ and $\epsilon \in D(0,\epsilon_0)$. We put

$$F(\boldsymbol{T},m,\epsilon) = \sum_{n_1,n_2 \geq 1} F_{n_1,n_2}(m,\epsilon) T_1^{n_1} T_2^{n_2}$$

which is a convergent series on $D(0,T_0/2) \times D(0,T_0/2)$ with values in $E_{(\beta,\mu)}$. We define the forcing term as a time rescaled version of the inverse Fourier transform of F, namely

$$f(\boldsymbol{t},z,\epsilon) := \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} F(\epsilon t_1, \epsilon t_2, m, \epsilon) \exp(imz) dm.$$

We make the additional assumption that there exist unbounded sectors

$$S_{Q_j,R_{D_j}} = \{ z \in \mathbb{C} : |z| \geq r_{Q_j,R_{D_j}} \quad , \quad |\arg(z) - d_{Q_j,R_{D_j}}| \leq \eta_{Q_j,R_{D_j}} \}$$

with direction $d_{Q_j,R_{D_j}} \in \mathbb{R}$, aperture $\eta_{Q_j,R_{D_j}} > 0$ for some radius $r_{Q_j,R_{D_j}} > 0$ such that

$$\frac{Q_j(im)}{R_{D_j}(im)} \in S_{Q_j,R_{D_j}} \tag{11}$$

for all $m \in \mathbb{R}$, and for $j=1,2$.

We search for the solutions of our main problem in the form

$$u(\boldsymbol{t},z,\epsilon) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} U(\epsilon t_1, \epsilon t_2, m, \epsilon) \exp(imz) dm,$$

for some expression $U(\boldsymbol{T},m,\epsilon)$.

The next result has already been mentioned in [6].

Proposition 1 *Let $f \in E_{(\beta,\mu)}$ with $\beta > 0$, $\mu > 1$. The inverse Fourier transform of f, defined by*

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} f(m)\exp(ixm)dm, \quad x \in \mathbb{R},$$

extends to an analytic function on the strip $H_\beta = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \beta\}$. Let $\phi(m) = imf(m) \in E_{(\beta,\mu-1)}$. Then, it holds $\partial_z \mathcal{F}^{-1}(f)(z) = \mathcal{F}^{-1}(\phi)(z), z \in H_\beta$.

Let $g \in E_{(\beta,\mu)}$ and put $\psi(m) = \frac{1}{(2\pi)^{1/2}} f * g(m)$, the convolution product of f and g, for all $m \in \mathbb{R}$. We have $\psi \in E_{(\beta,\mu)}$ and $\mathcal{F}^{-1}(f)(z)\mathcal{F}^{-1}(g)(z) = \mathcal{F}^{-1}(\psi)(z), z \in H_\beta$.

From the classical properties of the inverse Fourier transform recalled in Proposition 1, it holds that $U(\boldsymbol{T}, m, \epsilon)$ satisfies the auxiliary equation

(12)
$$\begin{aligned}\Big(Q_1(im)\partial_{T_1} - T_1^{(\delta_{D_1}-1)(k_1+1)}\partial_{T_1}^{\delta_{D_1}} R_{D_1}(im)\Big)\Big(Q_2(im)\partial_{T_2} - T_2^{(\tilde{\delta}_{D_2}-1)(k_2+1)}\partial_{T_2}^{\tilde{\delta}_{D_2}} R_{D_2}(im)\Big)U(\boldsymbol{T}, m, \epsilon)\\ = \epsilon^{-2}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} P_1(i(m-m_1),\epsilon)U(\boldsymbol{T}, m-m_1,\epsilon)P_2(im_1,\epsilon)U(\boldsymbol{T}, m_1,\epsilon)dm_1\\ + \sum_{1\le l_1\le D_1-1, 1\le l_2\le D_2-1} \epsilon^{\Delta_{l_1,l_2}-d_{l_1}-\tilde{d}_{l_2}+\delta_{l_1}+\tilde{\delta}_{l_2}-2} T_1^{d_{l_1}} T_2^{\tilde{d}_{l_2}} \partial_{T_1}^{\delta_{l_1}}\partial_{T_2}^{\tilde{\delta}_{l_2}} R_{\ell_1,\ell_2}(im)U(\boldsymbol{T}, m, \epsilon)\\ + \epsilon^{-2}F(\boldsymbol{T}, m, \epsilon).\end{aligned}$$

for given initial data $U(T_1, 0, m, \epsilon) \equiv U(0, T_2, m, \epsilon) \equiv 0$.

We seek for solutions of the auxiliary problem above in the form of a double Laplace transform

$$U_{p_1,p_2}(\boldsymbol{T}, m, \epsilon) = \frac{k_1k_2}{(2\pi)^{1/2}}\int_{L_{\gamma_{p_1}}}\int_{L_{\gamma_{p_2}}} \omega_{p_1,p_2}(\boldsymbol{u}, m, \epsilon)\exp\left(-\left(\frac{u_1}{T_1}\right)^{k_1} - \left(\frac{u_2}{T_2}\right)^{k_2}\right)\frac{du_2}{u_2}\frac{du_1}{u_1}, \tag{13}$$

where $L_{\gamma_{p_j}} = \mathbb{R}_+ e^{i\gamma_j} \subseteq S_{d_j} \cup \{0\}$, for $j = 1, 2$, where S_{d_j} is an infinite sector to be precised later. The function $\omega_{p_1,p_2}(\boldsymbol{u}, m, \epsilon)$ belongs to a Banach space of functions such that the integrals in the previous expression make sense. The explicit form of such Banach spaces will be described in Section 3.

This is motivated by the fact that the forcing term can be expressed in such a form. Indeed, we will show in the next section that

$$\psi_{\boldsymbol{k}}(\boldsymbol{\tau}, m, \epsilon) = \sum_{n_1,n_2\ge 1} F_{n_1,n_2}(m,\epsilon)\frac{\tau_1^{n_1}}{\Gamma\left(\frac{n_1}{k_1}\right)}\frac{\tau_2^{n_2}}{\Gamma\left(\frac{n_2}{k_2}\right)}$$

defines an entire function with values in $E_{(\beta,\mu)}$ that belongs to that Banach space. Direct computations show that

$$F(\boldsymbol{T}, m, \epsilon) = \frac{k_1k_2}{(2\pi)^{1/2}}\int_{L_{\gamma_{p_1}}}\int_{L_{\gamma_{p_2}}} \psi_{\boldsymbol{k}}(\boldsymbol{u}, m, \epsilon)\exp\left(-\left(\frac{u_1}{T_1}\right)^{k_1} - \left(\frac{u_2}{T_2}\right)^{k_2}\right)\frac{du_2}{u_2}\frac{du_1}{u_1}.$$

Our next goal is to provide a functional equation satisfied by $\omega_{p_1,p_2}(\boldsymbol{u}, m, \epsilon)$. For that purpose, we need first to expand the equation (12) by means of the following relations (see [12], p. 3630):

$$\begin{aligned}T_1^{\delta_{D_1}(k_1+1)}\partial_{T_1}^{\delta_{D_1}} &= (T_1^{k_1+1}\partial_{T_1})^{\delta_{D_1}} + \sum_{1\le p_1\le \delta_{D_1}-1} A_{\delta_{D_1},p_1} T_1^{k_1(\delta_{D_1}-p_1)}(T_1^{k_1+1}\partial_{T_1})^{p_1}\\ &= (T_1^{k_1+1}\partial_{T_1})^{\delta_{D_1}} + A_{\delta_{D_1}}(T_1, \partial_{T_1})\end{aligned} \tag{14}$$

$$(15)\quad T_2^{\tilde{\delta}_{D_2}(k_2+1)}\partial_{T_2}^{\tilde{\delta}_{D_2}} = (T_2^{k_2+1}\partial_{T_2})^{\tilde{\delta}_{D_2}} + \sum_{1\leq p_2\leq \tilde{\delta}_{D_2}-1} \tilde{A}_{\tilde{\delta}_{D_2},p_2} T_2^{k_2(\tilde{\delta}_{D_2}-p_2)}(T_2^{k_2+1}\partial_{T_2})^{p_2} = (T_2^{k_2+1}\partial_{T_2})^{\tilde{\delta}_{D_2}} + \tilde{A}_{\tilde{\delta}_{D_2}}(T_2,\partial_{T_2})$$

for some real numbers $A_{\delta_{D_1},p_1}$, $p_1 = 1,\ldots,\delta_{D_1}-1$ and $\tilde{A}_{\tilde{\delta}_{D_2},p_2}$, $p_2 = 1,\ldots,\tilde{\delta}_{D_2}-1$. We write A_{D_1} (resp. $\tilde{A}_{D_2}$) for $A_{\delta_{D_1}}$ (resp. $\tilde{A}_{\tilde{\delta}_{D_2}}$) for the sake of simplicity. Let $d_{l_1,k_1}, \tilde{d}_{l_1,k_2} \geq 0$ satisfying

$$(16)\qquad d_{l_1}+k_1+1 = \delta_{l_1}(k_1+1)+d_{l_1,k_1} \qquad \tilde{d}_{l_2}+k_2+1 = \tilde{\delta}_{l_2}(k_2+1)+\tilde{d}_{l_2,k_2}$$

for all $1 \leq l_1 \leq D_1-1$ and $1 \leq l_2 \leq D_2-1$. Multiplying the equation (12) by $T_1^{k_1+1}T_2^{k_2+1}$ and taking into account (14,15), we rewrite (12) in the form

$$(17)\quad \left(Q_1(im)T_1^{k_1+1}\partial_{T_1} - \left((T_1^{k_1+1}\partial_{T_1})^{\delta_{D_1}} + A_{D_1}(T_1,\partial_{T_1})\right)R_{D_1}(im)\right)$$
$$\times \left(Q_2(im)T_2^{k_2+1}\partial_{T_2} - \left((T_2^{k_2+1}\partial_{T_2})^{\tilde{\delta}_{D_2}} + \tilde{A}_{D_2}(T_2,\partial_{T_2})\right)R_{D_2}(im)\right)U(\boldsymbol{T},m,\epsilon)$$
$$= \epsilon^{-2}T_1^{k_1+1}T_2^{k_2+1}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} P_1(i(m-m_1),\epsilon)U(\boldsymbol{T},m-m_1,\epsilon)P_2(im_1,\epsilon)U(T,m_1,\epsilon)dm_1$$
$$+ \sum_{1\leq l_1\leq D_1-1, 1\leq l_2\leq D_2-1} \epsilon^{\Delta_{l_1,l_2}-d_{l_1}-d_{l_2}+\delta_{l_1}+\tilde{\delta}_{l_2}-2} T_1^{\delta_{l_1}(k_1+1)+d_{l_1,k_1}}\partial_{T_1}^{\delta_{l_1}}$$
$$\times T_2^{\tilde{\delta}_{l_2}(k_2+1)+\tilde{d}_{l_2,k_2}}\partial_{T_2}^{\tilde{\delta}_{l_2}} R_{l_1,l_2}(im)U(\boldsymbol{T},m,\epsilon)$$
$$+ \epsilon^{-2}T_1^{k_1+1}T_2^{k_2+1}F(\boldsymbol{T},m,\epsilon).$$

The proof of the following result, needed in the sequel, can be found in detail in Lemma 1 [9].

Lemma 1 *Let $U_{p_1,p_2}(\boldsymbol{T},m,\epsilon)$ be the function constructed in (13). Then, it holds that:*

$$T_j^{k_j+1}\partial_{T_j}U_{p_1,p_2}(\boldsymbol{T},m,\epsilon) = k_1k_2\int_{L_{\gamma_{p_1}}}\int_{L_{\gamma_{p_2}}}(k_ju_j^{k_j})\omega_{p_1,p_2}(\boldsymbol{u},m,\epsilon)e^{-\left(\frac{u_1}{T_1}\right)^{k_1}-\left(\frac{u_2}{T_2}\right)^{k_2}}\frac{du_2}{u_2}\frac{du_1}{u_1}, \quad j=1,2.$$

$$T_1^{m_1}U_{p_1,p_2}(\boldsymbol{T},m,\epsilon) = k_1k_2\int_{L_{\gamma_{p_1}}}\int_{L_{\gamma_{p_2}}}\left(\frac{u_1^{k_1}}{\Gamma\left(\frac{m_1}{k_1}\right)}\int_0^{u_1^{k_1}}(u_1^{k_1}-s_1)^{\frac{m_1}{k_1}-1}\omega_{p_1,p_2}(s_1^{1/k_1},u_2,m,\epsilon)\frac{ds_1}{s_1}\right)$$
$$\times e^{-\left(\frac{u_1}{T_1}\right)^{k_1}-\left(\frac{u_2}{T_2}\right)^{k_2}}\frac{du_2}{u_2}\frac{du_1}{u_1}, \qquad m_1\in\mathbb{N}.$$

$$T_2^{m_2}U_{p_1,p_2}(\boldsymbol{T},m,\epsilon) = k_1k_2\int_{L_{\gamma_{p_1}}}\int_{L_{\gamma_{p_2}}}\left(\frac{u_2^{k_2}}{\Gamma\left(\frac{m_2}{k_2}\right)}\int_0^{u_2^{k_2}}(u_2^{k_2}-s_2)^{\frac{m_2}{k_2}-1}\omega_{p_1,p_2}(u_1,s_2^{1/k_2},m,\epsilon)\frac{ds_2}{s_2}\right)$$
$$\times e^{-\left(\frac{u_1}{T_1}\right)^{k_1}-\left(\frac{u_2}{T_2}\right)^{k_2}}\frac{du_2}{u_2}\frac{du_1}{u_1}, \qquad m_2\in\mathbb{N}.$$

$$\int_{-\infty}^{\infty} U_{p_1,p_2}(\boldsymbol{T}, m-m_1, \epsilon) U_{p_1,p_2}(\boldsymbol{T}, m_1, \epsilon) dm_1$$
$$= k_1 k_2 \int_{L_{\gamma_{p_1}}} \int_{L_{\gamma_{p_2}}} \left(u_1^{k_1} u_2^{k_2} \int_{-\infty}^{\infty} \int_0^{u_1^{k_1}} \int_0^{u_2^{k_2}} \omega_{p_1,p_2}((u_1^{k_1}-s_1)^{\frac{1}{k_1}}, (u_2^{k_2}-s_2)^{\frac{1}{k_2}}, m-m_1, \epsilon) \right.$$
$$\left. \omega_{p_1,p_2}(s_1^{\frac{1}{k_1}}, s_2^{\frac{1}{k_2}}, m_1, \epsilon) \frac{1}{(u_1^{k_1}-s_1)s_1} \frac{1}{(u_2^{k_2}-s_2)s_2} ds_1 ds_2 \right) e^{-\left(\frac{u_1}{T_1}\right)^{k_1} - \left(\frac{u_2}{T_2}\right)^{k_2}} \frac{du_2}{u_2} \frac{du_1}{u_1}$$

We define the operators $\mathcal{A}_{D_1}$ (resp. $\tilde{\mathcal{A}}_{D_2}$) in the form

(18)
$$\mathcal{A}_{D_1}\omega_{\boldsymbol{k}}(\boldsymbol{\tau}, m, \epsilon) = \sum_{1 \le p_1 \le \delta_{D_1}-1} \frac{A_{\delta_{D_1},p_1} \tau_1^{k_1}}{\Gamma(\delta_{D_1}-p_1)} \int_0^{\tau_1^{k_1}} (\tau_1^{k_1}-s_1)^{\delta_{D_1}-p_1-1} k_1 s_1^{p_1} \omega_{\boldsymbol{k}}(s_1^{1/k_1}, \tau_2, m, \epsilon) \frac{ds_1}{s_1},$$

$$\tilde{\mathcal{A}}_{D_2}\omega_{\boldsymbol{k}}(\boldsymbol{\tau}, m, \epsilon) = \sum_{1 \le p_2 \le \tilde{\delta}_{D_2}-1} \frac{\tilde{A}_{\tilde{\delta}_{D_2},p_2} \tau_2^{k_2}}{\Gamma(\tilde{\delta}_{D_2}-p_2)} \int_0^{\tau_2^{k_2}} (\tau_2^{k_2}-s_2)^{\tilde{\delta}_{D_2}-p_2-1} k_2 s_2^{p_2} \omega_{\boldsymbol{k}}(\tau_1, s_2^{1/k_2}, m, \epsilon) \frac{ds_2}{s_2}.$$

Finally, we arrive at the next convolution equation

(19)
$$(Q_1(im)k_1\tau_1^{k_1} - (k_1\tau_1^{k_1})^{\delta_{D_1}} R_{D_1}(im))(Q_2(im)k_2\tau_2^{k_2} - (k_2\tau_2^{k_2})^{\tilde{\delta}_{D_2}} R_{D_2}(im))\omega_{\boldsymbol{k}}(\boldsymbol{\tau}, m, \epsilon)$$
$$= (Q_1(im)k_1\tau_1^{k_1} - (k_1\tau_1^{k_1})^{\delta_{D_1}} R_{D_1}(im))\tilde{\mathcal{A}}_{D_2} R_{D_2}(im)\omega_{\boldsymbol{k}}(\boldsymbol{\tau}, m, \epsilon)$$
$$+ (Q_2(im)k_2\tau_2^{k_2} - (k_2\tau_2^{k_2})^{\tilde{\delta}_{D_2}} R_{D_2}(im))\mathcal{A}_{D_1} R_{D_1}(im)\omega_{\boldsymbol{k}}(\boldsymbol{\tau}, m, \epsilon)$$
$$- \mathcal{A}_{D_1}\tilde{\mathcal{A}}_{D_2} R_{D_1}(im) R_{D_2}(im)\omega_{\boldsymbol{k}}(\boldsymbol{\tau}, m, \epsilon)$$
$$+ \epsilon^{-2} \frac{\tau_1^{k_1}\tau_2^{k_2}}{\Gamma(1+\frac{1}{k_1})\Gamma(1+\frac{1}{k_2})} \int_0^{\tau_1^{k_1}} \int_0^{\tau_2^{k_2}} (\tau_1^{k_1}-s_1)^{1/k_1} (\tau_2^{k_2}-s_2)^{1/k_2}$$
$$\times \left(\frac{1}{(2\pi)^{1/2}} s_1 s_2 \int_0^{s_1} \int_0^{s_2} \int_{-\infty}^{+\infty} P_1(i(m-m_1),\epsilon)\omega_{\boldsymbol{k}}((s_1-x_1)^{1/k_1}, (s_2-x_2)^{1/k_2}, m-m_1, \epsilon) \right.$$
$$\left. \times P_2(im_1,\epsilon)\omega_{\boldsymbol{k}}(x_1^{1/k_1}, x_2^{1/k_2}, m_1, \epsilon) \frac{1}{(s_1-x_1)x_1(s_2-x_2)x_2} dm_1 dx_2 dx_1 \right) \frac{ds_2}{s_2} \frac{ds_1}{s_1}$$
$$+ \sum_{1 \le l_1 \le D_1-1, 1 \le l_2 \le D_2-1} R_{l_1,l_2}(im) \epsilon^{\Delta_{l_1,l_2} - d_{l_1} - \tilde{d}_{l_2} + \delta_{l_1} + \tilde{\delta}_{l_2} - 2} \frac{\tau_1^{k_1}\tau_2^{k_2}}{\Gamma\left(\frac{d_{l_1,k_1}}{k_1}\right)\Gamma\left(\frac{\tilde{d}_{l_2,k_2}}{k_2}\right)}$$
$$\int_0^{\tau_1^{k_1}} \int_0^{\tau_2^{k_2}} (\tau_1^{k_1}-s_1)^{d_{l_1,k_1}/k_1-1} (\tau_2^{k_2}-s_2)^{\tilde{d}_{l_2,k_2}/k_2-1} k_1^{\delta_{l_1}} k_2^{\tilde{\delta}_{l_2}} s_1^{\delta_{l_1}} s_2^{\tilde{\delta}_{l_2}} \omega_{\boldsymbol{k}}(s_1^{1/k_1}, s_2^{1/k_2}, m, \epsilon) \frac{ds_2}{s_2} \frac{ds_1}{s_1}$$
$$+ \epsilon^{-2} \frac{\tau_1^{k_1}\tau_2^{k_2}}{\Gamma\left(1+\frac{1}{k_1}\right)\Gamma\left(1+\frac{1}{k_2}\right)} \int_0^{\tau_1^{k_1}} \int_0^{\tau_2^{k_2}} (\tau_1^{k_1}-s_1)^{1/k_1} (\tau_2^{k_2}-s_2)^{1/k_2} \psi_{\boldsymbol{k}}(s_1^{1/k_1}, s_2^{1/k_2}, m, \epsilon) \frac{ds_2}{s_2} \frac{ds_1}{s_1}.$$

3 Banach spaces of functions and solutions of the auxiliary problems

Let $D(0,r)$ be an open disc centered at 0 with radius $r>0$ in $\mathbb{C}$, and by $\bar{D}(0,r)$ its closure. Let S_{d_j} be open unbounded sectors with bisecting directions $d_j \in \mathbb{R}$ for $j=1,2$, and $\mathcal{E}$ be an open sector with finite radius $r_{\mathcal{E}}$, all centered at 0 in $\mathbb{C}$.

The definition of the following norm heavily rests on that considered in [6]. Here, the exponential growth is held with respect to the two time variables which are involved.

Definition 2 *Let $\nu_1,\nu_2,\beta,\mu>0$ and $\rho>0$ be positive real numbers. Let $k_1,k_2 \geq 1$ be integer numbers and let $\epsilon \in \mathcal{E}$. We put $\boldsymbol{\nu}=(\nu_1,\nu_2)$, $\boldsymbol{k}=(k_1,k_2)$, $\boldsymbol{d}=(d_1,d_2)$, and denote $F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$ the vector space of continuous functions $(\boldsymbol{\tau},m) \mapsto h(\boldsymbol{\tau},m)$ on the set $(\bar{D}(0,\rho)\cup S_{d_1}) \times (\bar{D}(0,\rho)\cup S_{d_2}) \times \mathbb{R}$, which are holomorphic with respect to (τ_1,τ_2) on $(D(0,\rho)\cup S_{d_1}) \times (D(0,\rho)\cup S_{d_2})$ and such that*

$$
(20)\quad ||h(\boldsymbol{\tau},m)||_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} = \sup_{\substack{\boldsymbol{\tau}\in(\bar{D}(0,\rho)\cup S_{d_1})\times(\bar{D}(0,\rho)\cup S_{d_2})\\ m\in\mathbb{R}}} (1+|m|)^{\mu} \frac{1+|\frac{\tau_1}{\epsilon}|^{2k_1}}{|\frac{\tau_1}{\epsilon}|} \frac{1+|\frac{\tau_2}{\epsilon}|^{2k_2}}{|\frac{\tau_2}{\epsilon}|} \exp(\beta|m|-\nu_1|\frac{\tau_1}{\epsilon}|^{k_1}-\nu_2|\frac{\tau_2}{\epsilon}|^{k_2})|h(\tau,m)|
$$

is finite. One can check that the normed space $(F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}, ||.||_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)})$ is a Banach space.

Throughout the whole section, we assume $\epsilon \in \mathcal{E}$, $\mu,\beta>0$ are fixed numbers. We also fix $\boldsymbol{\nu}=(\nu_1,\nu_2)$ for some positive numbers ν_1,ν_2, and $\boldsymbol{k}=(k_1,k_2)$ is a couple of positive integer numbers. Additionally, we take $d_1,d_2\in\mathbb{R}$, and write $\boldsymbol{d}$ for (d_1,d_2). The next results are stated without proofs, which are analogous to those in Section 2 of [6]. The integrals appearing in these results can be split accordingly, in order to apply the proof therein.

Lemma 2 *Let $(\boldsymbol{\tau},m)\mapsto a(\boldsymbol{\tau},m)$ be a bounded continuous function on $(\bar{D}(0,\rho)\cup S_{d_1}) \times (\bar{D}(0,\rho)\cup S_{d_2}) \times \mathbb{R}$, holomorphic with respect to $\boldsymbol{\tau}$ on $(D(0,\rho)\cup S_{d_1}) \times (D(0,\rho)\cup S_{d_2})$. Then,*

$$
(21)\quad ||a(\boldsymbol{\tau},m)h(\boldsymbol{\tau},m)||_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \leq \left(\sup_{\boldsymbol{\tau}\in(\bar{D}(0,\rho)\cup S_{d_1})\times(\bar{D}(0,\rho)\cup S_{d_2}), m\in\mathbb{R}} |a(\boldsymbol{\tau},m)| \right) ||h(\boldsymbol{\tau},m)||_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}
$$

for all $h(\boldsymbol{\tau},m)\in F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$.

Proposition 2 *Let $\gamma_{21},\gamma_{22}>0$ be real numbers. Assume that $k_1,k_2\geq 1$ are such that $1/k_j \leq \gamma_{2j} \leq 1$, for $j=1,2$. Then, a constant $C_1>0$ (depending on $\boldsymbol{\nu},\boldsymbol{k},\gamma_{21},\gamma_{22}$) exists with*

$$
(22)\quad || \int_0^{\tau_1^{k_1}} \int_0^{\tau_2^{k_2}} (\tau_1^{k_1}-s_1)^{\gamma_{21}} (\tau_2^{k_2}-s_2)^{\gamma_{22}} f(s_1^{1/k_1}, s_2^{1/k_2}, m) \frac{ds_2}{s_2}\frac{ds_1}{s_1} ||_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \leq C_1 |\epsilon|^{k_1\gamma_{21}+k_2\gamma_{22}} ||f(\tau,m)||_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}
$$

for all $f(\boldsymbol{\tau},m)\in F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$.

Proposition 2 in [6] is adapted to the Banach space considered in this work.

Proposition 3 *Let $\gamma_{11},\gamma_{12} \geq 0$ and $\chi_{21},\chi_{22} > -1$ be real numbers. Let $\xi_{21},\xi_{22} \geq 0$ be integer numbers. We write $\boldsymbol{\gamma}_1 = (\gamma_{11},\gamma_{12})$. We consider $a_{\boldsymbol{\gamma}_1,\boldsymbol{k}} \in \mathcal{O}((D(0,\rho) \cup S_{d_1}) \times (D(0,\rho)\cup S_{d_2}))$, continuous on $(\bar{D}(0,\rho)\cup S_{d_1}) \times (\bar{D}(0,\rho) \cup S_{d_2})$, such that*

$$|a_{\boldsymbol{\gamma}_1,\boldsymbol{k}}(\boldsymbol{\tau})| \leq \frac{1}{(1+|\tau_1|^{k_1})^{\gamma_{11}}(1+|\tau_2|^{k_2})^{\gamma_{12}}}, \quad \boldsymbol{\tau} \in (\bar{D}(0,\rho)\cup S_{d_1}) \times (\bar{D}(0,\rho)\cup S_{d_2}).$$

Assume that for $j=1$ and $j=2$, one of the following holds

- *$\chi_{2j} \geq 0$ and $\xi_{2j} + \chi_{2j} - \gamma_{1j} \leq 0$, or*
- *$\chi_{2j} = \frac{\tilde{\chi}_j}{k_j} - 1$, for some $\tilde{\chi}_j \geq 1$ and $\xi_{2j} + \frac{1}{k_j} - \gamma_{1j} \leq 0$.*

Then, there exists a constant $C_2 > 0$ (depending, eventually, on $\boldsymbol{\nu},\xi_{21},\xi_{22},\chi_{21},\chi_{22},\boldsymbol{\gamma}_1,\tilde{\chi}_1,\tilde{\chi}_2,\boldsymbol{k}$) such that

$$(23)\quad ||a_{\boldsymbol{\gamma}_1,\boldsymbol{k}}(\boldsymbol{\tau}) \int_0^{\tau_1^{k_1}} \int_0^{\tau_2^{k_2}} (\tau_1^{k_1} - s_1)^{\chi_{21}} (\tau_2^{k_2}-s_2)^{\chi_{22}} s_1^{\xi_{21}} s_2^{\xi_{22}} f(s_1^{1/k_1}, s_2^{1/k_2}, m) ds_2 ds_1||_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \\ \leq C_2 |\epsilon|^{k_1(1+\xi_{21}+\chi_{21}-\gamma_{11}) + k_2(1+\xi_{22}+\chi_{22}-\gamma_{12})} ||f(\boldsymbol{\tau},m)||_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$$

for all $f(\boldsymbol{\tau},m) \in F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$.

The previous result can also be particularized to each of the variables in time in the following manner. We write the result which corresponds to the first time variable, but one can reproduce the same arguments symmetrically on the second variable in time, τ_2.

Proposition 4 *Let $\gamma_1 \geq 0$ and $\chi_2 > -1$ be real numbers, and $\xi_2 \geq 0$ be an integer number. We consider $a_{\gamma_1,k_1} \in \mathcal{O}(D(0,\rho) \cup S_{d_1})$, continuous on $\bar{D}(0,\rho)\cup S_{d_1}$, such that*

$$|a_{\gamma_1,k_1}(\tau_1)| \leq \frac{1}{(1+|\tau_1|^{k_1})^{\gamma_1}}, \quad \tau_1 \in \bar{D}(0,\rho) \cup S_{d_1}.$$

Assume that

- *$\chi_2 \geq 0$ and $\xi_2 + \chi_2 - \gamma_1 \leq 0$, or*
- *$\chi_2 = \frac{\tilde{\chi}}{k_1} - 1$, for some $\tilde{\chi} \geq 1$ and $\xi_2 + \frac{1}{k_1} - \gamma_1 \leq 0$.*

Then, there exists a constant $C_2 > 0$ (depending, eventually, on $\boldsymbol{\nu},\xi_2,\chi_2,\gamma_1,\tilde{\chi},k_1$) such that

$$(24)\quad ||a_{\gamma_1,k_1}(\tau_1) \int_0^{\tau_1^{k_1}} (\tau_1^{k_1}-s_1)^{\chi_2} s_1^{\xi_2} f(s_1^{1/k_1},\tau_2,m) ds_1||_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \\ \leq C_2 |\epsilon|^{k_1(1+\xi_2+\chi_2-\gamma_1)} ||f(\boldsymbol{\tau},m)||_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$$

for all $f(\boldsymbol{\tau},m) \in F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$.

Proposition 5 *Let $Q_1(X), Q_2(X), R(X) \in \mathbb{C}[X]$ such that*

$$(25)\qquad \deg(R) \geq \deg(Q_1) \quad , \quad \deg(R) \geq \deg(Q_2) \quad , \quad R(im) \neq 0, \qquad m \in \mathbb{R}.$$

Assume that $\mu > \max(\deg(Q_1)+1, \deg(Q_2)+1)$. *Let* $m \mapsto b(m)$ *be a continuous function on* $\mathbb{R}$ *such that*

$$|b(m)| \leq \frac{1}{|R(im)|}, \quad m \in \mathbb{R}.$$

Then, there exists a constant $C_3 > 0$ *(depending on* $Q_1, Q_2, R, \mu, \boldsymbol{k}, \boldsymbol{\nu}$*) such that*

$$\begin{aligned} (26) \quad & ||b(m) \int_0^{\tau_1^{k_1}} \int_0^{\tau_2^{k_2}} (\tau_1^{k_1} - s_1)^{\frac{1}{k_1}} (\tau_2^{k_2} - s_2)^{\frac{1}{k_2}} \\ & \times \left(\int_0^{s_1} \int_0^{s_2} \int_{-\infty}^{+\infty} Q_1(i(m-m_1)) f((s_1-x_1)^{1/k_1}, (s_2-x_2)^{1/k_2}, m-m_1) \right. \\ & \left. \times Q_2(im_1) g(x_1^{1/k_1}, x_2^{1/k_2}, m_1) \frac{1}{(s_1-x_1)x_1} \frac{1}{(s_2-x_2)x_2} dm_1 dx_2 dx_1 \right) ds_2 ds_1 ||_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \\ & \leq C_3 |\epsilon|^2 ||f(\boldsymbol{\tau}, m)||_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} ||g(\boldsymbol{\tau}, m)||_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \end{aligned}$$

for all $f(\boldsymbol{\tau}, m), g(\boldsymbol{\tau}, m) \in F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$.

We consider the polynomial $P_{m,j}(\tau_j) = Q_j(im)k_j - R_{D_j}(im)k_j^{\delta_{D_j}} \tau_j^{(\delta_{D_j}-1)k_j}$ and assume that $\{q_{l,1}\}_{0 \leq l \leq (\delta_{D_1}-1)k_1-1}$ and $\{q_{l,2}\}_{0 \leq l \leq (\tilde{\delta}_{D_2}-1)k_2-1}$ are the complex roots of each polynomial, for $m \in \mathbb{R}$. Following an analogous manner as in the construction of [6], one can choose unbounded sectors S_{d_1} and S_{d_2}, with vertex at 0 and $\rho > 0$ such that

$$(27) \qquad |P_{m,1}(\tau_1)| \geq C_P (r_{Q_1,R_{D_1}})^{\frac{1}{(\delta_{D_1}-1)k_1}} |R_{D_1}(im)| (1 + |\tau_1|^{k_1})^{(\delta_{D_1}-1) - \frac{1}{k_1}},$$

for all $\tau_1 \in S_{d_1} \cup \overline{D}(0,\rho)$, and $m \in \mathbb{R}$; and

$$(28) \qquad |P_{m,2}(\tau_2)| \geq C_P (r_{Q_2,R_{D_2}})^{\frac{1}{(\tilde{\delta}_{D_2}-1)k_2}} |R_{D_2}(im)| (1 + |\tau_2|^{k_2})^{(\tilde{\delta}_{D_2}-1) - \frac{1}{k_2}},$$

for all $\tau_2 \in S_{d_2} \cup \overline{D}(0,\rho)$, and $m \in \mathbb{R}$. From now on, we write

$$C_{k_1} = C_P (r_{Q_1,R_{D_1}})^{\frac{1}{(\delta_{D_1}-1)k_1}}, \quad C_{k_2} := C_P (r_{Q_2,R_{D_2}})^{\frac{1}{(\tilde{\delta}_{D_2}-1)k_2}}$$

for a more compact writing.

Let $\boldsymbol{d} = (d_1, d_2)$. The next result guarantees the existence of an element in $F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$ which turns out to be a fixed point for certain operator to be described, solution of (19). Here β, μ are fixed at the beginning of this section.

We now give some words in order to guarantee that the forcing term belongs to the Banach space $F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$.

It holds that

$$\begin{aligned} (29) \quad & ||\psi_{\boldsymbol{k}}(\boldsymbol{\tau}, m, \epsilon)||_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \leq \sum_{n_1,n_2 \geq 1} ||F_{n_1,n_2}(m,\epsilon)||_{(\beta,\mu)} \\ & \times \Big(\sup_{\boldsymbol{\tau} \in (\bar{D}(0,\rho) \cup S_{d_1}) \times (\bar{D}(0,\rho) \cup S_{d_2})} \frac{1 + |\frac{\tau_1}{\epsilon}|^{2k_1}}{|\frac{\tau_1}{\epsilon}|} \frac{1 + |\frac{\tau_2}{\epsilon}|^{2k_2}}{|\frac{\tau_2}{\epsilon}|} \exp(-\nu_1 |\frac{\tau_1}{\epsilon}|^{k_1} - \nu_2 |\frac{\tau_2}{\epsilon}|^{k_2}) \frac{|\tau_1|^{n_1} |\tau_2|^{n_2}}{\Gamma(\frac{n_1}{k_1}) \Gamma(\frac{n_2}{k_2})} \Big) \end{aligned}$$

Using classical estimates and Stirling formula we guarantee the existence of $A_1, A_2 > 0$ depending on $\boldsymbol{\nu}, \boldsymbol{k}$ such that, if $\epsilon_0 A_2 < T_0$, then for all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$. One has

$$(30) \qquad ||\psi_{\boldsymbol{k}}(\boldsymbol{\tau},m,\epsilon)||_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \le \frac{A_1K_0}{\left(\frac{T_0}{\epsilon_0 A_2}-1\right)^2}.$$

We also refer to [6], Section 4, for further details.

Proposition 6 *Under the assumption that*

$$(31) \qquad \delta_{D_1} \ge \delta_{l_1} + \frac{2}{k_1}, \quad \tilde{\delta}_{D_2} \ge \tilde{\delta}_{l_2} + \frac{2}{k_2}, \quad \Delta_{l_1,l_2} + k_1(1-\delta_{D_1}) + k_2(1-\tilde{\delta}_{D_2}) + 2 \ge 0,$$

for all $1 \le l_1 \le D_1 - 1$, $1 \le l_2 \le D_2 - 1$, *there exists a constant* $\zeta_2 > 0$ *(depending on* $Q_1, Q_2, \boldsymbol{k}, C_P, \mu, \boldsymbol{\nu}, \epsilon_0, R_{l_1,l_2}, \Delta_{l_1,l_2}, \delta_{l_1}, \tilde{\delta}_{l_2}, d_{l_1}, \tilde{d}_{l_2}$ *for* $0 \le l_1 \le D_1$ *and* $0 \le l_2 \le D_2$*) such that if*

$$(32) \qquad ||\psi_{\boldsymbol{k}}(\boldsymbol{\tau},m,\epsilon)||_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \le \zeta_2$$

for all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$, *there exist* $r_{Q_j,R_{D_j}} > 0$, $j = 1,2$, *such that the equation (19) has a unique solution* $\omega_{\boldsymbol{k}}^{\boldsymbol{d}}(\boldsymbol{\tau},m,\epsilon)$ *in the space* $F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$, *where* $\beta, \mu > 0$ *are defined in construction of the forcing term* f, *which verifies* $||\omega_{\boldsymbol{k}}^{\boldsymbol{d}}(\boldsymbol{\tau},m,\epsilon)||_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \le \varpi$, *for all* $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$.

Proof

Let $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$. We consider the operator $\mathcal{H}_\epsilon$, defined by

$$(33) \qquad \mathcal{H}_\epsilon(\omega(\boldsymbol{\tau},m)) := \sum_{j=1}^{4} \mathcal{H}_\epsilon^j(\omega(\boldsymbol{\tau},m))$$

where

$$\mathcal{H}_\epsilon^1(\omega(\boldsymbol{\tau},m)) := \frac{R_{D_2}(im)}{P_{m,2}(\tau_2)}\frac{\tilde{\mathcal{A}}_{D_2}}{\tau_2^{k_2}}\omega(\boldsymbol{\tau},m) + \frac{R_{D_1}(im)}{P_{m,1}(\tau_1)}\frac{\mathcal{A}_{D_1}}{\tau_1^{k_1}}\omega(\boldsymbol{\tau},m) - \frac{R_{D_1}(im)}{P_{m,1}(\tau_1)}\frac{R_{D_2}(im)}{P_{m,2}(\tau_2)}\frac{\mathcal{A}_{D_1}}{\tau_1^{k_1}}\frac{\tilde{\mathcal{A}}_{D_2}}{\tau_2^{k_2}}\omega(\boldsymbol{\tau},m)$$

$$\mathcal{H}_\epsilon^2(\omega(\boldsymbol{\tau},m)) := \frac{\epsilon^{-2}}{P_{m,1}(\tau_1)P_{m,2}(\tau_2)\Gamma(1+\frac{1}{k_1})\Gamma(1+\frac{1}{k_2})}\int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}}(\tau_1^{k_1}-s_1)^{1/k_1}(\tau_2^{k_2}-s_2)^{1/k_2}$$
$$\times\left(\frac{1}{(2\pi)^{1/2}}s_1s_2\int_0^{s_1}\int_0^{s_2}\int_{-\infty}^{+\infty}P_1(i(m-m_1),\epsilon)\omega((s_1-x_1)^{1/k_1},(s_2-x_2)^{1/k_2},m-m_1)\right.$$
$$\left.\times P_2(im_1,\epsilon)\omega(x_1^{1/k_1},x_2^{1/k_2},m_1)\frac{1}{(s_1-x_1)x_1(s_2-x_2)x_2}dm_1dx_2dx_1\right)\frac{ds_2}{s_2}\frac{ds_1}{s_1}$$

$$\mathcal{H}_\epsilon^3(\omega(\boldsymbol{\tau},m)) := \sum_{1\le l_1\le D_1-1, 1\le l_2\le D_2-1}\frac{R_{l_1,l_2}(im)}{P_{m,1}(\tau_1)P_{m,2}(\tau_2)}\epsilon^{\Delta_{l_1,l_2}-d_{l_1}-\tilde{d}_{l_2}+\delta_{l_1}+\tilde{\delta}_{l_2}-2}\frac{1}{\Gamma\left(\frac{d_{l_1,k_1}}{k_1}\right)\Gamma\left(\frac{\tilde{d}_{l_2,k_2}}{k_2}\right)}$$
$$\int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}}(\tau_1^{k_1}-s_1)^{d_{l_1,k_1}/k_1-1}(\tau_2^{k_2}-s_2)^{\tilde{d}_{l_2,k_2}/k_2-1}k_1^{\delta_{l_1}}k_2^{\tilde{\delta}_{l_2}}s_1^{\delta_{l_1}}s_2^{\tilde{\delta}_{l_2}}\omega(s_1^{1/k_1},s_2^{1/k_2},m)\frac{ds_2}{s_2}\frac{ds_1}{s_1}$$

$$\mathcal{H}_{\epsilon}^{4}(\omega(\boldsymbol{\tau},m)) := \frac{\epsilon^{-2}}{P_{m,1}(\tau_1)P_{m,2}(\tau_2)\Gamma\left(1+\frac{1}{k_1}\right)\Gamma\left(1+\frac{1}{k_2}\right)} \times \int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}} (\tau_1^{k_1}-s_1)^{1/k_1}(\tau_2^{k_2}-s_2)^{1/k_2}\psi_{\boldsymbol{k}}(s_1^{1/k_1},s_2^{1/k_2},m,\epsilon)\frac{ds_2}{s_2}\frac{ds_1}{s_1}.$$

Let $\varpi>0$ and assume that $\omega(\boldsymbol{\tau},m)\in F^{\boldsymbol{d}}_{\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon}$. Assume that $\|\omega(\boldsymbol{\tau},m)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}\le\varpi$ for all $\epsilon\in D(0,\epsilon_0)\setminus\{0\}$. We first obtain the existence of $\varpi>0$ such that the operator $\mathcal{H}_\epsilon$ sends $\overline{B}(0,\varpi)\subseteq F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$ into itself. Here, $\overline{B}(0,\varpi)$ stands for the closed ball of radius ϖ, centered at 0, in the Banach space $F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$.

Using Lemma 2 and Proposition 4, with (27) and (28) we get

$$(34)\qquad \left\|\frac{R_{D_2}(im)}{P_{m,2}(\tau_2)}\frac{\tilde{\mathcal{A}}_{D_2}}{\tau_2^{k_2}}\omega(\boldsymbol{\tau},m)\right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \le \sum_{1\le p_2\le\tilde{\delta}_{D_2}-1}\frac{|\tilde{A}_{\tilde{\delta}_{D_2},p_2}|}{\Gamma(\tilde{\delta}_{D_2}-p_2)}\frac{C_2}{C_{k_2}}|\epsilon|\,\|\omega(\boldsymbol{\tau},m)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$$

$$(35)\qquad \left\|\frac{R_{D_1}(im)}{P_{m,1}(\tau_1)}\frac{\mathcal{A}_{D_1}}{\tau_1^{k_1}}\omega(\boldsymbol{\tau},m)\right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \le \sum_{1\le p_1\le\delta_{D_1}-1}\frac{|A_{\delta_{D_1},p_1}|}{\Gamma(\delta_{D_1}-p_1)}\frac{C_2}{C_{k_1}}|\epsilon|\,\|\omega(\boldsymbol{\tau},m)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$$

$$(36)\qquad \left\|\frac{R_{D_1}(im)}{P_{m,1}(\tau_1)}\frac{\mathcal{A}_{D_1}}{\tau_1^{k_1}}\frac{R_{D_2}(im)}{P_{m,2}(\tau_2)}\frac{\tilde{\mathcal{A}}_{D_2}}{\tau_2^{k_2}}\omega(\boldsymbol{\tau},m)\right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \le \sum_{1\le p_1\le\delta_{D_1}-1}\ \sum_{1\le p_2\le\tilde{\delta}_{D_2}-1}\frac{|A_{\delta_{D_1},p_1}|}{\Gamma(\delta_{D_1}-p_1)}\frac{|\tilde{A}_{\tilde{\delta}_{D_2},p_2}|}{\Gamma(\tilde{\delta}_{D_2}-p_2)}\frac{(C_2)^2}{C_{k_1}C_{k_2}}|\epsilon|^2\,\|\omega(\boldsymbol{\tau},m)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$$

Proposition 5 and Lemma 2 yield

$$(37)\qquad \Bigg\|\frac{\epsilon^{-2}}{P_{m,1}(\tau_1)P_{m,2}(\tau_2)}\int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}}(\tau_1^{k_1}-s_1)^{1/k_1}(\tau_2^{k_2}-s_2)^{1/k_2} \times\left(\int_0^{s_1}\int_0^{s_2}\int_{-\infty}^{+\infty}P_1(i(m-m_1),\epsilon)\omega((s_1-x_1)^{1/k_1},(s_2-x_2)^{1/k_2},m-m_1)\right. \times P_2(im_1,\epsilon)\omega(x_1^{1/k_1},x_2^{1/k_2},m_1)\left.\frac{1}{(s_1-x_1)x_1(s_2-x_2)x_2}dm_1dx_2dx_1\right)ds_2ds_1\Bigg\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \le\frac{C_3}{C_{k_1}C_{k_2}}\|\omega(\boldsymbol{\tau},m)\|^2_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$$

From Proposition 3 and Lemma 2, we get

$$(38)\qquad \Bigg\|\frac{R_{l_1,l_2}(im)}{P_{m,1}(\tau_1)P_{m,2}(\tau_2)}\epsilon^{\Delta_{l_1,l_2}-d_{l_1}-\tilde{d}_{l_2}+\delta_{l_1}+\tilde{\delta}_{l_2}-2} \int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}}(\tau_1^{k_1}-s_1)^{d_{l_1,k_1}/k_1-1}(\tau_2^{k_2}-s_2)^{\tilde{d}_{l_2,k_2}/k_2-1}s_1^{\delta_{l_1}}s_2^{\tilde{\delta}_{l_2}}\omega(s_1^{1/k_1},s_2^{1/k_2},m)\frac{ds_2}{s_2}\frac{ds_1}{s_1}\Bigg\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \le\frac{C_2}{C_{k_1}C_{k_2}}|\epsilon|^{\Delta_{l_1,l_2}+\delta_{l_1}(1+k_1)-k_1\delta_{D_1}+\tilde{\delta}_{l_2}(1+k_2)-k_2\tilde{\delta}_{D_2}}\|\omega(\boldsymbol{\tau},m)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$$

Finally, from Proposition 2 and Lemma 2 we get

$$\left\|\epsilon^{-2}\frac{1}{P_{m,1}(\tau_1)P_{m,2}(\tau_2)}\int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}}(\tau_1^{k_1}-s_1)^{1/k_1}(\tau_2^{k_2}-s_2)^{1/k_2} \times\psi_{\boldsymbol{k}}(s_1^{1/k_1},s_2^{1/k_2},m,\epsilon)\frac{ds_2}{s_2}\frac{ds_1}{s_1}\right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \leq \frac{C_1}{C_{k_1}C_{k_2}}\left\|\psi_{\boldsymbol{k}}(s_1^{1/k_1},s_2^{1/k_2},m,\epsilon)\right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \leq \frac{C_1}{C_{k_1}C_{k_2}}\zeta_2. \tag{39}$$

In view of (32) and by choosing $\epsilon_0,\varpi,\zeta_2>0$ such that

$$\sum_{1\leq p_2\leq\tilde{\delta}_{D_2}-1}\frac{C_{k_1}|\tilde{A}_{\tilde{\delta}_{D_2},p_2}|}{\Gamma(\tilde{\delta}_{D_2}-p_2)}C_2|\epsilon_0|\varpi+\sum_{1\leq p_1\leq\delta_{D_1}-1}\frac{C_{k_2}|A_{\delta_{D_1},p_1}|}{\Gamma(\delta_{D_1}-p_1)}C_2|\epsilon_0|\varpi + \sum_{1\leq p_1\leq\delta_{D_1}-1}\sum_{1\leq p_2\leq\tilde{\delta}_{D_2}-1}\frac{|A_{\delta_{D_1},p_1}|}{\Gamma(\delta_{D_1}-p_1)}\frac{|\tilde{A}_{\tilde{\delta}_{D_2},p_2}|}{\Gamma(\tilde{\delta}_{D_2}-p_2)}(C_2)^2|\epsilon_0|^2\varpi+\frac{C_3\varpi^2}{\Gamma(1+\frac{1}{k_1})\Gamma(1+\frac{1}{k_2})(2\pi)^{1/2}} + \sum_{1\leq l_1\leq D_1-1,1\leq l_2\leq D_2-1}C_2|\epsilon_0|^{\Delta_{l_1,l_2}+\delta_{l_1}(1+k_1)-k_1\delta_{D_1}+\tilde{\delta}_{l_2}(1+k_2)-k_2\tilde{\delta}_{D_2}}\frac{k_1^{\delta_{l_1}}k_2^{\tilde{\delta}_{l_2}}}{\Gamma(\frac{d_{l_1,k_1}}{k_1})\Gamma(\frac{\tilde{d}_{l_2,k_2}}{k_2})}\varpi + \frac{C_1\zeta_2}{\Gamma(1+\frac{1}{k_1})\Gamma(1+\frac{1}{k_2})\min_{m\in\mathbb{R}}|R_{D_1}(im)R_{D_2}(im)|}\leq\varpi C_{k_1}C_{k_2}. \tag{40}$$

In view of (34), (35), (36), (37), (38), and (40), one gets that the operator $\mathcal{H}_\epsilon$ is such that $\mathcal{H}_\epsilon(\overline{B}(0,\varpi))\subseteq\overline{B}(0,\varpi)$. The next stage of the proof is to show that, indeed, $\mathcal{H}_\epsilon$ is a contractive map in that ball. Let $\omega_1,\omega_2\in F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$ with $\|\omega_j\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}\leq\varpi$. Then, it holds that

$$\|\mathcal{H}_\epsilon(\omega_1)-\mathcal{H}_\epsilon(\omega_2)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}\leq\frac{1}{2}\|\omega_1-\omega_2\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}, \tag{41}$$

for all $\epsilon\in D(0,\epsilon_0)\setminus\{0\}$.

Analogous estimates as in (34), (35), (36), (38) yield

$$\left\|\frac{R_{D_2}(im)}{P_{m,2}(\tau_2)}\frac{\tilde{\mathcal{A}}_{D_2}}{\tau_2^{k_2}}(\omega_1(\boldsymbol{\tau},m)-\omega_2(\boldsymbol{\tau},m))\right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \leq\sum_{1\leq p_2\leq\tilde{\delta}_{D_2}-1}\frac{|\tilde{A}_{\tilde{\delta}_{D_2},p_2}|}{\Gamma(\tilde{\delta}_{D_2}-p_2)}\frac{C_2}{C_{k_2}}|\epsilon|\,\|\omega_1(\boldsymbol{\tau},m)-\omega_2(\boldsymbol{\tau},m)\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \tag{42}$$

$$\left\|\frac{R_{D_1}(im)}{P_{m,1}(\tau_1)}\frac{\mathcal{A}_{D_1}}{\tau_1^{k_1}}(\omega_1(\boldsymbol{\tau},m)-\omega_2(\boldsymbol{\tau},m))\right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \leq\sum_{1\leq p_1\leq\delta_{D_1}-1}\frac{|A_{\delta_{D_1},p_1}|}{\Gamma(\delta_{D_1}-p_1)}\frac{C_2}{C_{k_1}}|\epsilon|\,\|\omega_1(\boldsymbol{\tau},m)-\omega_2(\boldsymbol{\tau},m)\|_{\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon} \tag{43}$$

$$\left\| \frac{R_{D_1}(im)}{P_{m,1}(\tau_1)} \frac{\mathcal{A}_{D_1}}{\tau_1^{k_1}} \frac{R_{D_2}(im)}{P_{m,2}(\tau_2)} \frac{\tilde{A}_{D_2}}{\tau_2^{k_2}} (\omega_1(\boldsymbol{\tau},m) - \omega_2(\boldsymbol{\tau},m)) \right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)} \tag{44}$$
$$\leq \sum_{1 \leq p_1 \leq \delta_{D_1}-1} \sum_{1 \leq p_2 \leq \tilde{\delta}_{D_2}-1} \frac{|A_{\delta_{D_1},p_1}|}{\Gamma(\delta_{D_1}-p_1)} \frac{|\tilde{A}_{\tilde{\delta}_{D_2},p_2}|}{\Gamma(\tilde{\delta}_{D_2}-p_2)} \frac{(C_2)^2}{C_{k_1}C_{k_2}} |\epsilon|^2 \left\| \omega_1(\boldsymbol{\tau},m) - \omega_2(\boldsymbol{\tau},m) \right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$$

(45)
$$\left\| \frac{R_{l_1,l_2}(im)}{P_{m,1}(\tau_1)P_{m,2}(\tau_2)} \epsilon^{\Delta_{l_1,l_2}-d_{l_1}-\tilde{d}_{l_2}+\delta_{l_1}+\tilde{\delta}_{l_2}-2} \int_0^{\tau_1^{k_1}} \int_0^{\tau_2^{k_2}} (\tau_1^{k_1}-s_1)^{d_{l_1,k_1}/k_1-1} (\tau_2^{k_2}-s_2)^{\tilde{d}_{l_2,k_2}/k_2-1} \right.$$
$$\left. \times s_1^{\delta_{l_1}} s_2^{\tilde{\delta}_{l_2}} (\omega_1(s_1^{1/k_1},s_2^{1/k_2},m) - \omega_2(s_1^{1/k_1},s_2^{1/k_2},m)) \frac{ds_2}{s_2} \frac{ds_1}{s_1} \right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$$
$$\leq \frac{C_2}{C_{k_1}C_{k_2}} |\epsilon|^{\Delta_{l_1,l_2}+\delta_{l_1}(1+k_1)-k_1\delta_{D_1}+\tilde{\delta}_{l_2}(1+k_2)-k_2\tilde{\delta}_{D_2}} \left\| \omega_1(\boldsymbol{\tau},m) - \omega_2(\boldsymbol{\tau},m) \right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$$

Finally, put

$$W_1 := w_1((s_1-x_1)^{1/k_1},(s_2-x_2)^{1/k_2},m-m_1) - w_2((s_1-x_1)^{1/k_1},(s_2-x_2)^{1/k_2},m-m_1),$$

and $W_2 := w_1(x_1^{1/k_1},x_2^{1/k_2},m_1) - w_2(x_1^{1/k_1},x_2^{1/k_2},m_1)$. Then, taking into account that

$$P_1(i(m-m_1),\epsilon) w_1((s_1-x_1)^{1/k_1},(s_2-x_2)^{1/k_2},m-m_1) P_2(im_1,\epsilon) w_1(x_1^{1/k_1},x_2^{1/k_2},m_1) \tag{46}$$
$$- P_1(i(m-m_1),\epsilon) w_2((s_1-x_1)^{1/k_1},(s_2-x_2)^{1/k_2},m-m_1) P_2(im_1,\epsilon) w_2(x_1^{1/k_1},x_2^{1/k_2},m_1)$$
$$= P_1(i(m-m_1),\epsilon) W_1 P_2(im_1,\epsilon) w_1(x_1^{1/k_1},x_2^{1/k_2},m_1)$$
$$+ P_1(i(m-m_1),\epsilon) w_2((s_1-x_1)^{1/k_1},(s_2-x_2)^{1/k_2},m-m_1) P_2(im_1,\epsilon) W_2,$$

and by using Lemma 2 and analogous estimates as in (37) and (27), we get

$$\left\| \frac{\epsilon^{-2}}{P_{m,1}(\tau_1)P_{m,2}(\tau_2)} \int_0^{\tau_1^{k_1}} \int_0^{\tau_2^{k_2}} (\tau_1^{k_1}-s_1)^{1/k_1} (\tau_2^{k_2}-s_2)^{1/k_2} \right. \tag{47}$$
$$\times \left(\int_0^{s_1} \int_0^{s_2} \int_{-\infty}^{+\infty} P_1(i(m-m_1),\epsilon) W_1 \right.$$
$$\left. \left. \times P_2(im_1,\epsilon) W_2 \frac{1}{(s_1-x_1)x_1(s_2-x_2)x_2} dm_1 dx_2 dx_1 \right) ds_2 ds_1 \right\|_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$$
$$\leq \frac{2C_3\varpi}{C_{k_1}C_{k_2}} \left\| \omega_1(\boldsymbol{\tau},m) - \omega_2(\boldsymbol{\tau},m) \right\|^2_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$$

Let $\varpi, \epsilon_0, \zeta_1 > 0$ such that

$$\sum_{1\leq p_2\leq \tilde{\delta}_{D_2}-1} \frac{C_{k_1}|\tilde{A}_{\tilde{\delta}_{D_2},p_2}|}{\Gamma(\tilde{\delta}_{D_2}-p_2)}C_2|\epsilon_0| + \sum_{1\leq p_1\leq \delta_{D_1}-1} \frac{C_{k_2}|A_{\delta_{D_1},p_1}|}{\Gamma(\delta_{D_1}-p_1)}C_2|\epsilon_0|$$
$$+ \sum_{1\leq p_1\leq \delta_{D_1}-1}\sum_{1\leq p_2\leq \tilde{\delta}_{D_2}-1} \frac{|A_{\delta_{D_1},p_1}|}{\Gamma(\delta_{D_1}-p_1)}\frac{|\tilde{A}_{\tilde{\delta}_{D_2},p_2}|}{\Gamma(\tilde{\delta}_{D_2}-p_2)}(C_2)^2|\epsilon_0|^2$$
$$+ \frac{2C_3\varpi}{\Gamma(1+\frac{1}{k_1})\Gamma(1+\frac{1}{k_2})(2\pi)^{1/2}}$$
$$+ \sum_{1\leq l_1\leq D_1-1, 1\leq l_2\leq D_2-1} C_2|\epsilon_0|^{\Delta_{l_1,l_2}+\delta_{l_1}(1+k_1)-k_1\delta_{D_1}+\tilde{\delta}_{l_2}(1+k_2)-k_2\tilde{\delta}_{D_2}} \frac{k_1^{\delta_{l_1}}k_2^{\tilde{\delta}_{l_2}}}{\Gamma(\frac{d_{l_1,k_1}}{k_1})\Gamma(\frac{\tilde{d}_{l_2,k_2}}{k_2})} \leq \frac{1}{2}C_{k_1}C_{k_2}. \tag{48}$$

Then, (48) combined with (42), (43), (44), (45) and (47) yields (41).

We consider the ball $\bar{B}(0,\varpi) \subset F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$ constructed above. It turns out to be a complete metric space for the norm $||.||_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$. As $\mathcal{H}_\epsilon$ is a contractive map from $\bar{B}(0,\varpi)$ into itself, the classical contractive mapping theorem, guarantees the existence of a unique fixed point $\omega_{\boldsymbol{k}}(\boldsymbol{\tau},m,\epsilon) \in \bar{B}(0,\varpi) \subseteq F^{\boldsymbol{d}}_{(\boldsymbol{\nu},\beta,\mu,\boldsymbol{k},\epsilon)}$ for $\mathcal{H}_\epsilon$. The function $\omega_{\boldsymbol{k}}(\boldsymbol{\tau},m,\epsilon)$ depends holomorphically on ϵ in $D(0,\epsilon_0)\setminus\{0\}$. By construction, $\omega_{\boldsymbol{k}}(\boldsymbol{\tau},m,\epsilon)$ defines a solution of the equation (19). □

Regarding the construction of the auxiliary equations, one can obtain the analytic solutions of (12) by means of Laplace transform.

Proposition 7 *Under the hypotheses of Proposition 6, choose S_{d_1}, S_{d_2} and $S_{Q_1,R_{D_1}}, S_{Q_2,R_{D_2}}$ in such a way that the roots of $P_{m,1}(\tau_1)$ and $P_{m,2}(\tau_2)$ fall appart from S_{d_1} and S_{d_2}, respectively, as stated before (27).*

Notice that they apply for any small enough $\epsilon_0 > 0$, provided that (30) hold.

Let $S_{d_1,\theta_1,h'|\epsilon|}, S_{d_2,\theta_2,h'|\epsilon|}$ be bounded sectors with aperture $\pi/k_j < \theta_j < \pi/k_j + 2\delta_{S,j}$, for $j = 1,2$ (where $2\delta_{S,j}$ is the opening of S_{d_j}), with direction d_j and radius $h'|\epsilon|$ for some $h' > 0$ independent of ϵ. We choose $0 < \beta' < \beta$.

Then, equation (12) with initial condition $U(T_1,0,m,\epsilon) \equiv U(0,T_2,m,\epsilon) \equiv 0$ has a solution $(\boldsymbol{T},m) \mapsto U(\boldsymbol{T},m,\epsilon)$ defined on $S_{d_1,\theta_1,h'|\epsilon|} \times S_{d_2,\theta_2,h'|\epsilon|} \times \mathbb{R}$ for some $h' > 0$ and all $\epsilon \in D(0,\epsilon_0)\setminus\{0\}$. Let $\epsilon \in D(0,\epsilon_0)\setminus\{0\}$, then for $j = 1,2$ and all $T_j \in S_{d_j,\theta_j,h'|\epsilon|}$, the function $m \mapsto U(\boldsymbol{T},m,\epsilon)$ belongs to the space $E_{(\beta',\mu)}$ and for each $m \in \mathbb{R}$, the function $\boldsymbol{T} \mapsto U(\boldsymbol{T},m,\epsilon)$ is bounded and holomorphic on $S_{d_1,\theta_1,h'|\epsilon|} \times S_{d_2,\theta_2,h'|\epsilon|}$. Moreover, $U(\boldsymbol{T},m,\epsilon)$ can be written as a Laplace transform of order k_1 in the direction d_1 with respect to T_1 and the Laplace transform of order k_2 in the direction d_2 with respect to T_2,

$$U(\boldsymbol{T},m,\epsilon) = k_1k_2\int_{L_{\gamma_1}}\int_{L_{\gamma_2}} \omega^{\boldsymbol{d}}_{\boldsymbol{k}}(u_1,u_2,m,\epsilon)e^{-(\frac{u_1}{T_1})^{k_1}-(\frac{u_2}{T_2})^{k_2}}\frac{du_2}{u_2}\frac{du_1}{u_1} \tag{49}$$

where $L_{\gamma_j} = \mathbb{R}_+e^{i\gamma_j} \in S_{d_j}\cup\{0\}$, for $j = 1,2$ has bisecting direction which might depend on T_j. The function $\omega^{\boldsymbol{d}}_{\boldsymbol{k}}(\boldsymbol{\tau},m,\epsilon)$ defines a continuous function on $(\bar{D}(0,\rho)\cup S_{d_1})\times(\bar{D}(0,\rho)\cup S_{d_2})\times\mathbb{R}\times D(0,\epsilon_0)\setminus\{0\}$, holomorphic with respect to $(\boldsymbol{\tau},\epsilon)$ on $(D(0,\rho)\cup S_{d_1})\times(D(0,\rho)\cup S_{d_2})\times(D(0,\epsilon_0)\setminus\{0\})$. Moreover, there exists a constant $\varpi_{\boldsymbol{d}}$ (independent of ϵ) such that

$$|\omega^{\boldsymbol{d}}_{\boldsymbol{k}}(\boldsymbol{\tau},m,\epsilon)| \leq \varpi_{\boldsymbol{d}}(1+|m|)^{-\mu}e^{-\beta|m|}\frac{|\frac{\tau_1}{\epsilon}|}{1+|\frac{\tau_1}{\epsilon}|^{2k_1}}\frac{|\frac{\tau_2}{\epsilon}|}{1+|\frac{\tau_2}{\epsilon}|^{2k_2}}\exp(\nu_1|\frac{\tau_1}{\epsilon}|^{k_1}+\nu_2|\frac{\tau_2}{\epsilon}|^{k_2}) \tag{50}$$

for all $\boldsymbol{\tau} \in (\overline{D}(0,\rho) \cup S_{d_1}) \times (\overline{D}(0,\rho) \cup S_{d_2})$*, all* $m \in \mathbb{R}$*, and* $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$*.*

Proof Let $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$. We take the function $\omega_{\boldsymbol{k}}^{\boldsymbol{d}}(\boldsymbol{\tau}, m, \epsilon)$ constructed in Proposition 6

We follow backwards the details of the construction of Section 2. Since $\omega_{\boldsymbol{k}}^{\boldsymbol{d}}$ solves (19), we deduce that the function (49) is bounded on $S_{d_1,\theta_1,h'|\epsilon|} \times S_{d_2,\theta_2,h'|\epsilon|}$ with respect to (T_1, T_2) and fulfills the equation (17), in view of Lemma 1. Then, according to (14), (15) it holds that $U(\boldsymbol{T}, m, \epsilon)$ solves (12).

□

4 Construction of a finite set of genuine solutions of the main problem

We recall the definition of a good covering in $\mathbb{C}^\star$.

Definition 3 *Let* $\varsigma_1, \varsigma_2 \geq 2$ *be integer numbers. Let* $\{\mathcal{E}_{p_1,p_2}\}_{\substack{0 \leq p_1 \leq \varsigma_1 - 1 \\ 0 \leq p_2 \leq \varsigma_2 - 1}}$ *be a finite family of open sectors with vertex at* 0*, radius* ϵ_0 *and opening strictly larger than* $\frac{\pi}{k_2}$*. We assume that the intersection of three different sectors in the good covering is empty, and* $\cup_{\substack{0 \leq p_1 \leq \varsigma_1 - 1 \\ 0 \leq p_2 \leq \varsigma_2 - 1}} \mathcal{E}_{p_1,p_2} = \mathcal{U} \setminus \{0\}$*, for some neighborhood of 0,* $\mathcal{U} \in \mathbb{C}$*. Such set of sectors is called a good covering in* $\mathbb{C}^*$*.*

Definition 4 *Let* $\varsigma_1, \varsigma_2 \geq 2$ *and* $\{\mathcal{E}_{p_1,p_2}\}_{\substack{0 \leq p_1 \leq \varsigma_1 - 1 \\ 0 \leq p_2 \leq \varsigma_2 - 1}}$ *be a good covering in* $\mathbb{C}^*$*. Let* $\mathcal{T}_j$ *be open bounded sectors centered at 0 with radius* $r_{\mathcal{T}_j}$ *for* $j = 1, 2$*, and consider two families of open sectors as follows. The first one is given by*

$$S_{\mathfrak{d}_{p_1},\theta_1,\epsilon_0 r_{\mathcal{T}_1}} = \{T_1 \in \mathbb{C}^* : |T_1| < \epsilon_0 r_{\mathcal{T}_1} \quad , \quad |\mathfrak{d}_{p_1} - \arg(T_1)| < \theta_1/2\}$$

with opening $\theta_1 > \pi/k_1$*, and some* $\mathfrak{d}_{p_1} \in \mathbb{R}$*, for all* $0 \leq p_1 \leq \varsigma_1 - 1$*. This family is chosen to satisfy that:*

1) There exists a constant $M_1 > 0$ *such that*

$$|\tau_1 - q_{l,1}(m)| \geq M_1(1 + |\tau_1|) \tag{51}$$

for all $0 \leq l \leq (\delta_{D_1} - 1)k_1 - 1$*,* $m \in \mathbb{R}$*, and* $\tau_1 \in S_{\mathfrak{d}_{p_1}} \cup \bar{D}(0,\rho)$*, for all* $0 \leq p_1 \leq \varsigma_1 - 1$*, and every root* $q_{l,1}$ *of the polynomial* $P_{m,1}(\tau_1)$*.*

2) There exists a constant $M_2 > 0$ *such that*

$$|\tau_1 - q_{l_0,1}(m)| \geq M_2 |q_{l_0,1}(m)| \tag{52}$$

for some root $q_{l_0,1}$ *of* $P_{m,1}$*, all* $m \in \mathbb{R}$*,* $\tau_1 \in S_{\mathfrak{d}_{p_1}} \cup \bar{D}(0,\rho)$*, for all* $0 \leq p_1 \leq \varsigma_1 - 1$*.*

The second family is chosen in an analogous manner. It is given by

$$S_{\tilde{\mathfrak{d}}_{p_2},\theta_2,\epsilon_0 r_{\mathcal{T}_2}} = \{T_2 \in \mathbb{C}^* : |T_2| < \epsilon_0 r_{\mathcal{T}_2} \quad , \quad |\tilde{\mathfrak{d}}_{p_2} - \arg(T_2)| < \theta_2/2\}$$

with opening $\theta_2 > \pi/k_2$*, and some* $\tilde{\mathfrak{d}}_{p_2} \in \mathbb{R}$*, for all* $0 \leq p_2 \leq \varsigma_2 - 1$*. This family is chosen to satisfy analogous conditions with respect to the roots of the polynomial* $P_{m,2}(\tau_2)$*.*

In addition to the previous assumptions, we consider $S_{\mathfrak{d}_{p_1},\theta_1,\epsilon_0 r_{\mathcal{T}_1}}$ *and* $S_{\tilde{\mathfrak{d}}_{p_2},\theta_2,\epsilon_0 r_{\mathcal{T}_2}}$ *such that for all* $0 \leq p_1 \leq \varsigma_1 - 1$*,* $0 \leq p_2 \leq \varsigma_2 - 1$*,* $\boldsymbol{t} \in \mathcal{T}_1 \times \mathcal{T}_2$*, and* $\epsilon \in \mathcal{E}_{p_1,p_2}$*, one has*

$$\epsilon t_1 \in S_{\mathfrak{d}_{p_1},\theta_1,\epsilon_0 r_{\mathcal{T}_1}} \text{ and } \epsilon t_2 \in S_{\tilde{\mathfrak{d}}_{p_2},\theta_2,\epsilon_0 r_{\mathcal{T}_2}}.$$

We say that the family $\{(S_{\mathfrak{d}_{p_1},\theta_1,\epsilon_0 r_{\mathcal{T}_1}})_{0 \leq p_1 \leq \varsigma_1 - 1}, (S_{\tilde{\mathfrak{d}}_{p_2},\theta_2,\epsilon_0 r_{\mathcal{T}_2}})_{0 \leq p_2 \leq \varsigma_2 - 1}, \mathcal{T}_1 \times \mathcal{T}_2\}$ *is associated to the good covering* $\{\mathcal{E}_{p_1,p_2}\}_{\substack{0 \leq p_1 \leq \varsigma_1 - 1 \\ 0 \leq p_2 \leq \varsigma_2 - 1}}$*.*

The first main result of the present work is devoted to the construction of a family of actual holomorphic solutions to the equation (9) for null initial data. Each of the elements in the family of solutions is associated to an element of a good covering with respect to the complex parameter ϵ. The strategy leans on the control of the difference of two solutions defined in domains with nonempty intersection with respect to the perturbation parameter ϵ. The construction of each analytic solution in terms of two Laplace transforms in different time variables requires to distinguish different cases, depending on the coincidence of the integration paths or not.

Theorem 1 *We consider the equation (9) and we assume that (6-10) and (11) hold. We also make the additional assumption that*

$$\delta_{D_1} \geq \delta_{l_1} + \frac{2}{k_1}, \quad \tilde{\delta}_{D_2} \geq \tilde{\delta}_{l_2} + \frac{2}{k_2}, \quad \Delta_{l_1,l_2} + k_1(1-\delta_{D_1}) + k_2(1-\tilde{\delta}_{D_2}) + 2 \geq 0, \tag{53}$$

for all $1 \leq l_1 \leq D_1 - 1$ *and* $1 \leq l_2 \leq D_2 - 1$.

Let $\{\mathcal{E}_{p_1,p_2}\}_{\substack{0\leq p_1\leq\varsigma_1-1\\0\leq p_2\leq\varsigma_2-1}}$ *be a good covering in* $\mathbb{C}^*$ *with*

$$\{(S_{\mathfrak{d}_{p_1},\theta_1,\epsilon_0 r_{\mathcal{T}_1}})_{0\leq p_1\leq\varsigma_1-1}, (S_{\tilde{\mathfrak{d}}_{p_2},\theta_2,\epsilon_0 r_{\mathcal{T}_2}})_{0\leq p_2\leq\varsigma_2-1}, \mathcal{T}_1\times\mathcal{T}_2\}$$

being a family associated to this good covering can be considered.

Then, there exist $r_{Q_j,R_{D_j}}, \epsilon_0 > 0$, $j=1,2$, *such that for every* $0 \leq p_1 \leq \varsigma_1 - 1$ *and* $0 \leq p_2 \leq \varsigma_2 - 1$, *one can construct a solution* $u_{p_1,p_2}(\boldsymbol{t}, z, \epsilon)$ *of (9) with* $u_{p_1,p_2}(0,t_2,z,\epsilon) \equiv u_{p_1,p_2}(t_1,0,z,\epsilon) \equiv 0$ *which defines a bounded holomorphic function on the domain* $(\mathcal{T}_1 \cap D(0,h')) \times (\mathcal{T}_2 \cap D(0,h')) \times H_{\beta'} \times \mathcal{E}_{p_1,p_2}$ *for any given* $0 < \beta' < \beta$ *and for some* $h' > 0$.

Moreover, there exist constants $0 < h'' \leq h'$, $K_p, M_p > 0$ *(independent of* ϵ*), and sets* $\mathcal{U}_{k_1}\times\mathcal{U}_{k_2} \subseteq \{0,1,\ldots,\varsigma_1-1\}\times\{0,1,\ldots,\varsigma_2-1\}$ *such that for every* $(p_1,p_2),(p_1',p_2') \in \{0,1,\ldots,\varsigma_1-1\}\times\{0,1,\ldots,\varsigma_2-1\}$, *one of the following holds:*

- $\mathcal{E}_{p_1,p_2} \cap \mathcal{E}_{p_1',p_2'} = \emptyset$.
- $\mathcal{E}_{p_1,p_2} \cap \mathcal{E}_{p_1',p_2'} \neq \emptyset$ *and*

$$\sup_{\boldsymbol{t}\in(\mathcal{T}_1\cap D(0,h''))\times(\mathcal{T}_2\cap D(0,h'')), z\in H_{\beta'}} |u_{p_1,p_2}(\boldsymbol{t},z,\epsilon) - u_{p_1',p_2'}(\boldsymbol{t},z,\epsilon)| \leq K_p e^{-\frac{M_p}{|\epsilon|^{k_1}}} \tag{54}$$

  for all $\epsilon \in \mathcal{E}_{p_1,p_2} \cap \mathcal{E}_{p_1',p_2'}$. *In this situation, we say that* $\{(p_1,p_2),(p_1',p_2')\}$ *belong to* $\mathcal{U}_{k_1}$.

- $\mathcal{E}_{p_1,p_2} \cap \mathcal{E}_{p_1',p_2'} \neq \emptyset$ *and*

$$\sup_{\boldsymbol{t}\in(\mathcal{T}_1\cap D(0,h''))\times(\mathcal{T}_2\cap D(0,h'')), z\in H_{\beta'}} |u_{p_1,p_2}(\boldsymbol{t},z,\epsilon) - u_{p_1',p_2'}(\boldsymbol{t},z,\epsilon)| \leq K_p e^{-\frac{M_p}{|\epsilon|^{k_2}}} \tag{55}$$

  for all $\epsilon \in \mathcal{E}_{p_1,p_2} \cap \mathcal{E}_{p_1',p_2'}$. *In this situation, we say that* $\{(p_1,p_2),(p_1',p_2')\}$ *belong to* $\mathcal{U}_{k_2}$.

Proof Regarding Proposition 7, one can choose $r_{Q_j,R_{D_j}} > 0$, $j=1,2$, and $\epsilon_0 > 0$ in a way that for each pair (p_1,p_2), we fix the multidirection $(\mathfrak{d}_{p_1}, \tilde{\mathfrak{d}}_{p_2})$ with $0 \leq p_j \leq \varsigma_j - 1$ and construct

$U^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(\boldsymbol{T},m,\epsilon)$ such that $U^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(0,T_2,m,\epsilon) \equiv U^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(T_1,0,m,\epsilon) \equiv 0$ and is a solution of

(56)
$$\left(Q_1(im)\partial_{T_1} - T_1^{(\delta_{D_1}-1)(k_1+1)}\partial_{T_1}^{\delta_{D_1}}R_{D_1}(im)\right)\left(Q_2(im)\partial_{T_2} - T_2^{(\tilde{\delta}_{D_2}-1)(k_2+1)}\partial_{T_2}^{\tilde{\delta}_{D_2}}R_{D_2}(im)\right)U(\boldsymbol{T},m,\epsilon)$$
$$= \epsilon^{-2}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} P_1(i(m-m_1),\epsilon)U(\boldsymbol{T},m-m_1,\epsilon)P_2(im_1,\epsilon)U(\boldsymbol{T},m_1,\epsilon)dm_1$$
$$+ \sum_{1\leq l_1\leq D_1-1, 1\leq l_2\leq D_2-1} \epsilon^{\Delta_{l_1,l_2}-d_{l_1}-d_{l_2}+\delta_{l_1}+\tilde{d}_{l_2}-2}T_1^{d_{l_1}}T_2^{\tilde{d}_{l_2}}\partial_{T_1}^{\delta_{l_1}}\partial_{T_2}^{\tilde{\delta}_{l_2}}R_{\ell_1,\ell_2}(im)U(\boldsymbol{T},m,\epsilon)$$
$$+ \epsilon^{-2}F(\boldsymbol{T},m,\epsilon),$$

where
$$F(\boldsymbol{T},m,\epsilon) = \sum_{n_1,n_2\geq 1} F_{n_1,n_2}(m,\epsilon)T_1^{n_1}T_2^{n_2}$$
is a convergent series in $D(0,T_0/2)^2$ with values in $E_{(\beta,\mu)}$, for all $\epsilon \in D(0,\epsilon_0)\setminus\{0\}$. The function $(\boldsymbol{T},m) \mapsto U^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(\boldsymbol{T},m,\epsilon)$ is well defined on $S_{\mathfrak{d}_{p_1},\theta_1,h'|\epsilon|} \times S_{\tilde{\mathfrak{d}}_{p_2},\theta_2,h'|\epsilon|} \times \mathbb{R}$ where $h'>0$, for all $\epsilon \in D(0,\epsilon_0)\setminus\{0\}$. Moreover, $U^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(\boldsymbol{T},m,\epsilon)$ can be written as the iterated Laplace transform of order k_1 in the direction $\mathfrak{d}_{p_1}$, and the Laplace transform of order k_2 in the direction $\tilde{\mathfrak{d}}_{p_2}$

$$U^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(\boldsymbol{T},m,\epsilon) = k_1k_2\int_{L_{\gamma_{p_1}}}\int_{L_{\gamma_{p_2}}} \omega_{\boldsymbol{k}}^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(u_1,u_2,m,\epsilon)e^{-(\frac{u_1}{T_1})^{k_1}-(\frac{u_2}{T_2})^{k_2}}\frac{du_2}{u_2}\frac{du_1}{u_1} \tag{57}$$

along $L_{\gamma_{p_j}} = \mathbb{R}_+e^{i\gamma_{p_j}}$ which might depend on T_j. Here, $\omega_{\boldsymbol{k}}^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(\boldsymbol{\tau},m,\epsilon)$ defines a continuous function on $(\bar{D}(0,\rho)\cup S_{d_{p_1}}) \times (\bar{D}(0,\rho)\cup S_{d_{p_2}}) \times \mathbb{R} \times D(0,\epsilon_0)\setminus\{0\}$, holomorphic with respect to $(\boldsymbol{\tau},\epsilon)$ on $(D(0,\rho)\cup S_{\mathfrak{d}_{p_1}}) \times (D(0,\rho)\cup S_{\tilde{\mathfrak{d}}_{p_2}}) \times (D(0,\epsilon_0)\setminus\{0\})$ for all $m\in\mathbb{R}$. Moreover, there exists a constant $\varpi_{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}$ (independent of ϵ) such that

$$|\omega_{\boldsymbol{k}}^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(\boldsymbol{\tau},m,\epsilon)| \leq \varpi_{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(1+|m|)^{-\mu}e^{-\beta|m|}\frac{|\frac{\tau_1}{\epsilon}|}{1+|\frac{\tau_1}{\epsilon}|^{2k_1}}\frac{|\frac{\tau_2}{\epsilon}|}{1+|\frac{\tau_2}{\epsilon}|^{2k_2}}\exp(\nu_1|\frac{\tau_1}{\epsilon}|^{k_1}+\nu_2|\frac{\tau_2}{\epsilon}|^{k_2}) \tag{58}$$

for all $\boldsymbol{\tau} \in (D(0,\rho)\cup S_{\mathfrak{d}_{p_1}}) \times (D(0,\rho)\cup S_{\tilde{\mathfrak{d}}_{p_2}})$, all $m\in\mathbb{R}$ and $\epsilon \in D(0,\epsilon_0)\setminus\{0\}$. The function

$$(\boldsymbol{T},z) \mapsto \mathbf{U}^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(\boldsymbol{T},z,\epsilon) = \mathcal{F}^{-1}(m \mapsto U^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(\boldsymbol{T},m,\epsilon))(z)$$

turns out to be holomorphic on $S_{\mathfrak{d}_{p_1},\theta_1,h'|\epsilon|} \times S_{\tilde{\mathfrak{d}}_{p_2},\theta_2,h'|\epsilon|} \times H_{\beta'}$, for all $\epsilon \in D(0,\epsilon_0)\setminus\{0\}$ and $0<\beta'<\beta$. For all $0\leq p_j\leq \varsigma_j-1$, $j=1,2$ let

$$u_{p_1,p_2}(\boldsymbol{t},z,\epsilon) = \mathbf{U}^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(\epsilon t_1,\epsilon t_2,z,\epsilon)$$
$$= \frac{k_1k_2}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_{\gamma_{p_1}}}\int_{L_{\gamma_{p_2}}} \omega_{\boldsymbol{k}}^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(u_1,u_2,m,\epsilon)e^{-(\frac{u_1}{\epsilon t_1})^{k_1}-(\frac{u_2}{\epsilon t_2})^{k_2}}e^{izm}\frac{du_2}{u_2}\frac{du_1}{u_1}dm.$$

By construction (see Definition 4), the function $u_{p_1,p_2}(\boldsymbol{t},z,\epsilon)$ defines a bounded holomorphic function on $(\mathcal{T}_1\cap D(0,h')) \times (\mathcal{T}_2\cap D(0,h')) \times H_{\beta'} \times \mathcal{E}_{p_1,p_2}$. Moreover, $u_{p_1,p_2}(0,t_2,z,\epsilon) \equiv u_{p_1,p_2}(t_1,0,z,\epsilon) \equiv 0$. Moreover, the properties of inverse Fourier transform described in Proposition 1 guarantee that $u_{p_1,p_2}(\boldsymbol{t},z,\epsilon)$ is a solution of the main problem under study (9) on $(\mathcal{T}_1\cap D(0,h')) \times (\mathcal{T}_2\cap D(0,h')) \times H_{\beta'} \times \mathcal{E}_{p_1,p_2}$.

It is worth mentioning that all the functions $\boldsymbol{\tau} \mapsto \omega_{\boldsymbol{k}}^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(\boldsymbol{\tau},m,\epsilon)$ provide the analytic continuation of a common function $\boldsymbol{\tau} \mapsto \omega_{\boldsymbol{k}}(\boldsymbol{\tau},m,\epsilon) \in \mathcal{O}(D(0,\rho)^2, E_{(\beta,\mu)})$ to $S_{\mathfrak{d}_{p_1}} \times S_{\tilde{\mathfrak{d}}_{p_2}}$.

Different digressions are considered, due to the presence of two time variables. Let $p_j, p'_j \in \{0,\ldots,\varsigma_j - 1\}$ for $j=1,2$, and assume that $\mathcal{E}_{p_1,p_2} \cap \mathcal{E}_{p'_1,p'_2} \neq \emptyset$. Then, three different cases should be considered:

Case 1: Assume that the path $L_{\gamma_{p_1}}$ coincides with $L_{\gamma_{p'_1}}$, and $L_{\gamma_{p_2}}$ does not coincide with $L_{\gamma_{p'_2}}$. Then, using that $u_2 \mapsto \omega_{\boldsymbol{k}}^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(u_1,u_2,m,\epsilon)\exp(-(\frac{u_2}{\epsilon t_2})^{k_2})/u_2$ is holomorphic on $D(0,\rho)$ for all $(m,\epsilon) \in \mathbb{R} \times (D(0,\epsilon_0) \setminus \{0\})$, and every $u_1 \in L_{\gamma_{p_1}}$, one can deform one of the integration paths to write

$$I = \int_{L_{\gamma_{p_2}}} \omega_{\boldsymbol{k}}^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(u_1,u_2,m,\epsilon) e^{-\left(\frac{u_2}{\epsilon t_2}\right)^{k_2}} \frac{du_2}{u_2} - \int_{L_{\gamma_{p'_2}}} \omega_{\boldsymbol{k}}^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(u_1,u_2,m,\epsilon) e^{-\left(\frac{u_2}{\epsilon t_2}\right)^{k_2}} \frac{du_2}{u_2}$$

in the form

$$(59) \quad \int_{L_{\rho/2,\gamma_{p_2}}} \omega_{\boldsymbol{k}}^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(u_1,u_2,m,\epsilon) e^{-(\frac{u_2}{\epsilon t_2})^{k_2}} \frac{du_2}{u_2} - \int_{L_{\rho/2,\gamma_{p'_2}}} \omega_{\boldsymbol{k}}^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(u_1,u_2,m,\epsilon) e^{-(\frac{u_2}{\epsilon t_2})^{k_2}} \frac{du_2}{u_2} + \int_{C_{\rho/2,\gamma_{p'_2},\gamma_{p_2}}} \omega_{\boldsymbol{k}}^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(u_1,u_2,m,\epsilon) e^{-(\frac{u_2}{\epsilon t_2})^{k_2}} \frac{du_2}{u_2}.$$

where $L_{\rho/2,\gamma_{p_2}} = [\rho/2,+\infty)e^{i\gamma_{p_2}}$, $L_{\rho/2,\gamma_{p'_2}} = [\rho/2,+\infty)e^{i\gamma_{p'_2}}$ and $C_{\rho/2,\gamma_{p'_2},\gamma_{p_2}}$ is an arc of circle connecting $(\rho/2)e^{i\gamma_{p'_2}}$ and $(\rho/2)e^{i\gamma_{p_2}}$ with the adequate orientation.

The estimates for the previous expression can be found in detail in the proof of Theorem 1, [6]. Namely, we get the existence of constants $C_{p_2,p'_2}, M_{p_2,p'_2} > 0$ such that

$$|I| \leq C_{p_2,p'_2} \varpi_{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}} (1+|m|)^{-\mu} e^{-\beta|m|} \frac{|\frac{u_1}{\epsilon}|}{1+|\frac{u_1}{\epsilon}|^{2k_1}} \exp(\nu_1 |\frac{u_1}{\epsilon}|^{k_1}) e^{-\frac{M_{p_2,p'_2}}{|\epsilon|^{k_2}}},$$

for $t_2 \in \mathcal{T}_2 \cap D(0,h')$ and $\epsilon \in \mathcal{E}_{p_1,p_2} \cap \mathcal{E}_{p'_1,p'_2}$ and $u_1 \in L_{\gamma_{p_1}}$. We have

$$(60) \quad |u_{p_1,p_2}(\boldsymbol{t},z,\epsilon) - u_{p'_1,p'_2}(\boldsymbol{t},z,\epsilon)| \leq \frac{k_1 k_2}{(2\pi)^{1/2}} C_{p_2,p'_2} \left(\int_{-\infty}^{\infty} (1+|m|)^{-\mu} e^{-\beta|m|} e^{-m|\mathrm{Im}(z)|} dm \right) \times \int_{L_{\gamma_{p_1}}} \frac{|\frac{u_1}{\epsilon}|}{1+|\frac{u_1}{\epsilon}|^{2k_1}} \exp(\nu_1 |\frac{u_1}{\epsilon}|^{k_1}) \exp(-\left(\frac{u_1}{\epsilon t_1}\right)^{k_1}) \left|\frac{du_1}{u_1}\right| e^{-\frac{M_{p_2,p'_2}}{|\epsilon|^{k_2}}}.$$

The last integral is estimated via the reparametrization $u_1 = re^{i\gamma_{p_1}}$ and the change of variable $r = |\epsilon| s$ by

$$\int_0^\infty \frac{1}{1+s^2} e^{-\delta_1 s^{k_1}} ds,$$

for some $\delta_1 > 0$, whenever $t_1 \in \mathcal{T}_1 \cap D(0,h')$.

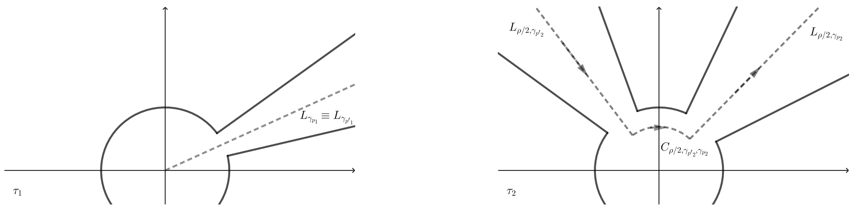

Figure 1: Path deformation in Case 1

From the fact that $z \in H_{\beta'}$, we get that $\{(p_1,p_2),(p_1',p_2')\}$ belong to $\mathcal{U}_{k_2}$.

Case 2: The path $L_{\gamma_{p_2}}$ coincides with $L_{\gamma_{p_2'}}$, and $L_{\gamma_{p_1}}$ does not coincide with $L_{\gamma_{p_1'}}$. It can be handled analogously as Case 1. We get that the set $\{(p_1,p_2),(p_1',p_2')\}$ belongs to $\mathcal{U}_{k_1}$. More precisely, we arrive at the expression

$$\begin{aligned}
&|u_{p_1,p_2}(\boldsymbol{t},z,\epsilon) - u_{p_1',p_2'}(\boldsymbol{t},z,\epsilon)| \\
&\qquad \leq \frac{k_1k_2}{(2\pi)^{1/2}} C_{p_1,p_1'} \left(\int_{-\infty}^{\infty} (1+|m|)^{-\mu} e^{-\beta|m|} e^{-m|\mathrm{Im}(z)|} dm \right) \\
&\qquad\qquad \times \int_{L_{\gamma_{p_2}}} \frac{|\frac{u_2}{\epsilon}|}{1+|\frac{u_2}{\epsilon}|^{2k_2}} \exp(\nu_2 |\frac{u_2}{\epsilon}|^{k_2}) \exp(-\left(\frac{u_2}{\epsilon t_2}\right)^{k_2}) \left|\frac{du_2}{u_2}\right| e^{-\frac{M_{p_1,p_1'}}{|\epsilon|^{k_1}}}.
\end{aligned}$$

Case 3: Assume that neither $L_{\gamma_{p_1}}$ coincides with $L_{\gamma_{p_1'}}$, nor $L_{\gamma_{p_2}}$ coincides with $L_{\gamma_{p_2'}}$.

We deform the integration paths with respect to the first time variable and write

$$u_{p_1,p_2}(\boldsymbol{t},z,\epsilon) - u_{p_1',p_2'}(\boldsymbol{t},z,\epsilon) = J_1 - J_2 + J_3,$$

where

$$J_1 = \frac{k_1k_2}{(2\pi)^{1/2}} \int_{L_{\gamma_{p_1},1}} \int_{L_{\gamma_{p_2}}} \int_{-\infty}^{\infty} \omega_{\boldsymbol{k}}^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(u_1,u_2,m,\epsilon) e^{-(\frac{u_1}{\epsilon t_1})^{k_1} - (\frac{u_2}{\epsilon t_2})^{k_2}} e^{izm} dm \frac{du_2}{u_2}\frac{du_1}{u_1}.$$

$$J_2 = \frac{k_1k_2}{(2\pi)^{1/2}} \int_{L_{\gamma_{p_1'},1}} \int_{L_{\gamma_{p_2'}}} \int_{-\infty}^{\infty} \omega_{\boldsymbol{k}}^{\mathfrak{d}_{p_1'},\tilde{\mathfrak{d}}_{p_2'}}(u_1,u_2,m,\epsilon) e^{-(\frac{u_1}{\epsilon t_1})^{k_1} - (\frac{u_2}{\epsilon t_2})^{k_2}} e^{izm} dm \frac{du_2}{u_2}\frac{du_1}{u_1}.$$

$$\begin{aligned}
J_3 = \frac{k_1k_2}{(2\pi)^{1/2}} \int_0^{\frac{\rho}{2}e^{i\theta}} &\left(\int_{-\infty}^{\infty} \left(\int_{L_{\gamma_{p_2}}} \omega_{\boldsymbol{k}}^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(u_1,u_2,m,\epsilon) e^{-(\frac{u_2}{\epsilon t_2})^{k_2}} \frac{du_2}{u_2} \right.\right. \\
&\left.\left. - \int_{L_{\gamma_{p_2'}}} \omega_{\boldsymbol{k}}^{\mathfrak{d}_{p_1'},\tilde{\mathfrak{d}}_{p_2'}}(u_1,u_2,m,\epsilon) e^{-(\frac{u_2}{\epsilon t_2})^{k_2}} \frac{du_2}{u_2} \right) e^{izm} dm \right) e^{-(\frac{u_1}{\epsilon t_1})^{k_1}} \frac{du_1}{u_1},
\end{aligned}$$

where $\frac{\rho}{2}e^{i\theta}$ is such that θ is an argument between γ_{p_1} and $\gamma_{p_1'}$. The path $L_{\gamma_{p_1},1}$ (resp. $L_{\gamma_{p_1'},1}$) consists of the concatenation of the arc of circle connecting $\frac{\rho}{2}e^{i\theta}$ with $\frac{\rho}{2}e^{i\gamma_{p_1}}$ (resp. with $\frac{\rho}{2}e^{i\gamma_{p_1'}}$) and the half line $[\frac{\rho}{2}e^{i\gamma_{p_1}},\infty)$ (resp. $[\frac{\rho}{2}e^{i\gamma_{p_1'}},\infty)$).

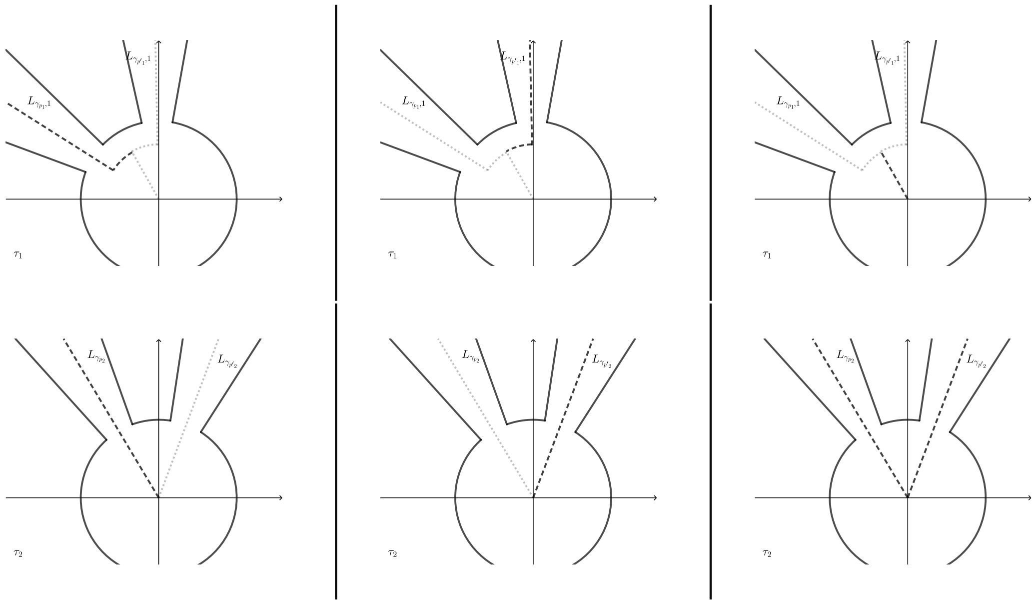

Figure 2: Path deformation in Case 3

We first give estimates for $|J_1|$. We have

$$\left| \int_{L_{\gamma_{p_2}}} \omega_{\boldsymbol{k}}^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(u_1,u_2,m,\epsilon) e^{-(\frac{u_2}{\epsilon t_2})^{k_2}} \frac{du_2}{u_2} \right| \leq \varpi_{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}} (1+|m|)^{-\mu} e^{-\beta|m|} \frac{|\frac{u_1}{\epsilon}|}{1+|\frac{u_1}{\epsilon}|^{2k_1}} \exp(\nu_1|\frac{u_1}{\epsilon}|^{k_1})$$
$$\times \int_{L_{\gamma_{p_2}}} \left(\frac{|\frac{u_2}{\epsilon}|}{1+|\frac{u_2}{\epsilon}|^{2k_2}} \exp(\nu_2 \left|\frac{u_2}{\epsilon}\right|^{k_2}) \right) |e^{-\left(\frac{u_2}{\epsilon t_2}\right)^{k_2}}| \left| \frac{du_2}{u_2} \right|$$
$$\leq \varpi_{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}} C_{p_2} (1+|m|)^{-\mu} e^{-\beta|m|} \frac{|\frac{u_1}{\epsilon}|}{1+|\frac{u_1}{\epsilon}|^{2k_1}} \exp(\nu_1|\frac{u_1}{\epsilon}|^{k_1}),$$

for some $C_{p_2} > 0$, and $t_2 \in \mathcal{T}_2 \cap D(0,h')$. Using the parametrization $u_2 = re^{i\gamma_{p_2}}$ and the change of variable $r = |\epsilon|s$. Using analogous estimations as in the Case 1, we arrive at

$$|J_1| \leq C_{p,1} e^{-\frac{M_{p,1}}{|\epsilon|^{k_1}}},$$

for some $C_{p,1}, M_{p,1} > 0$, for all $\epsilon \in \mathcal{E}_{p_1,p_2} \cap \mathcal{E}_{p_1',p_2'}$, where $t_1 \in \mathcal{T}_1 \cap D(0,h')$ and $t_2 \in \mathcal{T}_2 \cap D(0,h')$, $z \in H_{\beta'}$.

Analogous calculations yield to

$$|J_2| \leq C_{p,2} e^{-\frac{M_{p,2}}{|\epsilon|^{k_1}}},$$

for some $C_{p,2}, M_{p,2} > 0$, for all $\epsilon \in \mathcal{E}_{p_1,p_2} \cap \mathcal{E}_{p_1',p_2'}$, where $t_1 \in \mathcal{T}_1 \cap D(0,h')$ and $t_2 \in \mathcal{T}_2 \cap D(0,h')$, $z \in H_{\beta'}$.

In order to give upper bounds for $|J_3|$, we consider

$$\left| \int_{L_{\gamma_{p_2}}} \omega_{\boldsymbol{k}}^{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}}(u_1,u_2,m,\epsilon) e^{-(\frac{u_2}{\epsilon t_2})^{k_2}} \frac{du_2}{u_2} - \int_{L_{\gamma_{p_2'}}} \omega_{\boldsymbol{k}}^{\mathfrak{d}_{p_1'},\tilde{\mathfrak{d}}_{p_2'}}(u_1,u_2,m,\epsilon) e^{-(\frac{u_2}{\epsilon t_2})^{k_2}} \frac{du_2}{u_2} \right|.$$

We choose a deformation path in the form of that considered in Case 1. We get the previous expression is upper estimated by

$$\varpi_{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}} C_{p_2,p_2'}(1+|m|)^{-\mu} e^{-\beta|m|} \frac{|\frac{u_1}{\epsilon}|}{1+|\frac{u_1}{\epsilon}|^{2k_1}} \exp(\nu_1|\frac{u_1}{\epsilon}|^{k_1}) \exp\left(-\frac{M_{p_2,p_2'}}{|\epsilon|^{k_2}}\right),$$

for $\epsilon \in \mathcal{E}_{p_1,p_2} \cap \mathcal{E}_{p_1',p_2'}$, $t_2 \in \mathcal{T}_2 \cap D(0,h')$, $u_1 \in [0,\rho/2e^{i\theta}]$. We finally get

$$|J_3| \leq \frac{k_1k_2}{(2\pi)^{1/2}} C_{p_2,p_2'} \varpi_{\mathfrak{d}_{p_1},\tilde{\mathfrak{d}}_{p_2}} \left(\int_{-\infty}^{\infty} (1+|m|)^{-\mu} e^{-\beta|m|} e^{-m|\mathrm{Im}(z)|} dm\right) \times \left(\int_0^{\rho/2e^{i\theta}} \frac{|\frac{u_1}{\epsilon}|}{1+|\frac{u_1}{\epsilon}|^{2k_1}} \exp(\nu_1|\frac{u_1}{\epsilon}|^{k_1}) |e^{-\left(\frac{u_1}{\epsilon t_1}\right)^{k_1}}| \left|\frac{du_1}{u_1}\right|\right) \exp\left(-\frac{M_{p_2,p_2'}}{|\epsilon|^{k_2}}\right).$$

We conclude that

$$|J_3| \leq K_{p,3} e^{-\frac{M_{p,3}}{|\epsilon|^{k_2}}},$$

uniformly for $(t_1,t_2) \in (\mathcal{T}_1 \cap D(0,h'')) \times (\mathcal{T}_2 \cap D(0,h''))$ for some $h'' > 0$, and $z \in H_{\beta'}$ for any fixed $\beta' < \beta$, where $K_{\boldsymbol{p},3}, M_{\boldsymbol{p},3}$ are positive constants. □

5 Asymptotics of the problem in the perturbation parameter

5.1 $k-$Summable formal series and Ramis-Sibuya Theorem

For the sake of completeness, we recall the definition of $k-$Borel summability of formal series with coefficients in a Banach space, and Ramis-Sibuya Theorem. A reference for the details on the first part is [1], whilst the second part of this section can be found in [2], p. 121, and [5], Lemma XI-2-6.

Definition 5 *Let $k \geq 1$ be an integer. A formal series*

$$\hat{X}(\epsilon) = \sum_{j=0}^{\infty} \frac{a_j}{j!} \epsilon^j \in \mathbb{F}[[\epsilon]]$$

with coefficients in a Banach space $(\mathbb{F}, ||.||_{\mathbb{F}})$ is said to be $k-$summable with respect to ϵ in the direction $d \in \mathbb{R}$ if

i) *there exists $\rho \in \mathbb{R}_+$ such that the following formal series, called formal Borel transform of $\hat{X}$ of order k*

$$\mathcal{B}_k(\hat{X})(\tau) = \sum_{j=0}^{\infty} \frac{a_j \tau^j}{j!\Gamma(1+\frac{j}{k})} \in \mathbb{F}[[\tau]],$$

is absolutely convergent for $|\tau| < \rho$,

ii) *there exists $\delta > 0$ such that the series $\mathcal{B}_k(\hat{X})(\tau)$ can be analytically continued with respect to τ in a sector $S_{d,\delta} = \{\tau \in \mathbb{C}^* : |d - \arg(\tau)| < \delta\}$. Moreover, there exist $C > 0$, and $K > 0$ such that*

$$||\mathcal{B}(\hat{X})(\tau)||_{\mathbb{F}} \leq Ce^{K|\tau|^k}$$

for all $\tau \in S_{d,\delta}$.

If this is so, the vector valued Laplace transform of order k of $\mathcal{B}_k(\hat{X})(\tau)$ in the direction d is defined by

$$\mathcal{L}^d_k(\mathcal{B}_k(\hat{X}))(\epsilon) = \epsilon^{-k}\int_{L_\gamma} \mathcal{B}_k(\hat{X})(u)e^{-(u/\epsilon)^k}ku^{k-1}du,$$

along a half-line $L_\gamma = \mathbb{R}_+e^{i\gamma} \subset S_{d,\delta} \cup \{0\}$, where γ depends on ϵ and is chosen in such a way that $\cos(k(\gamma - \arg(\epsilon))) \geq \delta_1 > 0$, for some fixed δ_1, for all ϵ in a sector

$$S_{d,\theta,R^{1/k}} = \{\epsilon \in \mathbb{C}^* : |\epsilon| < R^{1/k} \quad , \quad |d - \arg(\epsilon)| < \theta/2\},$$

where $\frac{\pi}{k} < \theta < \frac{\pi}{k} + 2\delta$ and $0 < R < \delta_1/K$. The function $\mathcal{L}^d_k(\mathcal{B}_k(\hat{X}))(\epsilon)$ is called the $k-$sum of the formal series $\hat{X}(t)$ in the direction d. It is bounded and holomorphic on the sector $S_{d,\theta,R^{1/k}}$ and has the formal series $\hat{X}(\epsilon)$ as Gevrey asymptotic expansion of order $1/k$ with respect to ϵ on $S_{d,\theta,R^{1/k}}$. This means that for all $\frac{\pi}{k} < \theta_1 < \theta$, there exist $C, M > 0$ such that

$$||\mathcal{L}^d_k(\mathcal{B}_k(\hat{X}))(\epsilon) - \sum_{p=0}^{n-1}\frac{a_p}{p!}\epsilon^p||_{\mathbb{F}} \leq CM^n\Gamma(1+\frac{n}{k})|\epsilon|^n$$

for all $n \geq 1$, all $\epsilon \in S_{d,\theta_1,R^{1/k}}$.

Multisummability of a formal power series is a recursive process that allows to compute the sum of a formal power series in different Gevrey orders. One of the approaches to multisummability is that stated by W. Balser, which can be found in [1], Theorem 1, p.57. Roughly speaking, given a formal power series $\hat{f}$ which can be decomposed into a sum $\hat{f}(z) = \hat{f}_1(z) + \ldots + \hat{f}_m(z)$ such that each of the terms $\hat{f}_j(z)$ is k_j-summable, with sum given by f_j, then, $\hat{f}$ turns out to be multisummable, and its multisum is given by $f_1(z) + \ldots + f_m(z)$. More precisely, one has the following definition.

Definition 6 *Let $(\mathbb{F}, \|\cdot\|_{\mathbb{F}})$ be a complex Banach space and let $0 < k_1 < k_2$. Let $\mathcal{E}$ be a bounded open sector with vertex at 0, and opening $\frac{\pi}{k_2} + \delta_2$ for some $\delta_2 > 0$, and let $\mathcal{F}$ be a bounded open sector with vertex at the origin in $\mathbb{C}$, with opening $\frac{\pi}{k_1} + \delta_1$, for some $\delta_1 > 0$ and such that $\mathcal{E} \subseteq \mathcal{F}$ holds.*

A formal power series $\hat{f}(\epsilon) \in \mathbb{F}[[\epsilon]]$ is said to be $(k_2, k_1)-$summable on $\mathcal{E}$ if there exist $\hat{f}_2(\epsilon) \in \mathbb{F}[[\epsilon]]$ which is k_2-summable on $\mathcal{E}$, with k_2-sum given by $f_2 : \mathcal{E} \to \mathbb{F}$, and $\hat{f}_1(\epsilon) \in \mathbb{F}[[\epsilon]]$ which is k_1-summable on $\mathbb{E}$, with k_1-sum given by $f_1 : \mathcal{F} \to \mathbb{F}$, such that $\hat{f} = \hat{f}_1 + \hat{f}_2$. Furthermore, the holomorphic function $f(\epsilon) = f_1(\epsilon) + f_2(\epsilon)$ on $\mathcal{E}$ is called the $(k_2, k_1)-$sum of $\hat{f}$ on $\mathcal{E}$. In that situation, $f(\epsilon)$ can be obtained from the analytic continuation of the k_1-Borel transform of $\hat{f}$ by the successive application of accelerator operators and Laplace transform of order k_2, see Section 6.1 in [1].

A novel version of Ramis-Sibuya Theorem has been developed in [13], and has provided successful results in previous works by the authors, [7], [8]. A version of the result in two different levels which fits our needs is now given without proof, which can be found in [7], [8].

Theorem (multilevel-RS) *Let $(\mathbb{F}, ||.||_{\mathbb{F}})$ be a Banach space over $\mathbb{C}$ and $\{\mathcal{E}_{p_1,p_2}\}_{\substack{0\leq p_1\leq \varsigma_1-1\\ 0\leq p_2\leq\varsigma_2-1}}$ be a good covering in $\mathbb{C}^*$. Assume that $0 < k_1 < k_2$. For all $0 \leq p_1 \leq \varsigma_1 - 1$ and $0 \leq p_2 \leq \varsigma_2 - 1$, let G_{p_1,p_2} be a holomorphic function from $\mathcal{E}_{p_1,p_2}$ into the Banach space $(\mathbb{F}, ||.||_{\mathbb{F}})$ and for every $(p_1, p_2), (p_1', p_2') \in \{0, \ldots, \varsigma_1 - 1\} \times \{0, \ldots, \varsigma_2 - 1\}$ such that $\mathcal{E}_{p_1,p_2} \cap \mathcal{E}_{p_1',p_2'} \neq \emptyset$ we define $\Theta_{(p_1,p_2)(p_1',p_2')}(\epsilon) = G_{p_1,p_2}(\epsilon) - G_{p_1',p_2'}(\epsilon)$ be a holomorphic function from the sector $Z_{(p_1,p_2),(p_1',p_2')} = \mathcal{E}_{p_1,p_2} \cap \mathcal{E}_{p_1',p_2'}$ into $\mathbb{F}$. We make the following assumptions.*

1) *The functions $G_{p_1,p_2}(\epsilon)$ are bounded as $\epsilon \in \mathcal{E}_{p_1,p_2}$ tends to the origin in $\mathbb{C}$, for all $0 \leq p_1 \leq \varsigma_1 - 1$ and $0 \leq p_2 \leq \varsigma_2 - 1$.*

2) $(\{0,\ldots,\varsigma_1-1\}\times\{0,\ldots,\varsigma_2-1\})^2 = \mathcal{U}_0 \cup \mathcal{U}_{k_1} \cup \mathcal{U}_{k_2}$, *where*
$((p_1,p_2),(p'_1,p'_2)) \in \mathcal{U}_0$ *iff* $\mathcal{E}_{p_1,p_2} \cap \mathcal{E}_{p'_1,p'_2} = \emptyset$,
$((p_1,p_2),(p'_1,p'_2)) \in \mathcal{U}_{k_1}$ *iff* $\mathcal{E}_{p_1,p_2} \cap \mathcal{E}_{p'_1,p'_2} \neq \emptyset$ *and*

$$||\Theta_{(p_1,p_2),(p'_1,p'_2)}(\epsilon)||_{\mathbb{F}} \leq C_{p_1,p_2,p'_1,p'_2} e^{-A_{p_1,p_2,p'_1,p'_2}/|\epsilon|^{k_1}}$$

for all $\epsilon \in Z_{(p_1,p_2),(p'_1,p'_2)}$.
$((p_1,p_2),(p'_1,p'_2)) \in \mathcal{U}_{k_2}$ *iff* $\mathcal{E}_{p_1,p_2} \cap \mathcal{E}_{p'_1,p'_2} \neq \emptyset$ *and*

$$||\Theta_{(p_1,p_2),(p'_1,p'_2)}(\epsilon)||_{\mathbb{F}} \leq C_{p_1,p_2,p'_1,p'_2} e^{-A_{p_1,p_2,p'_1,p'_2}/|\epsilon|^{k_2}}$$

for all $\epsilon \in Z_{(p_1,p_2),(p'_1,p'_2)}$.

Then, there exists a convergent power series $a(\epsilon) \in \mathbb{F}\{\epsilon\}$ and two formal power series $\hat{G}^1(\epsilon), \hat{G}^2(\epsilon) \in \mathbb{F}[[\epsilon]]$ such that $G_{p_1,p_2}(\epsilon)$ can be split in the form

$$G_{p_1,p_2}(\epsilon) = a(\epsilon) + G^1_{p_1,p_2}(\epsilon) + G^2_{p_1,p_2}(\epsilon),$$

where $G^j_{p_1,p_2}(\epsilon) \in \mathcal{O}(\mathcal{E}_{p_1,p_2},\mathbb{F})$, and admits $\hat{G}^j(\epsilon)$ as its asymptotic expansion of Gevrey order $1/k_j$ on $\mathcal{E}_{p_1,p_2}$, for $j = 1,2$.

Moreover, assume that

$$\{((p_1^0,p_2^0),(p_1^1,p_2^1)),((p_1^1,p_2^1),(p_1^2,p_2^2)),\ldots,((p_1^{2y-1},p_2^{2y-1}),(p_1^{2y},p_2^{2y}))\}$$

is a subset of $\mathcal{U}_{k_2}$, for some positive integer y, and

$$\mathcal{E}_{p_1^y,p_2^y} \subseteq S_{\pi/k_1} \subseteq \bigcup_{0\leq j\leq 2y} \mathcal{E}_{p_1^j,p_2^j},$$

for some sector S_{π/k_1} with opening larger than π/k_1. Then, the formal power series $\hat{G}(\epsilon)$ is $(k_2,k_1)-$summable on $\mathcal{E}_{p_1^y,p_2^y}$ and its $(k_2,k_1)-$sum is $G_{p_1^y,p_2^y}(\epsilon)$ on $\mathcal{E}_{p_1^y,p_2^y}$.

5.2 Existence of formal power series solutions in the complex parameter and asymptotic behavior

The second main result of our work states the existence of a formal power series in the perturbation parameter ϵ, with coefficients in the Banach space $\mathbb{F}$ of holomorphic and bounded functions on $(\mathcal{T}_1 \cap D(0,h'')) \times (\mathcal{T}_2 \cap D(0,h'')) \times H_{\beta'}$, with the norm of the supremum. Here h'', $\mathcal{T}_1, \mathcal{T}_2$ are determined in Theorem 1.

The importance of this result compared to the main one in [6] lies on the fact that a multisummability phenomenon can be observed here, in contrast to [6]. This situation is attained due to the appearance of different Gevrey levels coming from the different variables in time.

Theorem 2 *Under the assumptions of Theorem 1, a formal power series*

$$\hat{u}(\boldsymbol{t},z,\epsilon) = \sum_{m\geq 0} H_m(\boldsymbol{t},z)\epsilon^m/m! \in \mathbb{F}[[\epsilon]]$$

exists, with the following properties. $\hat{u}$ is a formal solution of (9). In addition to that, $\hat{u}$ can be split in the form

$$\hat{u}(\boldsymbol{t},z,\epsilon) = a(\boldsymbol{t},z,\epsilon) + \hat{u}_1(\boldsymbol{t},z,\epsilon) + \hat{u}_2(\boldsymbol{t},z,\epsilon),$$

where $a(\boldsymbol{t},z,\epsilon) \in \mathbb{F}\{\epsilon\}$, and $\hat{u}_1, \hat{u}_2 \in \mathbb{F}[[\epsilon]]$. Moreover, for every $p_1 = 0,\dots,\varsigma_1 - 1$ and $p_2 = 0,\dots,\varsigma_2 - 1$, the function $u_{p_1,p_2}(\boldsymbol{t},z,\epsilon)$ can be written as

$$u_{p_1,p_2}(\boldsymbol{t},z,\epsilon) = a(\boldsymbol{t},z,\epsilon) + u^1_{p_1,p_2}(\boldsymbol{t},z,\epsilon) + u^2_{p_1,p_2}(\boldsymbol{t},z,\epsilon),$$

where $\epsilon \mapsto u^j_{p_1,p_2}(\boldsymbol{t},z,\epsilon)$ is an $\mathbb{F}-$valued function which admits $\hat{u}_j(\boldsymbol{t},z,\epsilon)$ as its k_j-Gevrey asymptotic expansion on $\mathcal{E}_{p_1,p_2}$, for $j = 1,2$.

Moreover, assume that

$$\{((p_1^0,p_2^0),(p_1^1,p_2^1)),((p_1^1,p_2^1),(p_1^2,p_2^2)),\dots,((p_1^{2y-1},p_2^{2y-1}),(p_1^{2y},p_2^{2y}))\}$$

is a subset of $\mathcal{U}_{k_2}$, for some positive integer y, and

$$\mathcal{E}_{p_1^y,p_2^y} \subseteq S_{\pi/k_1} \subseteq \bigcup_{0\le j\le 2y} \mathcal{E}_{p_1^j,p_2^j},$$

for some sector S_{π/k_1} with opening larger than π/k_1. Then, $\hat{u}(\boldsymbol{t},z,\epsilon)$ is $(k_2,k_1)-$summable on $\mathcal{E}_{p_1^y,p_2^y}$ and its $(k_2,k_1)-$sum is $u_{p_1^y,p_2^y}(\epsilon)$ on $\mathcal{E}_{p_1^y,p_2^y}$.

Proof Let $(u_{p_1,p_2}(\boldsymbol{t},z,\epsilon))_{\substack{0\le p_1\le\varsigma_1-1\\0\le p_2\le\varsigma_2-1}}$ be the family constructed in Theorem 1. We recall that $(\mathcal{E}_{p_1,p_2})_{\substack{0\le p_1\le\varsigma_1-1\\0\le p_2\le\varsigma_2-1}}$ is a good covering in $\mathbb{C}^\star$.

The function $G_{p_1,p_2}(\epsilon) := (t_1,t_2,z) \mapsto u_{p_1,p_2}(t_1,t_2,z,\epsilon)$ belongs to $\mathcal{O}(\mathcal{E}_{p_1,p_2},\mathbb{F})$. We consider $\{(p_1,p_2),(p_1',p_2')\}$ such that (p_1,p_2) and (p_1',p_2') belong to $\{0,\dots,\varsigma_1-1\}\times\{0,\dots,\varsigma_2-1\}$, and $\mathcal{E}_{p_1,p_2}$ and $\mathcal{E}_{p_1',p_2'}$ are consecutive sectors in the good covering, so their intersection is not empty. In view of (54) and (55), one has that $\Delta_{(p_1,p_2),(p_1',p_2')}(\epsilon) := G_{p_1,p_2}(\epsilon) - G_{p_1',p_2'}(\epsilon)$ satisfies exponentially flat bounds of certain Gevrey order, which is k_1 in the case that $\{(p_1,p_2),(p_1',p_2')\} \in \mathcal{U}_{k_1}$ and k_2 if $\{(p_1,p_2),(p_1',p_2')\} \in \mathcal{U}_{k_2}$. Multilevel-RS Theorem guarantees the existence of formal power series $\hat{G}(\epsilon), \hat{G}_1(\epsilon), \hat{G}_2(\epsilon) \in \mathbb{F}[[\epsilon]]$ such that

$$\hat{G}(\epsilon) = a(\epsilon) + \hat{G}_1(\epsilon) + \hat{G}_2(\epsilon),$$

and the splitting

$$G_{p_1,p_2}(\epsilon) = a(\epsilon) + G^1_{p_1,p_2}(\epsilon) + G^2_{p_1,p_2}(\epsilon),$$

for some $a \in \mathbb{F}\{\epsilon\}$, such that for every $(p_1,p_2) \in \{0,\dots,\varsigma_1-1\}\times\{0,\dots,\varsigma_2-1\}$, one has that $G^1_{p_1,p_2}(\epsilon)$ admits $\hat{G}^1_{p_1,p_2}(\epsilon)$ as its Gevrey asymptotic expansion of order k_1, and $G^2_{p_1,p_2}(\epsilon)$ admits $\hat{G}^2_{p_1,p_2}(\epsilon)$ as its Gevrey asymptotic expansion of order k_2. We define

$$\hat{G}(\epsilon) =: \hat{u}(\boldsymbol{t},z,\epsilon) = \sum_{m\ge 0} H_m(\boldsymbol{t},z)\frac{\epsilon^m}{m!}.$$

It only rests to prove that $\hat{u}(\boldsymbol{t},z,\epsilon)$ is a formal solution of (9). For every $0 \le p_1 \le \varsigma_1 - 1$, $0 \le p_2 \le \varsigma_2 - 1$ and $j = 1,2$, the existence of an asymptotic expansion concerning $G^j_{p_1,p_2}(\epsilon)$ and $\hat{G}^j(\epsilon)$ implies that

$$\lim_{\epsilon\to 0,\epsilon\in\mathcal{E}_{p_1,p_2}} \sup_{(\boldsymbol{t},z)\in(\tau_1\cap D(0,h''))\times(\tau_2\cap D(0,h''))\times H_{\beta'}} |\partial_\epsilon^\ell u_{p_1,p_2}(\boldsymbol{t},z,\epsilon) - H_\ell(\boldsymbol{t})| = 0, \quad (61)$$

for every $\ell \in \mathbb{N}$. By construction, the function $u_{p_1,p_2}(\boldsymbol{t}, z, \epsilon)$ is a solution of (9). Taking derivatives of order $m \geq 0$ with respect to ϵ on that equation yield

$$
\begin{aligned}
(62)\quad & Q_1(\partial_z)Q_2(\partial_z)\partial_{t_1}\partial_{t_2}\partial_\epsilon^m u_{p_1,p_2}(\boldsymbol{t}, z, \epsilon) \\
&= \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \left(\sum_{m_{11}+m_{12}=m_1} \frac{m_1!}{m_{11}!m_{12}!} \partial_\epsilon^{m_{11}} P_1(\partial_z, \epsilon) \partial_\epsilon^{m_{12}} u_{p_1,p_2}(\boldsymbol{t}, z, \epsilon) \right) \\
&\quad \times \left(\sum_{m_{21}+m_{22}=m_2} \frac{m_2!}{m_{21}!m_{22}!} \partial_\epsilon^{m_{21}} P_2(\partial_z, \epsilon) \partial_\epsilon^{m_{22}} u_{p_1,p_2}(\boldsymbol{t}, z, \epsilon) \right) \\
&+ \sum_{0 \leq l_1 \leq D_1, 0 \leq l_2 \leq D_2} \left(\sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \partial_\epsilon^{m_1} (\epsilon^{\Delta_{l_1,l_2}}) t_1^{d_{l_1}} \partial_{t_1}^{\delta_{l_1}} t_2^{\tilde{d}_{l_2}} \partial_{t_2}^{\tilde{\delta}_{l_2}} R_{l_1,l_2}(\partial_z) \partial_\epsilon^{m_2} u_{p_1,p_2}(\boldsymbol{t}, z, \epsilon) \right) \\
&\quad + \partial_\epsilon^m f(\boldsymbol{t}, z, \epsilon),
\end{aligned}
$$

for every $m \geq 0$ and $(\boldsymbol{t}, z, \epsilon) \in (\mathcal{T}_1 \cap D(0, h'')) \times (\mathcal{T}_2 \cap D(0, h'')) \times H_{\beta'} \times \mathcal{E}_{p_1,p_2}$. Tending $\epsilon \to 0$ in (62) together with (61), we obtain a recursion formula for the coefficients of the formal solution.

$$
\begin{aligned}
(63)\quad & Q_1(\partial_z)Q_2(\partial_z)\partial_{t_1}\partial_{t_2} H_m(\boldsymbol{t}, z) \\
&= \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \left(\sum_{m_{11}+m_{12}=m_1} \frac{m_1!}{m_{11}!m_{12}!} \partial_\epsilon^{m_{11}} P_1(\partial_z, 0) H_{m_{12}}(\boldsymbol{t}, z) \right) \\
&\quad \times \left(\sum_{m_{21}+m_{22}=m_2} \frac{m_2!}{m_{21}!m_{22}!} \partial_\epsilon^{m_{21}} P_2(\partial_z, 0) H_{m_{12}}(\boldsymbol{t}, z) \right) \\
&+ \sum_{0 \leq l_1 \leq D_1, 0 \leq l_2 \leq D_2} \frac{m!}{(m - \Delta_{l_1,l_2})!} t_1^{d_{l_1}} \partial_{t_1}^{\delta_{l_1}} t_2^{\tilde{d}_{l_2}} \partial_{t_2}^{\tilde{\delta}_{l_2}} R_{l_1,l_2}(\partial_z) H_{m-\Delta_{l_1,l_2}}(\boldsymbol{t}, z) \\
&\quad + \partial_\epsilon^m f(\boldsymbol{t}, z, 0),
\end{aligned}
$$

for every $m \geq \max_{1 \leq l_1 \leq D_1, 1 \leq l_2 \leq D_2} \Delta_{l_1,l_2}$, and $(\boldsymbol{t}, z, \epsilon) \in (\mathcal{T}_1 \cap D(0, h'')) \times (\mathcal{T}_2 \cap D(0, h'')) \times H_{\beta'}$. From the analyticity of f with respect to ϵ in a vicinity of the origin we get

$$
f(\boldsymbol{t}, z, \epsilon) = \sum_{m \geq 0} \frac{(\partial_\epsilon^m f)(\boldsymbol{t}, z, 0)}{m!} \epsilon^m, \tag{64}
$$

for every $\epsilon \in D(0, \epsilon_0)$ and $(\boldsymbol{t}, z)$ as above. On the other hand, a direct inspection from the recursion formula (63) and (64) allow us to affirm that the formal power series $\hat{u}(\boldsymbol{t}, z, \epsilon) = \sum_{m \geq 0} H_m(\boldsymbol{t}, z)\epsilon^m/m!$ solves the equation (9). □

Aknowledgements: The authors want to express their gratitude to the anonymous referee for his/her careful reading of our work and the helpful comments and suggestions that helped to improve the quality of the manuscript.

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