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Numerical Polynomial Reparametrization of Rational Curves

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Abstract

Given a real rational parametrization $\mathcal{P}(t)$ of a plane curve $\mathcal{C}$, we present an algorithm to compute polynomial curves to approximate $\mathcal{C}$ for the whole parameter domain. In this case, the denominators often have real roots in the whole interval. We decompose the interval as the union of finitely many intervals according to the real roots of the denominators. The key technique of the paper is to approximate the given curve by their asymptotes and error analysis at each interval is also presented. The asymptotes are associated with the infinity points corresponding to the real roots of the denominators. Numeric algorithms and examples are proposed to illustrate our results.

Keywords: Rational curves; Polynomial reparametrization; Asymptote; Error analysis

1. Introduction

Rational polynomial algebraic plane curves are defined by polynomial parametrizations. The non–existence of denominators avoids the possibly unstable behaviour of the parametrization when the parameter takes values close to the roots of the denominators, and the analysis of the differential and integral computations is much simpler. These curves play an important role in many applications, in particular in computer–aided design (CAD) and robotics (see Timar et al. (2005); Yang et al. (2015); Lin et al. (2019)).

Polynomial curves are characterized as those rational plane curves having only one place at infinity associated to the parameter (see Abhyankar (1990)). Thus, in general, one cannot deal globally with the problem of computing a polynomial parametrization of a given rational plane curve (i.e. curves having genus zero; see e.g. Sendra et al. (2007)). Nevertheless, we may try to approximate the given rational plane curve by some polynomial parametrizations. More precisely, let $\mathcal{P}(t) = (p_1(t), p_2(t)) \in \mathbb{R}(s)^2, p_i(t) = p_{i1}(t)/p_{i2}(t)$, $\deg(p_{i1}) = \deg(p_{i2}) = s_i$, $i = 1, 2$, be a real rational parametrization of a plane curve $\mathcal{C}$. The problem consists in:

(i) compute a rational plane curve $\overline{\mathcal{C}}$ defined by a rational parametrization

$$\overline{\mathcal{P}}(t) = \left(\frac{p_{11}(t)}{\overline{p}_{12}(t)}, \frac{p_{21}(t)}{\overline{p}_{22}(t)}\right) \in \mathbb{R}(t)^2, \qquad \overline{p}_{i2}(t) = (t-\alpha)^{s_i},\ i = 1, 2, \qquad \alpha \in \mathbb{R}$$

for $t \in \mathcal{I} := (-\infty, a) \cup (b, \infty)$, and $a, b \in \mathbb{R}$. In this case, we may compute a polynomial parametrization of $\overline{\mathcal{C}}$ by taking the reparametrization $\overline{\mathcal{P}}(1/t + \alpha)$ (see Section 6.2 in Sendra et al. (2007)).

(ii) decompose the interval $I = (a, b)$ as a union of finitely many intervals, $I_\xi = (\gamma, \beta)$, where $p_{j2}(\xi) = 0$, for $j = 1$ or $j = 2$, and $\gamma < \xi < \beta$. For each interval I_ξ of the partition, consider the asymptote $x_i = p_i(\xi)$, $i \neq j$, $i = 1$ or $i = 2$,

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(iii) analyze the error analysis and check whether the "curve pieces" defined by the input rational curve and the output polynomial curves are "close" (we refer to Pérez-Díaz et al. (2004) and Sonia Pérez-Díaz (2005) for the notion of closeness).

There exist methods to deal with the curve approximation problem. For instance, one may apply to both rational components of $\mathcal{P}(t)$ the well–known Approximation Theorem of Weierstrass in combination with Bernstein–polynomials (see e.g. H?mmerlin and Hoffman (1991)). In Sederberg and Kakimoto (1991), the authors present a Bézier–like approach, based on the hybrid polynomials. In Shen et al. (2012), the authors approximate an arbitrary parametric curve by a B-spline curve with certified error. Also, in Pérez-Díaz et al. (2007), authors present an approach based on polynomial sequences uniformly converging to the rational functions. For a given natural number N satisfying certain minimal requirements, the algorithm computes polynomial parametrizations which degrees are bounded by N.

These methods work well to approximate a curve segment in a finite interval where the curve segment does not have infinity points. However, rational curves obtained from some float computing circumstances, such as CAD and CNC (Computer Numerical Control), may have infinity points corresponding to the roots of their denominators. Then a problem is to approximate these rational curves for the whole parameter domain, especially keeping the important geometric features such as asymptotes associated to the infinity points. The problem usually comes from practical computations and few existing methods can deal straightforwardly with this problem.

In this paper, we present an easy approach to approximate a given rational parametrizations by a polynomial one. As an important property of our method is that, when the parameter takes values close to the roots of the denominators, our algorithm generates polynomial parametrizations that in fact are the asymptotes of the input rational plane curve. We present an error analysis and give an explicit priori bound of the closeness of the input and the output curves. Throughout this paper, we consider input plane curves but all the results and the algorithm can be easily generalized for space curves.

The structure of this paper is reproduced below. In Section 2, we present some preliminaries and in particular, we show how to decide whether a given affine rational parametrization can be reparametrized into a polynomial parametrization by using symbolically algorithms. In Section 3, we deal with the problem from the numerical point of view. In addition, we introduce the error analysis and we illustrate the method with some examples. Finally, in Section 4, we conclude our paper.

2. Previous Results

In this section, we first analyze, from the symbolic point of view, whether a given plane curve admits a polynomial parametrization, i.e. a rational affine parametrization where all components are polynomial. We follow the reasoning scheme in Sendra et al. (2007).

Definition 1. *A rational affine parametrization $\mathcal{P}(t)$ of a rational affine curve $\mathcal{C}$ is called a* polynomial parametrization *if all its components are polynomial. Furthermore, the affine curve $\mathcal{C}$ is called a* polynomial curve *if it is rational and can be parametrized by means of a polynomial parametrization.*

In Definition 1, we have introduced the notion of a polynomial affine curve but not imposed on the polynomial parametrization the condition of being proper (i.e. invertible). It is proved that properness can always be achieved simultaneously with polynomiality. More precisely, for every polynomial curve there exist proper polynomial parametrizations. Thus, in the following we assume the given rational curve is defined by a proper parametrization. Otherwise, one can compute a proper reparametrization (for this purpose, one may apply for instance the results in Pérez-Díaz (2006) or Sendra et al. (2007)).

In Theorem 1, polynomial curves are characterized and it is shown how to compute a proper polynomial parametrization (see Section 6.2 in Sendra et al. (2007)).

Theorem 1. *If $\mathcal{C}$ is a polynomial curve, then every proper non–polynomial rational parametrization in reduced form of $\mathcal{C}$ is of the type $\mathcal{P}(t) = \left(\frac{p_{11}(t)}{(bt-a)^r}, \frac{p_{21}(t)}{(bt-a)^s}\right)$, where $\deg(p_{11}) \leq r$ and $\deg(p_{21}) \leq s$, and $b \neq 0$.*

Furthermore, if $\mathcal{P}(t) = \left(\frac{p_{11}(t)}{(bt-a)^r}, \frac{p_{21}(t)}{(bt-a)^s}\right)$, *where* $\deg(p_{11}) \leq r$ *and* $\deg(p_{21}) \leq s$ *is a rational parametrization of an affine curve* $\mathcal{C}$*, then* $\mathcal{C}$ *is polynomial and can be polynomially parametrized as* $\mathcal{P}\left(a/b + 1/t\right)$.

In the following, we review a method for computing all the *generalized asymptotes* of a real plane algebraic curve $\mathcal{C}$ parametrically defined. The algorithm is based on the concepts and results presented in Blasco and Pérez-Díaz (2014a), Blasco and Pérez-Díaz (2014b) and Blasco and Pérez-Díaz (2015), and it can be easily generalized for space curves.

The notion of infinity branches which, intuitively speaking, reflect the status of a curve at the points with sufficiently large coordinates (for more details on this notion see Blasco and Pérez-Díaz (2014b)). The asymptotes of some branch, B, of a real plane algebraic curve $\mathcal{C}$ reflect the status of this branch at the points with sufficiently large coordinates. In analytic geometry, an asymptote of a curve is a line such that the distance between the curve and the line converges to zero as they tend to infinity. In some contexts, such as algebraic geometry, an asymptote is defined as a line which is tangent to a curve at infinity.

If B can be defined by some explicit equation of the form $x_2 = f(x_1)$ (or $x_1 = g(x_2)$), where f (or g) is a continuous function on an infinite interval, it is easy to decide whether $\mathcal{C}$ has an asymptote at B by analyzing the existence of the limits of certain functions when $x_1 \to \infty$ (or $x_2 \to \infty$). Moreover, if these limits can be computed, we may obtain the equation of the asymptote of $\mathcal{C}$ at B. However, if this branch B is implicitly defined and its equation cannot be converted into an explicit form, both the decision and the computation of the asymptote of $\mathcal{C}$ at B require some other tools.

An algebraic plane curve may have more general curves than lines describing the status of a branch at the points with sufficiently large coordinates. This motivates the notion of *generalized asymptotes*. We say that a curve $\widetilde{\mathcal{C}}$ is a *generalized asymptote* of another curve $\mathcal{C}$ if the distance between $\widetilde{\mathcal{C}}$ and $\mathcal{C}$ tends to zero as they tend to infinity, and $\mathcal{C}$ cannot be approximated by a new curve of lower degree.

In the following, we present an algorithm that computes the infinity branches of a given parametric curve and provides an asymptote for each of them. We assume that we have prepared the input curve $\mathcal{C}$, such that by means of a suitable linear change of coordinates, $(0:1:0)$ is not an infinity point of $\mathcal{C}$. For more details on this method and the concepts and results related with generalized asymptotes, we refer to Blasco and Pérez-Díaz (2014a), Blasco and Pérez-Díaz (2014b) and Blasco and Pérez-Díaz (2015).

Algorithm Asymptotes Construction-Parametric Case.

Given a rational algebraic curve $\mathcal{C}$ defined by a parametrization $\mathcal{P}(s) = (p_1(s), p_2(s)) \in \mathbb{C}(s)^2, p_j(s) = p_{j1}(s)/p(s)$, $j = 1, 2$, the algorithm outputs one asymptote for each of its infinity branches.
Step 1: Compute the Puiseux solutions of $p(s) - tp_{11}(s) = 0$ around $s = 0$. Let them be $\ell_1(t), \ell_2(t), \ldots, \ell_k(t) \in \mathbb{C} \ll t \gg$, where $\mathbb{C} \ll t \gg$ is the field of *formal Puiseux series*.
Step 2: For each $\ell_i(t) \in \mathbb{C} \ll t \gg$, $i = 1, \ldots, k$, do:

Step 2.1: Compute the corresponding infinity branch of $\mathcal{C}$, $B_i = \{(x_3, r_i(x_3)) \in \mathbb{C}^2 : x_3 \in \mathbb{C}, |x_3| > M_i\}$, where $r_i(x_3) = p_2(\ell_i(x_3^{-1}))$ is given as Puiseux series.

Step 2.2: Consider the series $\tilde{r}_i(x_3)$ obtained by eliminating the terms with negative exponent in $r_i(x_3)$. Let $\tilde{r}_i(x_3) = m_i x_3 + a_{1,i} x_3^{-n_{1,i}/n_i+1} + \cdots + a_{k_i,i} x_3^{-n_{k_i,i}/n_i+1}$.

Step 2.3: Return the asymptote $\widetilde{\mathcal{C}}_i$ defined by the proper parametrization $\widetilde{Q}_i(t) = (t^{n_i}, \tilde{r}_i(t^{n_i})) \in \mathbb{C}[t]^2$.

In the following example, we study a parametric plane curve with two infinity branches. We use algorithm *Asymptotes Construction-Parametric Case* to obtain the corresponding asymptote for each branch.

Example 1. *Let* $\mathcal{C}$ *be the plane curve defined by* $\mathcal{P}(s) = \left(\frac{p_{11}(s)}{p(s)}, \frac{p_{21}(s)}{p(s)}\right) = \left(\frac{s^2-1+s^3}{(s-1)s^3}, \frac{2s^2+1}{(s-1)s^3}\right) \in \mathbb{R}(s)^2$.
Step 1: *We compute the solutions of the equation* $p(s) - tp_{11}(s) = 0$ *around* $s = 0$*. For this purpose, we may use, for instance, the command* puiseux *included in the package* algcurves *of the computer algebra system*

Maple. *There are two solutions given by the Puiseux series:* $\ell_1(t) = 1 + 4t^5 - 13t^4 - t^3 + 2t^2 + t + \cdots$, *and* $\ell_2(t) = -4027/6561t^{7/3} - 2/3t^2 - 134/243t^{5/3} - 28/81t^{4/3} + 1/3t^{2/3} + t^{1/3} + \cdots$.

Step 2: *We apply Steps 2.1, 2.2 and 2.3 of the algorithm:*

> Step 2.1: *We compute* $r_1(x_3) = p_2(\ell_1(x_3^{-1})) = 3x_3 - 106x_3^{-4} - 120x_3^{-3} + 7x_3^{-2} + 23x_3^{-1} - 11 + \cdots$, *and* $r_2(x_3) = p_2(\ell_2(x_3^{-1})) = -3 - 3x_3^{1/3}(-1)^{2/3} - x_3 - 13/3x_3^{-1/3}(-1)^{1/3} - 23/3x_3^{-1} + \cdots$ *(we may use, for instance, the command* series *included in the computer algebra software* Maple*). The curve has two infinity branches given by* $B_i = \{(x_3, r_i(x_3)) \in \mathbb{C}^2 : x_3 \in \mathbb{C}, |x_3| > M_i\}$, $i = 1, 2$, *for some* $M_i \in \mathbb{R}^+$.
>
> Step 2.2: *We obtain* $\tilde{r}_1(x_3)$ *and* $\tilde{r}_2(x_3)$ *by eliminating the terms with negative exponent in* $r_1(x_3)$ *and* $r_2(x_3)$ *respectively:* $\tilde{r}_1(x_3) = -11 + 3x_3$ *and* $\tilde{r}_2(x_3) = -3 - 3x_3^{1/3}(-1)^{2/3} - x_3$.
>
> Step 2.3: *The input curve* $\mathcal{C}$ *has two asymptotes* $\widetilde{\mathcal{C}}_i$ *at* B_i *that can be polynomially parametrized by:* $\widetilde{Q}_1(t) = (t, -11 + 3t) \in \mathbb{R}[t]^2$, *and* $\widetilde{Q}_2(t) = (t^3, -3 - t^3 - 3t) \in \mathbb{R}[t]^2$.

In Figure 1, we plot the curve $\mathcal{C}$, *and the asymptotes* $\widetilde{\mathcal{C}}_1$ *and* $\widetilde{\mathcal{C}}_2$.

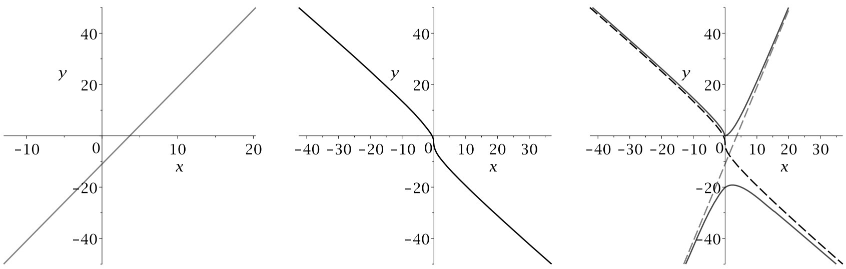

Figure 1: Asymptote $\widetilde{\mathcal{C}}_1$ (left), asymptote $\widetilde{\mathcal{C}}_2$ (center), and both asymptotes and the curve $\mathcal{C}$ (right)

3. Numerical Polynomial Reparametrization

In this section, for a rational plane curve $\mathcal{C}$ defined by a rational parametrization $\mathcal{P}(t)$, according the roots of the denominators of $\mathcal{P}(t)$, we decompose $\mathbb{R}$ as union of finitely many intervals I_j, $j = 0, \ldots, s$, and we approximate each curve piece $\mathcal{C}_{I_j} = \{\mathcal{P}(t), t \in I_j\}$ by a polynomial curve piece $\overline{\mathcal{C}}_{I_j} = \{\overline{\mathcal{P}}(t), t \in I_j\}$. We remind that polynomial curves are characterized as those rational plane curves having only one place at infinity (see Abhyankar (1990), and Theorem 1 in Section 2). Thus, if one considers the problem of computing "globally" a polynomial parametrization of a given rational plane curve, one may have to approximate the given rational plane curve by some polynomial parametrizations.

The error analysis proves that each curve piece $\mathcal{C}_I = \{\mathcal{P}(t), t \in I\}$ is in the "*vicinity*" of an output polynomial curve piece $\overline{\mathcal{C}}_I = \{\overline{\mathcal{P}}(t), t \in I\}$, and reciprocally. The notion of vicinity may be introduced as the offset region limited by the external and internal offset to $\mathcal{C}$ at certain distance (see Pérez-Díaz et al. (2004) for more details, and Arrondo et al. (1997) for basic concept on offsets), and therefore the problem consists in finding, for each interval I, a rational polynomial curve piece $\overline{\mathcal{C}}_I$ lying within the offset region of $\mathcal{C}_I$, and reciprocally. For this purpose, we study whether for almost every point P on the original curve piece, there exists a point Q on the output curve piece such that the Euclidean distance of P and Q is small. From this fact, and using Farouki and Rajan (1988), one may derive upper bounds for the distance of the offset region.

For instance, consider a plane curve $\mathcal{C}$ defined by the parametrization $\mathcal{P}(t) = (1/t^2, 1/(t - 0.001)^3)$. Note that $\mathcal{C}$ is not polynomial (see Theorem 1). Our method provides as an answer the plane curve $\overline{\mathcal{C}}$ defined by the parametrization $\overline{\mathcal{P}}(t) = (1/t^2, 1/t^3)$. Note that, by applying Theorem 1 (Section 2), we get that $\overline{\mathcal{P}}(1/t) = (t^2, t^3)$ is a polynomial parametrization of $\overline{\mathcal{C}}$.

In Figure 2, $\mathcal{C}$ and $\overline{\mathcal{C}}$ are close in the neighbour of the origin point. However, it is clear that at the infinity $\mathcal{C}$ and $\overline{\mathcal{C}}$ will be quite different (in fact, the infinity points of both curves are necessarily different and also

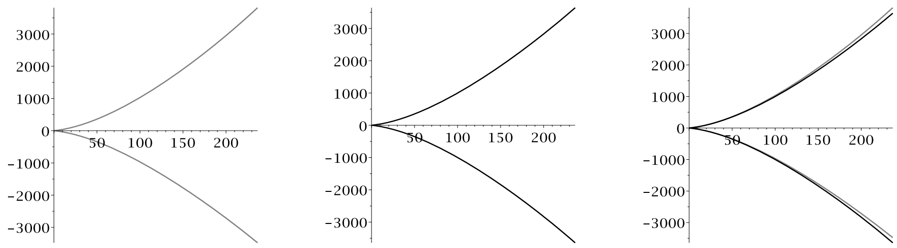

Figure 2: Curve $\mathcal{C}$ (left), curve $\overline{\mathcal{C}}$ (center) and both curves (right)

its asymptotes; see Section 2). Thus, at the infinity, the input curve has to be approximated by some other different polynomial curves.

In the following, in order to deal with this problem, we first consider a curve $\mathcal{C}$ defined parametrically by $(t, f(t))$, where $f(t) = \frac{p(t)}{\prod_{i=1}^{s}(t-r_i)} \in \mathbb{R}(t)$. We analyze whether in a certain interval $\mathcal{I} = (-\infty, a) \cup (b, \infty)$, $a, b \in \mathbb{R}$, the curve $\mathcal{C}$ can be approximated by a new curve $\overline{\mathcal{C}}$ defined parametrically by $(t, \overline{f}(t))$, where $\overline{f}(t) = \frac{p(t)}{(t-\alpha)^s} \in \mathbb{R}(t)$. We compute $\overline{f}(t)$ and also provide the error analysis.

Afterwards, we consider an input curve $\mathcal{C}$ defined by $\mathcal{P}(t) = (p_1(t), p_2(t)) = \left(\frac{p_{11}(t)}{p_{12}(t)}, \frac{p_{21}(t)}{p_{22}(t)}\right)$, $\deg(p_{i1}) = \deg(p_{i2}) = s_i$, $i = 1, 2$, with perturbed float coefficients. Using the ideas of above approach developed for the rational function $f(t)$, we show how to construct a new rational curve $\overline{\mathcal{C}}$ defined parametrically by $\overline{\mathcal{P}}(t) = \left(\frac{p_{11}(t)}{(t-\alpha)^{s_1}}, \frac{p_{21}(t)}{(t-\alpha)^{s_2}}\right) \in \mathbb{R}(t)^2$. We observe that $\overline{\mathcal{C}}$ is polynomial since $\overline{\mathcal{P}}(1/t + \alpha)$ is a polynomial reparametrization of $\overline{\mathcal{P}}(t)$ (see Theorem 1 in Section 2). Under these conditions, we prove that almost all points of the rational curve piece $\overline{\mathcal{C}}_{\mathcal{I}}$ are in the "*vicinity*" of $\mathcal{C}_{\mathcal{I}}$ (and reciprocally), where $\mathcal{I} = (-\infty, a) \cup (b, \infty)$, $a, b \in \mathbb{R}$.

In both cases, one has to deal with the approximation problem for the curve piece $\mathcal{C}_I$, where $I = (a, b)$. We will see that the roots of the denominators of the parametrization belong to I, and thus $\mathcal{C}_I$ corresponds to the curve piece at the "infinity". Here, the key consist in using the asymptotes of the given curve to approximate $\mathcal{C}_I$. Observe that for this purpose, one has to decompose the interval I as union of finitely many intervals according to the roots of the denominators of the parametrization. For each such interval, one different asymptote is used. We also provide the error analysis of this approximation. The results presented in this section can be easily generalized for space curves.

3.1. A particular case

Let $\mathbb{C}$ be the field of complex numbers and Consider a rational function $f(t) = \frac{p(t)}{q(t)} = \frac{p(t)}{\prod_{i=1}^{s}(t-r_i)} \in \mathbb{R}(t)$, $r_i \in \mathbb{C}$ with perturbed float coefficients. We assume that the approximate gcd of p and q is equal to one, i.e. ϵ-gcd$(p, q) = 1$ (otherwise, we simplify $f(t)$ using for instance Corless et al. (2001)), and $\deg(p) = \deg(q) = s$ (for our purposes in Subsection 3.2, we here only need to deal with this special case). Furthermore, we assume that at least one root of $q(t)$ is in $\mathbb{R}$. For the case of curve pieces without real roots, one can give a whole piecewise approximation method by using some traditional methods (see e.g. Sederberg and Kakimoto (1991) and Shen et al. (2012)). Finally, $|\cdot|$ denotes the modulus of a complex number.

We now deal with the problem of computing an approximation of the input rational curve piece $\mathcal{C}_{\mathcal{I}} = \{(t, f(t)),\, t \in \mathcal{I}\}$, with a curve piece of the form $\overline{\mathcal{C}}_{\mathcal{I}} = \{(t, \overline{f}(t)),\, t \in \mathcal{I}\}$, where $\overline{f}(t) = \frac{p(t)}{\overline{q}(t)} = \frac{p(t)}{(t-\alpha)^s} \in \mathbb{R}(t)$, and $\mathcal{I} = (-\infty, a) \cup (b, \infty), a, b \in \mathbb{R}$. To start with, we first show how to compute $a, b \in \mathbb{R}$ then, we construct $\overline{f}(t) \in \mathbb{R}(t)$, and finally we study how to approximate the curve pieces $\mathcal{C}_{\mathcal{I}}$ and $\mathcal{C}_I$ respectively. Afterwards, we present the error analysis and we illustrate the method with an example.

<u>Decomposition of $\mathbb{R}$</u>

In order to decompose $\mathbb{R}$, we compute the intervals $\mathcal{I} = (-\infty, a) \cup (b, \infty)$ and $I = (a, b)$ and thus, we first need to determine $a, b \in \mathbb{R}$. For this purpose, we distinguish some different cases:

1. Assume that among the roots r_i, $i = 1, \dots, s$, only one root is real, say r_1. Then, let $a, b \in \mathbb{R}$ be such that $|f(a)| = |f(b)| = \mu$, $a < r_1 < b$, where μ is any positive value. Under these conditions, we consider $\mathbb{R} = \mathcal{I} \cup I$.
2. Assume that among the roots r_i, $i = 1, \dots, s$, two roots are real, say r_1, r_2. Then, let $a, b \in \mathbb{R}$ be such that $|f(a)| = |f(b)| = \mu$, $a < r_1 < r_2 < b$, where μ is any positive value. Thus, we decompose the interval I as union of finitely many intervals according to the roots r_1 and r_2. More precisely, we consider $I = I_1 \cup I_2$, where $I_1 := (a, (r_1 + r_2)/2)$, and $I_2 := ((r_1 + r_2)/2, b)$. Under these conditions, we consider the decomposition $\mathbb{R} = \mathcal{I} \cup I_1 \cup I_2$.
   If there exist more than two real roots, we generalize the above process and we reason as before. More precisely, if $r_j \in \mathbb{R}$, $j = 1, \dots, \ell$, $r_1 < r_2 < \cdots < r_\ell$, we consider $I = \bigcup_{j=1}^{\ell} I_j$, where $I_j := ((r_{j-1} + r_j)/2, (r_j + r_{j+1})/2)$ (let $(r_{-1} + r_1)/2 := a$ and $(r_s + r_{s+1})/2 := b$). Under these conditions, we consider the decomposition $\mathbb{R} = \mathcal{I} \cup \bigcup_{j=1}^{\ell} I_j$.

Observe that we deal with the real part of the curve. In addition, we note that $a, b \in \mathbb{R}$ always exist since $f(t)$ has vertical asymptotes at $t = r_i$, $i = 1, \dots, s$. In Figure 3, we illustrate this reasoning.

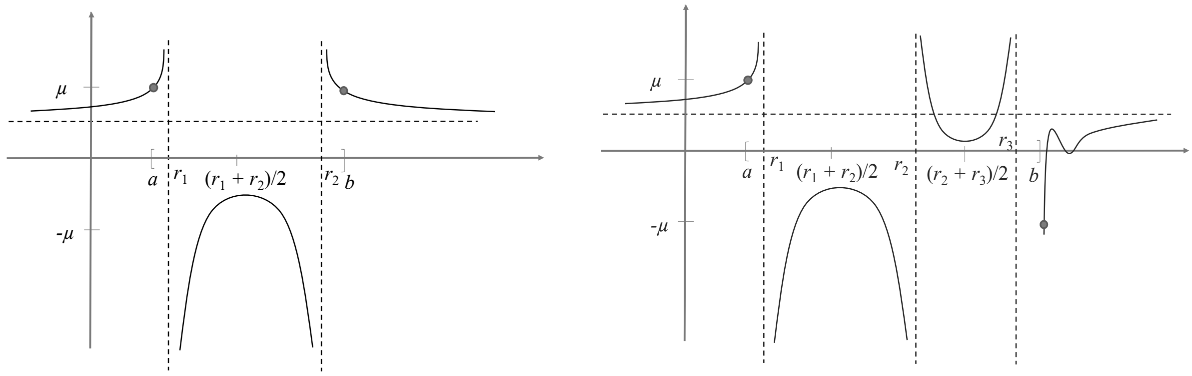

Figure 3: Rational function of degree 2 (left), and rational function of degree 3 (right)

<u>Approximation of the curve piece $\mathcal{C}_{\mathcal{I}}$</u>

Once $a, b \in \mathbb{R}$ are computed, we determine the curve piece $\overline{\mathcal{C}}_{\mathcal{I}}$ that will approximate $\mathcal{C}_{\mathcal{I}}$. For this purpose, we compute $\overline{f}(t) \in \mathbb{R}(t)$ as follows: we consider $\overline{f}(t) = \frac{p(t)}{\overline{q}(t)} = \frac{p(t)}{(t-\alpha)^s} \in \mathbb{R}(t)$, where $\alpha \in \mathbb{R}$ is such that $a < \alpha < b$ and it minimizes $|f(t) - \overline{f}(t)|, t \in \mathcal{I}$. For this purpose, since $|f(t) - \overline{f}(t)| \leq \max\{|f(a) - \overline{f}(a)|, |f(b) - \overline{f}(b)|\}$, we compute $\alpha \in \mathbb{R}$ such that $g(\alpha) = \sqrt{(f(a) - \overline{f}(a))^2 + (f(b) - \overline{f}(b))^2}$ is minimum. Let m_α be this minimum. Then, it holds that $|f(t) - \overline{f}(t)| \leq \max\{|f(a) - \overline{f}(a)|, |f(b) - \overline{f}(b)|\} \leq m_\alpha$, $t \in \mathcal{I}$. Note that $\alpha \in (a, b)$ always exists ($g(\alpha)$ has two vertical asymptotes at $\alpha = a$ and $\alpha = b$).

In Figure 4, we plot an example of a rational curve, $\mathcal{C}$, defined by $(t, f(t))$, and the output rational curve, $\overline{\mathcal{C}}$, defined by $(t, \overline{f}(t))$.

<u>Approximation of the curve piece $\mathcal{C}_I$</u>

In order to determine the curve piece $\overline{\mathcal{C}}_I$ that will approximate $\mathcal{C}_I$, we need to distinguish some different cases according the decomposition of the interval I. More precisely:

1. Assume that among the roots r_i, $i = 1, \dots, s$, only one root is real, say r_1. Once $a, b \in \mathbb{R}$ are computed, we approximate the curve piece $\mathcal{C}_I$, where $I := (a, b)$, by the asymptote $\mathcal{A}_1$ defined implicitly by $x_1 = r_1$.

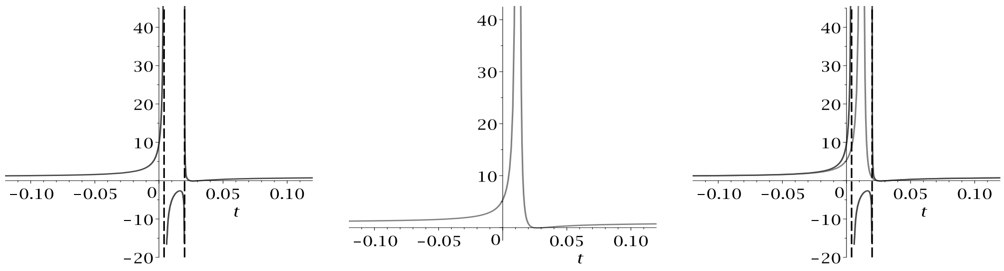

Figure 4: Rational curves $\mathcal{C}$ (left), $\overline{\mathcal{C}}$ (center), and both curves (right)

2. Assume that among the roots r_i, $i = 1, \ldots, s$, two roots are real, say r_1, r_2. Once $a, b \in \mathbb{R}$ are computed, we consider $I = I_1 \cup I_2$, where $I_1 := (a, (r_1 + r_2)/2)$, and $I_2 := ((r_1 + r_2)/2, b)$, and we approximate the curve piece $\mathcal{C}_{I_1}$ by the asymptote $\mathcal{A}_1$ defined implicitly by $x_1 = r_1$, and the curve piece $\mathcal{C}_{I_2}$ by the asymptote $\mathcal{A}_2$ defined implicitly by $x_1 = r_2$.
   If there exist more than two real roots, we generalize the above process and we reason as before. More precisely, once $a, b \in \mathbb{R}$ are computed, we consider $I = \bigcup_{j=1}^{\ell} I_j$, where $I_j := ((r_{j-1}+r_j)/2, (r_j+r_{j+1})/2)$ (let $(r_{-1}+r_1)/2 := a$ and $(r_s+r_{s+1})/2 := b$), and we approximate the curve piece $\mathcal{C}_{I_j}$ by the asymptote $\mathcal{A}_j$ defined implicitly by $x_1 = r_j$, for $j = 1, \ldots, \ell$. Note that $\mathcal{C}_I = \bigcup_{j=1}^{\ell} \mathcal{C}_{I_j}$.

We note that the curve is replaced by the asymptote in a small neighbourhood.

Error Analysis

In the following, we present the error analysis of the method developed above. The general strategy is to show that almost any affine real point on the curve piece $\overline{\mathcal{C}}_{\mathcal{I}}$ is at small (Euclidean) distance of an affine real point on curve piece $\mathcal{C}_{\mathcal{I}}$, and reciprocally. For this purpose, we compute the distance $|f(t) - \overline{f}(t)|$, $t \in \mathcal{I}$.

For the curve pieces $\mathcal{C}_{I_j}$, one reasons similarly by considering the asymptotes $\mathcal{A}_j$ as the output polynomial curves. In this case, in the proof of the theorem, we should use dehomogenizations provided by taking the axes as lines at infinity. More precisely, if $\mathcal{D}$ is an affine curve defined implicitly by the polynomial $f(x_1, x_2)$ (and parametrically by $(p_1(t)/p(t),\, p_2(t)/p(t))$), the corresponding projective curve is defined by the form $F(x_1, x_2, x_3)$ (and parametrically by $(p_1(t), p_2(t), p(t))$). Thus, we denote by $\mathcal{D}^1$ and $\mathcal{D}^2$, the affine plane curves defined by $F(1, x_2, x_3)$ (parametrically by $(p_2(t)/p_1(t),\, p(t)/p_1(t))$) and $F(x_1, 1, x_3)$ (parametrically by $(p_1(t)/p_2(t),\, p(t)/p_2(t))$), respectively. Note that any point on the projective curve corresponds to a point on a suitable affine version of $\mathcal{D}$.

Theorem 2. *The following statements hold:*

1. *For every point on the curve piece $\mathcal{C}_{\mathcal{I}}$, there exists a point on the curve piece $\overline{\mathcal{C}}_{\mathcal{I}}$ (and reciprocally) at distance at most m_α.*
2. *Let $I_{j_0} := (\gamma, \beta)$, where $\gamma := (r_{j_0-1} + r_{j_0})/2$, and $\beta := (r_{j_0} + r_{j_0+1})/2$. For every point on the curve piece $\mathcal{C}_{I_{j_0}}$, there exists a point on the asymptote $\mathcal{A}_{j_0}$ defined implicitly by $x_1 = r_{j_0}$ (and reciprocally) at distance at most* $\max\{|1/f(\gamma) - \gamma/(r_{j_0} f(\gamma))|, |1/f(\beta) - \beta/(r_{j_0} f(\beta))|\}$.

Proof. The first statement is proved by computing $|f(t) - \overline{f}(t)|$, for $t \in \mathcal{I}$. Taking into account the construction of $\overline{\mathcal{C}}_{\mathcal{I}}$, we have that for $t \in \mathcal{I}$, it holds that $|f(t) - \overline{f}(t)| \leq m_\alpha$ and then, for every point on the input curve piece $\mathcal{C}_{\mathcal{I}}$ there exists a point on the output curve piece $\overline{\mathcal{C}}_{\mathcal{I}}$ at distance at most m_α (and reciprocally).

In order to prove statement 2, we assume that there exists only one real root among r_i, $i = 1, \ldots, s$, namely r_1. Then, we approximate the input curve piece $\mathcal{C}_I$ by the asymptote $\mathcal{A}_1$ defined implicitly by $x_1 = r_1$. Note that $\mathcal{A}_1$ is defined parametrically by (r_1, t). For this purpose, since we are going to measure distances at the infinity (note that $f(t)$ is not defined at $t = r_1$), we use a dehomogenization to represent these points. More precisely, we consider the curve $\mathcal{C}^2$ and then, the input curve piece is defined by $(t/f(t), 1/f(t))$, $t \in I := (a, b)$, and the asymptote $\mathcal{A}_1^2$ is defined parametrically by $(r_1/t, 1/t)$.

Under these conditions, it holds that for every point on the input curve piece there exists a point on the asymptote (and reciprocally) at distance at most $\max\{|1/f(a) - a/(r_1 f(a))|, |1/f(b) - b/(r_1 f(b))|\}$. Indeed:

every point of the given curve piece is defined by $(t/f(t), 1/f(t))$, $t \in I$, and the vertical asymptote is defined by the parametrization $(r_1/s, 1/s)$, $s \in \mathbb{C}$. Thus, given $t_0 \in I$, there exists $s_0 \in \mathbb{C}$ ($s_0 = r_1 f(t_0)/t_0$) such that the distance between the point $(t_0/f(t_0), 1/f(t_0))$ of the given curve and the point $(r_1/s_0, 1/s_0)$ of the asymptote is $|1/f(t_0) - t_0/(r_1 f(t_0))| \leq \max\{|1/f(a) - a/(r_1 f(a))|, |1/f(b) - b/(r_1 f(b))|\}$.

One reasons similarly for the general case, and we obtain that for any interval $I_{j_0} := (\gamma, \beta)$, where $\gamma := (r_{j_0-1} + r_{j_0})/2$, and $\beta := (r_{j_0} + r_{j_0+1})/2$, it holds that for every point on the curve piece $\mathcal{C}_{I_{j_0}}$, there exists a point on the asymptote $\mathcal{A}_{j_0}$ defined implicitly by $x_1 = r_{j_0}$ (and reciprocally) at distance at most $\max\{|1/f(\gamma) - \gamma/(r_{j_0} f(\gamma))|, |1/f(\beta) - \beta/(r_{j_0} f(\beta))|\}$. □

Remark 1. *From the proof of Theorem 2, one deduces that for a point $(t_0, f(t_0)) \in \mathcal{C}_{\mathcal{I}}$, the point $(t_0, \overline{f}(t_0)) \in \overline{\mathcal{C}}_{\mathcal{I}}$ is at distance at most m_α, and reciprocally.*
In addition, for a point $(t_0/f(t_0), 1/f(t_0)) \in \mathcal{C}^2_{I_{j_0}}$, the point $(t_0/f(t_0), t_0/(r_{j_0} f(t_0))) \in \mathcal{A}^2_1$ is at distance at most $\max\{|1/f(\gamma) - \gamma/(r_{j_0} f(\gamma))|, |1/f(\beta) - \beta/(r_{j_0} f(\beta))|\}$, and reciprocally, where $\gamma := (r_{j_0-1} + r_{j_0})/2$ and $\beta := (r_{j_0} + r_{j_0+1})/2$.

Using Theorem 2, and applying the results in Section 2.2 in Farouki and Rajan (1988), we deduce the following corollary.

Corollary 1. *The following statements hold:*

1. *The input curve piece $\mathcal{C}_{\mathcal{I}}$ is contained in the offset region of the output curve piece $\overline{\mathcal{C}}_{\mathcal{I}}$ (and reciprocally) at distance at most $2m_\alpha$.*
2. *Let $I_{j_0} := (\gamma, \beta)$, where $\gamma := (r_{j_0-1} + r_{j_0})/2$ and $\beta := (r_{j_0} + r_{j_0+1})/2$. The curve piece $\mathcal{C}_{I_{j_0}}$ is contained in the offset region of the asymptote $x_1 = r_{j_0}$ (and reciprocally) at distance at most $2\max\{|1/f(\gamma) - \gamma/(r_{j_0} f(\gamma))|, |1/f(\beta) - \beta/(r_{j_0} f(\beta))|\}$.*

Remark 2. *We note that:*
1. Since we are working numerically, one may assume w.l.o.g that $r_i \neq 0$, $i = 1, \ldots, s$, $ab \neq 0$ and $f(\gamma)f(\beta) \neq 0$. Thus, all the above computed distances are well defined.
2. In order to control the error bound, one may reason as follows: given a tolerance $\epsilon > 0$, one needs that for every point on the input curve piece $\mathcal{C}_{\mathcal{I}}$, there exists a point on the output curve piece $\overline{\mathcal{C}}_{\mathcal{I}}$ (and reciprocally) at distance at most ϵ. Thus, one computes $a, b, \alpha \in \mathbb{R}$ such that $m_\alpha \leq \epsilon$.
3. The effectiveness of the method presented depends on the distance of the roots r_i, $i = 1, \ldots, s$, i.e. if these roots are "close enough", the method developed provides "good " approximations for $\mathcal{C}_I$. More precisely, let $I_{j_0} := (\gamma, \beta)$, where $\gamma := (r_{j_0-1} + r_{j_0})/2$ and $\beta := (r_{j_0} + r_{j_0+1})/2$. Given a tolerance $\epsilon > 0$, one needs that for every point on the input curve piece $\mathcal{C}_{I_{j_0}}$, there exists a point on the asymptote $\mathcal{A}_{j_0}$ (and reciprocally) at distance at most ϵ. Thus, it should be satisfied that $\max\{|1/f(\gamma) - \gamma/(r_{j_0} f(\gamma))|, |1/f(\beta) - \beta/(r_{j_0} f(\beta))|\} \leq \epsilon$ which is equivalent to $|r_{j_0} - r_{j_0-1}| \leq 2r_{j_0} f(\gamma)\epsilon$, and $|r_{j_0} - r_{j_0+1}| \leq 2r_{j_0} f(\beta)\epsilon$.
If there exist two roots, say r_1, r_2, that are not "close enough", one may think to approximate the given function in $\mathcal{I}^* := (r_1 + \rho_1, r_2 - \rho_2)$, for some $\rho_i \in \mathbb{R}$, $i = 1, 2$, similarly as in the interval $\mathcal{I}$. The values of $\rho_1, \rho_2 \in \mathbb{R}$, can be computed similarly as $a, b \in \mathbb{R}$.

Example 2. *Let $f(t) = \frac{p(t)}{q(t)} = \frac{(t-4)(t-7)}{(t-199/100)(t-1997/1000)} \in \mathbb{R}(t)$ be the input rational function. We apply the above reasoning to compute approximations to the pieces $\mathcal{C}_{\mathcal{I}} = \{(t, f(t)), t \in \mathcal{I}\}$ and $\mathcal{C}_I = \{(t, f(t)), t \in I\}$.*
Decomposition of $\mathbb{R}$: *We first solve the equations $|f(a)| = |f(b)| = \mu = 200$, where $a < r_1 = 199/100 < r_2 = 1997/1000 < b$, and we get that $a \approx 1.750484390$, $b \approx 2.201274404$.*

Now, we decompose the interval $I = (a, b)$ as union of finitely many intervals according to the roots r_1 and r_2. More precisely, we consider $I = I_1 \cup I_2$, where $I_1 := (a, (r_1 + r_2)/2) = (1.750484390, 1.9935)$, and $I_2 := ((r_1 + r_2)/2, b) = (1.9935, 2.201274404)$.
Under these conditions, let $\mathbb{R} = \mathcal{I} \cup I_1 \cup I_2$, where $\mathcal{I} = (-\infty, a) \cup (b, \infty)$.
Approximation of the curve piece $\mathcal{C}_{\mathcal{I}}$: *Let $\overline{f}(t) = \frac{p(t)}{\overline{q}(t)} = \frac{(t-4)(t-7)}{(t-\alpha)^2} \in \mathbb{R}(t)$, and we look for $\alpha \in \mathbb{R}$ such that $g(\alpha) = \sqrt{(f(a) - \overline{f}(a))^2 + (f(b) - \overline{f}(b))^2}$ is minimum. We get that $\alpha \approx 1.993506389$ and $m_\alpha \approx$*

0.06841260609. *Then, we approximate the curve piece* $\mathcal{C}_\mathcal{I}$ *by the curve piece* $\overline{\mathcal{C}}_\mathcal{I}$ *defined by* $(t, \overline{f}(t))$, $t \in \mathcal{I}$. *In Figure 5, we plot* $\mathcal{C}_\mathcal{I}$ *and* $\overline{\mathcal{C}}_\mathcal{I}$.
Approximation of the curve piece $\mathcal{C}_I$: *We approximate the curve piece* $\mathcal{C}_{I_1}$ *by the asymptote* $\mathcal{A}_1$ *defined implicitly by* $x_1 = r_1 \approx 1.99$, *and the curve piece* $\mathcal{C}_{I_2}$ *by the asymptote* $\mathcal{A}_2$ *defined implicitly by* $x_1 = r_2 \approx 1.997$.
Error analysis: *It holds that, for every point on the input curve piece* $\mathcal{C}_\mathcal{I}$, *there exists a point on the output curve piece* $\overline{\mathcal{C}}_\mathcal{I}$, *at distance at most* $|f(t)-\overline{f}(t)| \leq m_\alpha \approx 0.06841260609$. *In fact, given* $(t_0, f(t_0)) \in \mathcal{C},\ t_0 \in \mathcal{I}$, *the point satisfying the above statement is* $(t_0, \overline{f}(t_0)) \in \overline{\mathcal{C}}$ *(see Remark 1).*

In addition, it holds that for every point on the curve piece $\mathcal{C}_{I_1}$, *there exists a point in the asymptote* $\mathcal{A}_1$ *(and reciprocally) at distance at most*

$$\max\left\{\left|\frac{1}{f(a)} - \frac{a}{r_1 f(a)}\right|, \left|\frac{1}{f((r_1+r_2)/2)} - \frac{(r_1+r_2)/2}{r_1 f((r_1+r_2)/2)}\right|\right\} \approx 0.0006017980166.$$

In fact, from Remark 1, we deduce that given $(t_0/f(t_0), 1/f(t_0)) \in \mathcal{C}^2,\ t_0 \in I_1$, *the point satisfying the above statement is* $(t_0/f(t_0), t_0/(r_1 f(t_0))) \in \mathcal{A}_1^2$ *(we remind that since we are at the infinity, we are considering a dehomogenization of the input curve).*

Furthermore, for every point on the curve piece $\mathcal{C}_{I_2}$, *there exists a point in the asymptote* $\mathcal{A}_2$ *(and reciprocally) at distance at most*

$$\max\left\{\left|\frac{1}{f(b)} - \frac{b}{r_2 f(b)}\right|, \left|\frac{1}{f((r_1+r_2)/2)} - \frac{(r_1+r_2)/2}{r_2 f((r_1+r_2)/2)}\right|\right\} \approx 0.0005114531918.$$

Similarly as above, from Remark 1, we deduce that given $(t_0/f(t_0), 1/f(t_0)) \in \mathcal{C}^2,\ t_0 \in I_2$, *the point satisfying the above statement is* $(t_0/f(t_0), t_0/(r_2 f(t_0))) \in \mathcal{A}_2^2$.

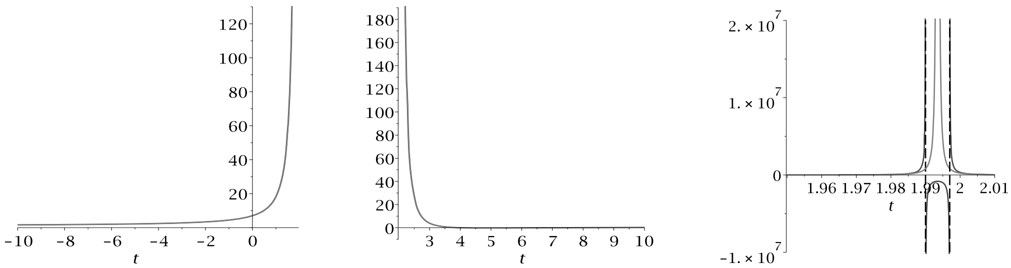

Figure 5: Curve pieces $\{(t, f(t)),\ t \in (-\infty, a)\}$ and $\{(t, \overline{f}(t)),\ t \in (-\infty, a)\}$ (left), $\{(t, f(t)),\ t \in (b, \infty)\}$ and $\{(t, \overline{f}(t)),\ t \in (b, \infty)\}$ (center), and both curve pieces (right)

3.2. The case of a rational parametrization

In the following, we consider the rational plane curve $\mathcal{C}$ defined by rational parametrization with perturbed float coefficients

$$\mathcal{P}(t) = (p_1(t), p_2(t)) = \left(\frac{p_{11}(t)}{p_{12}(t)}, \frac{p_{21}(t)}{p_{22}(t)}\right) \in \mathbb{R}(t)^2.$$

Let $p_{j2}(t) = \prod_{i=1}^{s_j}(t - r_{ij}),\ j = 1, 2$, where $r_{ij} \in \mathbb{C}$. We assume that ϵ-$\gcd(p_{i1}, p_{i2}) = 1,\ i = 1, 2$ (otherwise, we simplify the rational function $p_i(t)$), and $\deg(p_{i1}) = \deg(p_{i2}) = s_i,\ i = 1, 2$ (otherwise, we consider a linear change of variable). Furthermore, we assume that at least one root of the polynomial $p_{12}(t)p_{22}(t)$ is in $\mathbb{R}$. For the case of curve pieces without real roots, one can give a whole piecewise approximation method by using some traditional methods. Finally, $||\cdot||$ denotes the 2–norm.

Observe that we may assume w.l.o.g. that $\gcd(p_{12}, p_{22}) = 1$ since we are working numerically. In this case, using the results in Section 2, one may check that the asymptotes of the curve $\mathcal{C}$ are the vertical lines parametrized by $(p_1(r_{i2}), t)$, $i = 1, \dots, s_2$, and the horizontal lines parametrized by $(t, p_2(r_{i1}))$, $i = 1, \dots, s_1$.

The goal of this section is to compute an approximation of the input rational curve piece $\mathcal{C}_{\mathcal{I}} = \{\mathcal{P}(t),\, t \in \mathcal{I}\}$, by means of a curve piece of the form $\overline{\mathcal{C}}_{\mathcal{I}} = \{\overline{\mathcal{P}}(t),\, t \in \mathcal{I}\}$, where

$$\overline{\mathcal{P}}(t) = \left(\frac{p_{11}(t)}{(t-\alpha)^{s_1}}, \frac{p_{21}(t)}{(t-\alpha)^{s_2}} \right) \in \mathbb{R}(t)^2,$$

and $\mathcal{I} = (-\infty, a) \cup (b, \infty), a, b \in \mathbb{R}$. We denote by $\overline{\mathcal{C}}$ the output rational plane curve defined by the parametrization $\overline{\mathcal{P}}(t)$. We observe that $\overline{\mathcal{C}}$ is polynomial since $\overline{\mathcal{P}}(1/t + \alpha)$ is a polynomial reparametrization of $\overline{\mathcal{P}}(t)$ (see Section 2). Reasoning similarly as in Subsection 3.1, we first compute $a, b \in \mathbb{R}$ then, we construct $\overline{\mathcal{P}}(t)$, and then we study how to approximate the curve piece $\mathcal{C}_I = \{\mathcal{P}(t),\, t \in I\}$, where $I = (a, b)$. Afterwards, we present the error analysis and the algorithm derived from the method developed and we illustrate it with an example.

Decomposition of $\mathbb{R}$

In order to decompose $\mathbb{R}$, we compute the intervals $\mathcal{I} = (-\infty, a) \cup (b, \infty)$ and $I = (a, b)$ and thus, we first need to determine $a, b \in \mathbb{R}$. For this purpose, we distinguish some different cases:

1. Assume that among the roots r_{ij}, $i = 1, \dots, s_j$, $j = 1, 2$, only one root is real, say r_{12}. Then, let $a, b \in \mathbb{R}$ be such that $|p_2(a)| = |p_2(b)| = \mu$, $a < r_{12} < b$, where μ is any positive value. Under these conditions, we consider $\mathbb{R} = \mathcal{I} \cup I$.
2. Assume that among the roots r_{ij}, $i = 1, \dots, s_j$, $j = 1, 2$, two roots are real, say r_{11}, r_{12}, and assume that $r_{11} < r_{12}$. Then, let $a, b \in \mathbb{R}$ be such that $|p_i(a)| = |p_i(b)| = \mu$, $i = 1, 2$, $a < r_{11} < r_{12} < b$, where μ is any positive value. In addition, $|a|$ is the minimum and $|b|$ the maximum of all the values satisfying the above equations. Thus, we decompose the interval I as union of finitely many intervals according to the roots r_1 and r_2. More precisely, we consider $I = I_1 \cup I_2$, where $I_1 := (a, (r_{11} + r_{12})/2)$, and $I_2 := ((r_{11} + r_{12})/2, b)$. Under these conditions, we consider the decomposition $\mathbb{R} = \mathcal{I} \cup I_1 \cup I_2$.
   If there exist more than two real roots, we generalize the above process and we reason as before. More precisely, we assume that among the roots r_{ij}, $i = 1, \dots, s_j$, $j = 1, 2$, only ℓ roots are real. We denote these roots as $r_j \in \mathbb{R}$, $j = 1, \dots, \ell$ and we assume that $r_1 < r_2 < \cdots < r_\ell$. Then, let $I = \bigcup_{j=1}^{\ell} I_j$, where $I_j := ((r_{j-1} + r_j)/2, (r_j + r_{j+1})/2)$ (let $(r_{-1} + r_1)/2 := a$ and $(r_s + r_{s+1})/2 := b$). Under these conditions, we consider the decomposition $\mathbb{R} = \mathcal{I} \cup \bigcup_{j=1}^{\ell} I_j$.

Observe that we deal with the real part of the curve. In addition, we note that $a, b \in \mathbb{R}$ always exist since the rational functions $p_j(t)$ has vertical asymptotes at $t = r_{ij}$, $i = 1, \dots, s_j$, $j = 1, 2$. In Figure 6, we illustrate the above reasoning.

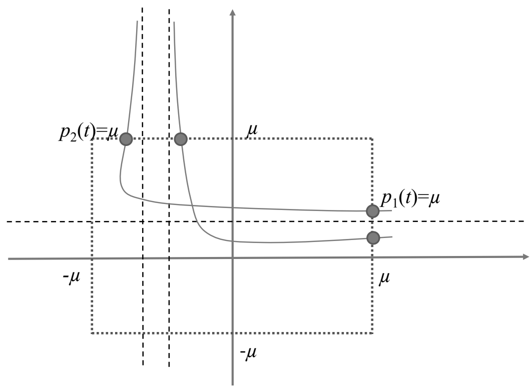

Figure 6: Decomposition of $\mathbb{R}$: rational curve and asymptotes

Approximation of the curve piece $\mathcal{C}_{\mathcal{I}}$

Once $a, b \in \mathbb{R}$ are computed, we determine the curve piece $\overline{\mathcal{C}}_{\mathcal{I}}$ that approximates $\mathcal{C}_{\mathcal{I}}$. For this purpose, we compute the output curve $\overline{\mathcal{C}}$ defined parametrically by $\overline{\mathcal{P}}(t)$, where

$$\overline{\mathcal{P}}(t) = (\overline{p}_1(t), \overline{p}_2(t)) = \left(\frac{p_{11}(t)}{(t-\alpha)^{s_1}}, \frac{p_{21}(t)}{(t-\alpha)^{s_2}} \right) \in \mathbb{R}(t)^2.$$

Reasoning as in Subsection 3.1, we compute $\alpha \in \mathbb{R}$, $a < \alpha < b$, that minimizes the Euclidean distance of $\mathcal{P}(t)$ and $\overline{\mathcal{P}}(t)$; that is, $||\mathcal{P}(t) - \overline{\mathcal{P}}(t)||$. Thus, we consider $\alpha \in \mathbb{R}$ such that

$$g(\alpha) = \sqrt{(p_1(a) - \overline{p}_1(a))^2 + (p_1(b) - \overline{p}_1(b))^2 + (p_2(a) - \overline{p}_2(a))^2 + (p_2(b) - \overline{p}_2(b))^2}$$

is minimum. Let m_α be this minimum. Then, $||\mathcal{P}(t) - \overline{\mathcal{P}}(t)|| \leq$

$$\max\{\sqrt{(p_1(a) - \overline{p}_1(a))^2 + (p_2(a) - \overline{p}_2(a))^2}, \sqrt{(p_1(b) - \overline{p}_1(b))^2 + (p_2(b) - \overline{p}_2(b))^2}\} \leq m_\alpha.$$

Note that $\alpha \in (a, b)$ always exists and that $\overline{\mathcal{C}}$ is polynomial since $\overline{\mathcal{P}}(1/t+\alpha)$ is a polynomial reparametrization of $\overline{\mathcal{P}}(t)$ (see Theorem 1 in Section 2).

In Figure 7, we plot an example of a rational curve, $\mathcal{C}$, defined by the parametrization $\mathcal{P}(t)$ and the new rational curve, $\overline{\mathcal{C}}$, parametrized by $\overline{\mathcal{P}}(t)$. We plot both curves for $(x_1, x_2) \in (-200, 200) \times (-200, 200)$.

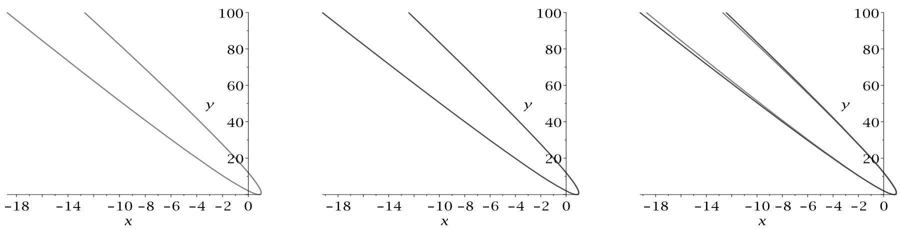

Figure 7: Curve $\mathcal{C}$ (left), curve $\overline{\mathcal{C}}$ (center), and both curves (right)

In addition, in Figure 8, we plot the input curve $\mathcal{C}$ for $(x_1, x_2) \in (-4 \cdot 10^5, 4 \cdot 10^5) \times (-6 \cdot 10^8, 6 \cdot 10^8)$ (left), and the curve $\mathcal{C}$ for $(x_1, x_2) \in (-2 \cdot 10^7, 2 \cdot 10^7) \times (-2 \cdot 10^7, 2 \cdot 10^7)$ (right). From these figures, one deduces that one has to use new approximations to the input curve, when one is far away, at the infinity.

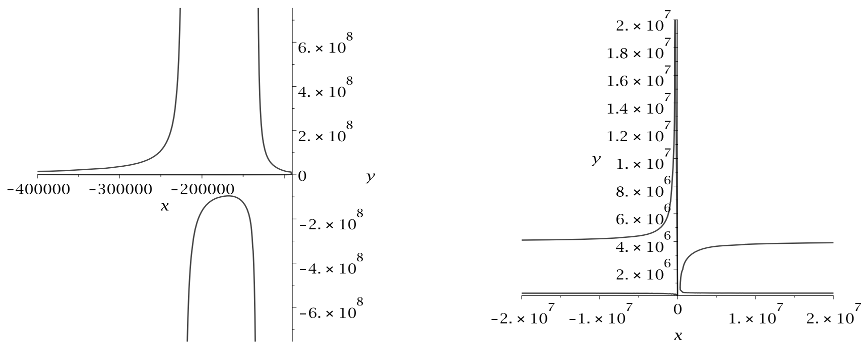

Figure 8: Curve $\mathcal{C}$ for $(x_1, x_2) \in (-4 \cdot 10^5, 4 \cdot 10^5) \times (-6 \cdot 10^8, 4 \cdot 10^8)$ (left), and curve $\mathcal{C}$ for $(x_1, x_2) \in (-2 \cdot 10^7, 2 \cdot 10^7) \times (-2 \cdot 10^7, 2 \cdot 10^7)$ (right)

Approximation of the curve piece $\mathcal{C}_I$

In order to determine the curve piece $\overline{\mathcal{C}}_I$ that will approximate $\mathcal{C}_I$, we need to distinguish some different cases according the decomposition of the interval I. More precisely:

1. Assume that among the roots $r_{ij},\ i=1,\dots,s_j,\ j=1,2$, only one root is real, say r_{12}. Once $a,b\in\mathbb{R}$ are computed, we approximate the curve piece $\mathcal{C}_I$ by the asymptote $\mathcal{A}_1$ defined implicitly by $x_1=p_1(r_{12})$.
2. Assume that among the roots $r_{ij},\ i=1,\dots,s_j,\ j=1,2$, two roots are real, say r_{11},r_{12}, and assume that $r_{11}<r_{12}$. Once $a,b\in\mathbb{R}$ are computed, we consider $I=I_1\cup I_2$, where $I_1:=(a,(r_{11}+r_{12})/2)$, and $I_2:=((r_{11}+r_{12})/2,b)$, and we approximate the curve piece $\mathcal{C}_{I_1}$ by the asymptote $\mathcal{A}_1$ defined implicitly by $x_2=p_2(r_{11})$, and the curve piece $\mathcal{C}_{I_2}$ by the asymptote $\mathcal{A}_2$ defined implicitly by $x_1=p_1(r_{12})$.
   If there exist more than two real roots, we generalize the above process and we reason as before. More precisely, we assume that among the roots $r_{ij},\ i=1,\dots,s_j,\ j=1,2$, only ℓ roots are real. We denote these roots as $r_j\in\mathbb{R},\ j=1,\dots,\ell$ and we assume that $r_1<r_2<\dots<r_\ell$. Once $a,b\in\mathbb{R}$ are computed, we consider $I=\bigcup_{j=1}^{\ell}I_j$, where $I_j:=((r_{j-1}+r_j)/2,(r_j+r_{j+1})/2)$ (let $(r_{-1}+r_1)/2:=a$ and $(r_s+r_{s+1})/2:=b$), and we approximate the curve piece $\mathcal{C}_{I_j}$ by the asymptote $\mathcal{A}_j$ defined implicitly by $x_i=p_i(r_j),\ j=1,\dots,\ell$, where $i=1$ (if $p_{22}(r_j)=0$) or $i=2$ (if $p_{12}(r_j)=0$). Note that $\mathcal{C}_I=\bigcup_{j=1}^{\ell}\mathcal{C}_{I_j}$.

We note that the curve is replaced by the asymptote in a small neighbourhood. In addition, since we are working numerically, we remind we are assuming that the denominators of both components of the parametrization do not have common roots. Thus, when t tends to a root of the denominator there are only horizontal or vertical asymptotes.

Error Analysis

In the following, we present the error analysis of the method developed above. The general strategy is to show that almost any affine real point on the curve piece $\overline{\mathcal{C}}_{\mathcal{I}}$ is at small (Euclidean) distance of an affine real point on curve piece $\mathcal{C}_{\mathcal{I}}$, and reciprocally. For this purpose, we compute the distance $||\mathcal{P}(t)-\overline{\mathcal{P}}(t)||$, $t\in\mathcal{I}$.

For the curve pieces $\mathcal{C}_{I_j}$, one reasons similarly by considering the asymptotes $\mathcal{A}_j$ as the output polynomial curves.

Theorem 3. *The following statements hold:*

1. *For every point on the curve piece $\mathcal{C}_{\mathcal{I}}$, there exists a point on the curve piece $\overline{\mathcal{C}}_{\mathcal{I}}$ (and reciprocally) at distance at most m_α.*
2. *Let $I_{j_0}:=(\gamma,\beta)$, where $\gamma:=(r_{j_0-1}+r_{j_0})/2$, $\beta:=(r_{j_0}+r_{j_0+1})/2$, and $p_{j2}(r_{j_0})=0$, for $j=1$ or $j=2$. For every point on the curve piece $\mathcal{C}_{I_{j_0}}$, there exists a point on the asymptote $\mathcal{A}_{j_0}$ defined implicitly by $x_i=p_i(r_{j_0}),\ i\neq j,\ i=1$ or $i=2$ (and reciprocally) at distance at most*

$$\max\{|1/p_j(\gamma)-p_i(\gamma)/(p_j(\gamma)p_i(r_{j_0}))|,\ |1/p_j(\beta)-p_i(\beta)/(p_j(\beta)p_i(r_{j_0}))|\}.$$

Proof. The first statement is proved by computing $||\mathcal{P}(t)-\overline{\mathcal{P}}(t)||$, $t\in\mathcal{I}$. Taking into account the above reasoning, we have that for $t\in\mathcal{I}$, it holds that $||\mathcal{P}(t)-\overline{\mathcal{P}}(t)||\leq m_\alpha$ and then, for every point on the input curve piece $\mathcal{C}_{\mathcal{I}}$ there exists a point on the output curve piece $\overline{\mathcal{C}}_{\mathcal{I}}$ at distance at most m_α (and reciprocally).

In order to prove statement 2, we assume that among the roots $r_{ij},\ i=1,\dots,s_j,\ j=1,2$, only one root is real, say r_{12}. Then, we approximate the input curve piece $\mathcal{C}_I$ by the asymptote $\mathcal{A}_1$ defined implicitly by $x_1=p_1(r_{12})$. Note that $\mathcal{A}_1$ is defined parametrically by $(p_1(r_{12}),t)$. Since we are going to measure distances at infinity (note that $p_2(t)$ is not defined at $t=r_{12}$), we use a dehomogenization to represent these points. More precisely, we consider the curve $\mathcal{C}^2$ defined by $\left(\frac{p_1(t)}{p_2(t)},\frac{1}{p_2(t)}\right)$, $t\in I$, and the asymptote $\mathcal{A}_1^2$ is defined parametrically by $(p_1(r_{12})/t,1/t)$.

Under these conditions, it holds that for every point on the input curve piece there exists a point on the asymptote (and reciprocally) at distance at most

$$\max\{|1/p_2(t)-p_1(t)/(p_2(t)p_1(r_{12}))|\,|\,t\in I\}\leq\max\{|1/p_2(a)-p_1(a)/(p_2(a)p_1(r_{12}))|,\ |1/p_2(b)-p_1(b)/(p_2(b)p_1(r_{12}))|\}.$$

Indeed: every point of the given curve is defined by $\left(\frac{p_1(t)}{p_2(t)},\frac{1}{p_2(t)}\right)$, and the vertical asymptote is defined by the parametrization $(p_1(r_{12})/s,1/s)$, $s\in\mathbb{C}$. Thus, given $t_0\in I$, there exists $s_0\in\mathbb{C}$ ($s_0=p_1(r_{12})\frac{p_2(t_0)}{p_1(t_0)}$)

such that the distance between the point $\left(\frac{p_1(t_0)}{p_2(t_0)}, \frac{1}{p_2(t_0)}\right)$ of the given curve and the point $(p_1(r_{12})/s_0, 1/s_0)$ of the asymptote is

$$|1/p_2(t_0) - p_1(t_0)/(p_2(t_0)p_1(r_{12}))| \leq \max\{|1/p_2(t_0) - p_1(t_0)/(p_2(t_0)p_1(r_{12}))| \,|\, t \in I\} \leq \max\{|1/p_2(a) - p_1(a)/(p_2(a)p_1(r_{12}))|, |1/p_2(b) - p_1(b)/(p_2(b)p_1(r_{12}))|\}.$$

On reasons similarly for the general case, and we get that for any interval $I_{j_0} := (\gamma, \beta)$, where $\gamma := (r_{j_0-1} + r_{j_0})/2$, and $\beta := (r_{j_0} + r_{j_0+1})/2$, and $p_{j2}(r_{j_0}) = 0$, for $j = 1$ or $j = 2$, it holds that for every point on the curve piece $\mathcal{C}_{I_{j_0}}$, there exists a point on the asymptote $\mathcal{A}_{j_0}$ defined implicitly by $x_i = p_i(r_{j_0})$, $i \neq j$, $i = 1$ or $i = 2$ (and reciprocally) at distance at most

$$\max\{|1/p_j(\gamma) - p_i(\gamma)/(p_j(\gamma)p_i(r_{j_0}))|, |1/p_j(\beta) - p_i(\beta)/(p_j(\beta)p_i(r_{j_0}))|\}.$$

□

Remark 3. *From the proof of Theorem 3, one deduces that for a point $\mathcal{P}(t_0) \in \mathcal{C}_{\mathcal{I}}$, the point $\overline{\mathcal{P}}(t_0) \in \overline{\mathcal{C}}_{\mathcal{I}}$ is at distance at most m_α, and reciprocally.*
In addition, for a point $(p_i(t_0)/p_j(t_0), 1/p_j(t_0)) \in \mathcal{C}^j_{I_{j_0}}$, $i \neq j$, the point $(p_i(t_0)/p_j(t_0), p_i(t_0)/(p_i(r_{j_0})p_j(t_0))) \in \mathcal{A}^j_{I_{j_0}}$ is at distance at most $\max\{|1/p_j(\gamma) - p_i(\gamma)/(p_j(\gamma)p_i(r_{j_0}))|, |1/p_j(\beta) - p_i(\beta)/(p_j(\beta)p_i(r_{j_0}))|\}$, and reciprocally, where $\gamma := (r_{j_0-1} + r_{j_0})/2$, $\beta := (r_{j_0} + r_{j_0+1})/2$, and $p_{j2}(r_{j_0}) = 0$, for $j = 1$ or $j = 2$.

Using Theorem 3, and applying the results in Section 2.2 in Farouki and Rajan (1988), we deduce the following corollary.

Corollary 2. *The following statements hold:*

1. *The input curve piece $\mathcal{C}_{\mathcal{I}}$ is contained in the offset region of the output curve piece $\overline{\mathcal{C}}_{\mathcal{I}}$ (and reciprocally) at distance at most $2m_\alpha$.*
2. *Let $I_{j_0} := (\gamma, \beta)$, where $\gamma := (r_{j_0-1} + r_{j_0})/2$, $\beta := (r_{j_0} + r_{j_0+1})/2$, and $p_{j2}(r_{j_0}) = 0$, for $j = 1$ or $j = 2$. The curve piece $\mathcal{C}_{I_{j_0}}$ is contained in the offset region of the asymptote defined implicitly by $x_i = p_i(r_{j_0})$, $i \neq j$, $i = 1$ or $i = 2$ (and reciprocally) at distance at most*

$$2\max\{|1/p_j(\gamma) - p_i(\gamma)/(p_j(\gamma)p_i(r_{j_0}))|, |1/p_j(\beta) - p_i(\beta)/(p_j(\beta)p_i(r_{j_0}))|\}.$$

We note that for the case of a given rational parametrization, Remark 2 applies similarly as in the case of a given rational function.

Algorithm and example

In the following, we propose the algorithm as well as the error bounds. We finally illustrate this algorithm with an example. Examples show that, in general, the approximation is better than the error bound provided.

Algorithm Approximate Polynomial Parametrization.

Given a rational algebraic plane curve $\mathcal{C}$ defined by a parametrization $\mathcal{P}(t) = (p_1(t), p_2(t)) \in \mathbb{R}(s)^2$, $p_j(t) = p_{j1}(t)/p_{j2}(t)$, $\deg(p_{j1}) = \deg(p_{j2}) = s_j$, $p_{j2}(r_{ij}) = 0$, $j = 1, 2$, $i = 1, \ldots, s_j$, the algorithm outputs polynomial curves approximating $\mathcal{C}$.
Step 1: Decomposition of $\mathbb{R}$. Assume that among the roots r_{ij}, $i = 1, \ldots, s_j$, $j = 1, 2$, only ℓ roots are real. Denote these roots as $r_j \in \mathbb{R}$, $j = 1, \ldots, \ell$ and let $r_1 < r_2 < \ldots < r_\ell$. Then, compute $a, b \in \mathbb{R}$ such that $|p_i(a)| = |p_i(b)| = \mu$, $i = 1, 2$, $a < r_1 < r_2 < \ldots < r_\ell < b$ (μ is a given real number). In addition, $|a|$ is the minimum and $|b|$ the maximum of all the values satisfying the above equations.

Thus, let $\mathbb{R} = \mathcal{I} \cup I$, where $\mathcal{I} = (-\infty, a) \cup (b, \infty)$ and $I = \bigcup_{j=1}^{\ell} I_j$, $I_j := ((r_{j-1} + r_j)/2, (r_j + r_{j+1})/2)$ (let $(r_{-1} + r_1)/2 := a$ and $(r_s + r_{s+1})/2 := b$).
Step 2: Approximation of the curve piece $\mathcal{C}_{\mathcal{I}}$. Approximate the input curve piece $\mathcal{C}_{\mathcal{I}}$ by the curve piece $\overline{\mathcal{C}}_{\mathcal{I}}$, where $\overline{\mathcal{C}}$ is defined by the rational parametrization

$$\overline{\mathcal{P}}(t) = \left(\frac{p_{11}(t)}{\overline{p}_{12}(t)}, \frac{p_{21}(t)}{\overline{p}_{22}(t)}\right) \in \mathbb{R}(t)^2, \qquad \overline{p}_{i2}(t) = (t - \alpha)^{s_i},\ i = 1, 2.$$

Compute $\alpha \in \mathbb{R}$ such that $g(\alpha) = \sqrt{(p_1(a) - \overline{p}_1(a))^2 + (p_1(b) - \overline{p}_1(b))^2 + (p_2(a) - \overline{p}_2(a))^2 + (p_2(b) - \overline{p}_2(b))^2}$ is minimum. Let m_α be this minimum. Apply Theorem 1 (see Section 2), to compute a polynomial parametrization of $\overline{\mathcal{C}}$.

Step 3: Approximation of the curve piece $\mathcal{C}_I$. Approximate the curve piece $\mathcal{C}_{I_j}$ by the asymptote $\mathcal{A}_j$ defined implicitly by $x_i = p_i(r_j)$, $j = 1, \ldots, \ell$, where $i = 1$ (if $p_{22}(r_j) = 0$) or $i = 2$ (if $p_{12}(r_j) = 0$).

Step 4: Error Analysis.

I. For every point on the curve piece $\mathcal{C}_{\mathcal{I}}$, there exists a point on the curve piece $\overline{\mathcal{C}}_{\mathcal{I}}$ (and reciprocally) at distance at most m_α.

II. Let $I_{j_0} := (\gamma, \beta)$, where $\gamma := (r_{j_0-1} + r_{j_0})/2$, $\beta := (r_{j_0} + r_{j_0+1})/2$, and $p_{j2}(r_{j_0}) = 0$, for $j = 1$ or $j = 2$. For every point on the curve piece $\mathcal{C}_{I_{j_0}}$, there exists a point on the asymptote $\mathcal{A}_{j_0}$ defined implicitly by $x_i = p_i(r_{j_0})$, $i \neq j$, $i = 1$ or $i = 2$ (and reciprocally) at distance at most

$$\max\{|1/p_j(\gamma) - p_i(\gamma)/(p_j(\gamma)p_i(r_{j_0}))|,\ |1/p_j(\beta) - p_i(\beta)/(p_j(\beta)p_i(r_{j_0}))|\}.$$

Example 3. *Let $\mathcal{C}$ be the rational plane curve defined by the rational parametrization $\mathcal{P}(t) = (p_1(t), p_2(t)) = \left(\frac{p_{11}(t)}{p_{12}(t)}, \frac{p_{21}(t)}{p_{22}(t)}\right) = \left(\frac{(t-1)(t-5)}{(t-2997/1000)(t-299/100)}, \frac{(t-7)(t-9)}{(t-2999/1000)(t-3)}\right) \in \mathbb{R}(t)^2$.*

In Step 1 *of the algorithm* Approximate Polynomial Parametrization, *we observe that the roots of the denominators of the parametrizations are all in $\mathbb{R}$. Thus, in order to compute $a, b \in \mathbb{R}$, we solve the equations $|p_i(a)| = |p_i(b)| = \mu = 100$, $i = 1, 2$, where $a < r_1 := r_{11} = 2.99 < r_2 := r_{21} = 2.997 < r_3 := r_{12} = 2.999 < r_4 := r_{22} = 3 < b$, and $|a|$ is the minimum and $|b|$ the maximum of all the values satisfying the above equations. We get that $a \approx 2.453988613$, $b \approx 3.443991185$. Then, we decompose the interval $I = (a, b)$ as $I = \bigcup_{j=1}^{4} I_j$, where $I_1 = (a, (r_1+r_2)/2)$, $I_2 = ((r_1+r_2)/2, (r_2+r_3)/2)$, $I_3 = ((r_2+r_3)/2, (r_3+r_4)/2)$, and $I_4 = ((r_3+r_4)/2, b)$, and $\mathbb{R} = \mathcal{I} \cup \bigcup_{j=1}^{\ell} I_j$, where $\mathcal{I} = (-\infty, a) \cup (b, \infty) = (-\infty, 2.453988613) \cup (3.443991185, \infty)$.*

In Step 2 *of the algorithm, we compute the rational plane curve $\overline{\mathcal{C}}$ defined by the parametrization*

$$\overline{\mathcal{P}}(t) = \left(\frac{p_{11}(t)}{\overline{p}_{12}(t)}, \frac{p_{21}(t)}{\overline{p}_{22}(t)}\right) = \left(\frac{(t-1)(t-5)}{(t-\alpha)^2}, \frac{(t-7)(t-8)}{(t-\alpha)^2}\right) \in \mathbb{R}(t)^2.$$

Now, we look for $\alpha \in \mathbb{R}$ that minimizes $||\mathcal{P}(t) - \overline{\mathcal{P}}(t)||$. Thus, we compute $\alpha \in \mathbb{R}$ such that $g(\alpha) = \sqrt{(p_1(a) - \overline{p}_1(a))^2 + (p_1(b) - \overline{p}_1(b))^2 + (p_2(a) - \overline{p}_2(a))^2 + (p_2(b) - \overline{p}_2(b))^2}$ is minimum. We get that $\alpha \approx 2.999336375$ and $m_\alpha \approx 0.5715548898$. We observe that by applying Theorem 1 (Section 2), one may compute a polynomial parametrization of $\overline{\mathcal{C}}$ by considering the reparametrization $\overline{\mathcal{P}}(1/t + \alpha)$. Hence, we approximate the input curve piece $\mathcal{C}_{\mathcal{I}}$ by curve piece $\overline{\mathcal{C}}_{\mathcal{I}}$, where $\overline{\mathcal{C}}$ is defined by $\overline{\mathcal{P}}(t)$. In Figure 9, we plot $\mathcal{C}_{\mathcal{I}}$ and $\overline{\mathcal{C}}_{\mathcal{I}}$.

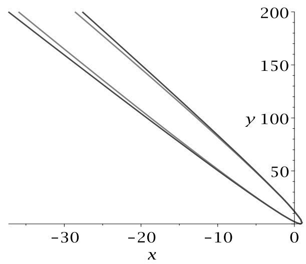

Figure 9: Curve pieces $\mathcal{C}_{\mathcal{I}}$ and $\overline{\mathcal{C}}_{\mathcal{I}}$

Now, we apply Step 3 *of the algorithm* Approximate Polynomial Parametrization. *Since $I = \bigcup_{j=1}^{4} I_j$, $I_1 = (a, (r_1 + r_2)/2)$, $I_2 = ((r_1 + r_2)/2, (r_2 + r_3)/2)$, $I_3 = ((r_2 + r_3)/2, (r_3 + r_4)/2)$, and $I_4 = ((r_3 + r_4)/2, b)$, for $j = 1, \ldots, 4$, we approximate the curve piece $\mathcal{C}_{I_j}$ by the asymptote $\mathcal{A}_j$ defined implicitly by $x_2 = p_2(r_1)$, $x_2 = p_2(r_2)$, $x_1 = p_1(r_3)$, $x_1 = p_1(r_4)$. In Figures 10 and 11, we plot the curve and the approximation with the asymptotes.*

We finally apply Step 4 *of the algorithm to analyze the error analysis:*

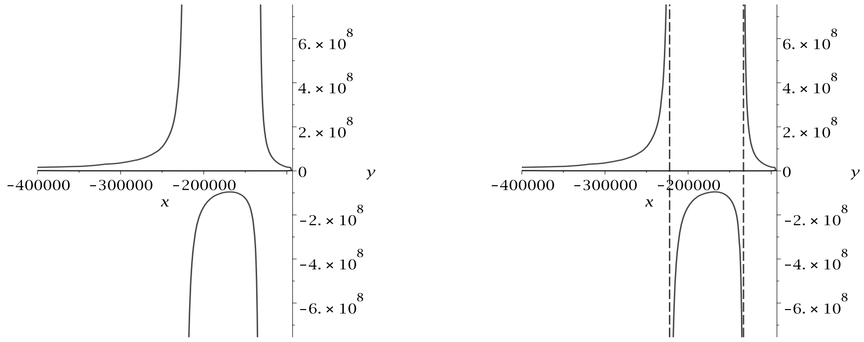

Figure 10: Curve $\mathcal{C}$ (left), and curve $\mathcal{C}$ and vertical asymptotes $\mathcal{A}_3$ and $\mathcal{A}_4$ (right)

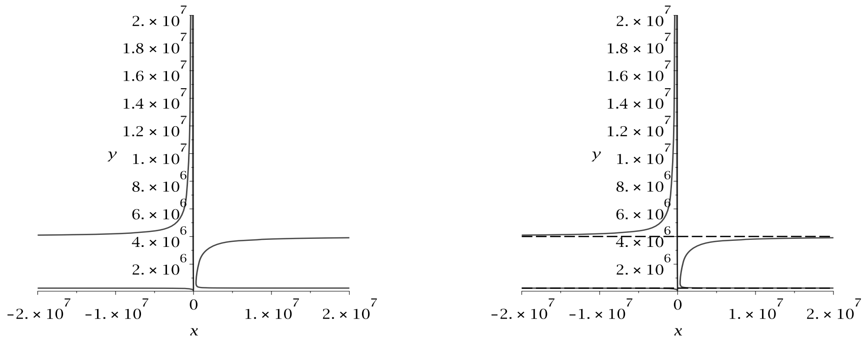

Figure 11: Curve $\mathcal{C}$ (left), and curve $\mathcal{C}$ and horizontal asymptotes $\mathcal{A}_1$ and $\mathcal{A}_2$ (right)

I. *For every point on the curve piece* $\mathcal{C}_{\mathcal{I}}$*, there exists a point on the curve piece* $\overline{\mathcal{C}}_{\mathcal{I}}$ *(and reciprocally) at distance at most* $m_\alpha \approx 0.5715548898$*. We note that given the point* $\mathcal{P}(t_0) \in \mathcal{C},\ t_0 \in \mathcal{I}$ *the point satisfying the above statement is* $\overline{\mathcal{P}}(t_0) \in \overline{\mathcal{C}}$ *(see Remark 3).*

II. *We analyze the error for every curve piece* $\mathcal{C}_{I_j},\ j = 1, \ldots, 4$.

(a) *For every point on the curve piece* $\mathcal{C}_{I_1}$*, there exists a point on the asymptote* $\mathcal{A}_1$ *(and reciprocally) at distance at most*

$$\max\left\{\left|\frac{1}{p_1(a)} - \frac{p_2(a)}{p_1(a)p_2(r_{11})}\right|, \left|\frac{1}{p_1((r_1+r_2)/2)} - \frac{p_2((r_1+r_2)/2)}{p_1((r_1+r_2)/2)p_2(r_1)}\right|\right\} \approx 0.07859580935.$$

From Remark 3, one deduces that given the point $(p_2(t_0)/p_1(t_0), 1/p_1(t_0)) \in \mathcal{C}^1$*, where* $t_0 \in I_1$*, the point on the asymptote satisfying the above statement is* $(p_2(t_0)/p_1(t_0), p_2(t_0)/(p_2(r_1)p_1(t_0))) \in \mathcal{A}_1^1$ *(we remind that since we are at the infinity, we are considering the dehomogenization of curves).*

(b) *For every point on the curve piece* $\mathcal{C}_{I_2}$*, there exists a point on the asymptote* $\mathcal{A}_2$ *(and reciprocally) at distance at most*

$$\max\left\{\left|\frac{1}{p_1((r_2+r_3)/2)} - \frac{p_2((r_2+r_3)/2)}{p_1((r_2+r_3)/2)p_2(r_2)}\right|, \left|\frac{1}{p_1((r_1+r_2)/2)} - \frac{p_2((r_1+r_2)/2)}{p_1((r_1+r_2)/2)p_2(r_2)}\right|\right\} \approx 3.997505871 \cdot 10^{-6}.$$

From Remark 3, one deduces that given $(p_2(t_0)/p_1(t_0), 1/p_1(t_0)) \in \mathcal{C}^1$*, where* $t_0 \in I_2$*, the point on the asymptote satisfying the above statement is* $(p_2(t_0)/p_1(t_0), p_2(t_0)/(p_2(r_2)p_1(t_0))) \in \mathcal{A}_2^1$.

(c) *For every point on the curve piece* $\mathcal{C}_{I_3}$*, there exists a point on the asymptote* $\mathcal{A}_3$ *(and reciprocally) at distance at most*

$$\max\left\{\left|\frac{1}{p_2((r_2+r_3)/2)} - \frac{p_1((r_2+r_3)/2)}{p_2((r_2+r_3)/2)p_1(r_3)}\right|, \left|\frac{1}{p_2((r_3+r_4)/2)} - \frac{p_1((r_3+r_4)/2)}{p_2((r_3+r_4)/2)p_1(r_3)}\right|\right\} \approx 1.040797756 \cdot 10^{-7}.$$

From Remark 3, one deduces that given the point $(p_1(t_0)/p_2(t_0),\ 1/p_2(t_0)) \in \mathcal{C}^2,\ t_0 \in I_3$*, the point on the asymptote satisfying the statement is* $(p_1(t_0)/p_2(t_0),\ p_1(t_0)/(p_1(r_3)p_2(t_0))) \in \mathcal{A}_3^2$.

(d) *For every point on the curve piece* $\mathcal{C}_{I_4}$, *there exists a point on the asymptote* $\mathcal{A}_4$ *(and reciprocally) at distance at most*

$$\max\left\{\left|\frac{1}{p_2(b)}-\frac{p_1(b)}{p_2(b)p_1(r_4)}\right|,\left|\frac{1}{p_2((r_3+r_4)/2)}-\frac{p_1((r_3+r_4)/2)}{p_2((r_3+r_4)/2)p_1(r_4)}\right|\right\}\approx 0.009998594345.$$

From Remark 3, one deduces that given the point $(p_1(t_0)/p_2(t_0),\, 1/p_2(t_0)) \in \mathcal{C}^2,\ t_0 \in I_4$, *the point on the asymptote satisfying the statement is* $(p_1(t_0)/p_2(t_0),\, p_1(t_0)/(p_1(r_4)p_2(t_0))) \in \mathcal{A}_4^2$.

4. Conclusion

We approximate a rational parametric curve using a polynomial one. More precisely, the method provides a parametric polynomial plane curve $\overline{\mathcal{C}}$ defined by a parametrization $\overline{\mathcal{P}}(t)$ that approximates $\mathcal{C}$ for $t \in \mathcal{I} := (-\infty, a) \cup (b, \infty), a, b \in \mathbb{R}$. For $t \in I := (a, b)$, we use the asymptotes of $\mathcal{C}$ to approximate the input curve. In addition, we present an error analysis where we prove that the curve piece defined by $\mathcal{P}(t)$, $t \in \mathcal{I}$, is in the offset region of the output curve $\overline{\mathcal{C}}$ at distance at most $2m_\alpha$, and conversely ($m_\alpha \in \mathbb{R}$, and $m_\alpha \geq 0$). The constant m_α is directly related with the values $a, b \in \mathbb{R}$ defining the interval $\mathcal{I}$. It is also shown that for $t \in I$, the approximation by asymptotes is good if the roots of the denominators of $\mathcal{P}(t)$ are close enough although the method can be adapted for the case of the existence of roots not being "*close enough*" (see Remark 2). All the algorithms can be easily generalized for space curves. The results are helpful in refining the rational curves computed from the CAD/CNC process.

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References

Abhyankar, S., 1990. Algebraic geometry for scientists and engineers. Ams Ebooks Program 35, 295.

Arrondo, E., Sendra, J., Sendra, J.R., 1997. Parametric generalized offsets to hypersurfaces. Journal of Symbolic Computation 23, 267 – 285. doi:https://doi.org/10.1006/jsco.1996.0088.

Blasco, A., Pérez-Díaz, 2015. Asymptotes of space curves. Journal of Computational and Applied Mathematics 278, 231 – 247. doi:https://doi.org/10.1016/j.cam.2014.10.013.

Blasco, A., Pérez-Díaz, S., 2014a. Asymptotes and perfect curves. Computer Aided Geometric Design 31, 81 – 96. doi:https://doi.org/10.1016/j.cagd.2013.12.004.

Blasco, A., Pérez-Díaz, S., 2014b. Asymptotic behavior of an implicit algebraic plane curve. Computer Aided Geometric Design 31, 345 – 357. doi:https://doi.org/10.1016/j.cagd.2014.04.002.

Corless, R.M., Giesbrecht, M.W., van Hoeij, M., Kotsireas, I.S., Watt, S.M., 2001. Towards factoring bivariate approximate polynomials, in: Proceedings of the 2001 International Symposium on Symbolic and Algebraic Computation, ACM, New York, NY, USA. pp. 85–92. doi:10.1145/384101.384114.

Farouki, R., Rajan, V., 1988. On the numerical condition of algebraic curves and surfaces 1. implicit equations. Computer Aided Geometric Design 5, 215 – 252. doi:https://doi.org/10.1016/0167-8396(88)90005-2.

H?mmerlin, G., Hoffman, K.H., 1991. Numerical Mathematics. Undergraduate Text in Mathematics, Springer New York.

Lin, F., Shen, L.Y., Yuan, C.M., Mi, Z., 2019. Certified space curve fitting and trajectory planning for cnc machining with cubic b-splines. Computer-Aided Design 106, 13 – 29. doi:https://doi.org/10.1016/j.cad.2018.08.001.

Pérez-Díaz, S., 2006. On the problem of proper reparametrization for rational curves and surfaces. Computer Aided Geometric Design 23, 307 – 323. doi:https://doi.org/10.1016/j.cagd.2006.01.001.

Pérez-Díaz, S., Sendra, J., Sendra, J., 2004. Parametrization of approximate algebraic curves by lines. Theoretical Computer Science 315, 627 – 650. doi:https://doi.org/10.1016/j.tcs.2004.01.010.

Pérez-Díaz, S., Sendra, J.R., Villarino, C., 2007. Finite piecewise polynomial parametrization of plane rational algebraic curves. Applicable Algebra in Engineering, Communication and Computing 18, 91–105. doi:10.1007/s00200-006-0029-2.

Sederberg, T.W., Kakimoto, M., 1991. Approximating rational curves using polynomial curves, in: Farin, G. (Ed.), G Farin Nurbs for Curve & Surface Design Siam Philadelphia Pa, SIAM.

Sendra, J.R., Winkler, F., Pérez-Díaz, S., 2007. Rational Algebraic Curves: A Computer Algebra Approach. 1st ed., Springer Publishing Company, Incorporated.

Shen, L.Y., Yuan, C.M., Gao, X.S., 2012. Certified approximation of parametric space curves with cubic b-spline curves. Computer Aided Geometric Design 29, 648 – 663. doi:https://doi.org/10.1016/j.cagd.2012.06.001.

Sonia Pérez-Díaz, Juana Sendra, J.R.S., 2005. Distance properties of epsilon points on algebraic curves, in: Computational methods for algebraic spline surfaces, Springer, Berlin, Heidelberg. pp. 45–61.

Timar, S.D., Farouki, R.T., Smith, T.S., Boyadjieff, C.L., 2005. Algorithms for timeCoptimal control of cnc machines along curved tool paths. Robotics and Computer-Integrated Manufacturing 21, 37 – 53. doi:https://doi.org/10.1016/j.rcim.2004.05.004.

Yang, Z., Shen, L.Y., Yuan, C.M., Gao, X.S., 2015. Curve fitting and optimal interpolation for cnc machining under confined error using quadratic b-splines. Computer-Aided Design 66, 62 – 72. doi:https://doi.org/10.1016/j.cad.2015.04.010.