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On parametric multisummable formal solutions to some nonlinear initial value Cauchy problems

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Abstract

We study a nonlinear initial value Cauchy problem depending upon a complex perturbation parameter ϵ whose coefficients depend holomorphically on (ϵ, t) near the origin in $\mathbb{C}^2$ and are bounded holomorphic on some horizontal strip in $\mathbb{C}$ w.r.t. the space variable. In our previous contribution (Lastra and Malek in Parametric Gevrey asymptotics for some nonlinear initial value Cauchy problems, arXiv:1403.2350), we assumed the forcing term of the Cauchy problem to be analytic near 0. Presently, we consider a family of forcing terms that are holomorphic on a common sector in time t and on sectors w.r.t. the parameter ϵ whose union form a covering of some neighborhood of 0 in $\mathbb{C}^*$, which are asked to share a common formal power series asymptotic expansion of some Gevrey order as ϵ tends to 0. We construct a family of actual holomorphic solutions to our Cauchy problem defined on the sector in time and on the sectors in ϵ mentioned above. These solutions are achieved by means of a version of the so-called accelero-summation method in the time variable and by Fourier inverse transform in space. It appears that these functions share a common formal asymptotic expansion in the perturbation parameter. Furthermore, this formal series expansion can be written as a sum of two formal series with a corresponding decomposition for the actual solutions which possess two different asymptotic Gevrey orders, one stemming from the shape of the equation and the other originating from the forcing terms. The special case of multisummability in ϵ is also analyzed thoroughly. The proof leans on a version of the so-called Ramis-Sibuya theorem which entails two distinct Gevrey orders. Finally, we give an application to the study of parametric multi-level Gevrey solutions for some nonlinear initial value Cauchy problems with holomorphic coefficients and forcing term in (ϵ, t) near 0 and bounded holomorphic on a strip in the complex space variable.

MSC: 35C10; 35C20

Keywords: asymptotic expansion; Borel-Laplace transform; Fourier transform; Cauchy problem; formal power series; nonlinear integro-differential equation; nonlinear partial differential equation; singular perturbation

1 Introduction

We consider a family of parameter depending nonlinear initial value Cauchy problems of the form

$$Q(\partial_z)\big(\partial_t u^{\mathfrak{d}_p}(t,z,\epsilon)\big) = c_{1,2}(\epsilon)\big(Q_1(\partial_z)u^{\mathfrak{d}_p}(t,z,\epsilon)\big)\big(Q_2(\partial_z)u^{\mathfrak{d}_p}(t,z,\epsilon)\big)$$

$$
\begin{aligned}
&+ \epsilon^{(\delta_D-1)(k_2+1)-\delta_D+1} t^{(\delta_D-1)(k_2+1)} \partial_t^{\delta_D} R_D(\partial_z) u^{\mathfrak{d}_p}(t,z,\epsilon) \\
&+ \sum_{l=1}^{D-1} \epsilon^{\Delta_l} t^{d_l} \partial_t^{\delta_l} R_l(\partial_z) u^{\mathfrak{d}_p}(t,z,\epsilon) \\
&+ c_0(t,z,\epsilon) R_0(\partial_z) u^{\mathfrak{d}_p}(t,z,\epsilon) + c_F(\epsilon) f^{\mathfrak{d}_p}(t,z,\epsilon)
\end{aligned}
\tag{1}
$$

for given vanishing initial data $u^{\mathfrak{d}_p}(0,z,\epsilon) \equiv 0$, where $D \geq 2$ and δ_D, k_2, Δ_l, d_l, δ_l, $1 \leq l \leq D-1$ are nonnegative integers and $Q(X)$, $Q_1(X)$, $Q_2(X)$, $R_l(X)$, $0 \leq l \leq D$, are polynomials belonging to $\mathbb{C}[X]$. The coefficient $c_0(t,z,\epsilon)$ is a bounded holomorphic function on a product $D(0,r) \times H_\beta \times D(0,\epsilon_0)$, where $D(0,r)$ (resp. $D(0,\epsilon_0)$) denotes a disc centered at 0 with small radius $r > 0$ (resp. $\epsilon_0 > 0$) and $H_\beta = \{z \in \mathbb{C}/|\operatorname{Im}(z)| < \beta\}$ is some strip of width $\beta > 0$. The coefficients $c_{1,2}(\epsilon)$ and $c_F(\epsilon)$ define bounded holomorphic functions on $D(0,\epsilon_0)$ vanishing at $\epsilon = 0$. The forcing terms $f^{\mathfrak{d}_p}(t,z,\epsilon)$, $0 \leq p \leq \varsigma - 1$, form a family of bounded holomorphic functions on products $\mathcal{T} \times H_\beta \times \mathcal{E}_p$, where $\mathcal{T}$ is a small sector centered at 0 contained in $D(0,r)$ and $\{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$ is a set of bounded sectors with aperture slightly larger than π/k_2 covering some neighborhood of 0 in $\mathbb{C}^*$. We make assumptions in order that all the functions $\epsilon \mapsto f^{\mathfrak{d}_p}(t,z,\epsilon)$, seen as functions from $\mathcal{E}_p$ into the Banach space $\mathbb{F}$ of bounded holomorphic functions on $\mathcal{T} \times H_\beta$ endowed with the supremum norm, share a common asymptotic expansion $\hat{f}(t,z,\epsilon) = \sum_{m \geq 0} f_m(t,z)\epsilon^m/m! \in \mathbb{F}[[\epsilon]]$ of Gevrey order $1/k_1$ on $\mathcal{E}_p$, for some integer $1 \leq k_1 < k_2$; see Lemma 11.

Our main purpose is the construction of actual holomorphic solutions $u^{\mathfrak{d}_p}(t,z,\epsilon)$ to the problem (1) on the domains $\mathcal{T} \times H_\beta \times \mathcal{E}_p$ and to analyze their asymptotic expansions as ϵ tends to 0.

This work is a continuation of the study initiated in [1] where the authors have studied initial value problems with a quadratic nonlinearity of the form

$$
\begin{aligned}
Q(\partial_z)\big(\partial_t u(t,z,\epsilon)\big) &= \big(Q_1(\partial_z)u(t,z,\epsilon)\big)\big(Q_2(\partial_z)u(t,z,\epsilon)\big) \\
&\quad + \epsilon^{(\delta_D-1)(k+1)-\delta_D+1} t^{(\delta_D-1)(k+1)} \partial_t^{\delta_D} R_D(\partial_z) u(t,z,\epsilon) \\
&\quad + \sum_{l=1}^{D-1} \epsilon^{\Delta_l} t^{d_l} \partial_t^{\delta_l} R_l(\partial_z) u(t,z,\epsilon) \\
&\quad + c_0(t,z,\epsilon) R_0(\partial_z) u(t,z,\epsilon) + f(t,z,\epsilon)
\end{aligned}
\tag{2}
$$

for given vanishing initial data $u(0,z,\epsilon) \equiv 0$, where D, Δ_l, d_l, δ_l are positive integers and $Q(X)$, $Q_1(X)$, $Q_2(X)$, $R_l(X)$, $0 \leq l \leq D$, are polynomials with complex coefficients. Under the assumption that the coefficients $c_0(t,z,\epsilon)$ and the forcing term $f(t,z,\epsilon)$ are bounded holomorphic functions on $D(0,r) \times H_\beta \times D(0,\epsilon_0)$, one can build, using some Borel-Laplace procedure and Fourier inverse transform, a family of holomorphic bounded functions $u_p(t,z,\epsilon)$, $0 \leq p \leq \varsigma - 1$, solutions of (2), defined on the products $\mathcal{T} \times H_\beta \times \mathcal{E}_p$, where $\mathcal{E}_p$ has an aperture slightly larger than π/k. Moreover, the functions $\epsilon \mapsto u_p(t,z,\epsilon)$ share a common formal power series $\hat{u}(t,z,\epsilon) = \sum_{m \geq 0} h_m(t,z)\epsilon^m/m!$ as asymptotic expansion of Gevrey order $1/k$ on $\mathcal{E}_p$. In other words, $u_p(t,z,\epsilon)$ is the k-sum of $\hat{u}(t,z,\epsilon)$ on $\mathcal{E}_p$; see Definition 9.

In this paper, we observe that the asymptotic expansion of the solutions $u^{\mathfrak{d}_p}(t,z,\epsilon)$ of (1) w.r.t. ϵ on $\mathcal{E}_p$, defined as $\hat{u}(t,z,\epsilon) = \sum_{m \geq 0} h_m(t,z)\epsilon^m/m! \in \mathbb{F}[[\epsilon]]$, inherits a finer structure which involves the two Gevrey orders $1/k_1$ and $1/k_2$. Namely, the order $1/k_2$ originates from (1) itself and its highest order term $\epsilon^{(\delta_D-1)(k_2+1)-\delta_D+1}t^{(\delta_D-1)(k_2+1)}\partial_t^{\delta_D}R_D(\partial_z)$ as was the case in the work [1] mentioned above and the scale $1/k_1$ arises, as a new feature, from the asymptotic expansion $\hat{f}$ of the forcing terms $f^{\mathfrak{d}_p}(t,z,\epsilon)$. We can also describe conditions for which $u^{\mathfrak{d}_p}(t,z,\epsilon)$ is the (k_2,k_1)-sum of $\hat{u}(t,z,\epsilon)$ on $\mathcal{E}_p$ for some $0 \le p \le \varsigma - 1$; see Definition 10. More specifically, we can present our two main statements and its application as follows.

Main results *Let $k_2 > k_1 \ge 1$ be integers. We choose a family $\{\mathcal{E}_p\}_{0\le p\le\varsigma-1}$ of bounded sectors with aperture slightly larger than π/k_2 which defines a good covering in $\mathbb{C}^*$ (see Definition 7) and a set of adequate directions $\mathfrak{d}_p \in \mathbb{R}$, $0 \le p \le \varsigma - 1$ for which the constraints (152) and (153) hold. We also take an open bounded sector $\mathcal{T}$ centered at 0 such that, for every $0 \le p \le \varsigma - 1$, the product ϵt belongs to a sector with direction $\mathfrak{d}_p$ and aperture slightly larger than π/k_2, for all $\epsilon \in \mathcal{E}_p$, all $t \in \mathcal{T}$. We make the assumption that the coefficient $c_0(t,z,\epsilon)$ can be written as a convergent series of the special form*

$$c_0(t,z,\epsilon) = c_0(\epsilon)\sum_{n\ge 0} c_{0,n}(z,\epsilon)(\epsilon t)^n$$

on a domain $D(0,r) \times H_{\beta'} \times D(0,\epsilon_0)$, where $H_{\beta'}$ is a strip of width β', such that $\mathcal{T} \subset D(0,r)$, $\bigcup_{0\le p\le\varsigma-1}\mathcal{E}_p \subset D(0,\epsilon_0)$ and $0 < \beta' < \beta$ are given positive real numbers. The coefficients $c_{0,n}(z,\epsilon)$, $n \ge 0$, are supposed to be inverse Fourier transform of functions $m \mapsto C_{0,n}(m,\epsilon)$ that belong to the Banach space $E_{(\beta,\mu)}$ (see Definition 2) for some $\mu > \max(\deg(Q_1) + 1, \deg(Q_2) + 1)$ and depend holomorphically on ϵ in $D(0,\epsilon_0)$ and $c_0(\epsilon)$ is a holomorphic function on $D(0,\epsilon_0)$ vanishing at 0. Since we have in view our principal application (Theorem 3), we choose the forcing term $f^{\mathfrak{d}_p}(t,z,\epsilon)$ as a m_{k_2}-Fourier-Laplace transform

$$f^{\mathfrak{d}_p}(t,z,\epsilon) = \frac{k_2}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_{\gamma_p}} \psi_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon)e^{-(\frac{u}{\epsilon t})^{k_2}}e^{izm}\frac{du}{u}\,dm,$$

where the inner integration is made along some half-line $L_{\gamma_p} \subset S_{\mathfrak{d}_p}$ and $S_{\mathfrak{d}_p}$ is an unbounded sector with bisecting direction $\mathfrak{d}_p$, with small aperture and where $\psi_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon)$ is a holomorphic function w.r.t. u on $S_{\mathfrak{d}_p}$, defined as an integral transform called acceleration operator with indices m_{k_2} and m_{k_1},

$$\psi_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon) = \int_{L_{\gamma_p^1}} \psi_{k_1}^{\mathfrak{d}_p}(h,m,\epsilon)G(u,h)\frac{dh}{h},$$

where $G(u,h)$ is a kernel function with exponential decay of order $\kappa = (\frac{1}{k_1} - \frac{1}{k_2})^{-1}$; see (114). The integration path $L_{\gamma_p^1}$ is a half-line in an unbounded sector $U_{\mathfrak{d}_p}$ with bisecting direction $\mathfrak{d}_p$ and $\psi_{k_1}^{\mathfrak{d}_p}(h,m,\epsilon)$ is a function with exponential growth of order k_1 w.r.t. h on $U_{\mathfrak{d}_p} \cup D(0,\rho)$ and exponential decay w.r.t. m on $\mathbb{R}$, satisfying the bounds (156). The function $f^{\mathfrak{d}_p}(t,z,\epsilon)$ represents a bounded holomorphic function on $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$. Actually, it turns out that $f^{\mathfrak{d}_p}(t,z,\epsilon)$ can be simply written as a m_{k_1}-Fourier-Laplace transform of $\psi_{k_1}^{\mathfrak{d}_p}(h,m,\epsilon)$,

$$f^{\mathfrak{d}_p}(t,z,\epsilon) = \frac{k_1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_{\gamma_p}} \psi_{k_1}^{\mathfrak{d}_p}(u,m,\epsilon)e^{-(\frac{u}{\epsilon t})^{k_1}}e^{izm}\frac{du}{u}\,dm.$$

Our first result stated in Theorem 1 reads as follows. We make the assumption that the integers δ_D, k_2, Δ_l, d_l, δ_l, $1 \le l \le D - 1$ satisfy the inequalities (147), (148), and (160). The *polynomials $Q(X)$, $Q_1(X)$, $Q_2(X)$, and $R_l(X)$, $0 \leq l \leq D$, are submitted to the constraints (149) on their degrees. We require the existence of constants $r_{Q,R_l} > 0$ such that*

$$\left|\frac{Q(im)}{R_l(im)}\right| \geq r_{Q,R_l}$$

for all $m \in \mathbb{R}$, all $1 \leq l \leq D$ (see (150)) and, moreover, that the quotient $Q(im)/R_D(im)$ belongs to some suitable unbounded sector S_{Q,R_D} for all $m \in \mathbb{R}$ (see (151)). Then, if the sup norms of the coefficients $c_{1,2}(\epsilon)/\epsilon$, $c_0(\epsilon)/\epsilon$, and $c_F(\epsilon)/\epsilon$ on $D(0,\epsilon_0)$ are chosen small enough and provided that the radii r_{Q,R_l}, $1 \leq l \leq D$, are taken large enough, we can construct a family of holomorphic bounded functions $u^{\mathfrak{d}_p}(t,z,\epsilon)$, $0 \leq p \leq \varsigma - 1$, defined on the products $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$, which solves the problem (1) with initial data $u^{\mathfrak{d}_p}(0,z,\epsilon) \equiv 0$. Similarly to the forcing term, $u^{\mathfrak{d}_p}(t,z,\epsilon)$ can be written as a m_{k_2}-Fourier-Laplace transform

$$u^{\mathfrak{d}_p}(t,z,\epsilon) = \frac{k_2}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\gamma_p}} \omega_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon) e^{-(\frac{u}{\epsilon t})^{k_2}} e^{izm} \frac{du}{u} \, dm,$$

where $\omega_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon)$ denotes a function with at most exponential growth of order k_2 in u on $S_{\mathfrak{d}_p}$ and exponential decay in $m \in \mathbb{R}$, satisfying (166). The function $\omega_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon)$ is shown to be the analytic continuation of a function $\mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_p}(\omega_{k_1}^{\mathfrak{d}_p})(u,m,\epsilon)$ defined only on a bounded sector $S_{\mathfrak{d}_p}^b$ with aperture slightly larger than π/κ w.r.t. u, for all $m \in \mathbb{R}$, with the help of an acceleration operator with indices m_{k_2} and m_{k_1},

$$\mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_p}(\omega_{k_1}^{\mathfrak{d}_p})(u,m,\epsilon) = \int_{L_{\gamma_p^1}} \omega_{k_1}^{\mathfrak{d}_p}(h,m,\epsilon) G(u,h) \frac{dh}{h}.$$

We show that, in general, $\omega_{k_1}^{\mathfrak{d}_p}(h,m,\epsilon)$ suffers an exponential growth of order larger than k_1 (and actually less than κ) w.r.t. h on $U_{\mathfrak{d}_p} \cup D(0,\rho)$, and obeys the estimates (168). At this point $u^{\mathfrak{d}_p}(t,z,\epsilon)$ cannot be merely expressed as a m_{k_1}-Fourier-Laplace transform of $\omega_{k_1}^{\mathfrak{d}_p}$ and is obtained by a version of the so-called accelero-summation procedure, as described in [2], Chapter 5.

Our second main result, described in Theorem 2, asserts that the functions $u^{\mathfrak{d}_p}$, seen as maps from $\mathcal{E}_p$ into $\mathbb{F}$, for $0 \leq p \leq \varsigma - 1$, turn out to share on $\mathcal{E}_p$ a common formal power series $\hat{u}(\epsilon) = \sum_{m \geq 0} h_m \epsilon^m / m! \in \mathbb{F}[[\epsilon]]$ as asymptotic expansion of Gevrey order $1/k_1$. The formal series $\hat{u}(\epsilon)$ formally solves (1) where the analytic forcing term $f^{\mathfrak{d}_p}(t,z,\epsilon)$ is replaced by its asymptotic expansion $\hat{f}(t,z,\epsilon) \in \mathbb{F}[[\epsilon]]$ of Gevrey order $1/k_1$ (see Lemma 11). Furthermore, the functions $u^{\mathfrak{d}_p}$ and the formal series $\hat{u}$ have a fine structure which actually involves two different Gevrey orders of asymptotics. Namely, $u^{\mathfrak{d}_p}$ and $\hat{u}$ can be written as sums

$$\hat{u}(\epsilon) = a(\epsilon) + \hat{u}_1(\epsilon) + \hat{u}_2(\epsilon), \qquad u^{\mathfrak{d}_p}(t,z,\epsilon) = a(\epsilon) + u_1^{\mathfrak{d}_p}(\epsilon) + u_2^{\mathfrak{d}_p}(\epsilon),$$

where $a(\epsilon)$ is a convergent series near $\epsilon = 0$ with coefficients in $\mathbb{F}$ and $\hat{u}_1(\epsilon)$ (resp. $\hat{u}_2(\epsilon)$) belongs to $\mathbb{F}[[\epsilon]]$ and is the asymptotic expansion of Gevrey order $1/k_1$ (resp. $1/k_2$) of the $\mathbb{F}$-valued function $u_1^{\mathfrak{d}_p}(\epsilon)$ (resp. $u_2^{\mathfrak{d}_p}(\epsilon)$) on $\mathcal{E}_p$. Besides, under a more restrictive assumption on the covering $\{\mathcal{E}_p\}_{0 \leq p \leq \varsigma - 1}$ and the unbounded sectors $\{U_{\mathfrak{d}_p}\}_{0 \leq p \leq \varsigma - 1}$ (see assumption (5) in Theorem 2), one gets that $u^{\mathfrak{d}_{p_0}}(t,z,\epsilon)$ is even the (k_2,k_1)-sum of $\hat{u}(\epsilon)$ on some sector $\mathcal{E}_{p_0}$, *with* $0 \leq p_0 \leq \varsigma - 1$, *meaning that* $u_1^{\mathfrak{d}_{p_0}}(\epsilon)$ *can be analytically continued on a larger sector* S_{π/k_1}, *containing* $\mathcal{E}_{p_0}$, *with aperture slightly larger than* π/k_1 *where it becomes the* k_1*-sum of* $\hat{u}_1(\epsilon)$ *and by construction* $u_2^{\mathfrak{d}_{p_0}}(\epsilon)$ *is already the* k_2*-sum of* $\hat{u}_2(\epsilon)$ *on* $\mathcal{E}_{p_0}$; *see Definition* 10.

As an important application (Theorem 3*), we deal with the special case when the forcing terms* $f^{\mathfrak{d}_p}(t,z,\epsilon)$ *themselves solve a linear partial differential equation with a similar shape as (*2*), see (*220*), whose coefficients are holomorphic functions on* $D(0,r) \times H_\beta \times D(0,\epsilon_0)$. *When this holds, it turns out that* $u^{\mathfrak{d}_p}(t,z,\epsilon)$ *and its asymptotic expansion* $\hat{u}(t,z,\epsilon)$ *solves a nonlinear singularly perturbed PDE with analytic coefficients and forcing term on* $D(0,r) \times H_\beta \times D(0,\epsilon_0)$, *see (*224*).*

We stress the fact that our application (Theorem 3) relies on the factorization of some nonlinear differential operator which is an approach that belongs to an active domain of research in symbolic computation these last years, see for instance [3–8].

We mention that a similar result has been recently obtained by Tahara and Yamazawa, see [9], for the multisummability of formal series $\hat{u}(t,x) = \sum_{n\geq 0} u_n(x)t^n \in \mathcal{O}(\mathbb{C}^N)[[t]]$ with entire coefficients on $\mathbb{C}^N$, $N \geq 1$, solutions of general non-homogeneous time depending linear PDEs of the form

$$\partial_t^m u + \sum_{j+|\alpha|\leq L} a_{j,\alpha}(t)\partial_t^j \partial_x^\alpha u = f(t,x)$$

for given initial data $(\partial_t^j u)(0,x) = \varphi_j(x)$, $0 \leq j \leq m-1$ (where $1 \leq m \leq L$), provided that the coefficients $a_{j,\alpha}(t)$ together with $t \mapsto f(t,x)$ are analytic near 0 and that $\varphi_j(x)$ with the forcing term $x \mapsto f(t,x)$ belong to a suitable class of entire functions of finite exponential order on $\mathbb{C}^N$. The different levels of multisummability are related to the slopes of a Newton polygon attached to the main equation and analytic acceleration procedures as described above are heavily used in their proof.

It is worthwhile noticing that the multisummable structure of formal solutions to linear and nonlinear meromorphic ODEs has been discovered two decades ago, see for instance [10–15], but in the framework of PDEs very few results are known. In the linear case in two complex variables with constant coefficients, we mention the important contributions of Balser *et al.* [16] and Michalik [17, 18]. Their strategy consists in the construction of a multisummable formal solution written as a sum of formal series, each of them associated to a root of the symbol attached to the PDE using the so-called Puiseux expansion for the roots of polynomial with holomorphic coefficients. In the linear and nonlinear context of PDEs that come from a perturbation of ordinary differential equations, we refer to the work of Ouchi [19, 20], which is based on a Newton polygon approach and accelero-summation technics as in [9]. Our result concerns more peculiarly multisummability and multiple scale analysis in the complex parameter ϵ. Also from this point of view, only few advances have been performed. Among them, we must mention two recent works by Suzuki and Takei [21] and Takei [22], for WKB solutions of the Schrödinger equation

$$\epsilon^2 \psi''(z) = \left(z - \epsilon^2 z^2\right)\psi(z)$$

which possesses 0 as fixed turning point and $z_\epsilon = \epsilon^{-2}$ as movable turning point tending to infinity as ϵ tends to 0.

In the sequel, we describe our main intermediate results and the sketch of the arguments needed in their proofs. In a first part, we depart from an auxiliary parameter depending initial value differential and convolution equation which is regularly perturbed in its parameter ϵ; see (70). This equation is formally constructed by making the change of variable $T = \epsilon t$ in (1) and by taking the Fourier transform w.r.t. the variable z (as done in our previous contribution [1]). We construct a formal power series $\hat{U}(T,m,\epsilon) = \sum_{n\geq 1} U_n(m,\epsilon)T^n$ solution of (70) whose coefficients $m \mapsto U_n(m,\epsilon)$ depend holomorphically on ϵ near 0 and belong to a Banach space $E_{(\beta,\mu)}$ of continuous functions with exponential decay on $\mathbb{R}$ introduced by Costin and Tanveer in [23].

As a first step, we follow the strategy recently developed by Tahara and Yamazawa in [9], namely we multiply each side of (70) by the power T^{k_1+1} which transforms it into an equation (75) which involves only differential operators in T of irregular type at $T = 0$ of the form $T^{\beta}\partial_T$ with $\beta \geq k_1 + 1$ due to the assumption (72) on the shape of (70). Then we apply a formal Borel transform of order k_1, which we call m_{k_1}-Borel transform in Definition 4, to the formal series $\hat{U}$ with respect to T, denoted by

$$\omega_{k_1}(\tau,m,\epsilon) = \sum_{n\geq 1} U_n(m,\epsilon)\frac{\tau^n}{\Gamma(n/k_1)}.$$

Then we show that $\omega_{k_1}(\tau,m,\epsilon)$ formally solves a convolution equation in both variables τ and m, see (83). Under some size constraints on the sup norm of the coefficients $c_{1,2}(\epsilon)/\epsilon$, $c_0(\epsilon)/\epsilon$ and $c_F(\epsilon)/\epsilon$ near 0, we show that $\omega_{k_1}(\tau,m,\epsilon)$ is actually convergent for τ on some fixed neighborhood of 0 and can be extended to a holomorphic function $\omega_{k_1}^d(\tau,m,\epsilon)$ on unbounded sectors U_d centered at 0 with bisecting direction d and tiny aperture, provided that the m_{k_1}-Borel transform of the formal forcing term $F(T,m,\epsilon)$, denoted by $\psi_{k_1}(\tau,m,\epsilon)$ is convergent near $\tau = 0$ and can be extended on U_d w.r.t. τ as a holomorphic function $\psi_{k_1}^d(\tau,m,\epsilon)$ with exponential growth of order less than k_1. Besides, the function $\omega_{k_1}^d(\tau,m,\epsilon)$ satisfies estimates of the form: there exist constants $\nu > 0$ and $\varpi_d > 0$ with

$$\left|\omega_{k_1}^d(\tau,m,\epsilon)\right| \leq \varpi_d\bigl(1+|m|\bigr)^{-\mu}e^{-\beta|m|}\frac{|\tau|}{1+|\tau|^{2k_1}}e^{\nu|\tau|^{\kappa}}$$

for all $\tau \in U_d$, all $m \in \mathbb{R}$, all $\epsilon \in D(0,\epsilon_0)$; see Proposition 11. The proof leans on a fixed point argument in a Banach space of holomorphic functions $F^d_{(\nu,\beta,\mu,k_1,\kappa)}$ studied in Section 2.1. Since the exponential growth order κ of $\omega_{k_1}^d$ is larger than k_1, we cannot take a m_{k_1}-Laplace transform of it in the direction d. We need to use a version of what is called an accelero-summation procedure as described in [2], Chapter 5, which is explained in Section 4.3.

In a second step, we go back to our seminal convolution equation (70) and we multiply each side by the power T^{k_2+1} which transforms it into (121). Then we apply a m_{k_2}-Borel transform to the formal series $\hat{U}$ w.r.t. T, denoted by $\hat{\omega}_{k_2}(\tau,m,\epsilon)$. We show that $\hat{\omega}_{k_2}(\tau,m,\epsilon)$ formally solves a convolution equation in both variables τ and m, see (123), where the formal m_{k_2}-Borel transform of the forcing term is set as $\hat{\psi}_{k_2}(\tau,m,\epsilon)$. Now, we observe that a version of the analytic acceleration transform with indices k_2 and k_1 constructed in Proposition 13 applied to $\psi_{k_1}^d(\tau,m,\epsilon)$, standing for $\psi_{k_2}^d(\tau,m,\epsilon)$, is the κ-sum of $\hat{\psi}_{k_2}(\tau,m,\epsilon)$ w.r.t. τ on some bounded sector $S^b_{d,\kappa}$ with aperture slightly larger than π/κ, viewed as a function with values in $E_{(\beta,\mu)}$. Furthermore, $\psi_{k_2}^d(\tau,m,\epsilon)$ can be extended as an analytic function on an unbounded sector $S_{d,\kappa}$ with aperture slightly larger than π/κ where it possesses an exponential growth of order less than k_2; see Lemma 4. In the sequel, we focus on the solution $\omega_{k_2}^d(\tau,m,\epsilon)$ of the convolution problem (129) which is similar to

(123) but with the formal forcing term $\hat{\psi}_{k_2}(\tau,m,\epsilon)$ replaced by $\psi^d_{k_2}(\tau,m,\epsilon)$. Under some size restriction on the sup norm of the coefficients $c_{1,2}(\epsilon)/\epsilon$, $c_0(\epsilon)/\epsilon$, and $c_F(\epsilon)/\epsilon$ near 0, we show that $\omega^d_{k_2}(\tau,m,\epsilon)$ defines a bounded holomorphic function for τ on the bounded sector $S^b_{d,\kappa}$ and can be extended to a holomorphic function on unbounded sectors S_d with direction d and tiny aperture, provided that S_d stays away from the roots of some polynomial $P_m(\tau)$ constructed with the help of $Q(X)$ and $R_D(X)$ in (1), see (131). Moreover, the function $\omega^d_{k_2}(\tau,m,\epsilon)$ satisfies estimates of the form: there exist constants $\nu'>0$ and $\upsilon_d>0$ with

$$\left|\omega^d_{k_2}(\tau,m,\epsilon)\right| \le \upsilon_d\bigl(1+|m|\bigr)^{-\mu}e^{-\beta|m|}\frac{|\tau|}{1+|\tau|^{2k_2}}e^{\nu'|\tau|^{k_2}}$$

for all $\tau\in S_d$, all $m\in\mathbb{R}$, all $\epsilon\in D(0,\epsilon_0)$; see Proposition 14. Again, the proof rests on a fixed point argument in a Banach space of holomorphic functions $F^d_{(\nu',\beta,\mu,k_2)}$ outlined in Section 2.2. In Proposition 15, we show that $\omega^d_{k_2}(\tau,m,\epsilon)$ actually coincides with the analytic acceleration transform with indices m_{k_2} and m_{k_1} applied to $\omega^d_{k_1}(\tau,m,\epsilon)$, denoted by $\mathrm{Acc}^d_{k_2,k_1}(\omega^d_{k_1})(\tau,m,\epsilon)$, as long as τ lies in the bounded sector $S^b_{d,\kappa}$. As a result, some m_{k_2}-Laplace transform of the analytic continuation of $\mathrm{Acc}^d_{k_2,k_1}(\omega^d_{k_1})(\tau,m,\epsilon)$, set as $U^d(T,m,\epsilon)$, can be considered for all T belonging to a sector $S_{d,\theta_{k_2},h}$ with bisecting direction d, aperture θ_{k_2} slightly larger than π/k_2, and radius $h>0$. Following the terminology of [2], Section 6.1, $U^d(T,m,\epsilon)$ can be called the (m_{k_2},m_{k_1})-sum of the formal series $\hat{U}(T,m,\epsilon)$ in the direction d. Additionally, $U^d(T,m,\epsilon)$ solves our primary convolution equation (70), where the formal forcing term $\hat{F}(T,m,\epsilon)$ is interchanged with $F^d(T,m,\epsilon)$, which denotes the (m_{k_2},m_{k_1})-sum of $\hat{F}$ in the direction d.

In Theorem 1, we construct a family of actual bounded holomorphic solutions $u^{\mathfrak{d}_p}(t,z,\epsilon)$, $0\le p\le\varsigma-1$, of our original problem (1) on domains of the form $\mathcal{T}\times H_{\beta'}\times\mathcal{E}_p$ described in the main results above. Namely, the functions $u^{\mathfrak{d}_p}(t,z,\epsilon)$ (resp. $f^{\mathfrak{d}_p}(t,z,\epsilon)$) are set as Fourier inverse transforms of $U^{\mathfrak{d}_p}$,

$$u^{\mathfrak{d}_p}(t,z,\epsilon)=\mathcal{F}^{-1}\bigl(m\mapsto U^{\mathfrak{d}_p}(\epsilon t,m,\epsilon)\bigr)(z),\qquad f^{\mathfrak{d}_p}(t,z,\epsilon)=\mathcal{F}^{-1}\bigl(m\mapsto F^{\mathfrak{d}_p}(\epsilon t,m,\epsilon)\bigr)(z),$$

where the definition of $\mathcal{F}^{-1}$ is pointed out in Proposition 9. One proves the crucial property that the difference of any two neighboring functions $u^{\mathfrak{d}_{p+1}}(t,z,\epsilon)-u^{\mathfrak{d}_p}(t,z,\epsilon)$ tends to zero as $\epsilon\to 0$ on $\mathcal{E}_{p+1}\cap\mathcal{E}_p$ faster than a function with exponential decay of order k, uniformly w.r.t. $t\in\mathcal{T}$, $z\in H_{\beta'}$, with $k=k_2$ when the intersection $U_{\mathfrak{d}_{p+1}}\cap U_{\mathfrak{d}_p}$ is not empty and with $k=k_1$, when this intersection is empty. The same estimates hold for the difference $f^{\mathfrak{d}_{p+1}}(t,z,\epsilon)-f^{\mathfrak{d}_p}(t,z,\epsilon)$.

Section 6 is devoted to the study of the asymptotic behavior of $u^{\mathfrak{d}_p}(t,z,\epsilon)$ as ϵ tends to zero. Using the decay estimates on the differences of the functions $u^{\mathfrak{d}_p}$ and $f^{\mathfrak{d}_p}$, we show the existence of a common asymptotic expansion $\hat{u}(\epsilon)=\sum_{m\ge 0}h_m\epsilon^m/m!\in\mathbb{F}[[\epsilon]]$ (resp. $\hat{f}(\epsilon)=\sum_{m\ge 0}f_m\epsilon^m/m!\in\mathbb{F}[[\epsilon]]$) of Gevrey order $1/k_1$ for all functions $u^{\mathfrak{d}_p}(t,z,\epsilon)$ (resp. $f^{\mathfrak{d}_p}(t,z,\epsilon)$) as ϵ tends to 0 on $\mathcal{E}_p$. We obtain also a double scale asymptotics for $u^{\mathfrak{d}_p}$ as explained in the main results above. The key tool in proving the result is a version of the Ramis-Sibuya theorem which entails two Gevrey asymptotics orders, described in Section 6.1. It is worthwhile noting that a similar version was recently brought into play by Takei and Suzuki in [21, 22], in order to study parametric multisummability for the complex Schrödinger equation.

In the last section, we study the particular situation when the formal forcing term $F(T,m,\epsilon)$ solves a linear differential and convolution initial value problem; see (204). We multiply each side of this equation by the power T^{k_1+1} which transforms it into (208). Then we show that the m_{k_1}-Borel transform $\psi_{k_1}(\tau,m,\epsilon)$ formally solves a convolution equation in both variables τ and m; see (209). Under a size control of the sup norm of the coefficients $\mathbf{c}_0(\epsilon)/\epsilon$ and $\mathbf{c}_{\mathbf{F}}(\epsilon)/\epsilon$ near 0, we show that $\psi_{k_1}(\tau,m,\epsilon)$ is actually convergent near 0 w.r.t. τ and can be holomorphically extended as a function $\psi_{k_1}^{\mathfrak{d}_p}(\tau,m,\epsilon)$ on any unbounded sectors $U_{\mathfrak{d}_p}$ with direction $\mathfrak{d}_p$ and small aperture, provided that $U_{\mathfrak{d}_p}$ stays away from the roots of some polynomial $\mathbf{P}_m(\tau)$ constructed with the help of $\mathbf{Q}(X)$ and $\mathbf{R}_{\mathbf{D}}(X)$ in (204). Additionally, the function $\psi_{k_1}^{\mathfrak{d}_p}(\tau,m,\epsilon)$ satisfies estimates of the form: there exists a constant $\boldsymbol{\upsilon}>0$ with

$$\left|\psi_{k_1}^{\mathfrak{d}_p}(\tau,m,\epsilon)\right| \le \boldsymbol{\upsilon}\left(1+|m|\right)^{-\mu}e^{-\beta|m|}\frac{|\tau|}{1+|\tau|^{2k_1}}e^{\nu|\tau|^{k_1}}$$

for all $\tau\in U_{\mathfrak{d}_p}$, all $m\in\mathbb{R}$, all $\epsilon\in D(0,\epsilon_0)$; see Proposition 18. The proof is once more based upon a fixed point argument in a Banach space of holomorphic functions $F^d_{(\nu,\beta,\mu,k_1,k_1)}$ defined in Section 2.1. The latter properties on $\psi_{k_1}^{\mathfrak{d}_p}(\tau,m,\epsilon)$ legitimize all the assumptions made above on the forcing term $F(T,m,\epsilon)$. Now, we can take the m_{k_1}-Laplace transform $\mathcal{L}^{\mathfrak{d}_p}_{m_{k_1}}(\psi_{k_1}^{\mathfrak{d}_p})(T)$ of $\psi_{k_1}^{\mathfrak{d}_p}(\tau,m,\epsilon)$ w.r.t. τ in the direction $\mathfrak{d}_p$, which yields an analytic solution of the initial linear equation (204) on some bounded sector $S_{\mathfrak{d}_p,\theta_{k_1},h}$ with aperture θ_{k_1} slightly larger than π/k_1. $\mathcal{L}^{\mathfrak{d}_p}_{m_{k_1}}(\psi_{k_1}^{\mathfrak{d}_p})(T)$ coincides with the analytic (m_{k_2},m_{k_1})-sum $F^{\mathfrak{d}_p}(T,m,\epsilon)$ of $\hat{F}$ in direction $\mathfrak{d}_p$ on the smaller sector $S_{\mathfrak{d}_p,\theta_{k_2},h}$ with aperture slightly larger than π/k_2. We deduce consequently that the analytic forcing term $f^{\mathfrak{d}_p}(t,z,\epsilon)$ solves the linear PDE (220) with analytic coefficients on $D(0,r)\times H_{\beta'}\times D(0,\epsilon_0)$, for all $t\in\mathcal{T}$, $z\in H_{\beta'}$, $\epsilon\in\mathcal{E}_p$. In our last main result (Theorem 3), we see that the latter issue implies that $u^{\mathfrak{d}_p}(t,z,\epsilon)$ itself solves a nonlinear PDE (224) with analytic coefficients and forcing term on $D(0,r)\times H_{\beta'}\times D(0,\epsilon_0)$, for all $t\in\mathcal{T}$, $z\in H_{\beta'}$, $\epsilon\in\mathcal{E}_p$.

The paper is organized as follows.

In Section 2, we define some weighted Banach spaces of continuous functions on $(D(0,\rho)\cup U)\times\mathbb{R}$ with exponential growths of different orders on unbounded sectors U w.r.t. the first variable and exponential decay on $\mathbb{R}$ w.r.t. the second one. We study the continuity properties of several kind of linear and nonlinear operators acting on these spaces that will be useful in Sections 4.2, 4.4 and 7.2.

In Section 3, we recall the definition and the main analytic and algebraic properties of the m_k-summability.

In Section 4.1, we introduce an auxiliary differential and convolution problem (70) for which we construct a formal solution.

In Section 4.2, we show that the m_{k_1}-Borel transform of this formal solution satisfies a convolution problem (83) that we can uniquely solve within the Banach spaces described in Section 2.

In Section 4.3, we describe the properties of a variant of the formal and analytic acceleration operators associated to the m_k-Borel and m_k-Laplace transforms.

In Section 4.4, we see that the m_{k_2}-Borel transform of the formal solution of (70) satisfies a convolution problem (123). We show that its formal forcing term is κ-summable and that its κ-sum is an acceleration of the m_{k_1}-Borel transform of the above formal forcing term. Then we construct an actual solution to the corresponding problem with the analytic continuation of this κ-sum as non-homogeneous term, within the Banach spaces defined in Section 2. We recognize that this actual solution is the analytic continuation of the acceleration of the m_{k_1}-Borel transform of the formal solution of (70). Finally, we take its m_{k_2}-Laplace transform in order to get an actual solution of (146).

In Section 5, with the help of Section 4, we build a family of actual holomorphic solutions to our initial Cauchy problem (1). We show that the difference of any two neighboring solutions is exponentially flat for some integer order in ϵ (Theorem 1).

In Section 6, we show that the actual solutions constructed in Section 5 share a common formal series as Gevrey asymptotic expansion as ϵ tends to 0 on sectors (Theorem 2). The result is built on a version of the Ramis-Sibuya theorem with two Gevrey orders stated in Section 6.1.

In Section 7, we inspect the special case when the forcing term itself solves a linear PDE. Then we notice that the solutions of (1) constructed in Section 5 actually solve a nonlinear PDE with holomorphic coefficients and forcing term near the origin (Theorem 3).

2 Banach spaces of functions with exponential growth and decay

The Banach spaces introduced in the next Section 2.1 (resp. Section 2.2) will be crucial in the construction of analytic solutions of a convolution problem investigated in the forthcoming Section 4.2 (resp. Section 4.4).

2.1 Banach spaces of functions with exponential growth κ and decay of exponential order 1

We denote $D(0,r)$ the open disc centered at 0 with radius $r > 0$ in $\mathbb{C}$ and $\bar{D}(0,r)$ its closure. Let U_d be an open unbounded sector in the direction $d \in \mathbb{R}$ centered at 0 in $\mathbb{C}$. By convention, the sectors we consider do not contain the origin in $\mathbb{C}$.

Definition 1 Let $\nu, \beta, \mu > 0$ and $\rho > 0$ be positive real numbers. Let $k \geq 1, \kappa \geq 1$ be integers and $d \in \mathbb{R}$. We denote $F^d_{(\nu,\beta,\mu,k,\kappa)}$ the vector space of continuous functions $(\tau, m) \mapsto h(\tau, m)$ on $(\bar{D}(0,\rho) \cup U_d) \times \mathbb{R}$, which are holomorphic with respect to τ on $D(0,\rho) \cup U_d$ and such that

$$\|h(\tau,m)\|_{(\nu,\beta,\mu,k,\kappa)} = \sup_{\tau \in \bar{D}(0,\rho)\cup U_d, m\in\mathbb{R}} \bigl(1+|m|\bigr)^{\mu} \frac{1+|\tau|^{2k}}{|\tau|} \exp\bigl(\beta|m| - \nu|\tau|^{\kappa}\bigr)\bigl|h(\tau,m)\bigr|$$

is finite. One can check that the normed space $(F^d_{(\nu,\beta,\mu,k,\kappa)}, \|.\|_{(\nu,\beta,\mu,k,\kappa)})$ is a Banach space.

Remark These norms are appropriate modifications of those introduced in the work [1], Section 2.

Throughout the whole subsection, we assume $\mu, \beta, \nu, \rho > 0$, $k, \kappa \geq 1$, and $d \in \mathbb{R}$ are fixed. In the next lemma, we check the continuity property under multiplication operation with bounded functions.

Lemma 1 *Let $(\tau, m) \mapsto a(\tau, m)$ be a bounded continuous function on $(\bar{D}(0,\rho) \cup U_d) \times \mathbb{R}$ by a constant $C_1 > 0$. We assume that $a(\tau, m)$ is holomorphic with respect to τ on $D(0,\rho) \cup U_d$.*

Then we have

$$\big\|a(\tau,m)h(\tau,m)\big\|_{(\nu,\beta,\mu,k,\kappa)} \leq C_1\big\|h(\tau,m)\big\|_{(\nu,\beta,\mu,k,\kappa)} \tag{3}$$

for all $h(\tau,m) \in F^d_{(\nu,\beta,\mu,k,\kappa)}$.

In the next proposition, we study the continuity property of some convolution operators acting on the latter Banach spaces.

Proposition 1 *Let* $\chi_2 > -1$ *be a real number. Let* $\nu_2 \geq -1$ *be an integer. We assume that* $1 + \chi_2 + \nu_2 \geq 0$.

If $\kappa \geq k(\frac{\nu_2}{\chi_2+1} + 1)$, *then there exists a constant* $C_2 > 0$ (*depending on* ν, ν_2, χ_2) *such that*

$$\left\|\int_0^{\tau^k} \big(\tau^k - s\big)^{\chi_2} s^{\nu_2} f\big(s^{1/k},m\big)\,ds\right\|_{(\nu,\beta,\mu,k,\kappa)} \leq C_2\big\|f(\tau,m)\big\|_{(\nu,\beta,\mu,k,\kappa)} \tag{4}$$

for all $f(\tau,m) \in F^d_{(\nu,\beta,\mu,k,\kappa)}$.

Proof Let $f(\tau,m) \in F^d_{(\nu,\beta,\mu,k,\kappa)}$. By definition, we have

$$\begin{aligned}
&\left\|\int_0^{\tau^k} \big(\tau^k - s\big)^{\chi_2} s^{\nu_2} f\big(s^{1/k},m\big)\,ds\right\|_{(\nu,\beta,\mu,k,\kappa)} \\
&\quad = \sup_{\tau\in\bar{D}(0,\rho)\cup U_d, m\in\mathbb{R}} \big(1+|m|\big)^{\mu} \frac{1+|\tau|^{2k}}{|\tau|} \exp\big(\beta|m| - \nu|\tau|^{\kappa}\big) \\
&\qquad \times \left|\int_0^{\tau^k} \left\{\big(1+|m|\big)^{\mu} e^{\beta|m|} \exp\big(-\nu|s|^{\kappa/k}\big) \frac{1+|s|^2}{|s|^{1/k}} f\big(s^{1/k},m\big)\right\} \mathcal{B}(\tau,s,m)\,ds\right|,
\end{aligned} \tag{5}$$

where

$$\mathcal{B}(\tau,s,m) = \frac{1}{(1+|m|)^{\mu}} e^{-\beta|m|} \frac{\exp(\nu|s|^{\kappa/k})}{1+|s|^2} |s|^{1/k} \big(\tau^k - s\big)^{\chi_2} s^{\nu_2}.$$

Therefore,

$$\left\|\int_0^{\tau^k} \big(\tau^k - s\big)^{\chi_2} s^{\nu_2} f\big(s^{1/k},m\big)\,ds\right\|_{(\nu,\beta,\mu,k,\kappa)} \leq C_2\big\|f(\tau,m)\big\|_{(\nu,\beta,\mu,k,\kappa)}, \tag{6}$$

where

$$\begin{aligned}
C_2 &= \sup_{\tau\in\bar{D}(0,\rho)\cup U_d} \frac{1+|\tau|^{2k}}{|\tau|} \exp\big(-\nu|\tau|^{\kappa}\big) \int_0^{|\tau|^k} \frac{\exp(\nu h^{\kappa/k})}{1+h^2} h^{\frac{1}{k}} \big(|\tau|^k - h\big)^{\chi_2} h^{\nu_2}\,dh \\
&= \sup_{x\geq 0} B(x),
\end{aligned} \tag{7}$$

where

$$B(x) = \frac{1+x^2}{x^{1/k}} \exp\big(-\nu x^{\kappa/k}\big) \int_0^x \frac{\exp(\nu h^{\kappa/k})}{1+h^2} h^{\frac{1}{k}+\nu_2} (x-h)^{\chi_2}\,dh.$$

We write $B(x) = B_1(x) + B_2(x)$, where

$$B_1(x) = \frac{1+x^2}{x^{1/k}} \exp\bigl(-\nu x^{\kappa/k}\bigr) \int_0^{x/2} \frac{\exp(\nu h^{\kappa/k})}{1+h^2} h^{\frac{1}{k}+\nu_2}(x-h)^{\chi_2}\, dh,$$
$$B_2(x) = \frac{1+x^2}{x^{1/k}} \exp\bigl(-\nu x^{\kappa/k}\bigr) \int_{x/2}^{x} \frac{\exp(\nu h^{\kappa/k})}{1+h^2} h^{\frac{1}{k}+\nu_2}(x-h)^{\chi_2}\, dh.$$

Now, we study the function $B_1(x)$. We first assume that $-1 < \chi_2 < 0$. In that case, we have $(x-h)^{\chi_2} \leq (x/2)^{\chi_2}$ for all $0 \leq h \leq x/2$ with $x > 0$. Since $\nu_2 \geq -1$, we deduce that

$$\begin{aligned} B_1(x) &\leq \frac{1+x^2}{x^{1/k}} \left(\frac{x}{2}\right)^{\chi_2} e^{-\nu x^{\kappa/k}} \int_0^{x/2} \frac{e^{\nu h^{\kappa/k}}}{1+h^2} h^{\frac{1}{k}+\nu_2}\, dh \\ &\leq \bigl(1+x^2\bigr) \frac{1}{2^{1/k}(\frac{1}{k}+\nu_2+1)} \left(\frac{x}{2}\right)^{1+\chi_2+\nu_2} \exp\left(-\nu\left(1-\frac{1}{2^{\kappa/k}}\right)x^{\kappa/k}\right) \end{aligned} \tag{8}$$

for all $x > 0$. Since $\kappa \geq k$ and $1+\chi_2+\nu_2 \geq 0$, we deduce that there exists a constant $K_1 > 0$ with

$$\sup_{x\geq 0} B_1(x) \leq K_1. \tag{9}$$

We assume now that $\chi_2 \geq 0$. In this situation, we know that $(x-h)^{\chi_2} \leq x^{\chi_2}$ for all $0 \leq h \leq x/2$, with $x \geq 0$. Hence, since $\nu_2 \geq -1$,

$$B_1(x) \leq \bigl(1+x^2\bigr) \frac{1}{2^{1/k}(\frac{1}{k}+\nu_2+1)} x^{\chi_2} (x/2)^{\nu_2+1} \exp\left(-\nu\left(1-\frac{1}{2^{\kappa/k}}\right)x^{\kappa/k}\right) \tag{10}$$

for all $x \geq 0$. Again, we deduce that there exists a constant $K_{1.1} > 0$ with

$$\sup_{x\geq 0} B_1(x) \leq K_{1.1}. \tag{11}$$

In the next step, we focus on the function $B_2(x)$. First, we observe that $1+h^2 \geq 1+(x/2)^2$ for all $x/2 \leq h \leq x$. Therefore, there exists a constant $K_2 > 0$ such that

$$\begin{aligned} B_2(x) &\leq \frac{1+x^2}{1+(\frac{x}{2})^2} \frac{1}{x^{1/k}} \exp\bigl(-\nu x^{\kappa/k}\bigr) \int_{x/2}^{x} \exp\bigl(\nu h^{\kappa/k}\bigr) h^{\frac{1}{k}+\nu_2}(x-h)^{\chi_2}\, dh \\ &\leq K_2 \frac{1}{x^{1/k}} \exp\bigl(-\nu x^{\kappa/k}\bigr) \int_0^{x} \exp\bigl(\nu h^{\kappa/k}\bigr) h^{\frac{1}{k}+\nu_2}(x-h)^{\chi_2}\, dh \end{aligned} \tag{12}$$

for all $x > 0$. It remains to study the function

$$B_{2.1}(x) = \int_0^x \exp\bigl(\nu h^{\kappa/k}\bigr) h^{\frac{1}{k}+\nu_2}(x-h)^{\chi_2}\, dh$$

for $x \geq 0$. By the uniform expansion $e^{\nu h^{\kappa/k}} = \sum_{n\geq 0}(\nu h^{\kappa/k})^n/n!$ on every compact interval $[0,x]$, $x \geq 0$, we can write

$$B_{2.1}(x) = \sum_{n\geq 0} \frac{\nu^n}{n!} \int_0^x h^{\frac{n\kappa}{k}+\frac{1}{k}+\nu_2}(x-h)^{\chi_2}\, dh. \tag{13}$$

Using the Beta integral formula (see [24], Appendix B3) and since $\chi_2 > -1$, $\frac{1}{k} + \nu_2 > -1$, we can write

$$B_{2.1}(x) = \sum_{n\geq 0} \frac{\nu^n}{n!} \frac{\Gamma(\chi_2+1)\Gamma(\frac{n\kappa}{k} + \frac{1}{k} + \nu_2 + 1)}{\Gamma(\frac{n\kappa}{k} + \frac{1}{k} + \nu_2 + \chi_2 + 2)} x^{\frac{n\kappa}{k} + \frac{1}{k} + \nu_2 + \chi_2 + 1} \tag{14}$$

for all $x \geq 0$. Bearing in mind that

$$\Gamma(x)/\Gamma(x+a) \sim 1/x^a \tag{15}$$

as $x \to +\infty$, for any $a > 0$ (see for instance, [24], Appendix B3), from (14), we get a constant $K_{2.1} > 0$ such that

$$B_{2.1}(x) \leq K_{2.1} x^{\frac{1}{k} + \nu_2 + \chi_2 + 1} \sum_{n\geq 0} \frac{1}{(n+1)^{\chi_2+1} n!} \left(\nu x^{\kappa/k}\right)^n \tag{16}$$

for all $x \geq 0$. Using again (15), we know that $1/(n+1)^{\chi_2+1} \sim \Gamma(n+1)/\Gamma(n+\chi_2+2)$ as $n \to +\infty$. Hence, from (16), there exists a constant $K_{2.2} > 0$ such that

$$B_{2.1}(x) \leq K_{2.2} x^{\frac{1}{k} + \nu_2 + \chi_2 + 1} \sum_{n\geq 0} \frac{1}{\Gamma(n+\chi_2+2)} \left(\nu x^{\kappa/k}\right)^n \tag{17}$$

for all $x \geq 0$.

Remembering the asymptotic properties of the generalized Mittag-Leffler function (known as Wiman function in the literature) $E_{\alpha,\beta}(z) = \sum_{n\geq 0} z^n/\Gamma(\beta + \alpha n)$, for any $\alpha, \beta > 0$ (see [24], Appendix B4 or [25], expansion (22), p.210), we get from (17) a constant $K_{2.3} > 0$ such that

$$B_{2.1}(x) \leq K_{2.3} x^{\frac{1}{k} + \nu_2 + \chi_2 + 1} x^{-\frac{\kappa}{k}(\chi_2+1)} e^{\nu x^{\kappa/k}} \tag{18}$$

for all $x \geq 1$. Under the assumption that $\nu_2 + \chi_2 + 1 \leq \frac{\kappa}{k}(\chi_2 + 1)$ and gathering (12), (18), we get a constant $K_{2.4} > 0$ such that

$$\sup_{x\geq 0} B_2(x) \leq K_{2.4}. \tag{19}$$

Finally, taking into account the estimates (6), (7), (9), (11), (19), the inequality (4) follows. □

Proposition 2 *Let $k, \kappa \geq 1$ be integers such that $\kappa \geq k$. Let $Q_1(X), Q_2(X), R(X) \in \mathbb{C}[X]$ such that*

$$\deg(R) \geq \deg(Q_1), \qquad \deg(R) \geq \deg(Q_2), \qquad R(im) \neq 0 \tag{20}$$

for all $m \in \mathbb{R}$. Assume that $\mu > \max(\deg(Q_1)+1, \deg(Q_2)+1)$. Let $m \mapsto b(m)$ be a continuous function on $\mathbb{R}$ such that

$$\left|b(m)\right| \le \frac{1}{|R(im)|}$$

for all $m \in \mathbb{R}$. *Then there exists a constant* $C_3 > 0$ (*depending on* Q_1, Q_2, R, μ, k, κ, ν) *such that*

$$\begin{aligned}
&\left\| b(m) \int_0^{\tau^k} \left(\tau^k - s\right)^{\frac{1}{k}} \left(\int_0^s \int_{-\infty}^{+\infty} Q_1\big(i(m-m_1)\big) f\big((s-x)^{1/k}, m-m_1\big) \right.\right.\\
&\qquad \left.\left. \times Q_2(im_1) g\big(x^{1/k}, m_1\big) \frac{1}{(s-x)x}\, dx\, dm_1 \right) ds \right\|_{(\nu,\beta,\mu,k,\kappa)}\\
&\quad \le C_3 \left\|f(\tau,m)\right\|_{(\nu,\beta,\mu,k,\kappa)} \left\|g(\tau,m)\right\|_{(\nu,\beta,\mu,k,\kappa)}
\end{aligned} \tag{21}$$

for all $f(\tau,m), g(\tau,m) \in F^d_{(\nu,\beta,\mu,k,\kappa)}$.

Proof Let $f(\tau,m), g(\tau,m) \in F^d_{(\nu,\beta,\mu,k,\kappa)}$. For any $\tau \in \bar{D}(0,\rho) \cup U_d$, the segment $[0,\tau^k]$ is such that, for any $s \in [0,\tau^k]$, any $x \in [0,s]$, the expressions $f((s-x)^{1/k}, m-m_1)$ and $g(x^{1/k}, m_1)$ are well defined, provided that $m, m_1 \in \mathbb{R}$. By definition, we can write

$$\begin{aligned}
&\left\| b(m) \int_0^{\tau^k} \left(\tau^k - s\right)^{\frac{1}{k}} \left(\int_0^s \int_{-\infty}^{+\infty} Q_1\big(i(m-m_1)\big) f\big((s-x)^{1/k}, m-m_1\big) \right.\right.\\
&\qquad \left.\left. \times Q_2(im_1) g\big(x^{1/k}, m_1\big) \frac{1}{(s-x)x}\, dx\, dm_1 \right) ds \right\|_{(\nu,\beta,\mu,k,\kappa)}\\
&\quad = \sup_{\tau \in \bar{D}(0,\rho)\cup U_d, m \in \mathbb{R}} \big(1+|m|\big)^{\mu} \frac{1+|\tau|^{2k}}{|\tau|} \exp\big(\beta|m| - \nu|\tau|^{\kappa}\big)\\
&\qquad \times \left| \int_0^{\tau^k} \left(\tau^k - s\right)^{1/k} \left(\int_0^s \int_{-\infty}^{+\infty} \left\{ \big(1+|m-m_1|\big)^{\mu} e^{\beta|m-m_1|} \frac{1+|s-x|^2}{|s-x|^{1/k}} \right.\right.\right.\\
&\qquad \left. \times \exp\big(-\nu|s-x|^{\kappa/k}\big) f\big((s-x)^{1/k}, m-m_1\big) \right\}\\
&\qquad \left.\left. \times \left\{ \big(1+|m_1|\big)^{\mu} e^{\beta|m_1|} \frac{1+|x|^2}{|x|^{1/k}} \exp\big(-\nu|x|^{\kappa/k}\big) g\big(x^{1/k}, m_1\big) \right\} \mathcal{C}(s,x,m,m_1)\, dx\, dm_1 \right) ds \right|,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{C}(s,x,m,m_1) &= \frac{\exp(-\beta|m_1|)\exp(-\beta|m-m_1|)}{(1+|m-m_1|)^{\mu}(1+|m_1|)^{\mu}} b(m) Q_1\big(i(m-m_1)\big) Q_2(im_1)\\
&\quad \times \frac{|s-x|^{1/k}|x|^{1/k}}{(1+|s-x|^2)(1+|x|^2)} \exp\big(\nu|s-x|^{\kappa/k}\big) \exp\big(\nu|x|^{\kappa/k}\big) \frac{1}{(s-x)x}.
\end{aligned}$$

Now, we know that there exist $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{R} > 0$ with

$$\begin{aligned}
&\left|Q_1\big(i(m-m_1)\big)\right| \le \mathfrak{Q}_1 \big(1+|m-m_1|\big)^{\deg(Q_1)}, \qquad \left|Q_2(im_1)\right| \le \mathfrak{Q}_2 \big(1+|m_1|\big)^{\deg(Q_2)},\\
&\left|R(im)\right| \ge \mathfrak{R} \big(1+|m|\big)^{\deg(R)}
\end{aligned} \tag{22}$$

for all $m, m_1 \in \mathbb{R}$. Therefore,

$$\left\| b(m)\int_0^{\tau^k}(\tau^k-s)^{\frac{1}{k}}\left(\int_0^s\int_{-\infty}^{+\infty}Q_1\big(i(m-m_1)\big)f\big((s-x)^{1/k},m-m_1\big)\right.\right.$$
$$\left.\left.\times Q_2(im_1)g\big(x^{1/k},m_1\big)\frac{1}{(s-x)x}\,dx\,dm_1\right)ds\right\|_{(\nu,\beta,\mu,k,\kappa)}$$
$$\leq C_3\big\|f(\tau,m)\big\|_{(\nu,\beta,\mu,k,\kappa)}\big\|g(\tau,m)\big\|_{(\nu,\beta,\mu,k,\kappa)}, \tag{23}$$

where

$$C_3=\sup_{\tau\in\bar{D}(0,\rho)\cup U_d,m\in\mathbb{R}}\big(1+|m|\big)^{\mu}\frac{1+|\tau|^{2k}}{|\tau|}\exp\big(\beta|m|-\nu|\tau|^{\kappa}\big)\frac{1}{\mathfrak{R}(1+|m|)^{\deg(R)}}$$
$$\times\int_0^{|\tau|^k}\big(|\tau|^k-h\big)^{1/k}\left(\int_0^h\int_{-\infty}^{+\infty}\frac{\exp(-\beta|m_1|)\exp(-\beta|m-m_1|)}{(1+|m-m_1|)^{\mu}(1+|m_1|)^{\mu}}\right.$$
$$\times\mathfrak{Q}_1\mathfrak{Q}_2\big(1+|m-m_1|\big)^{\deg(Q_1)}\big(1+|m_1|\big)^{\deg(Q_2)}\frac{(h-x)^{1/k}x^{1/k}}{(1+(h-x)^2)(1+x^2)}$$
$$\left.\times\exp\big(\nu(h-x)^{\kappa/k}\big)\exp\big(\nu x^{\kappa/k}\big)\frac{1}{(h-x)x}\,dx\,dm_1\right)dh. \tag{24}$$

Now, since $\kappa\geq k$, we have

$$h^{\kappa/k}\geq(h-x)^{\kappa/k}+x^{\kappa/k} \tag{25}$$

for all $h\geq 0$, all $x\in[0,h]$. Indeed, let $x=hu$ where $u\in[0,1]$. Then the inequality (25) is equivalent to show that

$$1\geq(1-u)^{\kappa/k}+u^{\kappa/k} \tag{26}$$

for all $u\in[0,1]$. Let $\varphi(u)=(1-u)^{\kappa/k}+u^{\kappa/k}$ on $[0,1]$. We have $\varphi'(u)=\frac{\kappa}{k}(u^{\frac{\kappa}{k}-1}-(1-u)^{\frac{\kappa}{k}-1})$. Since $\kappa\geq k$, the function $\psi(z)=z^{\frac{\kappa}{k}-1}$ is increasing on $[0,1]$, and therefore we find that $\varphi'(u)<0$ if $0\leq u<1/2$, $\varphi'(u)=0$, if $u=1/2$ and $\varphi'(u)>0$ if $1/2<u\leq 1$. Since $\varphi(0)=\varphi(1)=1$, we get that $\varphi(u)\leq 1$ for all $u\in[0,1]$. Therefore, (26) holds and (25) is proved.

Using the triangular inequality $|m|\leq|m_1|+|m-m_1|$, for all $m,m_1\in\mathbb{R}$, we find that $C_3\leq C_{3.1}C_{3.2}$ where

$$C_{3.1}=\frac{\mathfrak{Q}_1\mathfrak{Q}_2}{\mathfrak{R}}\sup_{m\in\mathbb{R}}\big(1+|m|\big)^{\mu-\deg(R)}$$
$$\times\int_{-\infty}^{+\infty}\frac{1}{(1+|m-m_1|)^{\mu-\deg(Q_1)}(1+|m_1|)^{\mu-\deg(Q_2)}}\,dm_1 \tag{27}$$

which is finite whenever $\mu>\max(\deg(Q_1)+1,\deg(Q_2)+1)$ under the assumption (20) using the same estimates as in Lemma 4 of [26] (see also Lemma 2.2 from [23]), and where

$$C_{3.2}=\sup_{\tau\in\bar{D}(0,\rho)\cup U_d}\frac{1+|\tau|^{2k}}{|\tau|}\exp\big(-\nu|\tau|^{\kappa}\big)$$
$$\times\int_0^{|\tau|^k}\big(|\tau|^k-h\big)^{1/k}\exp\big(\nu h^{\kappa/k}\big)\int_0^h\frac{(h-x)^{1/k}x^{1/k}}{(1+(h-x)^2)(1+x^2)}\frac{1}{(h-x)x}\,dx\,dh. \tag{28}$$

From (28) we find that $C_{3.2} \leq C_{3.3}$, where

$$C_{3.3} = \sup_{x\geq 0} \frac{1+x^2}{x^{1/k}} \exp(-\nu x^{\kappa/k}) \int_0^x (x-h')^{1/k} \exp(\nu h'^{\kappa/k}) \times \left(\int_0^{h'} \frac{1}{(1+(h'-x')^2)(1+x'^2)} \frac{1}{(h'-x')^{1-\frac{1}{k}} x'^{1-\frac{1}{k}}} \, dx' \right) dh'. \tag{29}$$

By the change of variable $x' = h'u$, for $u \in [0,1]$, we can write

$$\int_0^{h'} \frac{1}{(1+(h'-x')^2)(1+x'^2)} \frac{1}{(h'-x')^{1-\frac{1}{k}} x'^{1-\frac{1}{k}}} \, dx' = \frac{1}{h'^{1-\frac{2}{k}}} \int_0^1 \frac{1}{(1+h'^2(1-u)^2)(1+h'^2u^2)(1-u)^{1-\frac{1}{k}} u^{1-\frac{1}{k}}} \, du = J_k(h'). \tag{30}$$

Using a partial fraction decomposition, we can split $J_k(h') = J_{1,k}(h') + J_{2,k}(h')$, where

$$\begin{aligned} J_{1,k}(h') &= \frac{1}{h'^{1-\frac{2}{k}}(h'^2+4)} \int_0^1 \frac{3-2u}{(1+h'^2(1-u)^2)(1-u)^{1-\frac{1}{k}} u^{1-\frac{1}{k}}} \, du, \\ J_{2,k}(h') &= \frac{1}{h'^{1-\frac{2}{k}}(h'^2+4)} \int_0^1 \frac{2u+1}{(1+h'^2u^2)(1-u)^{1-\frac{1}{k}} u^{1-\frac{1}{k}}} \, du. \end{aligned} \tag{31}$$

From now on, we assume that $k \geq 2$. By construction of $J_{1,k}(h')$ and $J_{2,k}(h')$, we see that there exists a constant $j_k > 0$ such that

$$J_k(h') \leq \frac{j_k}{h'^{1-\frac{2}{k}}(h'^2+4)} \tag{32}$$

for all $h' > 0$. From (29) and (32), we deduce that $C_{3.3} \leq \sup_{x\geq 0} \tilde{C}_{3.3}(x)$, where

$$\tilde{C}_{3.3}(x) = (1+x^2) \exp(-\nu x^{\kappa/k}) \int_0^x \frac{j_k \exp(\nu h'^{\kappa/k})}{h'^{1-\frac{2}{k}}(h'^2+4)} \, dh'. \tag{33}$$

From L'Hospital rule, we know that

$$\lim_{x\to+\infty} \tilde{C}_{3.3}(x) = \lim_{x\to+\infty} \frac{j_k}{x^{1-\frac{2}{k}}} \frac{\frac{(1+x^2)^2}{x^2+4}}{\nu \frac{\kappa}{k} x^{\frac{\kappa}{k}-1}(1+x^2) - 2x}$$

which is finite if $\kappa \geq k$ and when $k \geq 2$. Therefore, we get a constant $\tilde{C}_{3.3} > 0$ such that

$$\sup_{x\geq 0} \tilde{C}_{3.3}(x) \leq \tilde{C}_{3.3}. \tag{34}$$

Taking into account the estimates for (24), (27), (28), (29), (33) and (34), we obtain the result (21).

It remains to consider the case $k = 1$. In that case, we know from Corollary 4.9 of [27] that there exists a constant $j_1 > 0$ such that

$$J_1(h') \leq \frac{j_1}{h'^2+1} \tag{35}$$

for all $h' \geq 0$. From (29) and (35), we deduce that $C_{3.3} \leq \sup_{x\geq 0} \tilde{C}_{3.3.1}(x)$, where

$$\tilde{C}_{3.3.1}(x) = (1+x^2)\exp(-\nu x^{\kappa}) \int_0^x \frac{j_1 \exp(\nu h'^{\kappa})}{h'^2+1}\, dh'. \tag{36}$$

From the L'Hospital rule, we know that

$$\lim_{x\to+\infty} \tilde{C}_{3.3.1}(x) = \lim_{x\to+\infty} \frac{(1+x^2)j_1}{\nu\kappa x^{\kappa-1}(1+x^2)-2x},$$

which is finite whenever $\kappa \geq 1$. Therefore, we get a constant $\tilde{C}_{3.3.1} > 0$ such that

$$\sup_{x\geq 0} \tilde{C}_{3.3.1}(x) \leq \tilde{C}_{3.3.1}. \tag{37}$$

Taking into account the estimates for (24), (27), (28), (29), (36), and (37), we obtain the result (21) for $k = 1$. □

Definition 2 Let $\beta, \mu \in \mathbb{R}$. We denote $E_{(\beta,\mu)}$ the vector space of continuous functions $h : \mathbb{R} \to \mathbb{C}$ such that

$$\|h(m)\|_{(\beta,\mu)} = \sup_{m\in\mathbb{R}} (1+|m|)^{\mu} \exp(\beta|m|)|h(m)|$$

is finite. The space $E_{(\beta,\mu)}$ equipped with the norm $\|\cdot\|_{(\beta,\mu)}$ is a Banach space.

Proposition 3 *Let $k, \kappa \geq 1$ be integers such that $\kappa \geq k$. Let $Q(X), R(X) \in \mathbb{C}[X]$ be polynomials such that*

$$\deg(R) \geq \deg(Q), \qquad R(im) \neq 0 \tag{38}$$

for all $m \in \mathbb{R}$. Assume that $\mu > \deg(Q) + 1$. Let $m \mapsto b(m)$ be a continuous function such that

$$|b(m)| \leq \frac{1}{|R(im)|}$$

for all $m \in \mathbb{R}$. Then there exists a constant $C_4 > 0$ (depending on Q, R, μ, k, κ, ν) such that

$$\begin{aligned} &\left\| b(m) \int_0^{\tau^k} (\tau^k - s)^{\frac{1}{k}} \int_{-\infty}^{+\infty} f(m-m_1) Q(im_1) g(s^{1/k}, m_1)\, dm_1 \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k,\kappa)} \\ &\quad \leq C_4 \|f(m)\|_{(\beta,\mu)} \|g(\tau,m)\|_{(\nu,\beta,\mu,k,\kappa)} \end{aligned} \tag{39}$$

for all $f(m) \in E_{(\beta,\mu)}$, all $g(\tau,m) \in F^d_{(\nu,\beta,\mu,k,\kappa)}$.

Proof The proof follows the same lines of arguments as those of Propositions 1 and 2. Let $f(m) \in E_{(\beta,\mu)}$, $g(\tau,m) \in F^d_{(\nu,\beta,\mu,k,\kappa)}$. We can write

$$
\begin{aligned}
N_2 &:= \left\| b(m) \int_0^{\tau^k} \left(\tau^k - s\right)^{\frac{1}{k}} \int_{-\infty}^{+\infty} f(m-m_1) Q(im_1) g\left(s^{1/k}, m_1\right) dm_1 \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k,\kappa)} \\
&= \sup_{\tau \in \bar{D}(0,\rho) \cup U_d, m \in \mathbb{R}} \left(1+|m|\right)^{\mu} \frac{1+|\tau|^{2k}}{|\tau|} \exp\left(\beta|m| - \nu|\tau|^{\kappa}\right) \\
&\quad \times \left| b(m) \int_0^{\tau^k} \int_{-\infty}^{+\infty} \left\{ \left(1+|m-m_1|\right)^{\mu} \exp\left(\beta|m-m_1|\right) f(m-m_1) \right\} \right. \\
&\quad \times \left\{ \left(1+|m_1|\right)^{\mu} \exp\left(\beta|m_1|\right) \exp\left(-\nu|s|^{\kappa/k}\right) \frac{1+|s|^2}{|s|^{1/k}} g\left(s^{1/k}, m_1\right) \right\} \\
&\quad \left. \times \mathcal{D}(\tau, s, m, m_1) \, dm_1 \, ds \right|,
\end{aligned}
\tag{40}
$$

where

$$
\mathcal{D}(\tau,s,m,m_1) = \frac{Q(im_1) e^{-\beta|m_1|} e^{-\beta|m-m_1|}}{(1+|m-m_1|)^{\mu} \left(1+|m_1|\right)^{\mu}} \frac{\exp(\nu|s|^{\kappa/k})}{1+|s|^2} |s|^{1/k} \left(\tau^k - s\right)^{1/k} \frac{1}{s}.
$$

Again, we know that there exist constants $\mathfrak{Q}, \mathfrak{R} > 0$ such that

$$
\left|Q(im_1)\right| \le \mathfrak{Q}\left(1+|m_1|\right)^{\deg(Q)}, \qquad \left|R(im)\right| \ge \mathfrak{R}\left(1+|m|\right)^{\deg(R)}
$$

for all $m, m_1 \in \mathbb{R}$. By means of the triangular inequality $|m| \le |m_1| + |m-m_1|$, we find that

$$
N_2 \le C_{4.1} C_{4.2} \left\| f(m) \right\|_{(\beta,\mu)} \left\| g(\tau,m) \right\|_{(\nu,\beta,\mu,k,\kappa)}, \tag{41}
$$

where

$$
C_{4.1} = \sup_{\tau \in \bar{D}(0,\rho) \cup U_d} \frac{1+|\tau|^{2k}}{|\tau|} \exp\left(-\nu|\tau|^{\kappa}\right) \int_0^{|\tau|^k} \frac{\exp(\nu h^{\kappa/k})}{1+h^2} h^{\frac{1}{k}-1} \left(|\tau|^k - h\right)^{1/k} dh
$$

and

$$
C_{4.2} = \frac{\mathfrak{Q}}{\mathfrak{R}} \sup_{m \in \mathbb{R}} \left(1+|m|\right)^{\mu - \deg(R)} \int_{-\infty}^{+\infty} \frac{1}{(1+|m-m_1|)^{\mu} (1+|m_1|)^{\mu - \deg(Q)}} dm_1.
$$

Under the hypothesis $\kappa \ge k$ and from the estimates (7), (11), and (19) in the special case $\chi_2 = 1/k$ and $\nu_2 = -1$, we know that $C_{4.1}$ is finite.

From the estimates for (27), we know that $C_{4.2}$ is finite under the assumption (38) provided that $\mu > \deg(Q) + 1$. Finally, gathering the latter bound estimates together with (41) yields the result (39). □

In the next proposition, we recall from [1], Proposition 5, that $(E_{(\beta,\mu)}, \|\cdot\|_{(\beta,\mu)})$ is a Banach algebra for some noncommutative product $\star$ introduced below.

Proposition 4 *Let $Q_1(X), Q_2(X), R(X) \in \mathbb{C}[X]$ be polynomials such that*

$$
\deg(R) \ge \deg(Q_1), \qquad \deg(R) \ge \deg(Q_2), \qquad R(im) \ne 0, \tag{42}
$$

for all $m \in \mathbb{R}$. *Assume that* $\mu > \max(\deg(Q_1)+1, \deg(Q_2)+1)$. *Then there exists a constant* $C_5 > 0$ (*depending on* Q_1, Q_2, R, μ) *such that*

$$\left\| \frac{1}{R(im)} \int_{-\infty}^{+\infty} Q_1\big(i(m-m_1)\big) f(m-m_1) Q_2(im_1) g(m_1)\, dm_1 \right\|_{(\beta,\mu)} \leq C_5 \big\|f(m)\big\|_{(\beta,\mu)} \big\|g(m)\big\|_{(\beta,\mu)} \tag{43}$$

for all $f(m), g(m) \in E_{(\beta,\mu)}$. *Therefore*, $(E_{(\beta,\mu)}, \|\cdot\|_{(\beta,\mu)})$ *becomes a Banach algebra for the product* $\star$ *defined by*

$$f \star g(m) = \frac{1}{R(im)} \int_{-\infty}^{+\infty} Q_1\big(i(m-m_1)\big) f(m-m_1) Q_2(im_1) g(m_1)\, dm_1.$$

As a particular case, when $f, g \in E_{(\beta,\mu)}$ *with* $\beta > 0$ *and* $\mu > 1$, *the classical convolution product*

$$f * g(m) = \int_{-\infty}^{+\infty} f(m-m_1) g(m_1)\, dm_1$$

belongs to $E_{(\beta,\mu)}$.

2.2 Banach spaces of functions with exponential growth *k* and decay of exponential order 1

In this subsection, we mainly recall some functional properties of the Banach spaces already introduced in the work [1], Section 2. The Banach spaces we consider here coincide with those introduced in [1] except the fact that they are not depending on a complex parameter ϵ and that the functions living in these spaces are not holomorphic on a disc centered at 0 but only on a bounded sector centered at 0. For this reason, all the propositions are given without proof except Proposition 5, which is an improved version of Propositions 1 and 2 of [1].

We denote by S_d^b an open bounded sector centered at 0 in the direction $d \in \mathbb{R}$ and $\bar{S}_d^b$ its closure. Let S_d be an open unbounded sector in the direction d. By convention, we recall, the sectors we consider throughout the paper do not contain the origin in $\mathbb{C}$.

Definition 3 Let $\nu, \beta, \mu > 0$ be positive real numbers. Let $k \geq 1$ be an integer and let $d \in \mathbb{R}$. We denote $F^d_{(\nu,\beta,\mu,k)}$ the vector space of continuous functions $(\tau, m) \mapsto h(\tau, m)$ on $(\bar{S}_d^b \cup S_d) \times \mathbb{R}$, which are holomorphic with respect to τ on $S_d^b \cup S_d$ and such that

$$\big\|h(\tau,m)\big\|_{(\nu,\beta,\mu,k)} = \sup_{\tau \in \bar{S}_d^b \cup S_d, m \in \mathbb{R}} \big(1+|m|\big)^{\mu} \frac{1+|\tau|^{2k}}{|\tau|} \exp\big(\beta|m| - \nu|\tau|^k\big) \big|h(\tau,m)\big|$$

is finite. One can check that the normed space $(F^d_{(\nu,\beta,\mu,k)}, \|\cdot\|_{(\nu,\beta,\mu,k)})$ is a Banach space.

Throughout the whole subsection, we assume that $\mu, \beta, \nu > 0$ and $k \geq 1$, $d \in \mathbb{R}$ are fixed. In the next lemma, we check the continuity property by multiplication operation with bounded functions.

Lemma 2 *Let $(\tau,m) \mapsto a(\tau,m)$ be a bounded continuous function on $(\bar{S}_d^b \cup S_d) \times \mathbb{R}$, which is holomorphic with respect to τ on $S_d^b \cup S_d$. Then we have*

$$\|a(\tau,m)h(\tau,m)\|_{(\nu,\beta,\mu,k)} \leq \Bigl(\sup_{\tau\in\bar{S}_d^b\cup S_d, m\in\mathbb{R}} |a(\tau,m)|\Bigr)\|h(\tau,m)\|_{(\nu,\beta,\mu,k)} \tag{44}$$

for all $h(\tau,m) \in F^d_{(\nu,\beta,\mu,k)}$.

In the next proposition, we study the continuity property of some convolution operators acting on the latter Banach spaces.

Proposition 5 *Let $\gamma_1 \geq 0$ and $\chi_2 > -1$ be real numbers. Let $\nu_2 \geq -1$ be an integer. We consider a holomorphic function $a_{\gamma_1,k}(\tau)$ on $S_d^b \cup S_d$, continuous on $\bar{S}_d^b \cup S_d$, such that*

$$|a_{\gamma_1,k}(\tau)| \leq \frac{1}{(1+|\tau|^k)^{\gamma_1}}$$

for all $\tau \in S_d^b \cup S_d$.

If $1+\chi_2+\nu_2 \geq 0$ and $\gamma_1 \geq \nu_2$, then there exists a constant $C_6 > 0$ (depending on ν, ν_2, χ_2, γ_1) such that

$$\left\|a_{\gamma_1,k}(\tau)\int_0^{\tau^k}(\tau^k-s)^{\chi_2}s^{\nu_2}f(s^{1/k},m)\,ds\right\|_{(\nu,\beta,\mu,k)} \leq C_6\|f(\tau,m)\|_{(\nu,\beta,\mu,k)} \tag{45}$$

for all $f(\tau,m) \in F^d_{(\nu,\beta,\mu,k)}$.

Proof The proof follows similar arguments to those in Proposition 1. Indeed, let $f(\tau,m) \in F^d_{(\nu,\beta,\mu,k)}$. By definition, we have

$$\begin{aligned}
&\left\|a_{\gamma_1,k}(\tau)\int_0^{\tau^k}(\tau^k-s)^{\chi_2}s^{\nu_2}f(s^{1/k},m)\,ds\right\|_{(\nu,\beta,\mu,k)} \\
&\quad= \sup_{\tau\in\bar{S}_d^b\cup S_d, m\in\mathbb{R}} (1+|m|)^{\mu}\frac{1+|\tau|^{2k}}{|\tau|}\exp(\beta|m|-\nu|\tau|^k) \\
&\qquad\times\left|a_{\gamma_1,k}(\tau)\int_0^{\tau^k}\left\{(1+|m|)^{\mu}e^{\beta|m|}\exp(-\nu|s|)\frac{1+|s|^2}{|s|^{1/k}}f(s^{1/k},m)\right\}\right. \\
&\qquad\left.\times\mathcal{F}(\tau,s,m)\,ds\right|,
\end{aligned} \tag{46}$$

where

$$\mathcal{F}(\tau,s,m) = \frac{1}{(1+|m|)^{\mu}}e^{-\beta|m|}\frac{\exp(\nu|s|)}{1+|s|^2}|s|^{1/k}(\tau^k-s)^{\chi_2}s^{\nu_2}.$$

Therefore,

$$\left\|a_{\gamma_1,k}(\tau)\int_0^{\tau^k}(\tau^k-s)^{\chi_2}s^{\nu_2}f(s^{1/k},m)\,ds\right\|_{(\nu,\beta,\mu,k)} \leq C_6\|f(\tau,m)\|_{(\nu,\beta,\mu,k)}, \tag{47}$$

where

$$C_6 = \sup_{\tau\in\bar{S}_d^b\cup S_d} \frac{1+|\tau|^{2k}}{|\tau|}\exp\bigl(-\nu|\tau|^k\bigr)\frac{1}{(1+|\tau|^k)^{\gamma_1}}\int_0^{|\tau|^k}\frac{\exp(\nu h)}{1+h^2}h^{\frac{1}{k}}\bigl(|\tau|^k-h\bigr)^{\chi_2}h^{\nu_2}\,dh$$
$$= \sup_{x\geq 0} F(x),$$

where

$$F(x) = \frac{1+x^2}{x^{1/k}}\exp(-\nu x)\frac{1}{(1+x)^{\gamma_1}}\int_0^x \frac{\exp(\nu h)}{1+h^2}h^{\frac{1}{k}+\nu_2}(x-h)^{\chi_2}\,dh.$$

We write $F(x) = F_1(x) + F_2(x)$, where

$$F_1(x) = \frac{1+x^2}{x^{1/k}}\exp(-\nu x)\frac{1}{(1+x)^{\gamma_1}}\int_0^{x/2} \frac{\exp(\nu h)}{1+h^2}h^{\frac{1}{k}+\nu_2}(x-h)^{\chi_2}\,dh,$$
$$F_2(x) = \frac{1+x^2}{x^{1/k}}\exp(-\nu x)\frac{1}{(1+x)^{\gamma_1}}\int_{x/2}^x \frac{\exp(\nu h)}{1+h^2}h^{\frac{1}{k}+\nu_2}(x-h)^{\chi_2}\,dh.$$

Now, we study the function $F_1(x)$. We first assume that $-1 < \chi_2 < 0$. In that case, we have $(x-h)^{\chi_2} \leq (x/2)^{\chi_2}$ for all $0 \leq h \leq x/2$ with $x > 0$. We deduce that

$$F_1(x) \leq \frac{1+x^2}{x^{1/k}}\left(\frac{x}{2}\right)^{\chi_2} e^{-\nu x}\frac{1}{(1+x)^{\gamma_1}}\int_0^{x/2}\frac{e^{\nu h}}{1+h^2}h^{\frac{1}{k}+\nu_2}\,dh$$
$$\leq \bigl(1+x^2\bigr)\frac{1}{2^{1/k}(\frac{1}{k}+\nu_2+1)}\left(\frac{x}{2}\right)^{1+\chi_2+\nu_2}\frac{1}{(1+x)^{\gamma_1}}\exp\left(-\frac{\nu x}{2}\right) \tag{48}$$

for all $x > 0$. Bearing in mind that $1+\chi_2+\nu_2 \geq 0$ and since $1+x \geq 1$ for all $x \geq 0$, we deduce that there exists a constant $K_1 > 0$ with

$$\sup_{x\geq 0} F_1(x) \leq K_1. \tag{49}$$

We assume now that $\chi_2 \geq 0$. In this situation, we know that $(x-h)^{\chi_2} \leq x^{\chi_2}$ for all $0 \leq h \leq x/2$, with $x \geq 0$. Hence,

$$F_1(x) \leq \bigl(1+x^2\bigr)\frac{1}{2^{1/k}(\frac{1}{k}+\nu_2+1)}x^{\chi_2}(x/2)^{\nu_2+1}\frac{1}{(1+x)^{\gamma_1}}\exp\left(-\frac{\nu x}{2}\right) \tag{50}$$

for all $x \geq 0$. Again, we deduce that there exists a constant $K_{1.1} > 0$ with

$$\sup_{x\geq 0} F_1(x) \leq K_{1.1}. \tag{51}$$

In the next step, we focus on the function $F_2(x)$. First, we observe that $1+h^2 \geq 1+(x/2)^2$ for all $x/2 \leq h \leq x$. Therefore, there exists a constant $K_2 > 0$ such that

$$F_2(x) \leq \frac{1+x^2}{1+(\frac{x}{2})^2}\frac{1}{x^{1/k}}\exp(-\nu x)\frac{1}{(1+x)^{\gamma_1}}\int_{x/2}^x \exp(\nu h)h^{\frac{1}{k}+\nu_2}(x-h)^{\chi_2}\,dh$$
$$\leq K_2\frac{1}{x^{1/k}}\frac{1}{(1+x)^{\gamma_1}}\exp(-\nu x)\int_0^x \exp(\nu h)h^{\frac{1}{k}+\nu_2}(x-h)^{\chi_2}\,dh \tag{52}$$

for all $x > 0$. Now, from the estimates (18), we know that there exists a constant $K_{2.3} > 0$ such that

$$F_{2.1}(x) = \int_0^x \exp(\nu h) h^{\frac{1}{k}+\nu_2}(x-h)^{\chi_2}\,dh \le K_{2.3} x^{\frac{1}{k}+\nu_2} e^{\nu x} \tag{53}$$

for all $x \ge 1$. From (52) we get the existence of a constant $\tilde{F}_2 > 0$ with

$$\sup_{x\in[0,1]} F_2(x) \le \tilde{F}_2. \tag{54}$$

On the other hand, we also have $1 + x \ge x$ for all $x \ge 1$. Since $\gamma_1 \ge \nu_2$ and due to (52) with (53), we get a constant $\check{F}_2 > 0$ with

$$\sup_{x\ge 1} F_2(x) \le \check{F}_2. \tag{55}$$

Gathering the estimates (47), (49), (51), (54), and (55), we finally obtain (45). □

The next two propositions are already stated as Propositions 3 and 4 in [1].

Proposition 6 *Let $k \ge 1$ be an integer. Let $Q_1(X), Q_2(X), R(X) \in \mathbb{C}[X]$ such that*

$$\deg(R) \ge \deg(Q_1), \qquad \deg(R) \ge \deg(Q_2), \qquad R(im) \neq 0 \tag{56}$$

for all $m \in \mathbb{R}$. Assume that $\mu > \max(\deg(Q_1)+1, \deg(Q_2)+1)$. Let $m \mapsto b(m)$ be a continuous function on $\mathbb{R}$ such that

$$\left|b(m)\right| \le \frac{1}{|R(im)|}$$

for all $m \in \mathbb{R}$. Then there exists a constant $C_7 > 0$ (depending on Q_1, Q_2, R, μ, k, ν) such that

$$\begin{aligned} &\left\| b(m) \int_0^{\tau^k} \left(\tau^k - s\right)^{\frac{1}{k}} \left(\int_0^s \int_{-\infty}^{+\infty} Q_1\big(i(m-m_1)\big) f\big((s-x)^{1/k}, m-m_1\big) \right.\right. \\ &\qquad \left.\left. \times Q_2(im_1) g\big(x^{1/k}, m_1\big) \frac{1}{(s-x)x}\,dx\,dm_1 \right) ds \right\|_{(\nu,\beta,\mu,k)} \\ &\quad \le C_7 \left\| f(\tau,m)\right\|_{(\nu,\beta,\mu,k)} \left\| g(\tau,m)\right\|_{(\nu,\beta,\mu,k)} \end{aligned} \tag{57}$$

for all $f(\tau,m), g(\tau,m) \in F^d_{(\nu,\beta,\mu,k)}$.

Proposition 7 *Let $k \ge 1$ be an integer. Let $Q(X), R(X) \in \mathbb{C}[X]$ be polynomials such that*

$$\deg(R) \ge \deg(Q), \qquad R(im) \neq 0 \tag{58}$$

for all $m \in \mathbb{R}$. Assume that $\mu > \deg(Q) + 1$. Let $m \mapsto b(m)$ be a continuous function such that

$$\left|b(m)\right| \le \frac{1}{|R(im)|}$$

for all $m \in \mathbb{R}$. Then there exists a constant $C_8 > 0$ (depending on Q, R, μ, k, ν) such that

$$\left\| b(m) \int_0^{\tau^k} \left(\tau^k - s\right)^{\frac{1}{k}} \int_{-\infty}^{+\infty} f(m-m_1) Q(im_1) g\left(s^{1/k}, m_1\right) dm_1 \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k)} \leq C_8 \left\| f(m) \right\|_{(\beta,\mu)} \left\| g(\tau,m) \right\|_{(\nu,\beta,\mu,k)} \tag{59}$$

for all $f(m) \in E_{(\beta,\mu)}$, all $g(\tau,m) \in F^d_{(\nu,\beta,\mu,k)}$.

3 Laplace transform, asymptotic expansions and Fourier transform

We recall the definition of k-Borel summability of formal series with coefficients in a Banach space which is a slightly modified version of the one given in [2], Section 3.2, that was introduced in [1]. All the properties stated in this section are already contained in our previous work [1].

Definition 4 Let $k \geq 1$ be an integer. Let $m_k(n)$ be the sequence defined by

$$m_k(n) = \Gamma\left(\frac{n}{k}\right) = \int_0^{+\infty} t^{\frac{n}{k}-1} e^{-t}\, dt$$

for all $n \geq 1$. A formal series

$$\hat{X}(T) = \sum_{n=1}^{\infty} a_n T^n \in T\mathbb{E}[\![T]\!]$$

with coefficients in a Banach space $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ is said to be m_k-summable with respect to T in the direction $d \in [0, 2\pi)$ if

(i) there exists $\rho \in \mathbb{R}_+$ such that the following formal series, called a formal m_k-Borel transform of $\hat{X}$,

$$\mathcal{B}_{m_k}(\hat{X})(\tau) = \sum_{n=1}^{\infty} \frac{a_n}{\Gamma(\frac{n}{k})} \tau^n \in \tau\mathbb{E}[\![\tau]\!],$$

is absolutely convergent for $|\tau| < \rho$;

(ii) there exists $\delta > 0$ such that the series $\mathcal{B}_{m_k}(\hat{X})(\tau)$ can be analytically continued with respect to τ in a sector $S_{d,\delta} = \{\tau \in \mathbb{C}^* : |d - \arg(\tau)| < \delta\}$. Moreover, there exist $C > 0$ and $K > 0$ such that

$$\left\| \mathcal{B}_{m_k}(\hat{X})(\tau) \right\|_{\mathbb{E}} \leq C e^{K|\tau|^k}$$

for all $\tau \in S_{d,\delta}$.

If this is so, the vector valued m_k-Laplace transform of $\mathcal{B}_{m_k}(\hat{X})(\tau)$ in the direction d is defined by

$$\mathcal{L}^d_{m_k}\left(\mathcal{B}_{m_k}(\hat{X})\right)(T) = k \int_{L_\gamma} \mathcal{B}_{m_k}(\hat{X})(u) e^{-(u/T)^k} \frac{du}{u},$$

along a half-line $L_\gamma = \mathbb{R}_+ e^{i\gamma} \subset S_{d,\delta} \cup \{0\}$, where γ depends on T and is chosen in such a way that $\cos(k(\gamma - \arg(T))) \geq \delta_1 > 0$, for some fixed δ_1. The function $\mathcal{L}^d_{m_k}(\mathcal{B}_{m_k}(\hat{X}))(T)$ is well defined, holomorphic and bounded in any sector

$$S_{d,\theta,R^{1/k}} = \{T \in \mathbb{C}^* : |T| < R^{1/k}, |d - \arg(T)| < \theta/2\},$$

where $\frac{\pi}{k} < \theta < \frac{\pi}{k} + 2\delta$ and $0 < R < \delta_1/K$. This function is called the m_k-sum of the formal series $\hat{X}(T)$ in the direction d.

In the next proposition, we give some identities for the m_k-Borel transform that will be useful in the sequel.

Proposition 8 *Let $\hat{f}(t) = \sum_{n\geq 1} f_n t^n$, $\hat{g}(t) = \sum_{n\geq 1} g_n t^n$ be formal series whose coefficients f_n, g_n belong to some Banach space $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$. We assume that $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ is a Banach algebra for some product $\star$. Let $k, m \geq 1$ be integers. The following formal identities hold.*

$$\mathcal{B}_{m_k}(t^{k+1}\partial_t \hat{f}(t))(\tau) = k\tau^k \mathcal{B}_{m_k}(\hat{f}(t))(\tau), \tag{60}$$

$$\mathcal{B}_{m_k}(t^m \hat{f}(t))(\tau) = \frac{\tau^k}{\Gamma(\frac{m}{k})} \int_0^{\tau^k} (\tau^k - s)^{\frac{m}{k}-1} \mathcal{B}_{m_k}(\hat{f}(t))(s^{1/k}) \frac{ds}{s} \tag{61}$$

and

$$\mathcal{B}_{m_k}(\hat{f}(t) \star \hat{g}(t))(\tau) = \tau^k \int_0^{\tau^k} \mathcal{B}_{m_k}(\hat{f}(t))((\tau^k - s)^{1/k}) \star \mathcal{B}_{m_k}(\hat{g}(t))(s^{1/k}) \frac{1}{(\tau^k - s)s} ds. \tag{62}$$

In the following proposition, we recall some properties of the inverse Fourier transform

Proposition 9 *Let $f \in E_{(\beta,\mu)}$ with $\beta > 0$, $\mu > 1$. The inverse Fourier transform of f is defined by*

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} f(m) \exp(ixm)\, dm$$

for all $x \in \mathbb{R}$. The function $\mathcal{F}^{-1}(f)$ extends to an analytic function on the strip

$$H_\beta = \{z \in \mathbb{C} / |\mathrm{Im}(z)| < \beta\}. \tag{63}$$

Let $\phi(m) = imf(m) \in E_{(\beta,\mu-1)}$. Then we have

$$\partial_z \mathcal{F}^{-1}(f)(z) = \mathcal{F}^{-1}(\phi)(z) \tag{64}$$

for all $z \in H_\beta$.

Let $g \in E_{(\beta,\mu)}$ and let $\psi(m) = \frac{1}{(2\pi)^{1/2}} f * g(m)$, the convolution product of f and g, for all $m \in \mathbb{R}$. From Proposition 4, we know that $\psi \in E_{(\beta,\mu)}$. Moreover, we have

$$\mathcal{F}^{-1}(f)(z)\mathcal{F}^{-1}(g)(z) = \mathcal{F}^{-1}(\psi)(z) \tag{65}$$

for all $z \in H_\beta$.

4 Formal and analytic solutions of convolution initial value problems with complex parameters

4.1 Formal solutions of the main convolution initial value problem

Let $k_1, k_2 \geq 1$, $D \geq 2$ be integers such that $k_2 > k_1$. Let $\delta_l \geq 1$ be integers such that

$$1 = \delta_1, \qquad \delta_l < \delta_{l+1}, \tag{66}$$

for all $1 \leq l \leq D-1$. For all $1 \leq l \leq D-1$, let $d_l, \Delta_l \geq 0$ be nonnegative integers such that

$$d_l > \delta_l, \qquad \Delta_l - d_l + \delta_l - 1 \geq 0. \tag{67}$$

Let $Q(X), Q_1(X), Q_2(X), R_l(X) \in \mathbb{C}[X]$, $0 \leq l \leq D$, be polynomials such that

$$\begin{aligned} &\deg(Q) \geq \deg(R_D) \geq \deg(R_l), \qquad \deg(R_D) \geq \deg(Q_1), \qquad \deg(R_D) \geq \deg(Q_2), \\ &Q(im) \neq 0, \qquad R_l(im) \neq 0, \qquad R_D(im) \neq 0 \end{aligned} \tag{68}$$

for all $m \in \mathbb{R}$, all $0 \leq l \leq D-1$. We consider sequences of functions $m \mapsto C_{0,n}(m,\epsilon)$, for all $n \geq 0$, and $m \mapsto F_n(m,\epsilon)$, for all $n \geq 1$, that belong to the Banach space $E_{(\beta,\mu)}$ for some $\beta > 0$ and $\mu > \max(\deg(Q_1)+1, \deg(Q_2)+1)$ and which depend holomorphically on $\epsilon \in D(0,\epsilon_0)$ for some $\epsilon_0 > 0$. We assume that there exist constants $K_0, T_0 > 0$ such that

$$\left\| C_{0,n}(m,\epsilon) \right\|_{(\beta,\mu)} \leq K_0 \left(\frac{1}{T_0} \right)^n \tag{69}$$

for all $n \geq 1$, for all $\epsilon \in D(0,\epsilon_0)$. We define $C_0(T,m,\epsilon) = \sum_{n \geq 1} C_{0,n}(m,\epsilon) T^n$ which is a convergent series on $D(0, T_0/2)$ with values in $E_{(\beta,\mu)}$ and $F(T,m,\epsilon) = \sum_{n\geq 1} F_n(m,\epsilon) T^n$, which is a formal series with coefficients in $E_{(\beta,\mu)}$. Let $c_{1,2}(\epsilon)$, $c_0(\epsilon)$, $c_{0,0}(\epsilon)$, and $c_F(\epsilon)$ be bounded holomorphic functions on $D(0,\epsilon_0)$ which vanish at the origin $\epsilon = 0$. We consider the following initial value problem:

$$\begin{aligned} &Q(im)\big(\partial_T U(T,m,\epsilon)\big) - T^{(\delta_D-1)(k_2+1)} \partial_T^{\delta_D} R_D(im) U(T,m,\epsilon) \\ &\quad = \epsilon^{-1} \frac{c_{1,2}(\epsilon)}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} Q_1\big(i(m-m_1)\big) U(T,m-m_1,\epsilon) Q_2(im_1) U(T,m_1,\epsilon)\, dm_1 \\ &\qquad + \sum_{l=1}^{D-1} R_l(im) \epsilon^{\Delta_l - d_l + \delta_l - 1} T^{d_l} \partial_T^{\delta_l} U(T,m,\epsilon) \\ &\qquad + \epsilon^{-1} \frac{c_0(\epsilon)}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} C_0(T,m-m_1,\epsilon) R_0(im_1) U(T,m_1,\epsilon)\, dm_1 \\ &\qquad + \epsilon^{-1} \frac{c_{0,0}(\epsilon)}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} C_{0,0}(m-m_1,\epsilon) R_0(im_1) U(T,m_1,\epsilon)\, dm_1 \\ &\qquad + \epsilon^{-1} c_F(\epsilon) F(T,m,\epsilon) \end{aligned} \tag{70}$$

for given initial data $U(0,m,\epsilon) \equiv 0$.

Proposition 10 *There exists a unique formal series*

$$\hat{U}(T,m,\epsilon) = \sum_{n \geq 1} U_n(m,\epsilon) T^n$$

solution of (70) *with initial data* $U(0,m,\epsilon) \equiv 0$, *where the coefficients* $m \mapsto U_n(m,\epsilon)$ *belong to* $E_{(\beta,\mu)}$ *for* $\beta > 0$ *and* $\mu > \max(\deg(Q_1)+1, \deg(Q_2)+1)$ *given above and depend holomorphically on* ϵ *in* $D(0,\epsilon_0)$.

Proof From Proposition 4 and the conditions stated above, we find that the coefficients $U_n(m,\epsilon)$ of $\hat{U}(T,m,\epsilon)$ are well defined, belong to $E_{(\beta,\mu)}$ for all $\epsilon \in D(0,\epsilon_0)$, all $n \geq 1$, and satisfy the following recursion relation:

$$
\begin{aligned}
&(n+1)U_{n+1}(m,\epsilon) \\
&\quad = \frac{R_D(im)}{Q(im)} \prod_{j=0}^{\delta_D-1} \bigl(n+\delta_D-(\delta_D-1)(k_2+1)-j\bigr) U_{n+\delta_D-(\delta_D-1)(k_2+1)}(m,\epsilon) \\
&\qquad + \frac{\epsilon^{-1}}{Q(im)} \sum_{n_1+n_2=n, n_1\geq 1, n_2\geq 1} \frac{c_{1,2}(\epsilon)}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} Q_1\bigl(i(m-m_1)\bigr) U_{n_1}(m-m_1,\epsilon) \\
&\qquad \times Q_2(im_1) U_{n_2}(m_1,\epsilon)\, dm_1 \\
&\qquad + \sum_{l=1}^{D-1} \frac{R_l(im)}{Q(im)} \Bigl(\epsilon^{\Delta_l-d_l+\delta_l-1} \prod_{j=0}^{\delta_l-1} (n+\delta_l-d_l-j)\Bigr) U_{n+\delta_l-d_l}(m,\epsilon) \\
&\qquad + \frac{\epsilon^{-1}}{Q(im)} \sum_{n_1+n_2=n, n_1\geq 1, n_2\geq 1} \frac{c_0(\epsilon)}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} C_{0,n_1}(m-m_1,\epsilon) R_0(im_1) U_{n_2}(m_1,\epsilon)\, dm_1 \\
&\qquad + \frac{\epsilon^{-1} c_{0,0}(\epsilon)}{(2\pi)^{1/2} Q(im)} \int_{-\infty}^{+\infty} C_{0,0}(m-m_1,\epsilon) R_0(im_1) U_n(m_1,\epsilon)\, dm_1 \\
&\qquad + \frac{\epsilon^{-1} c_F(\epsilon)}{Q(im)} F_n(m,\epsilon)
\end{aligned} \tag{71}
$$

for all $n \geq \max(\max_{1\leq l\leq D-1} d_l, (\delta_D-1)(k_2+1))$. □

4.2 Analytic solutions for an auxiliary convolution problem resulting from a m_{k_1}-Borel transform applied to the main convolution initial value problem

We make the additional assumption that

$$d_l > (\delta_l - 1)(k_1 + 1) \tag{72}$$

for all $1 \leq l \leq D-1$. Using (8.7) from [9], p.3630, we can expand the operators $T^{\delta_l(k_1+1)}\partial_T^{\delta_l}$ in the form

$$T^{\delta_l(k_1+1)}\partial_T^{\delta_l} = \bigl(T^{k_1+1}\partial_T\bigr)^{\delta_l} + \sum_{1\leq p\leq \delta_l-1} A_{\delta_l,p} T^{k_1(\delta_l-p)} \bigl(T^{k_1+1}\partial_T\bigr)^p, \tag{73}$$

where $A_{\delta_l,p}$, $p = 1,\ldots,\delta_l-1$ are real numbers, for all $1 \leq l \leq D$. We define integers $d^1_{l,k_1} > 0$ in order to satisfy

$$d_l + k_1 + 1 = \delta_l(k_1+1) + d^1_{l,k_1} \tag{74}$$

for all $1 \leq l \leq D-1$. We also rewrite $(\delta_D-1)(k_2+1) = (\delta_D-1)(k_1+1) + (\delta_D-1)(k_2-k_1)$.

Multiplying (70) by T^{k_1+1} and using (73), we can rewrite (70) in the form

$$
\begin{aligned}
&Q(im)\big(T^{k_1+1}\partial_T U(T,m,\epsilon)\big)\\
&\quad = R_D(im)T^{(\delta_D-1)(k_2-k_1)}\big(T^{k_1+1}\partial_T\big)^{\delta_D}U(T,m,\epsilon)\\
&\qquad + R_D(im)\sum_{1\le p\le\delta_D-1} A_{\delta_D,p}T^{(\delta_D-1)(k_2-k_1)}T^{k_1(\delta_D-p)}\big(T^{k_1+1}\partial_T\big)^{p}U(T,m,\epsilon)\\
&\qquad + \epsilon^{-1}T^{k_1+1}\frac{c_{1,2}(\epsilon)}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} Q_1\big(i(m-m_1)\big)U(T,m-m_1,\epsilon)Q_2(im_1)U(T,m_1,\epsilon)\,dm_1\\
&\qquad + \sum_{l=1}^{D-1} R_l(im)\bigg(\epsilon^{\Delta_l-d_l+\delta_l-1}T^{d^1_{l,k_1}}\big(T^{k_1+1}\partial_T\big)^{\delta_l}U(T,m,\epsilon)\\
&\qquad + \sum_{1\le p\le\delta_l-1} A_{\delta_l,p}\epsilon^{\Delta_l-d_l+\delta_l-1}T^{k_1(\delta_l-p)+d^1_{l,k_1}}\big(T^{k_1+1}\partial_T\big)^{p}U(T,m,\epsilon)\bigg)\\
&\qquad + \epsilon^{-1}T^{k_1+1}\frac{c_0(\epsilon)}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} C_0(T,m-m_1,\epsilon)R_0(im_1)U(T,m_1,\epsilon)\,dm_1\\
&\qquad + \epsilon^{-1}T^{k_1+1}\frac{c_{0,0}(\epsilon)}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} C_{0,0}(m-m_1,\epsilon)R_0(im_1)U(T,m_1,\epsilon)\,dm_1\\
&\qquad + \epsilon^{-1}c_F(\epsilon)T^{k_1+1}F(T,m,\epsilon). \qquad (75)
\end{aligned}
$$

We denote $\omega_{k_1}(\tau,m,\epsilon)$ the formal m_{k_1}-Borel transform of $\hat{U}(T,m,\epsilon)$ with respect to T, $\varphi_{k_1}(\tau,m,\epsilon)$ the formal m_{k_1}-Borel transform of $C_0(T,m,\epsilon)$ with respect to T and $\psi_{k_1}(\tau,m,\epsilon)$ the formal m_{k_1}-Borel transform of $F(T,m,\epsilon)$ with respect to T. More precisely,

$$
\omega_{k_1}(\tau,m,\epsilon)=\sum_{n\ge1}U_n(m,\epsilon)\frac{\tau^n}{\Gamma(\frac{n}{k_1})},\qquad \varphi_{k_1}(\tau,m,\epsilon)=\sum_{n\ge1}C_{0,n}(m,\epsilon)\frac{\tau^n}{\Gamma(\frac{n}{k_1})}
$$

$$
\psi_{k_1}(\tau,m,\epsilon)=\sum_{n\ge1}F_n(m,\epsilon)\frac{\tau^n}{\Gamma(\frac{n}{k_1})}.
$$

Using (69) we find that, for any $\kappa\ge k_1$, the function $\varphi_{k_1}(\tau,m,\epsilon)$ belongs to $F^d_{(\nu,\beta,\mu,k_1,\kappa)}$ for all $\epsilon\in D(0,\epsilon_0)$, any unbounded sector U_d centered at 0 with bisecting direction $d\in\mathbb{R}$, for some $\nu>0$. Indeed, we have

$$
\begin{aligned}
&\big\|\varphi_{k_1}(\tau,m,\epsilon)\big\|_{(\nu,\beta,\mu,k_1,\kappa)}\\
&\quad\le\sum_{n\ge1}\big\|C_{0,n}(m,\epsilon)\big\|_{(\beta,\mu)}\bigg(\sup_{\tau\in\bar{D}(0,\rho)\cup U_d}\frac{1+|\tau|^{2k_1}}{|\tau|}\exp\big(-\nu|\tau|^{\kappa}\big)\frac{|\tau|^n}{\Gamma(\frac{n}{k_1})}\bigg). \qquad (76)
\end{aligned}
$$

By using the classical estimates

$$
\sup_{x\ge0}x^{m_1}\exp(-m_2x)=\left(\frac{m_1}{m_2}\right)^{m_1}e^{-m_1} \qquad (77)
$$

for any real numbers $m_1\ge0$, $m_2>0$, and the Stirling formula $\Gamma(n/k_1)\sim(2\pi)^{1/2}\times(n/k_1)^{\frac{n}{k_1}-\frac{1}{2}}e^{-n/k_1}$ as n tends to $+\infty$, we get two constants $A_1,A_2>0$ depending on ν, k_1,

κ such that

$$\begin{aligned}
&\sup_{\tau\in\bar{D}(0,\rho)\cup U_d} \frac{1+|\tau|^{2k_1}}{|\tau|}\exp\bigl(-\nu|\tau|^{\kappa}\bigr)\frac{|\tau|^n}{\Gamma(\frac{n}{k_1})} \\
&\quad= \sup_{x\geq 0}\bigl(1+x^{2k_1/\kappa}\bigr)x^{\frac{n-1}{\kappa}}\frac{e^{-\nu x}}{\Gamma(\frac{n}{k_1})} \\
&\quad\leq \biggl(\biggl(\frac{n-1}{\nu\kappa}\biggr)^{\frac{n-1}{\kappa}}e^{-\frac{n-1}{\kappa}} + \biggl(\frac{n-1}{\nu\kappa}+\frac{2k_1}{\nu\kappa}\biggr)^{\frac{n-1}{\kappa}+\frac{2k_1}{\kappa}}e^{-(\frac{n-1}{\kappa}+\frac{2k_1}{\kappa})}\biggr)/\Gamma(n/k_1) \\
&\quad\leq A_1(A_2)^n
\end{aligned} \tag{78}$$

for all $n \geq 1$. Therefore, if the inequality $A_2 < T_0$ holds, we get the estimates

$$\bigl\|\varphi_{k_1}(\tau,m,\epsilon)\bigr\|_{(\nu,\beta,\mu,k_1,\kappa)} \leq A_1\sum_{n\geq 1}\bigl\|C_{0,n}(m,\epsilon)\bigr\|_{(\beta,\mu)}(A_2)^n \leq \frac{A_1A_2K_0}{T_0}\frac{1}{1-\frac{A_2}{T_0}}. \tag{79}$$

On the other hand, we make the assumption that $\psi_{k_1}(\tau,m,\epsilon) \in F^d_{(\nu,\beta,\mu,k_1,k_1)}$, for all $\epsilon \in D(0,\epsilon_0)$, for some unbounded sector U_d with bisecting direction $d \in \mathbb{R}$, where ν is chosen above. We will make the convention to denote $\psi^d_{k_1}$ the analytic continuation of the convergent power series ψ_{k_1} on the domain $U_d \cup D(0,\rho)$. In particular, we find that $\psi^d_{k_1}(\tau,m,\epsilon) \in F^d_{(\nu,\beta,\mu,k_1,\kappa)}$ for any $\kappa \geq k_1$. We also assume that there exists a constant $\zeta_{\psi_{k_1}} > 0$ such that

$$\bigl\|\psi^d_{k_1}(\tau,m,\epsilon)\bigr\|_{(\nu,\beta,\mu,k_1,k_1)} \leq \zeta_{\psi_{k_1}} \tag{80}$$

for all $\epsilon \in D(0,\epsilon_0)$. In particular, we notice that

$$\bigl\|\psi^d_{k_1}(\tau,m,\epsilon)\bigr\|_{(\nu,\beta,\mu,k_1,\kappa)} \leq \zeta_{\psi_{k_1}} \tag{81}$$

for any $\kappa \geq k_1$. We require that there exists a constant $r_{Q,R_l} > 0$ such that

$$\left|\frac{Q(im)}{R_l(im)}\right| \geq r_{Q,R_l} \tag{82}$$

for all $m \in \mathbb{R}$, all $1 \leq l \leq D$.

Using the computation rules for the formal m_{k_1}-Borel transform in Proposition 8, we deduce the following equation satisfied by $\omega_{k_1}(\tau,m,\epsilon)$:

$$\begin{aligned}
&Q(im)\bigl(k_1\tau^{k_1}\omega_{k_1}(\tau,m,\epsilon)\bigr) \\
&\quad= R_D(im)\frac{\tau^{k_1}}{\Gamma(\frac{(\delta_D-1)(k_2-k_1)}{k_1})}\int_0^{\tau^{k_1}}\bigl(\tau^{k_1}-s\bigr)^{\frac{(\delta_D-1)(k_2-k_1)}{k_1}-1}k_1^{\delta_D}s^{\delta_D}\omega_{k_1}\bigl(s^{1/k_1},m,\epsilon\bigr)\frac{ds}{s} \\
&\qquad+ R_D(im)\sum_{1\leq p\leq\delta_D-1}A_{\delta_D,p}\frac{\tau^{k_1}}{\Gamma(\frac{(\delta_D-1)(k_2-k_1)+k_1(\delta_D-p)}{k_1})} \\
&\qquad\times\int_0^{\tau^{k_1}}\bigl(\tau^{k_1}-s\bigr)^{\frac{(\delta_D-1)(k_2-k_1)+k_1(\delta_D-p)}{k_1}-1}k_1^p s^p\omega_{k_1}\bigl(s^{1/k_1},m,\epsilon\bigr)\frac{ds}{s}
\end{aligned}$$

$$
\begin{aligned}
&+\epsilon^{-1}\frac{\tau^{k_1}}{\Gamma(1+\frac{1}{k_1})}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{1/k_1}\\
&\times\left(\frac{c_{1,2}(\epsilon)}{(2\pi)^{1/2}}s\int_0^s\int_{-\infty}^{+\infty}Q_1\big(i(m-m_1)\big)\omega_{k_1}\big((s-x)^{1/k_1},m-m_1,\epsilon\big)\right.\\
&\left.\times Q_2(im_1)\omega_{k_1}\big(x^{1/k_1},m_1,\epsilon\big)\frac{1}{(s-x)x}\,dx\,dm_1\right)\frac{ds}{s}\\
&+\sum_{l=1}^{D-1}R_l(im)\left(\epsilon^{\Delta_l-d_l+\delta_l-1}\frac{\tau^{k_1}}{\Gamma(\frac{d_{l,k_1}^1}{k_1})}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\frac{d_{l,k_1}^1}{k_1}-1}\big(k_1^{\delta_l}s^{\delta_l}\omega_{k_1}\big(s^{1/k_1},m,\epsilon\big)\big)\frac{ds}{s}\right.\\
&+\sum_{1\le p\le\delta_l-1}A_{\delta_l,p}\epsilon^{\Delta_l-d_l+\delta_l-1}\frac{\tau^{k_1}}{\Gamma(\frac{d_{l,k_1}^1}{k_1}+\delta_l-p)}\\
&\left.\times\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\frac{d_{l,k_1}^1}{k_1}+\delta_l-p-1}\big(k_1^p s^p\omega_{k_1}\big(s^{1/k_1},m,\epsilon\big)\big)\frac{ds}{s}\right)\\
&+\epsilon^{-1}\frac{\tau^{k_1}}{\Gamma(1+\frac{1}{k_1})}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{1/k_1}\\
&\times\left(\frac{c_0(\epsilon)}{(2\pi)^{1/2}}s\int_0^s\int_{-\infty}^{+\infty}\varphi_{k_1}\big((s-x)^{1/k_1},m-m_1,\epsilon\big)\right.\\
&\left.\times R_0(im_1)\omega_{k_1}\big(x^{1/k_1},m_1,\epsilon\big)\frac{1}{(s-x)x}\,dx\,dm_1\right)\frac{ds}{s}\\
&+\epsilon^{-1}\frac{\tau^{k_1}}{\Gamma(1+\frac{1}{k_1})}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{1/k_1}\frac{c_{0,0}(\epsilon)}{(2\pi)^{1/2}}\left(\int_{-\infty}^{+\infty}C_{0,0}(m-m_1,\epsilon)\right.\\
&\left.\times R_0(im_1)\omega_{k_1}\big(s^{1/k_1},m_1,\epsilon\big)\,dm_1\right)\frac{ds}{s}\\
&+\epsilon^{-1}c_F(\epsilon)\frac{\tau^{k_1}}{\Gamma(1+\frac{1}{k_1})}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{1/k_1}\psi_{k_1}^d\big(s^{1/k_1},m,\epsilon\big)\frac{ds}{s}.
\end{aligned}
\tag{83}
$$

In the next proposition, we give sufficient conditions under which (83) has a solution $\omega_{k_1}^d(\tau,m,\epsilon)$ in the Banach space $F_{(\nu,\beta,\mu,k_1,\kappa)}^d$ where β, μ are defined above and for well chosen $\kappa > k_1$.

Proposition 11 *Under the assumption that*

$$\frac{1}{\kappa}=\frac{1}{k_1}-\frac{1}{k_2},\qquad \frac{k_2}{k_2-k_1}\ge\frac{d_l+(1-\delta_l)}{d_l+(1-\delta_l)(k_1+1)} \tag{84}$$

for all $1\le l\le D-1$, *there exist radii* $r_{Q,R_l}>0$, $1\le l\le D$, *a constant* $\varpi>0$, *and constants* $\zeta_{1,2},\zeta_{0,0},\zeta_0,\zeta_1,\zeta_{1,0},\zeta_F,\zeta_2>0$ (*depending on* Q_1, Q_2, k_1, μ, ν, ϵ_0, R_l, Δ_l, δ_l, d_l *for* $1\le l\le D-1$) *such that if*

$$
\begin{aligned}
&\sup_{\epsilon\in D(0,\epsilon_0)}\left|\frac{c_{1,2}(\epsilon)}{\epsilon}\right|\le\zeta_{1,2},\qquad \sup_{\epsilon\in D(0,\epsilon_0)}\left|\frac{c_0(\epsilon)}{\epsilon}\right|\le\zeta_{1,0},\qquad \big\|\varphi_{k_1}(\tau,m,\epsilon)\big\|_{(\nu,\beta,\mu,k_1,\kappa)}\le\zeta_1,\\
&\sup_{\epsilon\in D(0,\epsilon_0)}\left|\frac{c_{0,0}(\epsilon)}{\epsilon}\right|\le\zeta_{0,0},\qquad \big\|C_{0,0}(m,\epsilon)\big\|_{(\beta,\mu)}\le\zeta_0,
\end{aligned}
\tag{85}
$$

$$\sup_{\epsilon \in D(0,\epsilon_0)} \left| \frac{c_F(\epsilon)}{\epsilon} \right| \le \zeta_F, \qquad \left\| \psi_{k_1}^d(\tau,m,\epsilon) \right\|_{(\nu,\beta,\mu,k_1,\kappa)} \le \zeta_2$$

for all $\epsilon \in D(0,\epsilon_0)$, (83) *has a unique solution* $\omega_{k_1}^d(\tau,m,\epsilon)$ *in the space* $F^d_{(\nu,\beta,\mu,k_1,\kappa)}$ *where* $\beta,\mu > 0$ *are defined in Proposition* 10 *which verifies* $\|\omega_{k_1}^d(\tau,m,\epsilon)\|_{(\nu,\beta,\mu,k_1,\kappa)} \le \varpi$, *for all* $\epsilon \in D(0,\epsilon_0)$.

Proof We start the proof with a lemma which provides appropriate conditions in order to apply a fixed point theorem.

Lemma 3 *One can choose the constants* $r_{Q,R_l} > 0$, *for* $1 \le l \le D$, *a small enough constant* ϖ, *and constants* $\zeta_{1,2}, \zeta_0, \zeta_{0,0}, \zeta_1, \zeta_{1,0}, \zeta_F, \zeta_2 > 0$ (*depending on* Q_1, Q_2, k_1, μ, ν, ϵ_0, R_l, Δ_l, δ_l, d_l *for* $1 \le l \le D-1$) *such that if* (85) *holds for all* $\epsilon \in D(0,\epsilon_0)$, *the map* $\mathcal{H}_\epsilon^{k_1}$ *defined by*

$$\begin{aligned}
&\mathcal{H}_\epsilon^{k_1}\big(w(\tau,m)\big) \\
&= \frac{R_D(im)}{Q(im)} \frac{1}{k_1\Gamma(\frac{(\delta_D-1)(k_2-k_1)}{k_1})} \int_0^{\tau^{k_1}} \big(\tau^{k_1}-s\big)^{\frac{(\delta_D-1)(k_2-k_1)}{k_1}-1} k_1^{\delta_D} s^{\delta_D} w\big(s^{1/k_1},m\big) \frac{ds}{s} \\
&\quad + \frac{R_D(im)}{Q(im)} \sum_{1 \le p \le \delta_D-1} A_{\delta_D,p} \frac{1}{k_1\Gamma(\frac{(\delta_D-1)(k_2-k_1)+k_1(\delta_D-p)}{k_1})} \\
&\quad \times \int_0^{\tau^{k_1}} \big(\tau^{k_1}-s\big)^{\frac{(\delta_D-1)(k_2-k_1)+k_1(\delta_D-p)}{k_1}-1} k_1^p s^p w\big(s^{1/k_1},m\big) \frac{ds}{s} \\
&\quad + \epsilon^{-1} \frac{1}{Q(im)k_1\Gamma(1+\frac{1}{k_1})} \int_0^{\tau^{k_1}} \big(\tau^{k_1}-s\big)^{1/k_1} \\
&\quad \times \left(\frac{c_{1,2}(\epsilon)}{(2\pi)^{1/2}} s \int_0^s \int_{-\infty}^{+\infty} Q_1\big(i(m-m_1)\big) w\big((s-x)^{1/k_1}, m-m_1\big) \right. \\
&\quad \left. \times Q_2(im_1) w\big(x^{1/k_1},m_1\big) \frac{1}{(s-x)x}\, dx\, dm_1 \right) \frac{ds}{s} \\
&\quad + \sum_{l=1}^{D-1} \frac{R_l(im)}{Q(im)} \left(\epsilon^{\Delta_l-d_l+\delta_l-1} \frac{1}{k_1\Gamma(\frac{d_{l,k_1}^1}{k_1})} \int_0^{\tau^{k_1}} \big(\tau^{k_1}-s\big)^{\frac{d_{l,k_1}^1}{k_1}-1} \big(k_1{}^{\delta_l} s^{\delta_l} w\big(s^{1/k_1},m\big)\big) \frac{ds}{s} \right. \\
&\quad + \sum_{1 \le p \le \delta_l-1} A_{\delta_l,p} \epsilon^{\Delta_l-d_l+\delta_l-1} \frac{1}{k_1\Gamma(\frac{d_{l,k_1}^1}{k_1}+\delta_l-p)} \\
&\quad \left. \times \int_0^{\tau^{k_1}} \big(\tau^{k_1}-s\big)^{\frac{d_{l,k_1}^1}{k_1}+\delta_l-p-1} \big(k_1^p s^p w\big(s^{1/k_1},m\big)\big) \frac{ds}{s} \right) \\
&\quad + \epsilon^{-1} \frac{c_0(\epsilon)}{Q(im)k_1\Gamma(1+\frac{1}{k_1})} \int_0^{\tau^{k_1}} \big(\tau^{k_1}-s\big)^{1/k_1} \\
&\quad \times \left(\frac{1}{(2\pi)^{1/2}} s \int_0^s \int_{-\infty}^{+\infty} \varphi_{k_1}\big((s-x)^{1/k_1}, m-m_1, \epsilon\big) \right. \\
&\quad \left. \times R_0(im_1) w\big(x^{1/k_1},m_1\big) \frac{1}{(s-x)x}\, dx\, dm_1 \right) \frac{ds}{s} \\
&\quad + \epsilon^{-1} \frac{c_{0,0}(\epsilon)}{Q(im)k_1\Gamma(1+\frac{1}{k_1})} \int_0^{\tau^{k_1}} \big(\tau^{k_1}-s\big)^{1/k_1}
\end{aligned}$$

$$\times \frac{1}{(2\pi)^{1/2}} \left(\int_{-\infty}^{+\infty} C_{0,0}(m-m_1,\epsilon) R_0(im_1) w\bigl(s^{1/k_1}, m_1\bigr)\, dm_1 \right) \frac{ds}{s}$$

$$+ \epsilon^{-1} \frac{c_F(\epsilon)}{Q(im)k_1\Gamma(1+\frac{1}{k_1})} \int_0^{\tau^{k_1}} \bigl(\tau^{k_1}-s\bigr)^{1/k_1} \psi_{k_1}^d\bigl(s^{1/k_1}, m, \epsilon\bigr) \frac{ds}{s} \tag{86}$$

satisfies the next properties.

(i) *The following inclusion holds*:

$$\mathcal{H}_\epsilon^{k_1}\bigl(\bar{B}(0,\varpi)\bigr) \subset \bar{B}(0,\varpi), \tag{87}$$

where $\bar{B}(0,\varpi)$ *is the closed ball of radius* $\varpi > 0$ *centered at* 0 *in* $F^d_{(\nu,\beta,\mu,k_1,\kappa)}$, *for all* $\epsilon \in D(0,\epsilon_0)$.

(ii) *We have*

$$\bigl\| \mathcal{H}_\epsilon^{k_1}(w_1) - \mathcal{H}_\epsilon^{k_1}(w_2) \bigr\|_{(\nu,\beta,\mu,k_1,\kappa)} \le \frac{1}{2} \|w_1 - w_2\|_{(\nu,\beta,\mu,k_1,\kappa)} \tag{88}$$

for all $w_1, w_2 \in \bar{B}(0,\varpi)$, *for all* $\epsilon \in D(0,\epsilon_0)$.

Proof We first check the property (87). Let $\epsilon \in D(0,\epsilon_0)$ and $w(\tau,m)$ be in $F^d_{(\nu,\beta,\mu,k_1,\kappa)}$. We take $\zeta_{1,2}, \zeta_0, \zeta_{0,0}, \zeta_1, \zeta_{1,0}, \zeta_2, \zeta_F, \varpi > 0$ such that (85) holds and $\|w(\tau,m)\|_{(\nu,\beta,\mu,k_1,\kappa)} \le \varpi$ for all $\epsilon \in D(0,\epsilon_0)$.

Since $\kappa \ge k_1$ and (68) hold, using Proposition 2, we find that

$$\begin{aligned}
&\left\| \epsilon^{-1} \frac{c_{1,2}(\epsilon)}{Q(im)k_1\Gamma(1+\frac{1}{k_1})} \int_0^{\tau^{k_1}} \bigl(\tau^{k_1}-s\bigr)^{1/k_1} \right. \\
&\quad \times \left(\frac{1}{(2\pi)^{1/2}} s \int_0^s \int_{-\infty}^{+\infty} Q_1\bigl(i(m-m_1)\bigr) w\bigl((s-x)^{1/k_1}, m-m_1\bigr) \right. \\
&\quad \left. \left. \times Q_2(im_1) w\bigl(x^{1/k_1}, m_1\bigr) \frac{1}{(s-x)x}\, dx\, dm_1 \right) \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k_1,\kappa)} \\
&\le \frac{C_3\zeta_{1,2}}{(2\pi)^{1/2}k_1\Gamma(1+\frac{1}{k_1})} \bigl\| w(\tau,m) \bigr\|^2_{(\nu,\beta,\mu,k_1,\kappa)} \le \frac{C_3\zeta_{1,2}\varpi^2}{(2\pi)^{1/2}k_1\Gamma(1+\frac{1}{k_1})}.
\end{aligned} \tag{89}$$

Due to the lower bound assumption (82) and taking into account the definition of κ in (84), we get from Lemma 1 and Proposition 1

$$\begin{aligned}
&\left\| \frac{R_D(im)}{Q(im)} \frac{1}{k_1\Gamma(\frac{(\delta_D-1)(k_2-k_1)}{k_1})} \int_0^{\tau^{k_1}} \bigl(\tau^{k_1}-s\bigr)^{\frac{(\delta_D-1)(k_2-k_1)}{k_1}-1} k_1^{\delta_D} s^{\delta_D} w\bigl(s^{1/k_1}, m\bigr) \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k_1,\kappa)} \\
&\le \frac{C_2 k_1^{\delta_D}}{r_{Q,R_D} k_1 \Gamma(\frac{(\delta_D-1)(k_2-k_1)}{k_1})} \bigl\| w(\tau,m) \bigr\|_{(\nu,\beta,\mu,k_1,\kappa)} \le \frac{C_2 k_1^{\delta_D}}{r_{Q,R_D} k_1 \Gamma(\frac{(\delta_D-1)(k_2-k_1)}{k_1})} \varpi
\end{aligned} \tag{90}$$

and

$$\begin{aligned}
&\left\| \frac{R_D(im)}{Q(im)} A_{\delta_D,p} \frac{1}{k_1\Gamma(\frac{(\delta_D-1)(k_2-k_1)+k_1(\delta_D-p)}{k_1})} \right. \\
&\quad \left. \times \int_0^{\tau^{k_1}} \bigl(\tau^{k_1}-s\bigr)^{\frac{(\delta_D-1)(k_2-k_1)+k_1(\delta_D-p)}{k_1}-1} k_1^p s^p w\bigl(s^{1/k_1}, m\bigr) \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k_1,\kappa)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{|A_{\delta_D,p}|C_2 k_1^p}{r_{Q,R_D} k_1 \Gamma(\frac{(\delta_D-1)(k_2-k_1)+k_1(\delta_D-p)}{k_1})} \|w(\tau,m)\|_{(\nu,\beta,\mu,k_1,\kappa)} \\
&\leq \frac{|A_{\delta_D,p}|C_2 k_1^p}{r_{Q,R_D} k_1 \Gamma(\frac{(\delta_D-1)(k_2-k_1)+k_1(\delta_D-p)}{k_1})} \varpi
\end{aligned}
\tag{91}
$$

for all $1 \leq p \leq \delta_D - 1$.

From assumption (68) and due to the second constraint in (84), we get from Lemma 1 and Proposition 1

$$
\begin{aligned}
&\left\| \frac{R_l(im)}{Q(im)} \epsilon^{\Delta_l - d_l + \delta_l - 1} \frac{1}{k_1 \Gamma(\frac{d_{l,k_1}^1}{k_1})} \int_0^{\tau^{k_1}} (\tau^{k_1} - s)^{\frac{d_{l,k_1}^1}{k_1} - 1} (k_1^{\delta_l} s^{\delta_l} w(s^{1/k_1}, m)) \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k_1,\kappa)} \\
&\quad \leq |\epsilon|^{\Delta_l - d_l + \delta_l - 1} \frac{1}{r_{Q,R_l}} \frac{C_2 k_1^{\delta_l}}{k_1 \Gamma(\frac{d_{l,k_1}^1}{k_1})} \|w(\tau,m)\|_{(\nu,\beta,\mu,k_1,\kappa)} \\
&\quad \leq |\epsilon|^{\Delta_l - d_l + \delta_l - 1} \frac{1}{r_{Q,R_l}} \frac{C_2 k_1^{\delta_l}}{k_1 \Gamma(\frac{d_{l,k_1}^1}{k_1})} \varpi
\end{aligned}
\tag{92}
$$

for all $1 \leq l \leq D - 1$ and

$$
\begin{aligned}
&\left\| \frac{R_l(im)}{Q(im)} A_{\delta_l,p} \epsilon^{\Delta_l - d_l + \delta_l - 1} \frac{1}{k_1 \Gamma(\frac{d_{l,k_1}^1}{k_1} + \delta_l - p)} \right. \\
&\qquad \left. \times \int_0^{\tau^{k_1}} (\tau^{k_1} - s)^{\frac{d_{l,k_1}^1}{k_1} + \delta_l - p - 1} (k_1^p s^p w(s^{1/k_1}, m)) \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k_1,\kappa)} \\
&\quad \leq |\epsilon|^{\Delta_l - d_l + \delta_l - 1} \frac{1}{r_{Q,R_l}} |A_{\delta_l,p}| \frac{C_2 k_1^p}{k_1 \Gamma(\frac{d_{l,k_1}^1}{k_1} + \delta_l - p)} \|w(\tau,m)\|_{(\nu,\beta,\mu,k_1,\kappa)} \\
&\quad \leq |\epsilon|^{\Delta_l - d_l + \delta_l - 1} \frac{1}{r_{Q,R_l}} |A_{\delta_l,p}| \frac{C_2 k_1^p}{k_1 \Gamma(\frac{d_{l,k_1}^1}{k_1} + \delta_l - p)} \varpi
\end{aligned}
\tag{93}
$$

for all $1 \leq p \leq \delta_l - 1$. Since $\kappa \geq k_1$ and (68) we get from Proposition 2 that

$$
\begin{aligned}
&\left\| \epsilon^{-1} \frac{c_0(\epsilon)}{Q(im) k_1 \Gamma(1 + \frac{1}{k_1})} \int_0^{\tau^{k_1}} (\tau^{k_1} - s)^{1/k_1} \right. \\
&\qquad \times \left(\frac{1}{(2\pi)^{1/2}} s \int_0^s \int_{-\infty}^{+\infty} \varphi_{k_1}((s-x)^{1/k_1}, m - m_1, \epsilon) \right. \\
&\qquad \left. \left. \times R_0(im_1) w(x^{1/k_1}, m_1) \frac{1}{(s-x)x} \, dx \, dm_1 \right) \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k_1,\kappa)} \\
&\quad \leq \frac{C_3 \zeta_{1,0}}{(2\pi)^{1/2} k_1 \Gamma(1 + \frac{1}{k_1})} \|\varphi_{k_1}(\tau,m,\epsilon)\|_{(\nu,\beta,\mu,k_1,\kappa)} \|w(\tau,m)\|_{(\nu,\beta,\mu,k_1,\kappa)} \\
&\quad \leq \frac{C_3 \zeta_{1,0}}{(2\pi)^{1/2} k_1 \Gamma(1 + \frac{1}{k_1})} \zeta_1 \varpi.
\end{aligned}
\tag{94}
$$

Since $\kappa \geq k_1$ and (68) we deduce from Proposition 3 that

$$\begin{aligned} &\left\| \epsilon^{-1} \frac{c_{0,0}(\epsilon)}{Q(im)k_1\Gamma(1+\frac{1}{k_1})} \int_0^{\tau^{k_1}} \left(\tau^{k_1}-s\right)^{1/k_1} \frac{1}{(2\pi)^{1/2}} \left(\int_{-\infty}^{+\infty} C_{0,0}(m-m_1,\epsilon) \right.\right. \\ &\qquad \left.\left. \times R_0(im_1) w\left(s^{1/k_1}, m_1\right) dm_1 \right) \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k_1,\kappa)} \\ &\quad \leq \frac{C_4 \zeta_{0,0}}{(2\pi)^{1/2} k_1 \Gamma(1+\frac{1}{k_1})} \left\| C_{0,0}(m,\epsilon) \right\|_{(\beta,\mu)} \left\| w(\tau,m) \right\|_{(\nu,\beta,\mu,k_1,\kappa)} \\ &\quad \leq \frac{C_4 \zeta_{0,0}}{(2\pi)^{1/2} k_1 \Gamma(1+\frac{1}{k_1})} \zeta_0 \varpi \end{aligned} \tag{95}$$

and finally bearing in mind Proposition 1 we find that

$$\begin{aligned} &\left\| \epsilon^{-1} \frac{c_F(\epsilon)}{Q(im)k_1\Gamma(1+\frac{1}{k_1})} \int_0^{\tau^{k_1}} \left(\tau^{k_1}-s\right)^{1/k_1} \psi_{k_1}^d \left(s^{1/k_1}, m, \epsilon\right) \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k_1,\kappa)} \\ &\quad \leq \sup_{m\in\mathbb{R}} \frac{1}{|Q(im)|} \frac{C_2 \zeta_F}{k_1 \Gamma(1+\frac{1}{k_1})} \left\| \psi_{k_1}^d(\tau,m,\epsilon) \right\|_{(\nu,\beta,\mu,k_1,\kappa)} \\ &\quad \leq \sup_{m\in\mathbb{R}} \frac{1}{|Q(im)|} \frac{C_2 \zeta_F}{k_1 \Gamma(1+\frac{1}{k_1})} \zeta_2. \end{aligned} \tag{96}$$

Now, we choose $r_{Q,R_l} > 0$, for $1 \leq l \leq D$, $\zeta_{1,2}, \zeta_{0,0}, \zeta_0, \zeta_F, \zeta_{1,0}, \zeta_1, \zeta_2 > 0$ and $\varpi > 0$ such that

$$\begin{aligned} &\frac{C_3 \zeta_{1,2} \varpi^2}{(2\pi)^{1/2} k_1 \Gamma(1+\frac{1}{k_1})} + \frac{C_2 k_1^{\delta_D}}{r_{Q,R_D} k_1 \Gamma(\frac{(\delta_D-1)(k_2-k_1)}{k_1})} \varpi \\ &\qquad + \sum_{1\leq p\leq \delta_D-1} \frac{|A_{\delta_D,p}| C_2 k_1^p}{r_{Q,R_D} k_1 \Gamma(\frac{(\delta_D-1)(k_2-k_1)+k_1(\delta_D-p)}{k_1})} \varpi + \sum_{l=1}^{D-1} \epsilon_0^{\Delta_l-d_l+\delta_l-1} \frac{1}{r_{Q,R_l}} \frac{C_2 k_1^{\delta_l}}{k_1 \Gamma(\frac{d_{l,k_1}^1}{k_1})} \varpi \\ &\qquad + \sum_{1\leq p\leq \delta_l-1} \epsilon_0^{\Delta_l-d_l+\delta_l-1} \frac{1}{r_{Q,R_l}} |A_{\delta_l,p}| \frac{C_2 k_1^p}{k_1 \Gamma(\frac{d_{l,k_1}^1}{k_1} + \delta_l - p)} \varpi + \frac{C_3 \zeta_{1,0}}{(2\pi)^{1/2} k_1 \Gamma(1+\frac{1}{k_1})} \zeta_1 \varpi \\ &\qquad + \frac{C_4 \zeta_{0,0}}{(2\pi)^{1/2} k_1 \Gamma(1+\frac{1}{k_1})} \zeta_0 \varpi + \sup_{m\in\mathbb{R}} \frac{1}{|Q(im)|} \frac{C_2 \zeta_F}{k_1 \Gamma(1+\frac{1}{k_1})} \zeta_2 \leq \varpi. \end{aligned} \tag{97}$$

Gathering all the norm estimates (89), (90), (91), (92), (93), (94), (95), and (96) together with the constraint (97), one gets (87).

Now, we check the second property (88). Let $w_1(\tau,m), w_2(\tau,m)$ be in $F^d_{(\nu,\beta,\mu,k_1,\kappa)}$. We take $\varpi > 0$ such that

$$\left\| w_l(\tau,m) \right\|_{(\nu,\beta,\mu,k_1,\kappa)} \leq \varpi,$$

for $l = 1, 2$, for all $\epsilon \in D(0,\epsilon_0)$. One can write

$$\begin{aligned} &Q_1\left(i(m-m_1)\right) w_1\left((s-x)^{1/k_1}, m-m_1\right) Q_2(im_1) w_1\left(x^{1/k_1}, m_1\right) \\ &\qquad - Q_1\left(i(m-m_1)\right) w_2\left((s-x)^{1/k_1}, m-m_1\right) Q_2(im_1) w_2\left(x^{1/k_1}, m_1\right) \end{aligned}$$

$$
\begin{aligned}
&= Q_1\big(i(m-m_1)\big)\big(w_1\big((s-x)^{1/k_1},m-m_1\big)-w_2\big((s-x)^{1/k_1},m-m_1\big)\big)\\
&\quad\times Q_2(im_1)w_1\big(x^{1/k_1},m_1\big)\\
&\quad+Q_1\big(i(m-m_1)\big)w_2\big((s-x)^{1/k_1},m-m_1\big)\\
&\quad\times Q_2(im_1)\big(w_1\big(x^{1/k_1},m_1\big)-w_2\big(x^{1/k_1},m_1\big)\big)
\end{aligned}
\tag{98}
$$

and taking into account that $\kappa \geq k_1$, (68), (98) and using Proposition 2, we find that

$$
\begin{aligned}
&\left\|\epsilon^{-1}\frac{c_{1,2}(\epsilon)}{Q(im)k_1\Gamma(1+\frac{1}{k_1})}\int_0^{\tau^{k_1}}\big(\tau^{k_1}-s\big)^{1/k_1}\right.\\
&\quad\times\left(\frac{1}{(2\pi)^{1/2}}s\int_0^s\int_{-\infty}^{+\infty}\big(Q_1\big(i(m-m_1)\big)w_1\big((s-x)^{1/k_1},m-m_1\big)\right.\\
&\quad\times Q_2(im_1)w_1\big(x^{1/k_1},m_1\big)-Q_1\big(i(m-m_1)\big)w_2\big((s-x)^{1/k_1},m-m_1\big)\\
&\quad\left.\left.\times Q_2(im_1)w_2\big(x^{1/k_1},m_1\big)\big)\frac{1}{(s-x)x}\,dx\,dm_1\right)\frac{ds}{s}\right\|_{(\nu,\beta,\mu,k_1,\kappa)}\\
&\leq\frac{C_3\zeta_{1,2}}{(2\pi)^{1/2}k_1\Gamma(1+\frac{1}{k_1})}\big\|w_1(\tau,m)-w_2(\tau,m)\big\|_{(\nu,\beta,\mu,k_1,\kappa)}\big(\big\|w_1(\tau,m)\big\|_{(\nu,\beta,\mu,k_1,\kappa)}\\
&\quad+\big\|w_2(\tau,m)\big\|_{(\nu,\beta,\mu,k_1,\kappa)}\big)\\
&\leq\frac{C_3\zeta_{1,2}2\varpi}{(2\pi)^{1/2}k_1\Gamma(1+\frac{1}{k_1})}\big\|w_1(\tau,m)-w_2(\tau,m)\big\|_{(\nu,\beta,\mu,k_1,\kappa)}.
\end{aligned}
\tag{99}
$$

On the other hand, from the estimates (90), (91), (92), (93), (94), (95) and under the constraints (68), (84), and the lower bound assumption (82), we deduce that

$$
\begin{aligned}
&\left\|\frac{R_D(im)}{Q(im)}\frac{1}{k_1\Gamma(\frac{(\delta_D-1)(k_2-k_1)}{k_1})}\int_0^{\tau^{k_1}}\big(\tau^{k_1}-s\big)^{\frac{(\delta_D-1)(k_2-k_1)}{k_1}-1}k_1^{\delta_D}s^{\delta_D}\right.\\
&\quad\left.\times\big(w_1\big(s^{1/k_1},m\big)-w_2\big(s^{1/k_1},m\big)\big)\frac{ds}{s}\right\|_{(\nu,\beta,\mu,k_1,\kappa)}\\
&\leq\frac{C_2k_1^{\delta_D}}{r_{Q,R_D}k_1\Gamma(\frac{(\delta_D-1)(k_2-k_1)}{k_1})}\big\|w_1(\tau,m)-w_2(\tau,m)\big\|_{(\nu,\beta,\mu,k_1,\kappa)}
\end{aligned}
\tag{100}
$$

and that

$$
\begin{aligned}
&\left\|\frac{R_D(im)}{Q(im)}A_{\delta_D,p}\frac{1}{k_1\Gamma(\frac{(\delta_D-1)(k_2-k_1)+k_1(\delta_D-p)}{k_1})}\right.\\
&\quad\left.\times\int_0^{\tau^{k_1}}\big(\tau^{k_1}-s\big)^{\frac{(\delta_D-1)(k_2-k_1)+k_1(\delta_D-p)}{k_1}-1}k_1^p s^p\big(w_1\big(s^{1/k_1},m\big)-w_2\big(s^{1/k_1},m\big)\big)\frac{ds}{s}\right\|_{(\nu,\beta,\mu,k_1,\kappa)}\\
&\leq\frac{|A_{\delta_D,p}|C_2k_1^p}{r_{Q,R_D}k_1\Gamma(\frac{(\delta_D-1)(k_2-k_1)+k_1(\delta_D-p)}{k_1})}\big\|w_1(\tau,m)-w_2(\tau,m)\big\|_{(\nu,\beta,\mu,k_1,\kappa)}
\end{aligned}
\tag{101}
$$

for all $1 \leq p \leq \delta_D - 1$ and also

$$\left\| \frac{R_l(im)}{Q(im)} \epsilon^{\Delta_l - d_l + \delta_l - 1} \frac{1}{k_1 \Gamma(\frac{d^1_{l,k_1}}{k_1})} \int_0^{\tau^{k_1}} \left(\tau^{k_1} - s\right)^{\frac{d^1_{l,k_1}}{k_1} - 1} \times \left(k_1^{\delta_l} s^{\delta_l} \left(w_1\left(s^{1/k_1}, m\right) - w_2\left(s^{1/k_1}, m\right)\right)\right) \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k_1,\kappa)}$$
$$\leq |\epsilon|^{\Delta_l - d_l + \delta_l - 1} \frac{1}{r_{Q,R_l}} \frac{C_2 k_1^{\delta_l}}{k_1 \Gamma(\frac{d^1_{l,k_1}}{k_1})} \left\| w_1(\tau, m) - w_2(\tau, m) \right\|_{(\nu,\beta,\mu,k_1,\kappa)} \tag{102}$$

for all $1 \leq l \leq D - 1$ together with

$$\left\| \frac{R_l(im)}{Q(im)} A_{\delta_l,p} \epsilon^{\Delta_l - d_l + \delta_l - 1} \frac{1}{k_1 \Gamma(\frac{d^1_{l,k_1}}{k_1} + \delta_l - p)} \times \int_0^{\tau^{k_1}} \left(\tau^{k_1} - s\right)^{\frac{d^1_{l,k_1}}{k_1} + \delta_l - p - 1} \left(k_1^p s^p \left(w_1\left(s^{1/k_1}, m\right) - w_2\left(s^{1/k_1}, m\right)\right)\right) \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k_1,\kappa)}$$
$$\leq |\epsilon|^{\Delta_l - d_l + \delta_l - 1} \frac{1}{r_{Q,R_l}} |A_{\delta_l,p}| \frac{C_2 k_1^p}{k_1 \Gamma(\frac{d^1_{l,k_1}}{k_1} + \delta_l - p)} \left\| w_1(\tau, m) - w_2(\tau, m) \right\|_{(\nu,\beta,\mu,k_1,\kappa)} \tag{103}$$

for all $1 \leq p \leq \delta_l - 1$. Finally, we also obtain

$$\left\| \epsilon^{-1} \frac{c_0(\epsilon)}{Q(im) k_1 \Gamma(1 + \frac{1}{k_1})} \int_0^{\tau^{k_1}} \left(\tau^{k_1} - s\right)^{1/k_1} \left(\frac{1}{(2\pi)^{1/2}} s \int_0^s \int_{-\infty}^{+\infty} \varphi_{k_1}\left((s - x)^{1/k_1}, m - m_1, \epsilon\right) \times R_0(im_1) \left(w_1\left(x^{1/k_1}, m_1\right) - w_2\left(x^{1/k_1}, m_1\right)\right) \frac{1}{(s - x)x} \, dx \, dm_1 \right) \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k_1,\kappa)}$$
$$\leq \frac{C_3 \zeta_{1,0}}{(2\pi)^{1/2} k_1 \Gamma(1 + \frac{1}{k_1})} \zeta_1 \left\| w_1(\tau, m) - w_2(\tau, m) \right\|_{(\nu,\beta,\mu,k_1,\kappa)} \tag{104}$$

and

$$\left\| \epsilon^{-1} \frac{c_{0,0}(\epsilon)}{Q(im) k_1 \Gamma(1 + \frac{1}{k_1})} \int_0^{\tau^{k_1}} \left(\tau^{k_1} - s\right)^{1/k_1} \frac{1}{(2\pi)^{1/2}} \left(\int_{-\infty}^{+\infty} C_{0,0}(m - m_1, \epsilon) \times R_0(im_1) \left(w_1\left(s^{1/k_1}, m_1\right) - w_2\left(s^{1/k_1}, m_1\right)\right) dm_1 \right) \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k_1,\kappa)}$$
$$\leq \frac{C_4 \zeta_{0,0}}{(2\pi)^{1/2} k_1 \Gamma(1 + \frac{1}{k_1})} \zeta_0 \left\| w_1(\tau, m) - w_2(\tau, m) \right\|_{(\nu,\beta,\mu,k_1,\kappa)}. \tag{105}$$

Now, we take $\varpi, r_{Q,R_l} > 0$, for $1 \leq l \leq D$, and $\zeta_{1,2}, \zeta_{0,0}, \zeta_0, \zeta_{1,0}, \zeta_1 > 0$ such that

$$\frac{C_3 \zeta_{1,2} 2\varpi}{(2\pi)^{1/2} k_1 \Gamma(1 + \frac{1}{k_1})} + \frac{C_2 k_1^{\delta_D}}{r_{Q,R_D} k_1 \Gamma(\frac{(\delta_D - 1)(k_2 - k_1)}{k_1})} + \sum_{1 \leq p \leq \delta_D - 1} \frac{|A_{\delta_D,p}| C_2 k_1^p}{r_{Q,R_D} k_1 \Gamma(\frac{(\delta_D - 1)(k_2 - k_1) + k_1(\delta_D - p)}{k_1})} + \sum_{1 \leq l \leq D-1} \epsilon_0^{\Delta_l - d_l + \delta_l - 1} \frac{1}{r_{Q,R_l}} \frac{C_2 k_1^{\delta_l}}{k_1 \Gamma(\frac{d^1_{l,k_1}}{k_1})}$$

$$+ \sum_{1\leq p\leq \delta_l-1} \epsilon_0^{\Delta_l-d_l+\delta_l-1}\frac{1}{r_{Q,R_l}}|A_{\delta_l,p}|\frac{C_2k_1^p}{k_1\Gamma(\frac{d_{l,k_1}^1}{k_1}+\delta_l-p)} + \frac{C_3\zeta_{1,0}}{(2\pi)^{1/2}k_1\Gamma(1+\frac{1}{k_1})}\zeta_1$$

$$+ \frac{C_4\zeta_{0,0}}{(2\pi)^{1/2}k_1\Gamma(1+\frac{1}{k_1})}\zeta_0 \leq 1/2. \tag{106}$$

Bearing in mind the estimates (99), (100), (101), (102), (103), (104), (105) with the constraint (106), one gets (88). Finally, we choose $r_{Q,R_l} > 0$, for $1 \leq l \leq D$, $\zeta_{1,2}, \zeta_{0,0}, \zeta_0, \zeta_F, \zeta_{1,0}, \zeta_1, \zeta_2 > 0$ and $\varpi > 0$ such that both (97), (106) are fulfilled. This yields our lemma. □

We consider the ball $\bar{B}(0,\varpi) \subset F^d_{(\nu,\beta,\mu,k_1,\kappa)}$ constructed in Lemma 3 which is a complete metric space for the norm $\|\cdot\|_{(\nu,\beta,\mu,k_1,\kappa)}$. From the lemma above, we find that $\mathcal{H}_\epsilon^{k_1}$ is a contractive map from $\bar{B}(0,\varpi)$ into itself. Due to the classical contractive mapping theorem, we deduce that the map $\mathcal{H}_\epsilon^{k_1}$ has a unique fixed point denoted by $\omega_{k_1}^d(\tau,m,\epsilon)$ (*i.e.* $\mathcal{H}_\epsilon^{k_1}(\omega_{k_1}^d(\tau,m,\epsilon)) = \omega_{k_1}^d(\tau,m,\epsilon)$) in $\bar{B}(0,\varpi)$, for all $\epsilon \in D(0,\epsilon_0)$. Moreover, the function $\omega_{k_1}^d(\tau,m,\epsilon)$ depends holomorphically on ϵ in $D(0,\epsilon_0)$. By construction, $\omega_{k_1}^d(\tau,m,\epsilon)$ defines a solution of (83). This yields Proposition 11. □

4.3 Formal and analytic acceleration operators

In this section, we give a definition of the formal and analytic acceleration operator which is a slightly modified version of the one given in [2], Chapter 5, adapted to our definitions of m_k-Laplace and m_k-Borel transforms. First we give a definition for the formal transform.

Definition 5 Let $\tilde{k} > k > 0$ be real numbers. Let $\hat{f}(\tau) = \sum_{n\geq 1} f_n\tau^n \in \tau\mathbb{C}[[\tau]]$ be a formal series. We define the formal acceleration operator with indices $m_{\tilde{k}}$, m_k by

$$\hat{\mathcal{A}}_{m_{\tilde{k}},m_k}\hat{f}(\xi) = \sum_{n\geq 1} f_n\frac{\Gamma(\frac{n}{k})}{\Gamma(\frac{n}{\tilde{k}})}\xi^n \in \xi\mathbb{C}[[\xi]].$$

Notice that if one defines the formal m_k-Laplace transform $\hat{\mathcal{L}}_{m_k}(\hat{f})$ and the formal $m_{\tilde{k}}$-Borel transform $\hat{\mathcal{B}}_{m_{\tilde{k}}}(\hat{f})$ of $\hat{f}(\tau)$ by

$$\hat{\mathcal{L}}_{m_k}(\hat{f})(T) = \sum_{n\geq 1} f_n\Gamma\left(\frac{n}{k}\right)T^n, \qquad \hat{\mathcal{B}}_{m_{\tilde{k}}}(\hat{f})(Z) = \sum_{n\geq 1}\frac{f_n}{\Gamma(\frac{n}{\tilde{k}})}Z^n,$$

then the formal acceleration operator $\hat{\mathcal{A}}_{m_{\tilde{k}},m_k}$ can also be defined as

$$\hat{\mathcal{A}}_{m_{\tilde{k}},m_k}\hat{f}(\xi) = (\hat{\mathcal{B}}_{m_{\tilde{k}}} \circ \hat{\mathcal{L}}_{m_k})(\hat{f})(\xi).$$

In the next definition, we define the analytic transforms.

Proposition 12 *Let $\tilde{k} > k > 0$ be real numbers. Let $S(d, \frac{\pi}{\tilde{k}} + \delta, \rho)$ be a bounded sector of radius ρ with aperture $\frac{\pi}{\tilde{k}} + \delta$, for some $\delta > 0$, and with direction d. Let $F : S(d, \frac{\pi}{\tilde{k}} + \delta, \rho) \to \mathbb{C}$ be a bounded analytic function such that there exist a formal series $\hat{F}(z) = \sum_{n\geq 1} F_nz^n \in \mathbb{C}[[z]]$ and two constants $C, K > 0$ with*

$$\left|F(z) - \sum_{n=1}^{N-1} F_nz^n\right| \leq CK^N\Gamma(1+N/k)|z|^N \tag{107}$$

for all $z \in S(d, \frac{\pi}{\tilde{k}} + \delta, \rho)$, *all* $N \geq 2$. *The analytic* $m_{\tilde{k}}$*-Borel transform of* F *in the direction* d *is defined as*

$$\big(\mathcal{B}^d_{m_{\tilde{k}}} F\big)(Z) = \frac{-\tilde{k}}{2i\pi} \int_{\gamma_{\tilde{k}}} F(u) \exp\left(\left(\frac{Z}{u}\right)^{\tilde{k}}\right) \frac{Z^{\tilde{k}}}{u^{\tilde{k}+1}}\, du, \tag{108}$$

where $\gamma_{\tilde{k}}$ *is the closed Hankel path starting from the origin along the segment* $[0, (\rho/2) \times e^{i(d+\frac{\pi}{2\tilde{k}}+\frac{\delta'}{2})}]$, *following the arc of circle* $[(\rho/2)e^{i(d+\frac{\pi}{2\tilde{k}}+\frac{\delta'}{2})}, (\rho/2)e^{i(d-\frac{\pi}{2\tilde{k}}-\frac{\delta'}{2})}]$ *and going back to the origin along the segment* $[(\rho/2)e^{i(d-\frac{\pi}{2\tilde{k}}-\frac{\delta'}{2})}, 0]$ *where* $0 < \delta' < \delta$, *which can be chosen as close to* δ *as needed. Then the function* $(\mathcal{B}^d_{m_{\tilde{k}}} F)(Z)$ *is analytic on the unbounded sector* $S(d, \delta'')$ *with direction* d *and aperture* δ'' *where* $0 < \delta'' < \delta'$, *which can be chosen as close to* δ' *as needed. Moreover, if* $(\hat{\mathcal{B}}_{m_{\tilde{k}}} \hat{F})(Z) = \sum_{n\geq 1} F_n Z^n/\Gamma(n/\tilde{k})$ *denotes the formal* $m_{\tilde{k}}$*-Borel transform of* $\hat{F}$, *then for any given* $\rho' > 0$, *there exist two constants* $C, K > 0$ *with*

$$\left| \big(\mathcal{B}^d_{m_{\tilde{k}}} F\big)(Z) - \sum_{n=1}^{N-1} \frac{F_n}{\Gamma(\frac{n}{\tilde{k}})} Z^n \right| \leq CK^N \Gamma(1 + N/\kappa)|Z|^N \tag{109}$$

for all $Z \in S(d, \delta'') \cap D(0, \rho')$, *all* $N \geq 2$, *where* κ *is defined as* $1/\kappa = 1/k - 1/\tilde{k}$. *Finally, the* $m_{\tilde{k}}$*-Borel operator* $\mathcal{B}^d_{m_{\tilde{k}}}$ *is the right inverse operator of the* $m_{\tilde{k}}$*-Laplace transform, namely we have*

$$\mathcal{L}^d_{m_{\tilde{k}}}\big(v \mapsto \big(\mathcal{B}^d_{m_{\tilde{k}}} F\big)(v)\big)(T) = F(T), \tag{110}$$

for all $T \in S(d, \frac{\pi}{\tilde{k}} + \delta', \rho/2)$.

Proof The proof follows the same lines of arguments as Theorem 2, Section 2.3 in [2]. Namely, one can check that if $F(z) = z^n$, for an integer $n \geq 0$, then

$$\mathcal{B}^d_{m_{\tilde{k}}} F(Z) = Z^n/\Gamma(n/\tilde{k}) \tag{111}$$

for all $Z \in S(d, \delta'')$ by using the change of variable $u = z/w^{1/\tilde{k}}$ in the integral (108) and a path deformation, bearing in mind the Hankel formula

$$\frac{1}{\Gamma(\frac{n}{\tilde{k}})} = \frac{1}{2i\pi} \int_{\gamma} w^{-\frac{n}{\tilde{k}}} e^w \, dw,$$

where γ is the path of integration from infinity along the ray $\arg(w) = -\pi$ to the unit disc, then around the circle and back to infinity along the ray $\arg(w) = \pi$. From the asymptotic expansion (107) and using the same integrals estimates as in Theorem 2, Section 2.3 in [2], together with the Stirling formula, for any given $\rho' > 0$, we get two constants $\check{C}, \check{K} > 0$ such that

$$\left| \mathcal{B}^d_{m_{\tilde{k}}} F(Z) - \sum_{n=1}^{N-1} \frac{F_n}{\Gamma(\frac{n}{\tilde{k}})} Z^n \right| = \left| \mathcal{B}^d_{m_{\tilde{k}}} (R_{N-1} F)(Z) \right| \leq \check{C}\check{K}^N \frac{\Gamma(1 + N/k)}{\Gamma(1 + N/\tilde{k})} |Z|^N,$$

where $R_{N-1}F(u) = F(u) - \sum_{n=1}^{N-1} F_n u^n$, for all $N \geq 2$, for all $Z \in S(d, \delta'') \cap D(0, \rho')$. Therefore (109) follows.

In the last part of the proof, we show the identity (110). We follow the same lines of arguments as Theorem 3 in Section 2.4 from [2]. Using Fubini's theorem, we can write

$$\begin{aligned}
&\mathcal{L}^d_{m_{\tilde{k}}}\big(v \mapsto \big(\mathcal{B}^d_{m_{\tilde{k}}}F\big)(v)\big)(T) \\
&= \tilde{k}\int_{L_d}\left(-\frac{\tilde{k}}{2i\pi}\int_{\gamma_{\tilde{k}}} F(u)e^{(\frac{v}{u})^{\tilde{k}}}\frac{v^{\tilde{k}}}{u^{\tilde{k}+1}}\,du\right)e^{-(\frac{v}{T})^{\tilde{k}}}\frac{dv}{v} \\
&= -\frac{\tilde{k}}{2i\pi}\int_{\gamma_{\tilde{k}}}\frac{F(u)}{u^{\tilde{k}+1}}\left(\int_{L_d}\exp\left(v^{\tilde{k}}\left(\frac{1}{u^{\tilde{k}}}-\frac{1}{T^{\tilde{k}}}\right)\right)\tilde{k}v^{\tilde{k}-1}\,dv\right)du. \qquad (112)
\end{aligned}$$

Therefore, by direct integration, we deduce that

$$\mathcal{L}^d_{m_{\tilde{k}}}\big(v \mapsto \big(\mathcal{B}^d_{m_{\tilde{k}}}F\big)(v)\big)(T) = \frac{\tilde{k}}{2i\pi}\int_{\gamma_{\tilde{k}}}\frac{F(u)}{u}\frac{T^{\tilde{k}}}{T^{\tilde{k}}-u^{\tilde{k}}}\,du. \qquad (113)$$

Now, the function $u \mapsto \frac{F(u)}{u}\frac{T^{\tilde{k}}}{T^{\tilde{k}}-u^{\tilde{k}}}$ has in the interior of $\gamma_{\tilde{k}}$ exactly one singularity at $u = T$ (since T is assumed to belong to $S(d, \frac{\pi}{\tilde{k}} + \delta', \rho/2)$), this being a pole of order one, with residue $-F(T)/\tilde{k}$. The residue theorem completes the proof of (110). □

Proposition 13 *Let $S(d,\alpha)$ be an unbounded sector with direction $d \in \mathbb{R}$ and aperture α. Let $\tilde{k} > k > 0$ be given real numbers and let $\kappa > 0$ be the real number defined by $1/\kappa = 1/k - 1/\tilde{k}$. Let $f : S(d,\alpha) \cup D(0,r) \to \mathbb{C}$ be an analytic function with $f(0) = 0$ and such that there exist $C, M > 0$ with*

$$\big|f(h)\big| \le Ce^{M|h|^{\kappa}}$$

for all $h \in S(d,\alpha) \cup D(0,r)$.

For all $0 < \delta' < \pi/\kappa$ (which can be chosen close to π/κ), we define the kernel function

$$G(\xi,h) = -\frac{\tilde{k}k}{2i\pi}\xi^{\tilde{k}}\int_{V_{d,\tilde{k},\delta'}}\exp\left(-\left(\frac{h}{u}\right)^{k}+\left(\frac{\xi}{u}\right)^{\tilde{k}}\right)\frac{du}{u^{\tilde{k}+1}},$$

where $V_{d,\tilde{k},\delta'}$ is the path starting from 0 along the half-line $\mathbb{R}_+e^{i(d+\frac{\pi}{2\tilde{k}}+\frac{\delta'}{2})}$ and back to the origin along the half-line $\mathbb{R}_+e^{i(d-\frac{\pi}{2\tilde{k}}-\frac{\delta'}{2})}$. The function $G(\xi,h)$ is well defined and satisfies the following estimates: there exist $c_1, c_2 > 0$ such that

$$\big|G(\xi,h)\big| \le c_1\exp\left(-c_2\left(\frac{|h|}{|\xi|}\right)^{\kappa}\right) \qquad (114)$$

for all $h \in L_d = \mathbb{R}_+e^{id}$ and all $\xi \in S(d,\delta'')$ for $0 < \delta'' < \delta'$ (which can be chosen close to δ').

Then, for any $0 < \rho < (c_2/M)^{1/\kappa}$, the function

$$\mathcal{A}^d_{m_{\tilde{k}},m_k}f(\xi) = \int_{L_d} f(h)G(\xi,h)\frac{dh}{h} = g(\xi)$$

defines an analytic function on the bounded sector $S_{d,\kappa,\delta,\rho}$ with aperture $\frac{\pi}{\kappa} + \delta$, for any $0 < \delta < \alpha$, in the direction d, and with radius ρ and which satisfies the requirement that there *exist* $C, K > 0$ *with*

$$\left| g(\xi) - \sum_{n=1}^{N-1} f_n \frac{\Gamma(n/k)}{\Gamma(n/\tilde{k})} \xi^n \right| \leq CK^N \Gamma(1 + N/\kappa)|\xi|^N \tag{115}$$

for all $\xi \in S_{d,\kappa,\delta,\rho}$, *all* $N \geq 2$, *where* $\hat{g}(\xi) = \sum_{n\geq 1} f_n \frac{\Gamma(n/k)}{\Gamma(n/\tilde{k})}\xi^n$ *is the formal acceleration* $\hat{\mathcal{A}}_{m_{\tilde{k}},m_k}\hat{f}(\xi)$ *where* $\hat{f}(h) = \sum_{n\geq 1} f_n h^n$ *is the (convergent) Taylor expansion at* $h = 0$ *of* f *on* $D(0,r)$.

In other words, $g(\xi)$ *is the* κ*-sum of* $\hat{g}(\xi)$ *on* $S_{d,\kappa,\delta,\rho}$ *in the sense of the definition* [2] *from Section* 3.2.

Proof We first show the estimates (114). We follow the idea of proof of Lemma 1, Section 5.1 in [2]. We make the change of variable $u = h\tilde{u}$ in the integral $G(\xi,h)$ and we deform the path of the integration in order to get the expression

$$G(\xi,h) = -\frac{\tilde{k}k}{2i\pi}\left(\frac{\xi}{h}\right)^{\tilde{k}} \int_{\gamma_{\tilde{k}}} e^{-(1/\tilde{u})^k} e^{(\xi/h)^{\tilde{k}}(\frac{1}{\tilde{u}})^{\tilde{k}}} \frac{1}{\tilde{u}^{\tilde{k}+1}}\, d\tilde{u},$$

where $\gamma_{\tilde{k}}$ is the closed Hankel path defined in Proposition 12 with the direction $d = 0$. Hence, we recognize that $G(\xi,h)$ can be written as an analytic Borel transform $G(\xi,h) = k(\mathcal{B}^0_{m_{\tilde{k}}} e_k)(\xi/h)$ where $e_k(u) = e^{-(1/u)^k}$. From Exercise 1 in Section 2.2 from [2], we know that $e_k(u)$ has $\hat{0}$ as formal power series expansion of Gevrey order k on any sector $S_{0,\frac{\pi}{k}+\delta}$ with direction 0 for any $0 < \delta < \pi/\kappa$. From Proposition 12, we deduce that $(\mathcal{B}^0_{m_{\tilde{k}}} e_k)(Z)$ has $\hat{0}$ as formal series expansion of Gevrey order κ on any unbounded sector $S_{0,\delta''}$ where $0 < \delta'' < \delta' < \delta < \pi/\kappa$ (where δ'' can be chosen close to π/κ). Finally, using Exercise 3 in Section 2.2 from [2], we get two constants $c_1, c_2 > 0$ such that

$$\left|(\mathcal{B}^0_{m_{\tilde{k}}} e_k)(Z)\right| \leq c_1 e^{-c_2|Z|^{-\kappa}}$$

for all $Z \in S_{0,\delta''}$. The estimates (114) follow.

In order to show the asymptotic expansion with bound estimates (115), we first check that if $f(h) = h^n$, for an integer $n \geq 0$, then

$$\mathcal{A}^d_{m_{\tilde{k}},m_k} f(\xi) = \frac{\Gamma(n/k)}{\Gamma(n/\tilde{k})}\xi^n \tag{116}$$

on $S_{d,\kappa,\delta,\rho}$. Indeed using Fubini's theorem, we can write

$$\mathcal{A}^d_{m_{\tilde{k}},m_k} f(\xi) = -\frac{\tilde{k}}{2i\pi}\int_{V_{d,\tilde{k},\delta'}} \left(k\int_{L_d} h^n e^{-(\frac{h}{u})^k}\frac{dh}{h}\right) e^{(\frac{\xi}{u})^{\tilde{k}}}\frac{\xi^{\tilde{k}}}{u^{\tilde{k}+1}}\, du.$$

From the definition of the Gamma function we know that

$$k\int_{L_d} h^n e^{-(\frac{h}{u})^k}\frac{dh}{h} = \mathcal{L}^d_{m_k}(h^n)(u) = \Gamma\left(\frac{n}{k}\right)u^n,$$

and bearing in mind (111), after a path deformation, we recognize that

$$\mathcal{A}^d_{m_{\tilde{k}},m_k} f(\xi) = \Gamma\left(\frac{n}{k}\right)\mathcal{B}^d_{m_{\tilde{k}}}\left(u^n\right)(\xi) = \frac{\Gamma(n/k)}{\Gamma(n/\tilde{k})}\xi^n.$$

Since the Taylor expansion of f at $h = 0$ is convergent, there exist two constants $C_f, K_f > 0$ such that

$$\left| f(h) - \sum_{n=1}^{N-1} f_n h^n \right| \leq C_f K_f^N |h|^N \tag{117}$$

for all $h \in D(0,r)$, all $N \geq 2$. Taking the expansion (117) and the exponential growth estimates (114), using the same integrals estimates as in Exercise 3 in Section 2.1 of [2], we get two constants $C, K > 0$ such that

$$\left| \mathcal{A}^d_{m_{\tilde{k}},m_k} f(\xi) - \sum_{n=1}^{N-1} f_n \frac{\Gamma(\frac{n}{k})}{\Gamma(\frac{n}{\tilde{k}})}\xi^n \right| = \left|\mathcal{A}^d_{m_{\tilde{k}},m_k}(R_{N-1}f)(\xi)\right| \leq CK^N\Gamma(1+N/\kappa)|\xi|^N,$$

where $R_{N-1}f(h) = f(h) - \sum_{n=1}^{N-1} f_n h^n$, for all $N \geq 2$, all $\xi \in S_{d,\kappa,\delta,\rho}$. □

4.4 Analytic solutions for an auxiliary convolution problem resulting from a m_{k_2}-Borel transform applied to the main convolution initial value problem

We keep the notations of Sections 4.1 and 4.2. For the integers d_l, δ_l, for $1 \leq l \leq D-1$, that satisfy the constraints (66), (67), and (72), we make the additional assumption that there exist integers $d^2_{l,k_2} > 0$ such that

$$d_l + k_2 + 1 = \delta_l(k_2+1) + d^2_{l,k_2} \tag{118}$$

for all $1 \leq l \leq D-1$. In order to ensure the positivity of the integers d^2_{l,k_2}, we impose the following assumption on the integers d^1_{l,k_1}:

$$d^1_{l,k_1} > (\delta_l - 1)(k_2 - k_1), \tag{119}$$

for all $1 \leq l \leq D-1$. Indeed, by the definition of d^1_{l,k_1} in (74), the constraint (118) can be rewritten $d^2_{l,k_2} = d^1_{l,k_1} + k_2 - k_1 - \delta_l(k_2 - k_1)$. Using (8.7) from [9], p.3630, we can expand the operators $T^{\delta_l(k_2+1)}\partial_T^{\delta_l}$ in the form

$$T^{\delta_l(k_2+1)}\partial_T^{\delta_l} = \left(T^{k_2+1}\partial_T\right)^{\delta_l} + \sum_{1\leq p\leq \delta_l-1} A_{\delta_l,p}T^{k_2(\delta_l-p)}\left(T^{k_2+1}\partial_T\right)^p, \tag{120}$$

where $A_{\delta_l,p}$, $p = 1,\ldots,\delta_l - 1$ are real numbers, for all $1 \leq l \leq D$.

Multiplying (70) by T^{k_2+1} and using (120), we can rewrite (70) in the form

$$\begin{aligned}
&Q(im)\left(T^{k_2+1}\partial_T U(T,m,\epsilon)\right) - R_D(im)\left(T^{k_2+1}\partial_T\right)^{\delta_D}U(T,m,\epsilon)\\
&\quad= R_D(im)\sum_{1\leq p\leq \delta_D-1} A_{\delta_D,p}T^{k_2(\delta_D-p)}\left(T^{k_2+1}\partial_T\right)^p U(T,m,\epsilon)\\
&\qquad+ \epsilon^{-1}T^{k_2+1}\frac{c_{1,2}(\epsilon)}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} Q_1\left(i(m-m_1)\right)U(T,m-m_1,\epsilon)Q_2(im_1)U(T,m_1,\epsilon)\,dm_1
\end{aligned}$$

$$+\sum_{l=1}^{D-1} R_l(im)\Big(\epsilon^{\Delta_l-d_l+\delta_l-1} T^{d_{l,k_2}^2}\big(T^{k_2+1}\partial_T\big)^{\delta_l} U(T,m,\epsilon)$$

$$+\sum_{1\le p\le\delta_l-1} A_{\delta_l,p}\epsilon^{\Delta_l-d_l+\delta_l-1} T^{k_2(\delta_l-p)+d_{l,k_2}^2}\big(T^{k_2+1}\partial_T\big)^{p} U(T,m,\epsilon)\Big)$$

$$+\epsilon^{-1}T^{k_2+1}\frac{c_0(\epsilon)}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} C_0(T,m-m_1,\epsilon)R_0(im_1)U(T,m_1,\epsilon)\,dm_1$$

$$+\epsilon^{-1}T^{k_2+1}\frac{c_{0,0}(\epsilon)}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} C_{0,0}(m-m_1,\epsilon)R_0(im_1)U(T,m_1,\epsilon)\,dm_1$$

$$+\epsilon^{-1}c_F(\epsilon)T^{k_2+1}F(T,m,\epsilon). \tag{121}$$

We denote $\hat{\omega}_{k_2}(\tau,m,\epsilon)$ the formal m_{k_2}-Borel transform of $\hat{U}(T,m,\epsilon)$ with respect to T, $\varphi_{k_2}(\tau,m,\epsilon)$ the formal m_{k_2}-Borel transform of $C_0(T,m,\epsilon)$ with respect to T and $\hat{\psi}_{k_2}(\tau,m,\epsilon)$ the formal m_{k_2}-Borel transform of $F(T,m,\epsilon)$ with respect to T,

$$\hat{\omega}_{k_2}(\tau,m,\epsilon)=\sum_{n\ge1}U_n(m,\epsilon)\frac{\tau^n}{\Gamma(\frac{n}{k_2})},\qquad \varphi_{k_2}(\tau,m,\epsilon)=\sum_{n\ge1}C_{0,n}(m,\epsilon)\frac{\tau^n}{\Gamma(\frac{n}{k_2})}$$
$$\hat{\psi}_{k_2}(\tau,m,\epsilon)=\sum_{n\ge1}F_n(m,\epsilon)\frac{\tau^n}{\Gamma(\frac{n}{k_2})}. \tag{122}$$

Using the computation rules for the formal m_{k_2}-Borel transform in Proposition 8, we deduce the following equation satisfied by $\hat{\omega}_{k_2}(\tau,m,\epsilon)$:

$$Q(im)\big(k_2\tau^{k_2}\hat{\omega}_{k_2}(\tau,m,\epsilon)\big)-\big(k_2\tau^{k_2}\big)^{\delta_D}R_D(im)\hat{\omega}_{k_2}(\tau,m,\epsilon)$$

$$=R_D(im)\sum_{1\le p\le\delta_D-1}A_{\delta_D,p}\frac{\tau^{k_2}}{\Gamma(\delta_D-p)}\int_0^{\tau^{k_2}}\big(\tau^{k_2}-s\big)^{\delta_D-p-1}\big(k_2^p s^p\hat{\omega}_{k_2}\big(s^{1/k_2},m,\epsilon\big)\big)\frac{ds}{s}$$

$$+\epsilon^{-1}\frac{\tau^{k_2}}{\Gamma(1+\frac{1}{k_2})}\int_0^{\tau^{k_2}}\big(\tau^{k_2}-s\big)^{1/k_2}$$

$$\times\bigg(\frac{c_{1,2}(\epsilon)}{(2\pi)^{1/2}}s\int_0^s\int_{-\infty}^{+\infty}Q_1\big(i(m-m_1)\big)\hat{\omega}_{k_2}\big((s-x)^{1/k_2},m-m_1,\epsilon\big)$$

$$\times Q_2(im_1)\hat{\omega}_{k_2}\big(x^{1/k_2},m_1,\epsilon\big)\frac{1}{(s-x)x}\,dx\,dm_1\bigg)\frac{ds}{s}$$

$$+\sum_{l=1}^{D-1}R_l(im)\bigg(\epsilon^{\Delta_l-d_l+\delta_l-1}\frac{\tau^{k_2}}{\Gamma(\frac{d_{l,k_2}^2}{k_2})}\int_0^{\tau^{k_2}}\big(\tau^{k_2}-s\big)^{\frac{d_{l,k_2}^2}{k_2}-1}\big(k_2{}^{\delta_l}s^{\delta_l}\hat{\omega}_{k_2}\big(s^{1/k_2},m,\epsilon\big)\big)\frac{ds}{s}$$

$$+\sum_{1\le p\le\delta_l-1}A_{\delta_l,p}\epsilon^{\Delta_l-d_l+\delta_l-1}\frac{\tau^{k_2}}{\Gamma(\frac{d_{l,k_2}^2}{k_2}+\delta_l-p)}$$

$$\times\int_0^{\tau^{k_2}}\big(\tau^{k_2}-s\big)^{\frac{d_{l,k_2}^2}{k_2}+\delta_l-p-1}\big(k_2{}^p s^p\hat{\omega}_{k_2}\big(s^{1/k_2},m,\epsilon\big)\big)\frac{ds}{s}\bigg)$$

$$+\epsilon^{-1}\frac{\tau^{k_2}}{\Gamma(1+\frac{1}{k_2})}\int_0^{\tau^{k_2}}\big(\tau^{k_2}-s\big)^{1/k_2}\bigg(\frac{c_0(\epsilon)}{(2\pi)^{1/2}}s\int_0^s\int_{-\infty}^{+\infty}\varphi_{k_2}\big((s-x)^{1/k_2},m-m_1,\epsilon\big)$$

$$\times R_0(im_1)\hat{\omega}_{k_2}\big(x^{1/k_2},m_1,\epsilon\big)\frac{1}{(s-x)x}\,dx\,dm_1\bigg)\frac{ds}{s}$$

$$+ \epsilon^{-1} \frac{\tau^{k_2}}{\Gamma(1+\frac{1}{k_2})} \int_0^{\tau^{k_2}} \left(\tau^{k_2} - s\right)^{1/k_2} \frac{c_{0,0}(\epsilon)}{(2\pi)^{1/2}}$$
$$\times \left(\int_{-\infty}^{+\infty} C_{0,0}(m-m_1,\epsilon) R_0(im_1) \hat{\omega}_{k_2}\left(s^{1/k_2}, m_1, \epsilon\right) dm_1 \right) \frac{ds}{s}$$
$$+ \epsilon^{-1} c_F(\epsilon) \frac{\tau^{k_2}}{\Gamma(1+\frac{1}{k_2})} \int_0^{\tau^{k_2}} \left(\tau^{k_2} - s\right)^{1/k_2} \hat{\psi}_{k_2}\left(s^{1/k_2}, m, \epsilon\right) \frac{ds}{s}. \tag{123}$$

We recall from [1] that $\varphi_{k_2}(\tau,m,\epsilon) \in F^d_{(\nu,\beta,\mu,k_2)}$ for all $\epsilon \in D(0,\epsilon_0)$, any unbounded sector S_d and any bounded sector S^b_d centered at 0 with bisecting direction $d \in \mathbb{R}$, for some $\nu > 0$.

From Section 4.2, we recall that $\psi^d_{k_1}(\tau,m,\epsilon) \in F^d_{(\nu,\beta,\mu,k_1,k_1)}$, for all $\epsilon \in D(0,\epsilon_0)$, for some unbounded sector U_d with bisecting direction $d \in \mathbb{R}$, where ν is chosen in that section.

Lemma 4 *The function*

$$\psi^d_{k_2}(\tau,m,\epsilon) := \mathcal{A}^d_{m_{k_2},m_{k_1}}\left(h \mapsto \psi^d_{k_1}(h,m,\epsilon)\right)(\tau) = \int_{L_d} \psi^d_{k_1}(h,m,\epsilon) G(\tau,h) \frac{dh}{h}$$

is analytic on an unbounded sector $S_{d,\kappa,\delta}$ with aperture $\frac{\pi}{\kappa} + \delta$ in the direction d, for any $0 < \delta < \mathrm{ap}(U_d)$ where $\mathrm{ap}(U_d)$ denotes the aperture of the sector U_d and has estimates of the form: there exist constants $C_{\psi_{k_2}} > 0$ and $\nu' > 0$ such that

$$\left|\psi^d_{k_2}(\tau,m,\epsilon)\right| \le C_{\psi_{k_2}} \left(1+|m|\right)^{-\mu} e^{-\beta|m|} \frac{|\tau|}{1+|\tau|^{2k_2}} \exp\left(\nu'|\tau|^{k_2}\right) \tag{124}$$

for all $\tau \in S_{d,\kappa,\delta}$, all $m \in \mathbb{R}$, all $\epsilon \in D(0,\epsilon_0)$. In particular, we find that $\mathcal{A}^d_{m_{k_2},m_{k_1}}(h \mapsto \psi^d_{k_1}(h,m,\epsilon))(\tau) \in F^d_{(\nu',\beta,\mu,k_2)}$ for any unbounded sector S_d and bounded sector S^b_d with aperture $\frac{\pi}{\kappa} + \delta$, with δ as above, and we carry a constant $\zeta_{\psi_{k_2}} > 0$ with

$$\left\|\psi^d_{k_2}(\tau,m,\epsilon)\right\|_{(\nu',\beta,\mu,k_2)} \le \zeta_{\psi_{k_2}} \tag{125}$$

for all $\epsilon \in D(0,\epsilon_0)$.

Proof Bearing in mind the inclusion (81) we already know from Proposition 13 that the function $\tau \mapsto \psi^d_{k_2}(\tau,m,\epsilon)$ defines a holomorphic and bounded function (with bound independent of $\epsilon \in D(0,\epsilon_0)$) on a sector $S_{d,\kappa,\delta,(c_2/\nu)^{1/\kappa}/2}$ with direction d, aperture $\frac{\pi}{\kappa} + \delta$, and radius $(c_2/\nu)^{1/\kappa}/2$, for some $\delta > 0$ and the constant c_2 introduced in (114), for all $m \in \mathbb{R}$, all $\epsilon \in D(0,\epsilon_0)$.

From the assumption that the function $\psi^d_{k_1}(\tau,m,\epsilon)$ belongs to the space $F^d_{(\nu,\beta,\mu,k_1,k_1)}$, see (80), we know that the m_{k_1}-Laplace transform

$$\mathcal{L}^d_{m_{k_1}}\left(h \mapsto \psi^d_{k_1}(h,m,\epsilon)\right)(u) = k_1 \int_{L_d} \psi^d_{k_1}(h,m,\epsilon) \exp\left(-\left(\frac{h}{u}\right)^{k_1}\right) \frac{dh}{h}$$

defines a holomorphic and bounded function (by a constant that does not depend on $\epsilon \in D(0,\epsilon_0)$) on a sector $S_{d,\theta,\sigma'}$ in the direction d, with radius σ' and aperture θ which satisfies $\frac{\pi}{k_2} + \frac{\pi}{\kappa} < \theta < \frac{\pi}{k_2} + \frac{\pi}{\kappa} + \mathrm{ap}(U_d)$, where $\mathrm{ap}(U_d)$ is the aperture of U_d, for some $\sigma' > 0$.

Hence, by using a path deformation and the Fubini theorem, we can rewrite the function $\psi_{k_2}^d(\tau,m,\epsilon)$ in the form

$$\begin{aligned}\psi_{k_2}^d(\tau,m,\epsilon) &= -\frac{k_2}{2i\pi}\int_{V_{d,k_2,\delta',\sigma'/2}} \mathcal{L}_{m_{k_1}}^d\big(h\mapsto \psi_{k_1}^d(h,m,\epsilon)\big)(u)e^{(\frac{\tau}{u})^{k_2}}\frac{\tau^{k_2}}{u^{k_2+1}}\,du \\ &= \mathcal{B}_{m_{k_2}}^d\big(\mathcal{L}_{m_{k_1}}^d\big(h\mapsto\psi_{k_1}^d(h,m,\epsilon)\big)(u)\big)(\tau),\end{aligned} \tag{126}$$

where $V_{d,k_2,\delta',\sigma'/2}$ is the closed Hankel path starting from the origin along the segment

$$\big[0,(\sigma'/2)e^{i(d+\frac{\pi}{2k_2}+\frac{\delta'}{2})}\big]$$

following the arc of circle $[(\sigma'/2)e^{i(d+\frac{\pi}{2k_2}+\frac{\delta'}{2})},(\sigma'/2)e^{i(d-\frac{\pi}{2k_2}-\frac{\delta'}{2})}]$ and going back to the origin along the segment $[(\sigma'/2)e^{i(d-\frac{\pi}{2k_2}-\frac{\delta'}{2})},0]$, where $0<\delta'<\frac{\pi}{\kappa}+\mathrm{ap}(U_d)$ that can be chosen close to $\frac{\pi}{\kappa}+\mathrm{ap}(U_d)$.

Therefore, from Proposition 12, we know that $\tau\mapsto\psi_{k_2}^d(\tau,m,\epsilon)$ defines a holomorphic function on the unbounded sector $S(d,\delta'')$ where $0<\delta''<\delta'$, which can be chosen close to δ', for all $m\in\mathbb{R}$, all $\epsilon\in D(0,\epsilon_0)$. Now, we turn to the estimates (124). From the representation (126), we get the following estimates: there exist constants $E_1,E_2,E_3>0$ such that

$$\begin{aligned}\big|\psi_{k_2}^d(\tau,m,\epsilon)\big| &\leq \frac{E_1e^{-\beta|m|}}{(1+|m|)^{\mu}}\bigg(e^{E_2|\tau|^{k_2}}|\tau|^{k_2}+\int_0^{\frac{\sigma'}{2}}e^{-E_3(\frac{|\tau|}{s})^{k_2}}\frac{|\tau|^{k_2}}{s^{k_2+1}}\,ds\bigg) \\ &\leq \frac{E_1e^{-\beta|m|}}{(1+|m|)^{\mu}}\bigg(e^{E_2|\tau|^{k_2}}|\tau|^{k_2}+\frac{1}{E_3k_2}e^{-E_3(\frac{2}{\sigma'})^{k_2}|\tau|^{k_2}}\bigg)\end{aligned} \tag{127}$$

for all $\tau\in S(d,\delta'')$, all $m\in\mathbb{R}$, all $\epsilon\in D(0,\epsilon_0)$. Besides, from the asymptotic expansion (109), we get in particular the existence of a constant $E_0>0$ such that

$$\big|\psi_{k_2}^d(\tau,m,\epsilon)\big|\leq\frac{E_0e^{-\beta|m|}}{(1+|m|)^{\mu}}|\tau| \tag{128}$$

for all $\tau\in S(d,\delta'')\cap D(0,\rho')$ and some $\rho'>0$. Finally, combining the estimates (127) and (128) yields (124). □

We consider now the following problem:

$$\begin{aligned}&Q(im)\big(k_2\tau^{k_2}\omega_{k_2}(\tau,m,\epsilon)\big)-\big(k_2\tau^{k_2}\big)^{\delta_D}R_D(im)\omega_{k_2}(\tau,m,\epsilon) \\ &\quad= R_D(im)\sum_{1\leq p\leq\delta_D-1}A_{\delta_D,p}\frac{\tau^{k_2}}{\Gamma(\delta_D-p)}\int_0^{\tau^{k_2}}\big(\tau^{k_2}-s\big)^{\delta_D-p-1}\big(k_2^ps^p\omega_{k_2}\big(s^{1/k_2},m,\epsilon\big)\big)\frac{ds}{s} \\ &\qquad+\epsilon^{-1}\frac{\tau^{k_2}}{\Gamma(1+\frac{1}{k_2})}\int_0^{\tau^{k_2}}\big(\tau^{k_2}-s\big)^{1/k_2} \\ &\qquad\times\bigg(\frac{c_{1,2}(\epsilon)}{(2\pi)^{1/2}}s\int_0^s\int_{-\infty}^{+\infty}Q_1\big(i(m-m_1)\big)\omega_{k_2}\big((s-x)^{1/k_2},m-m_1,\epsilon\big) \\ &\qquad\times Q_2(im_1)\omega_{k_2}\big(x^{1/k_2},m_1,\epsilon\big)\frac{1}{(s-x)x}\,dx\,dm_1\bigg)\frac{ds}{s}\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{l=1}^{D-1} R_l(im)\Bigg(\epsilon^{\Delta_l-d_l+\delta_l-1}\frac{\tau^{k_2}}{\Gamma(\frac{d_{l,k_2}^2}{k_2})}\int_0^{\tau^{k_2}}\big(\tau^{k_2}-s\big)^{\frac{d_{l,k_2}^2}{k_2}-1}\big(k_2^{\delta_l}s^{\delta_l}\omega_{k_2}\big(s^{1/k_2},m,\epsilon\big)\big)\frac{ds}{s} \\
&+ \sum_{1\le p\le \delta_l-1} A_{\delta_l,p}\epsilon^{\Delta_l-d_l+\delta_l-1}\frac{\tau^{k_2}}{\Gamma(\frac{d_{l,k_2}^2}{k_2}+\delta_l-p)} \\
&\times \int_0^{\tau^{k_2}}\big(\tau^{k_2}-s\big)^{\frac{d_{l,k_2}^2}{k_2}+\delta_l-p-1}\big(k_2^{p}s^{p}\omega_{k_2}\big(s^{1/k_2},m,\epsilon\big)\big)\frac{ds}{s}\Bigg) \\
&+ \epsilon^{-1}\frac{\tau^{k_2}}{\Gamma(1+\frac{1}{k_2})}\int_0^{\tau^{k_2}}\big(\tau^{k_2}-s\big)^{1/k_2}\bigg(\frac{c_0(\epsilon)}{(2\pi)^{1/2}}s\int_0^s\int_{-\infty}^{+\infty}\varphi_{k_2}\big((s-x)^{1/k_2},m-m_1,\epsilon\big) \\
&\times R_0(im_1)\omega_{k_2}\big(x^{1/k_2},m_1,\epsilon\big)\frac{1}{(s-x)x}\,dx\,dm_1\bigg)\frac{ds}{s} \\
&+ \epsilon^{-1}\frac{\tau^{k_2}}{\Gamma(1+\frac{1}{k_2})}\int_0^{\tau^{k_2}}\big(\tau^{k_2}-s\big)^{1/k_2}\frac{c_{0,0}(\epsilon)}{(2\pi)^{1/2}} \\
&\times \bigg(\int_{-\infty}^{+\infty}C_{0,0}(m-m_1,\epsilon)R_0(im_1)\omega_{k_2}\big(s^{1/k_2},m_1,\epsilon\big)\,dm_1\bigg)\frac{ds}{s} \\
&+ \epsilon^{-1}c_F(\epsilon)\frac{\tau^{k_2}}{\Gamma(1+\frac{1}{k_2})}\int_0^{\tau^{k_2}}\big(\tau^{k_2}-s\big)^{1/k_2}\psi_{k_2}^d\big(s^{1/k_2},m,\epsilon\big)\frac{ds}{s}
\end{aligned} \tag{129}
$$

for vanishing initial data $\omega_{k_2}(0,m,\epsilon)\equiv 0$, where $\psi_{k_2}^d(\tau,m,\epsilon)$ has been constructed in Lemma 4.

We make the additional assumption that there exists an unbounded sector

$$
S_{Q,R_D} = \big\{z\in\mathbb{C}/|z|\ge r_{Q,R_D}, \big|\arg(z)-d_{Q,R_D}\big|\le \eta_{Q,R_D}\big\}
$$

with direction $d_{Q,R_D}\in\mathbb{R}$, aperture $\eta_{Q,R_D}>0$ for some radius $r_{Q,R_D}>0$ such that

$$
\frac{Q(im)}{R_D(im)}\in S_{Q,R_D} \tag{130}
$$

for all $m\in\mathbb{R}$. We factorize the polynomial $P_m(\tau)=Q(im)k_2-R_D(im)k_2^{\delta_D}\tau^{(\delta_D-1)k_2}$ in the form

$$
P_m(\tau) = -R_D(im)k_2^{\delta_D}\prod_{l=0}^{(\delta_D-1)k_2-1}\big(\tau-q_l(m)\big), \tag{131}
$$

where

$$
\begin{aligned}
q_l(m) = {}&\bigg(\frac{|Q(im)|}{|R_D(im)|k_2^{\delta_D-1}}\bigg)^{\frac{1}{(\delta_D-1)k_2}} \\
&\times \exp\bigg(\sqrt{-1}\bigg(\arg\bigg(\frac{Q(im)}{R_D(im)k_2^{\delta_D-1}}\bigg)\frac{1}{(\delta_D-1)k_2}+\frac{2\pi l}{(\delta_D-1)k_2}\bigg)\bigg)
\end{aligned} \tag{132}
$$

for all $0\le l\le(\delta_D-1)k_2-1$, all $m\in\mathbb{R}$.

We choose an unbounded sector S_d centered at 0, a small closed disc $\bar{D}(0,\rho)$ and we prescribe the sector S_{Q,R_D} in such a way that the following conditions hold.

(1) There exists a constant $M_1 > 0$ such that

$$\left|\tau - q_l(m)\right| \geq M_1\left(1 + |\tau|\right) \tag{133}$$

for all $0 \leq l \leq (\delta_D - 1)k_2 - 1$, all $m \in \mathbb{R}$, all $\tau \in S_d \cup \bar{D}(0,\rho)$. Indeed, from (130) and the explicit expression (132) of $q_l(m)$, we first observe that $|q_l(m)| > 2\rho$ for every $m \in \mathbb{R}$, all $0 \leq l \leq (\delta_D - 1)k_2 - 1$ for an appropriate choice of r_{Q,R_D} and of $\rho > 0$. We also see that, for all $m \in \mathbb{R}$, all $0 \leq l \leq (\delta_D - 1)k_2 - 1$, the roots $q_l(m)$ remain in a union $\mathcal{U}$ of unbounded sectors centered at 0 that do not cover a full neighborhood of the origin in $\mathbb{C}^*$ provided that η_{Q,R_D} is small enough. Therefore, one can choose an adequate sector S_d such that $S_d \cap \mathcal{U} = \emptyset$ with the property that, for all $0 \leq l \leq (\delta_D - 1)k_2 - 1$, the quotients $q_l(m)/\tau$ lie outside some small disc centered at 1 in $\mathbb{C}$ for all $\tau \in S_d$, all $m \in \mathbb{R}$. This yields (133) for some small constant $M_1 > 0$.

(2) There exists a constant $M_2 > 0$ such that

$$\left|\tau - q_{l_0}(m)\right| \geq M_2\left|q_{l_0}(m)\right| \tag{134}$$

for some $l_0 \in \{0,\ldots,(\delta_D - 1)k_2 - 1\}$, all $m \in \mathbb{R}$, all $\tau \in S_d \cup \bar{D}(0,\rho)$. Indeed, for the sector S_d and the disc $\bar{D}(0,\rho)$ chosen as above in (1), we notice that, for any fixed $0 \leq l_0 \leq (\delta_D - 1)k_2 - 1$, the quotient $\tau/q_{l_0}(m)$ stays outside a small disc centered at 1 in $\mathbb{C}$ for all $\tau \in S_d \cup \bar{D}(0,\rho)$, all $m \in \mathbb{R}$. Hence (134) must hold for some small constant $M_2 > 0$.

By construction of the roots (132) in the factorization (131) and using the lower bound estimates (133), (134), we get a constant $C_P > 0$ such that

$$\begin{aligned}
\left|P_m(\tau)\right| &\geq M_1^{(\delta_D-1)k_2-1} M_2 \left|R_D(im)\right| k_2^{\delta_D} \left(\frac{|Q(im)|}{|R_D(im)| k_2^{\delta_D-1}}\right)^{\frac{1}{(\delta_D-1)k_2}} \left(1 + |\tau|\right)^{(\delta_D-1)k_2-1} \\
&\geq M_1^{(\delta_D-1)k_2-1} M_2 \frac{k_2^{\delta_D}}{(k_2^{\delta_D-1})^{\frac{1}{(\delta_D-1)k_2}}} (r_{Q,R_D})^{\frac{1}{(\delta_D-1)k_2}} \left|R_D(im)\right| \\
&\quad \times \left(\min_{x \geq 0} \frac{(1+x)^{(\delta_D-1)k_2-1}}{(1+x^{k_2})^{(\delta_D-1)-\frac{1}{k_2}}}\right) \left(1 + |\tau|^{k_2}\right)^{(\delta_D-1)-\frac{1}{k_2}} \\
&= C_P (r_{Q,R_D})^{\frac{1}{(\delta_D-1)k_2}} \left|R_D(im)\right| \left(1 + |\tau|^{k_2}\right)^{(\delta_D-1)-\frac{1}{k_2}}
\end{aligned} \tag{135}$$

for all $\tau \in S_d \cup \bar{D}(0,\rho)$, all $m \in \mathbb{R}$.

In the next proposition, we give sufficient conditions under which (129) has a solution $\omega_{k_2}^d(\tau,m,\epsilon)$ in the Banach space $F_{(\nu',\beta,\mu,k_2)}^d$ where ν', β, μ are defined above.

Proposition 14 *Under the assumption that*

$$\delta_D \geq \delta_l + \frac{1}{k_2} \tag{136}$$

for all $1 \leq l \leq D-1$, there exist a radius $r_{Q,R_D} > 0$, a constant $\upsilon > 0$, and constants $\varsigma_{1,2}, \varsigma_{0,0}, \varsigma_0, \varsigma_1, \varsigma_{1,0}, \varsigma_F, \varsigma_2 > 0$ (depending on Q_1, Q_2, k_2, C_P, μ, ν, ϵ_0, R_l, Δ_l, δ_l, d_l for $1 \leq l \leq D-1$) such that if

$$\sup_{\epsilon \in D(0,\epsilon_0)} \left|\frac{c_{1,2}(\epsilon)}{\epsilon}\right| \leq \varsigma_{1,2}, \qquad \sup_{\epsilon \in D(0,\epsilon_0)} \left|\frac{c_0(\epsilon)}{\epsilon}\right| \leq \varsigma_{1,0}, \qquad \left\|\varphi_{k_2}(\tau,m,\epsilon)\right\|_{(\nu',\beta,\mu,k_2)} \leq \varsigma_1,$$

$$\sup_{\epsilon\in D(0,\epsilon_0)}\left|\frac{c_{0,0}(\epsilon)}{\epsilon}\right| \leq \varsigma_{0,0}, \qquad \left\|C_{0,0}(m,\epsilon)\right\|_{(\beta,\mu)} \leq \varsigma_0, \tag{137}$$

$$\sup_{\epsilon\in D(0,\epsilon_0)}\left|\frac{c_F(\epsilon)}{\epsilon}\right| \leq \varsigma_F, \qquad \left\|\psi_{k_2}^d(\tau,m,\epsilon)\right\|_{(\nu',\beta,\mu,k_2)} \leq \varsigma_2$$

for all $\epsilon \in D(0,\epsilon_0)$, (129) *has a unique solution* $\omega_{k_2}^d(\tau,m,\epsilon)$ *in the space* $F_{(\nu',\beta,\mu,k_2)}^d$ *with the property that* $\|\omega_{k_2}^d(\tau,m,\epsilon)\|_{(\nu',\beta,\mu,k_2)} \leq \upsilon$, *for all* $\epsilon \in D(0,\epsilon_0)$, *where* $\beta,\mu > 0$ *are defined above, for any unbounded sector* S_d *that satisfies the constraints* (133), (134) *and for any bounded sector* S_d^b *with aperture strictly larger than* $\frac{\pi}{\kappa}$ *such that*

$$S_d^b \subset D(0,\rho), \qquad S_d^b \subset S_{d,\kappa,\delta}, \tag{138}$$

where $D(0,\rho)$ *fulfills the constraints* (133), (134) *and where the sector* $S_{d,\kappa,\delta}$ *with aperture* $\frac{\pi}{\kappa}+\delta$ *is defined in Lemma* 4, *where* $0<\delta<\mathrm{ap}(U_d)$.

Proof We start the proof with a lemma which provides appropriate conditions in order to apply a fixed point theorem.

Lemma 5 *One can choose the constant* $r_{Q,R_D}>0$, *a constant* υ *small enough and constants* $\varsigma_{1,2}, \varsigma_{0,0}, \varsigma_0, \varsigma_1, \varsigma_{1,0}, \varsigma_F, \varsigma_2 > 0$ (*depending on* Q_1, Q_2, k_2, C_P, μ, ν, ϵ_0, R_l, Δ_l, δ_l, d_l *for* $1\leq l\leq D-1$) *such that if* (137) *holds for all* $\epsilon\in D(0,\epsilon_0)$, *the map* $\mathcal{H}_\epsilon^{k_2}$ *defined by*

$$\begin{aligned}
&\mathcal{H}_\epsilon^{k_2}\big(w(\tau,m)\big)\\
&\quad= \frac{R_D(im)}{P_m(\tau)}\sum_{1\leq p\leq\delta_D-1} A_{\delta_D,p}\frac{1}{\Gamma(\delta_D-p)}\int_0^{\tau^{k_2}}\big(\tau^{k_2}-s\big)^{\delta_D-p-1}\big(k_2^p s^p w\big(s^{1/k_2},m\big)\big)\frac{ds}{s}\\
&\qquad+\epsilon^{-1}\frac{1}{P_m(\tau)\Gamma(1+\frac{1}{k_2})}\int_0^{\tau^{k_2}}\big(\tau^{k_2}-s\big)^{1/k_2}\\
&\qquad\times\bigg(\frac{c_{1,2}(\epsilon)}{(2\pi)^{1/2}}s\int_0^s\int_{-\infty}^{+\infty}Q_1\big(i(m-m_1)\big)w\big((s-x)^{1/k_2},m-m_1\big)\\
&\qquad\times Q_2(im_1)w\big(x^{1/k_2},m_1\big)\frac{1}{(s-x)x}\,dx\,dm_1\bigg)\frac{ds}{s}\\
&\qquad+\sum_{l=1}^{D-1}\frac{R_l(im)}{P_m(\tau)}\bigg(\epsilon^{\Delta_l-d_l+\delta_l-1}\frac{1}{\Gamma(\frac{d_{l,k_2}^2}{k_2})}\int_0^{\tau^{k_2}}\big(\tau^{k_2}-s\big)^{\frac{d_{l,k_2}^2}{k_2}-1}\big(k_2^{\delta_l}s^{\delta_l}w\big(s^{1/k_2},m\big)\big)\frac{ds}{s}\\
&\qquad+\sum_{1\leq p\leq\delta_l-1}A_{\delta_l,p}\epsilon^{\Delta_l-d_l+\delta_l-1}\frac{1}{\Gamma(\frac{d_{l,k_2}^2}{k_2}+\delta_l-p)}\\
&\qquad\times\int_0^{\tau^{k_2}}\big(\tau^{k_2}-s\big)^{\frac{d_{l,k_2}^2}{k_2}+\delta_l-p-1}\big(k_2^p s^p w\big(s^{1/k_2},m\big)\big)\frac{ds}{s}\bigg)\\
&\qquad+\epsilon^{-1}\frac{1}{P_m(\tau)\Gamma(1+\frac{1}{k_2})}\int_0^{\tau^{k_2}}\big(\tau^{k_2}-s\big)^{1/k_2}\\
&\qquad\times\bigg(\frac{c_0(\epsilon)}{(2\pi)^{1/2}}s\int_0^s\int_{-\infty}^{+\infty}\varphi_{k_2}\big((s-x)^{1/k_2},m-m_1,\epsilon\big)\\
&\qquad\times R_0(im_1)w\big(x^{1/k_2},m_1\big)\frac{1}{(s-x)x}\,dx\,dm_1\bigg)\frac{ds}{s}
\end{aligned}$$

$$+ \epsilon^{-1} \frac{1}{P_m(\tau)\Gamma(1+\frac{1}{k_2})} \int_0^{\tau^{k_2}} \left(\tau^{k_2} - s\right)^{1/k_2} \frac{c_{0,0}(\epsilon)}{(2\pi)^{1/2}}$$
$$\times \left(\int_{-\infty}^{+\infty} C_{0,0}(m-m_1,\epsilon) R_0(im_1) w\left(s^{1/k_2}, m_1\right) dm_1 \right) \frac{ds}{s}$$
$$+ \epsilon^{-1} c_F(\epsilon) \frac{1}{P_m(\tau)\Gamma(1+\frac{1}{k_2})} \int_0^{\tau^{k_2}} \left(\tau^{k_2} - s\right)^{1/k_2} \psi_{k_2}^d\left(s^{1/k_2}, m, \epsilon\right) \frac{ds}{s} \tag{139}$$

satisfies the next properties.

(i) *The following inclusion holds*:

$$\mathcal{H}_\epsilon^{k_2}\left(\bar{B}(0,\upsilon)\right) \subset \bar{B}(0,\upsilon), \tag{140}$$

where $\bar{B}(0,\upsilon)$ *is the closed ball of radius* $\upsilon > 0$ *centered at* 0 *in* $F^d_{(\nu',\beta,\mu,k_2)}$, *for all* $\epsilon \in D(0,\epsilon_0)$.

(ii) *We have*

$$\left\| \mathcal{H}_\epsilon^{k_2}(w_1) - \mathcal{H}_\epsilon^{k_2}(w_2) \right\|_{(\nu',\beta,\mu,k_2)} \leq \frac{1}{2} \|w_1 - w_2\|_{(\nu',\beta,\mu,k_2)} \tag{141}$$

for all $w_1, w_2 \in \bar{B}(0,\upsilon)$, *for all* $\epsilon \in D(0,\epsilon_0)$.

The proof of Lemma 5 follows the same lines of arguments as Lemma 2 in Proposition 9 of [1] and rests on Lemma 2, Propositions 5, 6, and 7 given in Section 2.2. Therefore, we omit the details.

We consider the ball $\bar{B}(0,\upsilon) \subset F^d_{(\nu',\beta,\mu,k_2)}$ constructed in Lemma 5 which is a complete metric space for the norm $\|\cdot\|_{(\nu',\beta,\mu,k_2)}$. From the lemma above, we find that $\mathcal{H}_\epsilon^{k_2}$ is a contractive map from $\bar{B}(0,\upsilon)$ into itself. Due to the classical contractive mapping theorem, we deduce that the map $\mathcal{H}_\epsilon^{k_2}$ has a unique fixed point denoted by $\omega_{k_2}^d(\tau,m,\epsilon)$ (*i.e.* $\mathcal{H}_\epsilon^{k_2}(\omega_{k_2}^d(\tau,m,\epsilon)) = \omega_{k_2}^d(\tau,m,\epsilon)$) in $\bar{B}(0,\upsilon)$, for all $\epsilon \in D(0,\epsilon_0)$. Moreover, the function $\omega_{k_2}^d(\tau,m,\epsilon)$ depends holomorphically on ϵ in $D(0,\epsilon_0)$. By construction, $\omega_{k_2}^d(\tau,m,\epsilon)$ defines a solution of (129). This yields the proposition. □

In the next proposition, we present the link, by means of the analytic acceleration operator defined in Proposition 13, between the holomorphic solution of the problem (83) constructed in Proposition 11 and the solution of the problem (129) found in Proposition 14.

Proposition 15 *Let us consider the function* $\omega_{k_1}^d(\tau,m,\epsilon)$ *constructed in Proposition* 11 *and which solves* (83). *The function*

$$\tau \mapsto \mathrm{Acc}^d_{k_2,k_1}\left(\omega^d_{k_1}\right)(\tau,m,\epsilon) := \mathcal{A}^d_{m_{k_2},m_{k_1}}\left(h \mapsto \omega^d_{k_1}(h,m,\epsilon)\right)(\tau)$$
$$= \int_{L_d} \omega^d_{k_1}(h,m,\epsilon) G(\tau,h) \frac{dh}{h}$$

defines an analytic function on a sector $S_{d,\kappa,\delta,(c_2/\nu)^{1/\kappa}/2}$ *with direction* d, *aperture* $\frac{\pi}{\kappa} + \delta$, *and radius* $(c_2/\nu)^{1/\kappa}/2$, *for any* $0 < \delta < \mathrm{ap}(U_d)$ *and for a constant* c_2 *introduced in* (114), *with the property that* $\mathrm{Acc}^d_{k_2,k_1}(\omega^d_{k_1})(0,m,\epsilon) \equiv 0$, *for all* $m \in \mathbb{R}$, *all* $\epsilon \in D(0,\epsilon_0)$.

Moreover, for all fixed $\epsilon \in D(0,\epsilon_0)$, *the identity*

$$\mathrm{Acc}^{d}_{k_2,k_1}\big(\omega^{d}_{k_1}\big)(\tau,m,\epsilon) = \omega^{d}_{k_2}(\tau,m,\epsilon) \tag{142}$$

holds for all $\tau \in S_{d,\kappa,\delta,(c_2/\nu)^{1/\kappa}/2}$, *all* $m \in \mathbb{R}$, *provided that* $\nu > 0$ *is chosen in such a way that* $S_{d,\kappa,\delta,(c_2/\nu)^{1/\kappa}/2} \subset S^{b}_{d}$ *holds where* S^{b}_{d} *is the bounded sector introduced in Proposition* 14.

As a consequence of Proposition 14, *the function* $\tau \mapsto \mathrm{Acc}^{d}_{k_2,k_1}(\omega^{d}_{k_1})(\tau,m,\epsilon)$ *has an analytic continuation on the union* $S^{b}_{d} \cup S_d$, *where the sector* S_d *has been described in Proposition* 14, *denoted again by* $\mathrm{Acc}^{d}_{k_2,k_1}(\omega^{d}_{k_1})(\tau,m,\epsilon)$ *which satisfies estimates of the form*: *there exists a constant* $C_{\omega_{k_2}} > 0$ *with*

$$\big|\mathrm{Acc}^{d}_{k_2,k_1}\big(\omega^{d}_{k_1}\big)(\tau,m,\epsilon)\big| \le C_{\omega_{k_2}}\big(1+|m|\big)^{-\mu} e^{-\beta|m|} \frac{|\tau|}{1+|\tau|^{2k_2}} \exp\big(\nu'|\tau|^{k_2}\big) \tag{143}$$

for all $\tau \in S^{b}_{d} \cup S_d$, *all* $m \in \mathbb{R}$, *all* $\epsilon \in D(0,\epsilon_0)$.

Proof From Proposition 11, we point out that $\omega^{d}_{k_1}(\tau,m,\epsilon)$ belongs to the space $F^{d}_{(\nu,\beta,\mu,k_1,\kappa)}$ and that $\|\omega^{d}_{k_1}\|_{(\nu,\beta,\mu,k_1,\kappa)} \le \varpi$ for all $\epsilon \in D(0,\epsilon_0)$. Due to Proposition 13, we deduce that the function $\tau \mapsto \mathrm{Acc}^{d}_{k_2,k_1}(\omega^{d}_{k_1})(\tau,m,\epsilon)$ defines a holomorphic and bounded function with values in the Banach space $E_{(\beta,\mu)}$ (with bound independent of ϵ) on a sector $S_{d,\kappa,\delta,(c_2/\nu)^{1/\kappa}/2}$ with direction d, aperture $\frac{\pi}{\kappa} + \delta$, and radius $(c_2/\nu)^{1/\kappa}/2$, for any $0 < \delta < \mathrm{ap}(U_d)$ and for a constant c_2 introduced in (114), for all $\epsilon \in D(0,\epsilon_0)$.

Now, as a result of Proposition 13, we also know that the function $\tau \mapsto \mathrm{Acc}^{d}_{k_2,k_1}(\omega^{d}_{k_1})(\tau, m,\epsilon)$ is the κ-sum of the formal series

$$\hat{\mathcal{A}}_{m_{k_2},m_{k_1}}\big(h \mapsto \omega_{k_1}(h,m,\epsilon)\big)(\tau) = \hat{\omega}_{k_2}(\tau,m,\epsilon)$$

viewed as a formal series in τ with coefficients in the Banach space $E_{(\beta,\mu)}$, on $S_{d,\kappa,\delta,(c_2/\nu)^{1/\kappa}/2}$, for all $\epsilon \in D(0,\epsilon_0)$. In particular, one sees that $\mathrm{Acc}^{d}_{k_2,k_1}(\omega^{d}_{k_1})(0,m,\epsilon) \equiv 0$, for all $\epsilon \in D(0,\epsilon_0)$.

Likewise, we notice from Lemma 4 that the function $\tau \mapsto \psi^{d}_{k_2}(\tau,m,\epsilon)$ is the κ-sum on $S_{d,\kappa,\delta,(c_2/\nu)^{1/\kappa}/2}$ of the formal series $\hat{\psi}_{k_2}(\tau,m,\epsilon)$ defined in (122), viewed as a formal series in τ with coefficients in the Banach space $E_{(\beta,\mu)}$, for all $\epsilon \in D(0,\epsilon_0)$. We recall that $\hat{\omega}_{k_2}(\tau,m,\epsilon)$ formally solves (123) for vanishing initial data $\hat{\omega}_{k_2}(0,m,\epsilon) \equiv 0$. Using the standard stability properties of the κ-sums of the formal series with respect to algebraic operations and integration (see [2], Section 3.3, Theorem 2, p.28), we deduce that the function $\mathrm{Acc}^{d}_{k_2,k_1}(\omega^{d}_{k_1})(\tau,m,\epsilon)$ satisfies (129) for all $\tau \in S_{d,\kappa,\delta,(c_2/\nu)^{1/\kappa}/2}$, all $m \in \mathbb{R}$, all $\epsilon \in D(0,\epsilon_0)$, for vanishing initial data $\mathrm{Acc}^{d}_{k_2,k_1}(\omega^{d}_{k_1})(0,m,\epsilon) \equiv 0$.

In order to justify the identity (142), we need to define some additional Banach space. We keep the aforementioned notations.

Definition 6 Let $h' = (c_2/\nu)^{1/\kappa}/2$. We denote $H_{(\nu',\beta,\mu,k_2,h')}$ the vector space of continuous functions $(\tau,m) \mapsto h(\tau,m)$ on $\bar{S}_{d,\kappa,\delta,h'} \times \mathbb{R}$, holomorphic with respect to τ on $S_{d,\kappa,\delta,h'}$ such that

$$\begin{aligned}&\big\|h(\tau,m)\big\|_{(\nu',\beta,\mu,k_2,h')}\\ &\quad= \sup_{\tau\in\bar{S}_{d,\kappa,\delta,h'},m\in\mathbb{R}} \big(1+|m|\big)^{\mu} \frac{1+|\tau|^{2k_2}}{|\tau|} \exp\big(\beta|m| - \nu'|\tau|^{k_2}\big)\big|h(\tau,m)\big|\end{aligned} \tag{144}$$

is finite. One can check that $H_{(\nu',\beta,\mu,k_2,h')}$ endowed with the norm $\|\cdot\|_{(\nu',\beta,\mu,k_2,h')}$ is a Banach space.

Remark Notice that if a function $h(\tau,m)$ belongs to the space $F^d_{(\nu',\beta,\mu,k_2)}$ for the sectors S_d and S^b_d described in Proposition 14, then it belongs to the space $H_{(\nu',\beta,\mu,k_2,h')}$ (provided that $\nu > 0$ is chosen such that $S_{d,\kappa,\delta,h'} \subset S^b_d$) and, moreover,

$$\|h(\tau,m)\|_{(\nu',\beta,\mu,k_2,h')} \le \|h(\tau,m)\|_{(\nu',\beta,\mu,k_2)}$$

holds.

From the remark above, one deduces that the functions $\varphi_{k_2}(\tau,m,\epsilon)$ and $\psi^d_{k_2}(\tau,m,\epsilon)$ belong to the space $H_{(\nu',\beta,\mu,k_2,h')}$.

In the following, one can reproduce the same lines of arguments as in the proof of Proposition 14 just by replacing the Banach space $F^d_{(\nu',\beta,\mu,k_2)}$ by $H_{(\nu',\beta,\mu,k_2,h')}$, one gets the following.

Lemma 6 *Under the assumption that* (136) *holds, for the radius* $r_{Q,R_D} > 0$, *the constants* υ *and* $\varsigma_{1,2}$, $\varsigma_{0,0}$, ς_0, ς_1, $\varsigma_{1,0}$, ς_F, ς_2 *given in Proposition* 14 *for which the constraints* (137) *hold,* (129) *has a unique solution* $\omega_{k_2,h'}(\tau,m,\epsilon)$ *in the space* $H_{(\nu',\beta,\mu,k_2,h')}$ *with the property that* $\|\omega_{k_2,h'}(\tau,m,\epsilon)\|_{(\nu',\beta,\mu,k_2,h')} \le \upsilon$, *for all* $\epsilon \in D(0,\epsilon_0)$.

Taking into account Proposition 14, since $\omega^d_{k_2}(\tau,m,\epsilon)$ belongs to $F^d_{(\nu',\beta,\mu,k_2)}$, it also belongs to the space $H_{(\nu',\beta,\mu,k_2,h')}$. Moreover, since $\tau \mapsto \mathrm{Acc}^d_{k_2,k_1}(\omega^d_{k_1})(\tau,m,\epsilon)$ defines a holomorphic and bounded function with values in the Banach space $E_{(\beta,\mu)}$ (with bound independent of ϵ) on $S_{d,\kappa,\delta,h'}$ that vanishes at $\tau = 0$, we also find that $\mathrm{Acc}^d_{k_2,k_1}(\omega^d_{k_1})(\tau,m,\epsilon)$ belongs to $H_{(\nu',\beta,\mu,k_2,h')}$.

As a summary, we have seen that both $\omega^d_{k_2}(\tau,m,\epsilon)$ and $\mathrm{Acc}^d_{k_2,k_1}(\omega^d_{k_1})(\tau,m,\epsilon)$ solve the same equation (129) for vanishing initial data and belong to $H_{(\nu',\beta,\mu,k_2,h')}$. Moreover, one can check that the constant $\upsilon > 0$ in Lemma 6 and Proposition 14 can be chosen sufficiently large such that $\|\mathrm{Acc}^d_{k_2,k_1}(\omega^d_{k_1})(\tau,m,\epsilon)\|_{(\nu',\beta,\mu,k_2,h')} \le \upsilon$ holds, if the constants $\varsigma_{1,2}$, $\varsigma_{0,0}$, $\varsigma_{1,0}$, $\varsigma_F > 0$ are chosen small enough and $r_{Q,R_D} > 0$ is taken large enough. By construction, we already know that $\|\omega^d_{k_2}(\tau,m,\epsilon)\|_{(\nu',\beta,\mu,k_2,h')} \le \upsilon$. Therefore, from Lemma 6, we find that they must be equal. Proposition 15 follows. □

Now, we define the m_{k_2}-Laplace transforms

$$\begin{aligned} F^d(T,m,\epsilon) &:= k_2 \int_{L_d} \psi^d_{k_2}(u,m,\epsilon) e^{-(\frac{u}{T})^{k_2}} \frac{du}{u}, \\ U^d(T,m,\epsilon) &:= k_2 \int_{L_d} \omega^d_{k_2}(u,m,\epsilon) e^{-(\frac{u}{T})^{k_2}} \frac{du}{u}, \end{aligned} \tag{145}$$

which, according to the estimates (124) and (143), are $E_{(\beta,\mu)}$-valued bounded holomorphic functions on the sector $S_{d,\theta,h'}$ with bisecting direction d, aperture $\frac{\pi}{k_2} < \theta < \frac{\pi}{k_2} + \mathrm{ap}(S_d)$, and radius h', where $h' > 0$ is some positive real number, for all $\epsilon \in D(0,\epsilon_0)$.

Remark The analytic functions $F^d(T,m,\epsilon)$ (resp. $U^d(T,m,\epsilon)$) can be called the (m_{k_2}, m_{k_1})-sums in the direction d of the formal series $F(T,m,\epsilon)$ (resp. $U(T,m,\epsilon)$) introduced in the Section 4.1, following the terminology of [2], Section 6.1.

In the next proposition, we construct analytic solutions to the problem (70) with analytic forcing term and for vanishing initial data.

Proposition 16 *The function $U^d(T,m,\epsilon)$ solves the following equation*:

$$
\begin{aligned}
&Q(im)\big(\partial_T U^d(T,m,\epsilon)\big) - T^{(\delta_D-1)(k_2+1)}\partial_T^{\delta_D} R_D(im)U^d(T,m,\epsilon)\\
&\quad = \epsilon^{-1}\frac{c_{1,2}(\epsilon)}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} Q_1\big(i(m-m_1)\big)U^d(T,m-m_1,\epsilon)Q_2(im_1)U^d(T,m_1,\epsilon)\,dm_1\\
&\qquad + \sum_{l=1}^{D-1} R_l(im)\epsilon^{\Delta_l-d_l+\delta_l-1}T^{d_l}\partial_T^{\delta_l}U^d(T,m,\epsilon)\\
&\qquad + \epsilon^{-1}\frac{c_0(\epsilon)}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} C_0(T,m-m_1,\epsilon)R_0(im_1)U^d(T,m_1,\epsilon)\,dm_1\\
&\qquad + \epsilon^{-1}\frac{c_{0,0}(\epsilon)}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} C_{0,0}(m-m_1,\epsilon)R_0(im_1)U^d(T,m_1,\epsilon)\,dm_1\\
&\qquad + \epsilon^{-1}c_F(\epsilon)F^d(T,m,\epsilon)
\end{aligned}
\tag{146}
$$

for given initial data $U^d(0,m,\epsilon)=0$, for all $T \in S_{d,\theta,h'}$, $m \in \mathbb{R}$, all $\epsilon \in D(0,\epsilon_0)$.

Proof Since the function $\omega_{k_2}^d(u,m,\epsilon)$ solves the integral equation (129), one can check by direct computations similar to those described in Proposition 8, using the integral representations (145) that $U^d(T,m,\epsilon)$ solves (121) where the formal series $F(T,m,\epsilon)$ is replaced by $F^d(T,m,\epsilon)$ and hence solves (70) where $F^d(T,m,\epsilon)$ must be put in place of $F(T,m,\epsilon)$. □

5 Analytic solutions of a nonlinear initial value Cauchy problem with analytic forcing term on sectors and with complex parameter

Let $k_1,k_2 \geq 1$, $D \geq 2$ be integers such that $k_2 > k_1$. Let $\delta_l \geq 1$ be integers such that

$$
1 = \delta_1, \qquad \delta_l < \delta_{l+1}, \tag{147}
$$

for all $1 \leq l \leq D-1$. For all $1 \leq l \leq D-1$, let $d_l, \Delta_l \geq 0$ be nonnegative integers such that

$$
d_l > \delta_l, \qquad \Delta_l - d_l + \delta_l - 1 \geq 0. \tag{148}
$$

Let $Q(X), Q_1(X), Q_2(X), R_l(X) \in \mathbb{C}[X]$, $0 \leq l \leq D$, be polynomials such that

$$
\begin{aligned}
&\deg(Q) \geq \deg(R_D) \geq \deg(R_l), \qquad \deg(R_D) \geq \deg(Q_1), \qquad \deg(R_D) \geq \deg(Q_2),\\
&Q(im) \neq 0, \qquad R_l(im) \neq 0, \qquad R_D(im) \neq 0
\end{aligned}
\tag{149}
$$

for all $m \in \mathbb{R}$, all $0 \leq l \leq D-1$.

We require that there exists a constant $r_{Q,R_l} > 0$ such that

$$
\left|\frac{Q(im)}{R_l(im)}\right| \geq r_{Q,R_l} \tag{150}
$$

for all $m \in \mathbb{R}$, all $1 \leq l \leq D$. We make the additional assumption that there exists an unbounded sector

$$S_{Q,R_D} = \{ z \in \mathbb{C}/|z| \geq r_{Q,R_D}, |\arg(z) - d_{Q,R_D}| \leq \eta_{Q,R_D} \}$$

with direction $d_{Q,R_D} \in \mathbb{R}$, aperture $\eta_{Q,R_D} > 0$ for the radius $r_{Q,R_D} > 0$ given above, such that

$$\frac{Q(im)}{R_D(im)} \in S_{Q,R_D} \tag{151}$$

for all $m \in \mathbb{R}$.

Definition 7 Let $\varsigma \geq 2$ be an integer. For all $0 \leq p \leq \varsigma - 1$, we consider open sectors $\mathcal{E}_p$ centered at 0, with radius ϵ_0, and opening $\frac{\pi}{k_2} + \kappa_p$, with $\kappa_p > 0$ small enough such that $\mathcal{E}_p \cap \mathcal{E}_{p+1} \neq \emptyset$, for all $0 \leq p \leq \varsigma - 1$ (with the convention that $\mathcal{E}_\varsigma = \mathcal{E}_0$). Moreover, we assume that the intersection of any three different elements in $\{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$ is empty and that $\bigcup_{p=0}^{\varsigma-1} \mathcal{E}_p = \mathcal{U} \setminus \{0\}$, where $\mathcal{U}$ is some neighborhood of 0 in $\mathbb{C}$. Such a set of sectors $\{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$ is called a good covering in $\mathbb{C}^*$.

Definition 8 Let $\{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$ be a good covering in $\mathbb{C}^*$. Let $\mathcal{T}$ be an open bounded sector centered at 0 with radius $r_{\mathcal{T}}$ and consider a family of open sectors

$$S_{\mathfrak{d}_p,\theta,\epsilon_0 r_{\mathcal{T}}} = \{ T \in \mathbb{C}^*/|T| < \epsilon_0 r_{\mathcal{T}}, |\mathfrak{d}_p - \arg(T)| < \theta/2 \}$$

with aperture $\theta > \pi/k_2$ and where $\mathfrak{d}_p \in \mathbb{R}$, for all $0 \leq p \leq \varsigma - 1$, are directions which satisfy the following constraints: Let $q_l(m)$ be the roots of the polynomials (131) defined by (132) and $S_{\mathfrak{d}_p}$, $0 \leq p \leq \varsigma - 1$, be unbounded sectors centered at 0 with directions $\mathfrak{d}_p$ and with small aperture. Let $\rho > 0$ be a positive real number. We assume that:

(1) There exists a constant $M_1 > 0$ such that

$$|\tau - q_l(m)| \geq M_1(1 + |\tau|) \tag{152}$$

for all $0 \leq l \leq (\delta_D - 1)k_2 - 1$, all $m \in \mathbb{R}$, all $\tau \in S_{\mathfrak{d}_p} \cup \bar{D}(0, \rho)$, for all $0 \leq p \leq \varsigma - 1$.

(2) There exists a constant $M_2 > 0$ such that

$$|\tau - q_{l_0}(m)| \geq M_2 |q_{l_0}(m)| \tag{153}$$

for some $l_0 \in \{0, \dots, (\delta_D - 1)k_2 - 1\}$, all $m \in \mathbb{R}$, all $\tau \in S_{\mathfrak{d}_p} \cup \bar{D}(0, \rho)$, for all $0 \leq p \leq \varsigma - 1$.

(3) There exist a family of unbounded sectors $U_{\mathfrak{d}_p}$ with bisecting direction $\mathfrak{d}_p$ and bounded sectors $S^b_{\mathfrak{d}_p}$ with bisecting direction $\mathfrak{d}_p$, with radius less than ρ, with aperture $\frac{\pi}{\kappa} + \delta_p$, with $0 < \delta_p < \mathrm{ap}(U_{\mathfrak{d}_p})$, for all $0 \leq p \leq \varsigma - 1$, with the property that $S^b_{\mathfrak{d}_p} \cap S^b_{\mathfrak{d}_{p+1}} \neq \emptyset$ for all $0 \leq p \leq \varsigma - 1$ (with the convention that $\mathfrak{d}_\varsigma = \mathfrak{d}_0$).

(4) For all $0 \leq p \leq \varsigma - 1$, for all $t \in \mathcal{T}$, all $\epsilon \in \mathcal{E}_p$, we have $\epsilon t \in S_{\mathfrak{d}_p,\theta,\epsilon_0 r_{\mathcal{T}}}$.

We say that the family $\{(S_{\mathfrak{d}_p,\theta,\epsilon_0 r_{\mathcal{T}}})_{0 \leq p \leq \varsigma-1}, \mathcal{T}\}$ is associated to the good covering $\{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$.

We consider a good covering $\{\mathcal{E}_p\}_{0\leq p\leq\varsigma-1}$ and a family of sectors $\{(S_{\mathfrak{d}_p,\theta,\epsilon_0 r_{\mathcal{T}}})_{0\leq p\leq\varsigma-1},\mathcal{T}\}$ associated to it. For all $0\leq p\leq\varsigma-1$, we consider the following nonlinear initial value problem with forcing term:

$$\begin{aligned}&Q(\partial_z)\big(\partial_t u^{\mathfrak{d}_p}(t,z,\epsilon)\big)\\&\quad= c_{1,2}(\epsilon)\big(Q_1(\partial_z)u^{\mathfrak{d}_p}(t,z,\epsilon)\big)\big(Q_2(\partial_z)u^{\mathfrak{d}_p}(t,z,\epsilon)\big)\\&\qquad+\epsilon^{(\delta_D-1)(k_2+1)-\delta_D+1}t^{(\delta_D-1)(k_2+1)}\partial_t^{\delta_D}R_D(\partial_z)u^{\mathfrak{d}_p}(t,z,\epsilon)+\sum_{l=1}^{D-1}\epsilon^{\Delta_l}t^{d_l}\partial_t^{\delta_l}R_l(\partial_z)u^{\mathfrak{d}_p}(t,z,\epsilon)\\&\qquad+c_0(t,z,\epsilon)R_0(\partial_z)u^{\mathfrak{d}_p}(t,z,\epsilon)+c_F(\epsilon)f^{\mathfrak{d}_p}(t,z,\epsilon)\end{aligned}\tag{154}$$

for given initial data $u^{\mathfrak{d}_p}(0,z,\epsilon)\equiv 0$.

The functions $c_{1,2}(\epsilon)$ and $c_F(\epsilon)$ are holomorphic and bounded on the disc $D(0,\epsilon_0)$ and are such that $c_{1,2}(0)=c_F(0)=0$. The coefficient $c_0(t,z,\epsilon)$ and the forcing term $f^{\mathfrak{d}_p}(t,z,\epsilon)$ are constructed as follows. Let $c_0(\epsilon)$ and $c_{0,0}(\epsilon)$ be holomorphic and bounded functions on the disc $D(0,\epsilon_0)$ which satisfy $c_0(0)=c_{0,0}(0)=0$. We consider sequences of functions $m\mapsto C_{0,n}(m,\epsilon)$, for $n\geq 0$, and $m\mapsto F_n(m,\epsilon)$, for $n\geq 1$, that belong to the Banach space $E_{(\beta,\mu)}$ for some $\beta>0$, $\mu>\max(\deg(Q_1)+1,\deg(Q_2)+1)$, and which depend holomorphically on $\epsilon\in D(0,\epsilon_0)$. We assume that there exist constants $K_0,T_0>0$ such that (69) holds for all $n\geq 1$, for all $\epsilon\in D(0,\epsilon_0)$. We deduce that the function

$$\mathfrak{C}_0(T,z,\epsilon)=c_{0,0}(\epsilon)\mathcal{F}^{-1}\big(m\mapsto C_{0,0}(m,\epsilon)\big)(z)+\sum_{n\geq 1}c_0(\epsilon)\mathcal{F}^{-1}\big(m\mapsto C_{0,n}(m,\epsilon)\big)(z)T^n$$

represents a bounded holomorphic function on $D(0,T_0/2)\times H_{\beta'}\times D(0,\epsilon_0)$ for any $0<\beta'<\beta$ (where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform defined in Proposition 9). We define the coefficient $c_0(t,z,\epsilon)$ as

$$c_0(t,z,\epsilon)=\mathfrak{C}_0(\epsilon t,z,\epsilon).\tag{155}$$

The function c_0 is holomorphic and bounded on $D(0,r)\times H_{\beta'}\times D(0,\epsilon_0)$ where $r\epsilon_0<T_0/2$.

We make the assumption that the formal m_{k_1}-Borel transform

$$\psi_{k_1}(\tau,m,\epsilon)=\sum_{n\geq 1}F_n(m,\epsilon)\frac{\tau^n}{\Gamma(\frac{n}{k_1})}$$

is convergent on the disc $D(0,\rho)$ given in Definition 8 and can be analytically continued w.r.t. τ as a function $\tau\mapsto\psi_{k_1}^{\mathfrak{d}_p}(\tau,m,\epsilon)$ on the domain $U_{\mathfrak{d}_p}\cup D(0,\rho)$, where $U_{\mathfrak{d}_p}$ is the unbounded sector given in Definition 8, with $\psi_{k_1}^{\mathfrak{d}_p}(\tau,m,\epsilon)\in F^{\mathfrak{d}_p}_{(\nu,\beta,\mu,k_1,k_1)}$ and such that there exists a constant $\zeta_{\psi_{k_1}}>0$ such that

$$\big\|\psi_{k_1}^{\mathfrak{d}_p}(\tau,m,\epsilon)\big\|_{(\nu,\beta,\mu,k_1,k_1)}\leq\zeta_{\psi_{k_1}}\tag{156}$$

for all $\epsilon\in D(0,\epsilon_0)$.

From Lemma 4, we know that the accelerated function

$$\psi_{k_2}^{\mathfrak{d}_p}(\tau,m,\epsilon):=\mathcal{A}^{\mathfrak{d}_p}_{m_{k_2},m_{k_1}}\big(h\mapsto\psi_{k_1}^{\mathfrak{d}_p}(h,m,\epsilon)\big)(\tau)$$

defines a function that belongs to the space $F^{\mathfrak{d}_p}_{(\nu',\beta,\mu,k_2)}$ for the unbounded sector $S_{\mathfrak{d}_p}$ and the bounded sector $S^b_{\mathfrak{d}_p}$ given in Definition 8. Moreover, we get a constant $\zeta_{\psi_{k_2}} > 0$ with

$$\left\| \psi^{\mathfrak{d}_p}_{k_2}(\tau,m,\epsilon)\right\|_{(\nu',\beta,\mu,k_2)} \leq \zeta_{\psi_{k_2}} \tag{157}$$

for all $\epsilon \in D(0,\epsilon_0)$. We take the m_{k_2}-Laplace transform

$$F^{\mathfrak{d}_p}(T,m,\epsilon) := k_2 \int_{L_{\mathfrak{d}_p}} \psi^{\mathfrak{d}_p}_{k_2}(u,m,\epsilon) e^{-(\frac{u}{T})^{k_2}} \frac{du}{u}, \tag{158}$$

which exists for all $T \in S_{\mathfrak{d}_p,\theta,h'}$, $m \in \mathbb{R}$, $\epsilon \in D(0,\epsilon_0)$, where $S_{\mathfrak{d}_p,\theta,h'}$ is a sector with bisecting direction $\mathfrak{d}_p$, aperture $\frac{\pi}{k_2} < \theta < \frac{\pi}{k_2} + \mathrm{ap}(S_{\mathfrak{d}_p})$, and radius h', where $h' > 0$ is some positive real number, for all $\epsilon \in D(0,\epsilon_0)$.

We define the forcing term $f^{\mathfrak{d}_p}(t,z,\epsilon)$ as

$$f^{\mathfrak{d}_p}(t,z,\epsilon) := \mathcal{F}^{-1}\big(m \mapsto F^{\mathfrak{d}_p}(\epsilon t,m,\epsilon)\big)(z). \tag{159}$$

By construction, $f^{\mathfrak{d}_p}(t,z,\epsilon)$ represents a bounded holomorphic function on $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$ (provided that the radius $r_{\mathcal{T}}$ of $\mathcal{T}$ satisfies the inequality $\epsilon_0 r_{\mathcal{T}} \leq h'$, which will be assumed in the sequel).

In the next first main result, we construct a family of actual holomorphic solutions to (154) for given initial data at $t = 0$ being identically equal to zero, defined on the sectors $\mathcal{E}_p$ with respect to the complex parameter ϵ. We can also control the difference between any two neighboring solutions on the intersection of sectors $\mathcal{E}_p \cap \mathcal{E}_{p+1}$.

Theorem 1 *We consider* (154) *and we assume that the constraints* (147), (148), (149), (150), *and* (151) *hold. We also make the additional assumption that*

$$\begin{gathered} d_l + k_1 + 1 = \delta_l(k_1+1) + d^1_{l,k_1}, \qquad d^1_{l,k_1} > 0, \qquad \frac{1}{\kappa} = \frac{1}{k_1} - \frac{1}{k_2}, \\ \frac{k_2}{k_2 - k_1} \geq \frac{d_l + (1-\delta_l)}{d_l + (1-\delta_l)(k_1+1)}, \qquad d^1_{l,k_1} > (\delta_l - 1)(k_2 - k_1), \qquad \delta_D \geq \delta_l + \frac{1}{k_2}, \end{gathered} \tag{160}$$

for $1 \leq l \leq D-1$. *Let the coefficient* $c_0(t,z,\epsilon)$ *and the forcing terms* $f^{\mathfrak{d}_p}(t,z,\epsilon)$ *be constructed as in* (155), (159). *Let a good covering* $\{\mathcal{E}_p\}_{0\leq p\leq\varsigma-1}$ *in* $\mathbb{C}^*$ *be given, for which a family of sectors* $\{(S_{\mathfrak{d}_p,\theta,\epsilon_0 r_{\mathcal{T}}})_{0\leq p\leq\varsigma-1}, \mathcal{T}\}$ *associated to this good covering can be considered.*

Then there exist radii $r_{Q,R_l} > 0$ *large enough, for* $1 \leq l \leq D$ *and constants* $\zeta_{1,2}, \zeta_{0,0}, \zeta_{1,0}, \zeta_F > 0$ *small enough, such that if*

$$\begin{gathered} \sup_{\epsilon\in D(0,\epsilon_0)} \left|\frac{c_{1,2}(\epsilon)}{\epsilon}\right| \leq \zeta_{1,2}, \qquad \sup_{\epsilon\in D(0,\epsilon_0)} \left|\frac{c_0(\epsilon)}{\epsilon}\right| \leq \zeta_{1,0}, \\ \sup_{\epsilon\in D(0,\epsilon_0)} \left|\frac{c_{0,0}(\epsilon)}{\epsilon}\right| \leq \zeta_{0,0}, \qquad \sup_{\epsilon\in D(0,\epsilon_0)} \left|\frac{c_F(\epsilon)}{\epsilon}\right| \leq \zeta_F, \end{gathered} \tag{161}$$

and also for every $0 \leq p \leq \varsigma - 1$, *one can construct a solution* $u^{\mathfrak{d}_p}(t,z,\epsilon)$ *of* (154) *with* $u^{\mathfrak{d}_p}(0,z,\epsilon) \equiv 0$, *which defines a bounded holomorphic function on the domain* $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$ *for any given* $0 < \beta' < \beta$.

Moreover, the next estimates hold for the solution $u^{\mathfrak{d}_p}$ *and the forcing term* $f^{\mathfrak{d}_p}$: *there exist constants* $0 < h'' \leq r_{\mathcal{T}}$, $K_p, M_p > 0$ (*independent of* ϵ) *with the following properties*:

(1) *Assume that the unbounded sectors* $U_{\mathfrak{d}_p}$ *and* $U_{\mathfrak{d}_{p+1}}$ *have a sufficiently large aperture in such a way that* $U_{\mathfrak{d}_p} \cap U_{\mathfrak{d}_{p+1}}$ *contains the sector* $U_{\mathfrak{d}_p,\mathfrak{d}_{p+1}} = \{\tau \in \mathbb{C}^*/\arg(\tau) \in [\mathfrak{d}_p, \mathfrak{d}_{p+1}]\}$, *then*

$$\begin{aligned} &\sup_{t\in\mathcal{T}\cap D(0,h''),z\in H_{\beta'}} \left|u^{\mathfrak{d}_{p+1}}(t,z,\epsilon) - u^{\mathfrak{d}_p}(t,z,\epsilon)\right| \le K_p e^{-\frac{M_p}{|\epsilon|^{k_2}}}, \\ &\sup_{t\in\mathcal{T}\cap D(0,h''),z\in H_{\beta'}} \left|f^{\mathfrak{d}_{p+1}}(t,z,\epsilon) - f^{\mathfrak{d}_p}(t,z,\epsilon)\right| \le K_p e^{-\frac{M_p}{|\epsilon|^{k_2}}} \end{aligned} \tag{162}$$

for all $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$.

(2) *Assume that the unbounded sectors* $U_{\mathfrak{d}_p}$ *and* $U_{\mathfrak{d}_{p+1}}$ *have an empty intersection, then*

$$\begin{aligned} &\sup_{t\in\mathcal{T}\cap D(0,h''),z\in H_{\beta'}} \left|u^{\mathfrak{d}_{p+1}}(t,z,\epsilon) - u^{\mathfrak{d}_p}(t,z,\epsilon)\right| \le K_p e^{-\frac{M_p}{|\epsilon|^{k_1}}}, \\ &\sup_{t\in\mathcal{T}\cap D(0,h''),z\in H_{\beta'}} \left|f^{\mathfrak{d}_{p+1}}(t,z,\epsilon) - f^{\mathfrak{d}_p}(t,z,\epsilon)\right| \le K_p e^{-\frac{M_p}{|\epsilon|^{k_1}}} \end{aligned} \tag{163}$$

for all $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$.

Proof Let $0 \le p \le \varsigma - 1$. Under the assumptions of Theorem 1, using Proposition 16, one can construct a function $U^{\mathfrak{d}_p}(T,m,\epsilon)$ which satisfies $U^{\mathfrak{d}_p}(0,m,\epsilon) \equiv 0$ and solves the equation

$$\begin{aligned} &Q(im)\big(\partial_T U^{\mathfrak{d}_p}(T,m,\epsilon)\big) - T^{(\delta_D-1)(k_2+1)}\partial_T^{\delta_D} R_D(im)U^{\mathfrak{d}_p}(T,m,\epsilon) \\ &\quad= \epsilon^{-1}\frac{c_{1,2}(\epsilon)}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} Q_1\big(i(m-m_1)\big)U^{\mathfrak{d}_p}(T,m-m_1,\epsilon)Q_2(im_1)U^{\mathfrak{d}_p}(T,m_1,\epsilon)\,dm_1 \\ &\qquad+ \sum_{l=1}^{D-1} R_l(im)\epsilon^{\Delta_l-d_l+\delta_l-1}T^{d_l}\partial_T^{\delta_l}U^{\mathfrak{d}_p}(T,m,\epsilon) \\ &\qquad+ \epsilon^{-1}\frac{c_0(\epsilon)}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} C_0(T,m-m_1,\epsilon)R_0(im_1)U^{\mathfrak{d}_p}(T,m_1,\epsilon)\,dm_1 \\ &\qquad+ \epsilon^{-1}\frac{c_{0,0}(\epsilon)}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} C_{0,0}(m-m_1,\epsilon)R_0(im_1)U^{\mathfrak{d}_p}(T,m_1,\epsilon)\,dm_1 \\ &\qquad+ \epsilon^{-1}c_F(\epsilon)F^{\mathfrak{d}_p}(T,m,\epsilon), \end{aligned} \tag{164}$$

where $C_0(T,m,\epsilon) = \sum_{n\ge1} C_{0,n}(m,\epsilon)T^n$ is a convergent series on $D(0,T_0/2)$ with values in $E_{(\beta,\mu)}$ and $F^{\mathfrak{d}_p}(T,m,\epsilon)$ is given by (158), for all $\epsilon \in D(0,\epsilon_0)$. The function $(T,m) \mapsto U^{\mathfrak{d}_p}(T,m,\epsilon)$ is well defined on the domain $S_{\mathfrak{d}_p,\theta,h'} \times \mathbb{R}$.

Moreover, $U^{\mathfrak{d}_p}(T,m,\epsilon)$ can be written as m_{k_2}-Laplace transform

$$U^{\mathfrak{d}_p}(T,m,\epsilon) = k_2\int_{L_{\gamma_p}} \omega_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon)\exp\left(-\left(\frac{u}{T}\right)^{k_2}\right)\frac{du}{u} \tag{165}$$

along a half-line $L_{\gamma_p} = \mathbb{R}_+ e^{\sqrt{-1}\gamma_p} \subset S_{\mathfrak{d}_p} \cup \{0\}$ (the direction γ_p may depend on T), where $\omega_{k_2}^{\mathfrak{d}_p}(\tau,m,\epsilon)$ defines a continuous function on $(S_{\mathfrak{d}_p}^b \cup S_{\mathfrak{d}_p}) \times \mathbb{R} \times D(0,\epsilon_0)$, which is holomorphic with respect to (τ,ϵ) on $(S_{\mathfrak{d}_p}^b \cup S_{\mathfrak{d}_p}) \times D(0,\epsilon_0)$ for any $m \in \mathbb{R}$ and satisfies the estimates: there exists a constant $C_{\omega_{k_2}^{\mathfrak{d}_p}} > 0$ with

$$\left|\omega_{k_2}^{\mathfrak{d}_p}(\tau,m,\epsilon)\right| \le C_{\omega_{k_2}^{\mathfrak{d}_p}}\left(1+|m|\right)^{-\mu}e^{-\beta|m|}\frac{|\tau|}{1+|\tau|^{2k_2}}\exp\left(\nu'|\tau|^{k_2}\right) \tag{166}$$

for all $\tau \in S_{\mathfrak{d}_p}^b \cup S_{\mathfrak{d}_p}$, all $m \in \mathbb{R}$, all $\epsilon \in D(0,\epsilon_0)$. Besides, the function $\omega_{k_2}^{\mathfrak{d}_p}(\tau,m,\epsilon)$ is the analytic continuation w.r.t. τ of the function

$$\tau \mapsto \mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_p}\left(\omega_{k_1}^{\mathfrak{d}_p}\right)(\tau,m,\epsilon) = \int_{L_{\gamma_p^1}} \omega_{k_1}^{\mathfrak{d}_p}(h,m,\epsilon)G(\tau,h)\frac{dh}{h}, \tag{167}$$

where the path of integration is a half-line $L_{\gamma_p^1} = \mathbb{R}_+e^{\sqrt{-1}\gamma_p^1} \subset U_{\mathfrak{d}_p}$ (the direction γ_p^1 may depend on τ), which defines an analytic function on $S_{\mathfrak{d}_p,\kappa,\delta_p,(c_2/\nu)^{1/\kappa}/2} \subset S_{\mathfrak{d}_p}^b$ which is a sector with bisecting direction $\mathfrak{d}_p$, aperture $\frac{\pi}{\kappa}+\delta_p$, and radius $(c_2/\nu)^{1/\kappa}/2$. We recall that $\omega_{k_1}^{\mathfrak{d}_p}(h,m,\epsilon)$ defines a continuous function on $(U_{\mathfrak{d}_p} \cup D(0,\rho)) \times \mathbb{R} \times D(0,\epsilon_0)$, which is holomorphic w.r.t. (τ,ϵ) on $(U_{\mathfrak{d}_p} \cup D(0,\rho)) \times D(0,\epsilon_0)$, for any $m \in \mathbb{R}$, and satisfies the estimates: there exists a constant $C_{\omega_{k_1}^{\mathfrak{d}_p}} > 0$ with

$$\left|\omega_{k_1}^{\mathfrak{d}_p}(\tau,m,\epsilon)\right| \le C_{\omega_{k_1}^{\mathfrak{d}_p}}\left(1+|m|\right)^{-\mu}e^{-\beta|m|}\frac{|\tau|}{1+|\tau|^{2k_1}}\exp\left(\nu|\tau|^{\kappa}\right) \tag{168}$$

for all $\tau \in U_{\mathfrak{d}_p} \cup D(0,\rho)$, all $m \in \mathbb{R}$, all $\epsilon \in D(0,\epsilon_0)$.

Using the estimates (166), we find that the function

$$(T,z) \mapsto \mathbf{U}^{\mathfrak{d}_p}(T,z,\epsilon) = \mathcal{F}^{-1}\left(m \mapsto U^{\mathfrak{d}_p}(T,m,\epsilon)\right)(z)$$

defines a bounded holomorphic function on $S_{\mathfrak{d}_p,\theta,h'} \times H_{\beta'}$, for all $\epsilon \in D(0,\epsilon_0)$ and any $0 < \beta' < \beta$. For all $0 \le p \le \varsigma - 1$, we define

$$u^{\mathfrak{d}_p}(t,z,\epsilon) = \mathbf{U}^{\mathfrak{d}_p}(\epsilon t,z,\epsilon) = \frac{k_2}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_{\gamma_p}} \omega_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon)\exp\left(-\left(\frac{u}{\epsilon t}\right)^{k_2}\right)e^{izm}\frac{du}{u}\,dm.$$

Taking into account the construction provided in (4) from Definition 8, the function $u^{\mathfrak{d}_p}(t,z,\epsilon)$ defines a bounded holomorphic function on the domain $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$. Moreover, we have $u^{\mathfrak{d}_p}(0,z,\epsilon) \equiv 0$ and using the properties of the Fourier inverse transform from Proposition 9, we deduce that $u^{\mathfrak{d}_p}(t,z,\epsilon)$ solves the main equation (154) on $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$.

Now, we proceed to the proof of the estimates (162). We detail only the arguments for the functions $u^{\mathfrak{d}_p}$ since the estimates for the forcing terms $f^{\mathfrak{d}_p}$ follow the same line of discourse as below with the help of the estimates (157) instead of (166).

Let $0 \le p \le \varsigma - 1$ such that $U_{\mathfrak{d}_p} \cap U_{\mathfrak{d}_{p+1}}$ contains the sector $U_{\mathfrak{d}_p,\mathfrak{d}_{p+1}}$. First of all, from the integral representation (167) by using a path deformation between $L_{\gamma_p^1}$ and $L_{\gamma_{p+1}^1}$, we observe that the functions $\mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_p}(\omega_{k_1}^{\mathfrak{d}_p})(\tau,m,\epsilon)$ and $\mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_{p+1}}(\omega_{k_1}^{\mathfrak{d}_{p+1}})(\tau,m,\epsilon)$ must coincide on the domain $(S_{\mathfrak{d}_p,\kappa,\delta_p,(c_2/\nu)^{1/\kappa}/2} \cap S_{\mathfrak{d}_{p+1},\kappa,\delta_{p+1},(c_2/\nu)^{1/\kappa}/2}) \times \mathbb{R} \times D(0,\epsilon_0)$. Hence, there exists a function that we denote $\omega_{k_2}^{\mathfrak{d}_p,\mathfrak{d}_{p+1}}(\tau,m,\epsilon)$ which is holomorphic w.r.t. τ on $S_{\mathfrak{d}_p,\kappa,\delta_p,(c_2/\nu)^{1/\kappa}/2} \cup S_{\mathfrak{d}_{p+1},\kappa,\delta_{p+1},(c_2/\nu)^{1/\kappa}/2}$, continuous w.r.t. m on $\mathbb{R}$, holomorphic w.r.t. ϵ on

$D(0,\epsilon_0)$ which coincides with $\mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_p}(\omega_{k_1}^{\mathfrak{d}_p})(\tau,m,\epsilon)$ on $S_{\mathfrak{d}_p,\kappa,\delta_p,(c_2/\nu)^{1/\kappa}/2} \times \mathbb{R} \times D(0,\epsilon_0)$ and with $\mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_{p+1}}(\omega_{k_1}^{\mathfrak{d}_{p+1}})(\tau,m,\epsilon)$ on $S_{\mathfrak{d}_{p+1},\kappa,\delta_{p+1},(c_2/\nu)^{1/\kappa}/2} \times \mathbb{R} \times D(0,\epsilon_0)$.

Now, we put $\rho_{\nu,\kappa} = (c_2/\nu)^{1/\kappa}/2$. Using the fact that the function

$$u \mapsto \omega_{k_2}^{\mathfrak{d}_p,\mathfrak{d}_{p+1}}(u,m,\epsilon)\exp\left(-\left(\frac{u}{\epsilon t}\right)^{k_2}\right)/u$$

is holomorphic on $S_{\mathfrak{d}_p,\kappa,\delta_p,\rho_{\nu,\kappa}} \cup S_{\mathfrak{d}_{p+1},\kappa,\delta_{p+1},\rho_{\nu,\kappa}}$ for all $(m,\epsilon) \in \mathbb{R} \times D(0,\epsilon_0)$, its integral along the union of a segment starting from 0 to $(\rho_{\nu,\kappa}/2)e^{i\gamma_{p+1}}$, an arc of circle with radius $\rho_{\nu,\kappa}/2$ which connects $(\rho_{\nu,\kappa}/2)e^{i\gamma_{p+1}}$ and $(\rho_{\nu,\kappa}/2)e^{i\gamma_p}$, and a segment starting from $(\rho_{\nu,\kappa}/2)e^{i\gamma_p}$ to 0, is equal to zero. Therefore, we can write the difference $u^{\mathfrak{d}_{p+1}} - u^{\mathfrak{d}_p}$ as a sum of three integrals,

$$\begin{aligned}
&u^{\mathfrak{d}_{p+1}}(t,z,\epsilon) - u^{\mathfrak{d}_p}(t,z,\epsilon) \\
&\quad= \frac{k_2}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\rho_{\nu,\kappa}/2,\gamma_{p+1}}} \omega_{k_2}^{\mathfrak{d}_{p+1}}(u,m,\epsilon) e^{-(\frac{u}{\epsilon t})^{k_2}} e^{izm} \frac{du}{u}\,dm \\
&\qquad- \frac{k_2}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\rho_{\nu,\kappa}/2,\gamma_p}} \omega_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon) e^{-(\frac{u}{\epsilon t})^{k_2}} e^{izm} \frac{du}{u}\,dm \\
&\qquad+ \frac{k_2}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{C_{\rho_{\nu,\kappa}/2,\gamma_p,\gamma_{p+1}}} \omega_{k_2}^{\mathfrak{d}_p,\mathfrak{d}_{p+1}}(u,m,\epsilon) e^{-(\frac{u}{\epsilon t})^{k_2}} e^{izm} \frac{du}{u}\,dm,
\end{aligned} \tag{169}$$

where $L_{\rho_{\nu,\kappa}/2,\gamma_{p+1}} = [\rho_{\nu,\kappa}/2,+\infty)e^{i\gamma_{p+1}}$, $L_{\rho_{\nu,\kappa}/2,\gamma_p} = [\rho_{\nu,\kappa}/2,+\infty)e^{i\gamma_p}$, and $C_{\rho_{\nu,\kappa}/2,\gamma_p,\gamma_{p+1}}$ is an arc of circle with radius $\rho_{\nu,\kappa}/2$ connecting $(\rho_{\nu,\kappa}/2)e^{i\gamma_p}$ and $(\rho_{\nu,\kappa}/2)e^{i\gamma_{p+1}}$ with a well chosen orientation.

We give estimates for the quantity

$$I_1 = \left| \frac{k_2}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\rho_{\nu,\kappa}/2,\gamma_{p+1}}} \omega_{k_2}^{\mathfrak{d}_{p+1}}(u,m,\epsilon) e^{-(\frac{u}{\epsilon t})^{k_2}} e^{izm} \frac{du}{u}\,dm \right|.$$

By construction, the direction γ_{p+1} (which depends on ϵt) is chosen in such a way that $\cos(k_2(\gamma_{p+1} - \arg(\epsilon t))) \geq \delta_1$, for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, all $t \in \mathcal{T}$, for some fixed $\delta_1 > 0$. From the estimates (166), we find that

$$\begin{aligned}
I_1 &\leq \frac{k_2}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{\rho_{\nu,\kappa}/2}^{+\infty} C_{\omega_{k_2}^{\mathfrak{d}_{p+1}}} \bigl(1+|m|\bigr)^{-\mu} e^{-\beta|m|} \frac{r}{1+r^{2k_2}} \\
&\quad\times \exp\bigl(\nu' r^{k_2}\bigr) \exp\left(-\frac{\cos(k_2(\gamma_{p+1} - \arg(\epsilon t)))}{|\epsilon t|^{k_2}} r^{k_2}\right) e^{-m\,\mathrm{Im}(z)} \frac{dr}{r}\,dm \\
&\leq \frac{k_2 C_{\omega_{k_2}^{\mathfrak{d}_{p+1}}}}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|}\,dm \int_{\rho_{\nu,\kappa}/2}^{+\infty} \exp\left(-\left(\frac{\delta_1}{|t|^{k_2}} - \nu'|\epsilon|^{k_2}\right)\left(\frac{r}{|\epsilon|}\right)^{k_2}\right) dr \\
&\leq \frac{2k_2 C_{\omega_{k_2}^{\mathfrak{d}_{p+1}}}}{(2\pi)^{1/2}} \int_0^{+\infty} e^{-(\beta-\beta')m}\,dm \int_{\rho_{\nu,\kappa}/2}^{+\infty} \frac{|\epsilon|^{k_2}}{(\frac{\delta_1}{|t|^{k_2}} - \nu'|\epsilon|^{k_2}) k_2 (\frac{\rho_{\nu,\kappa}}{2})^{k_2-1}} \\
&\quad\times \frac{(\frac{\delta_1}{|t|^{k_2}} - \nu'|\epsilon|^{k_2}) k_2 r^{k_2-1}}{|\epsilon|^{k_2}} \exp\left(-\left(\frac{\delta_1}{|t|^{k_2}} - \nu'|\epsilon|^{k_2}\right)\left(\frac{r}{|\epsilon|}\right)^{k_2}\right) dr
\end{aligned}$$

$$\leq \frac{2k_2 C_{\omega_{k_2}^{\mathfrak{d}_{p+1}}}}{(2\pi)^{1/2}} \frac{|\epsilon|^{k_2}}{(\beta-\beta')(\frac{\delta_1}{|t|^{k_2}} - \nu'|\epsilon|^{k_2})k_2(\frac{\rho_{\nu,\kappa}}{2})^{k_2-1}}$$
$$\times \exp\left(-\left(\frac{\delta_1}{|t|^{k_2}} - \nu'|\epsilon|^{k_2}\right)\frac{(\rho_{\nu,\kappa}/2)^{k_2}}{|\epsilon|^{k_2}}\right)$$
$$\leq \frac{2k_2 C_{\omega_{k_2}^{\mathfrak{d}_{p+1}}}}{(2\pi)^{1/2}} \frac{|\epsilon|^{k_2}}{(\beta-\beta')\delta_2 k_2(\frac{\rho_{\nu,\kappa}}{2})^{k_2-1}} \exp\left(-\delta_2 \frac{(\rho_{\nu,\kappa}/2)^{k_2}}{|\epsilon|^{k_2}}\right) \tag{170}$$

for all $t \in \mathcal{T}$ and $|\mathrm{Im}(z)| \leq \beta'$ with $|t| < (\frac{\delta_1}{\delta_2+\nu'\epsilon_0^{k_2}})^{1/k_2}$, for some $\delta_2 > 0$, for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$.

In the same way, we also give estimates for the integral

$$I_2 = \left|\frac{k_2}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\rho_{\nu,\kappa}/2,\gamma_p}} \omega_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon) e^{-(\frac{u}{\epsilon t})^{k_2}} e^{izm} \frac{du}{u}\, dm\right|.$$

Namely, the direction γ_p (which depends on ϵt) is chosen in such a way that $\cos(k_2(\gamma_p - \arg(\epsilon t))) \geq \delta_1$, for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, all $t \in \mathcal{T}$, for some fixed $\delta_1 > 0$. Again from the estimates (166) and following the same steps as in (170), we find that

$$I_2 \leq \frac{2k_2 C_{\omega_{k_2}^{\mathfrak{d}_p}}}{(2\pi)^{1/2}} \frac{|\epsilon|^{k_2}}{(\beta-\beta')\delta_2 k_2(\frac{\rho_{\nu,\kappa}}{2})^{k_2-1}} \exp\left(-\delta_2 \frac{(\rho_{\nu,\kappa}/2)^{k_2}}{|\epsilon|^{k_2}}\right) \tag{171}$$

for all $t \in \mathcal{T}$ and $|\mathrm{Im}(z)| \leq \beta'$ with $|t| < (\frac{\delta_1}{\delta_2+\nu'\epsilon_0^{k_2}})^{1/k_2}$, for some $\delta_2 > 0$, for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$.

Finally, we give upper bound estimates for the integral

$$I_3 = \left|\frac{k_2}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{C_{\rho_{\nu,\kappa}/2,\gamma_p,\gamma_{p+1}}} \omega_{k_2}^{\mathfrak{d}_p,\mathfrak{d}_{p+1}}(u,m,\epsilon) e^{-(\frac{u}{\epsilon t})^{k_2}} e^{izm} \frac{du}{u}\, dm\right|.$$

By construction, the arc of circle $C_{\rho_{\nu,\kappa}/2,\gamma_p,\gamma_{p+1}}$ is chosen in such a way that $\cos(k_2(\theta - \arg(\epsilon t))) \geq \delta_1$, for all $\theta \in [\gamma_p, \gamma_{p+1}]$ (if $\gamma_p < \gamma_{p+1}$), $\theta \in [\gamma_{p+1}, \gamma_p]$ (if $\gamma_{p+1} < \gamma_p$), for all $t \in \mathcal{T}$, all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, for some fixed $\delta_1 > 0$. Bearing in mind (166), we find that

$$I_3 \leq \frac{k_2}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \left|\int_{\gamma_p}^{\gamma_{p+1}} \max\{C_{\omega_{k_2}^{\mathfrak{d}_p}}, C_{\omega_{k_2}^{\mathfrak{d}_{p+1}}}\}\big(1+|m|\big)^{-\mu} e^{-\beta|m|} \frac{\rho_{\nu,\kappa}/2}{1+(\rho_{\nu,\kappa}/2)^{2k_2}}\right.$$
$$\left.\times \exp\big(\nu'(\rho_{\nu,\kappa}/2)^{k_2}\big) \exp\left(-\frac{\cos(k_2(\theta - \arg(\epsilon t)))}{|\epsilon t|^{k_2}}\left(\frac{\rho_{\nu,\kappa}}{2}\right)^{k_2}\right) e^{-m\,\mathrm{Im}(z)}\, d\theta\right| dm$$
$$\leq \frac{k_2(\max\{C_{\omega_{k_2}^{\mathfrak{d}_p}}, C_{\omega_{k_2}^{\mathfrak{d}_{p+1}}}\})}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|}\, dm$$
$$\times |\gamma_p - \gamma_{p+1}| \frac{\rho_{\nu,\kappa}}{2} \exp\left(-\left(\frac{\delta_1}{|t|^{k_2}} - \nu'|\epsilon|^{k_2}\right)\left(\frac{\rho_{\nu,\kappa}/2}{|\epsilon|}\right)^{k_2}\right)$$
$$\leq \frac{2k_2(\max\{C_{\omega_{k_2}^{\mathfrak{d}_p}}, C_{\omega_{k_2}^{\mathfrak{d}_{p+1}}}\})}{(2\pi)^{1/2}(\beta-\beta')} |\gamma_p - \gamma_{p+1}| \frac{\rho_{\nu,\kappa}}{2} \exp\left(-\delta_2\left(\frac{\rho_{\nu,\kappa}/2}{|\epsilon|}\right)^{k_2}\right) \tag{172}$$

for all $t \in \mathcal{T}$ and $|\mathrm{Im}(z)| \leq \beta'$ with $|t| < (\frac{\delta_1}{\delta_2+\nu'\epsilon_0^{k_2}})^{1/k_2}$, for some $\delta_2 > 0$, for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$.

Finally, gathering the three above inequalities, (170), (171), and (172), we deduce from the decomposition (169) that

$$\begin{aligned}
&\left|u^{\mathfrak{d}_{p+1}}(t,z,\epsilon)-u^{\mathfrak{d}_p}(t,z,\epsilon)\right| \\
&\quad\leq \frac{2k_2(C_{\omega_{k_2}^{\mathfrak{d}_p}}+C_{\omega_{k_2}^{\mathfrak{d}_{p+1}}})}{(2\pi)^{1/2}}\frac{|\epsilon|^{k_2}}{(\beta-\beta')\delta_2 k_2(\frac{\rho_{\nu,\kappa}}{2})^{k_2-1}}\exp\left(-\delta_2\frac{(\rho_{\nu,\kappa}/2)^{k_2}}{|\epsilon|^{k_2}}\right) \\
&\qquad+\frac{2k_2(\max\{C_{\omega_{k_2}^{\mathfrak{d}_p}},C_{\omega_{k_2}^{\mathfrak{d}_{p+1}}}\})}{(2\pi)^{1/2}(\beta-\beta')}|\gamma_p-\gamma_{p+1}|\frac{\rho_{\nu,\kappa}}{2}\exp\left(-\delta_2\left(\frac{\rho_{\nu,\kappa}/2}{|\epsilon|}\right)^{k_2}\right)
\end{aligned}$$

for all $t\in\mathcal{T}$ and $|\mathrm{Im}(z)|\leq\beta'$ with $|t|<(\frac{\delta_1}{\delta_2+\nu'\epsilon_0^{k_2}})^{1/k_2}$, for some $\delta_2>0$, for all $\epsilon\in\mathcal{E}_p\cap\mathcal{E}_{p+1}$. Therefore, the inequality (162) holds.

In the last part of the proof, we show the estimates (163). Again, we only describe the arguments for the functions $u^{\mathfrak{d}_p}$ since exactly the same analysis can be made for the forcing term $f^{\mathfrak{d}_p}$ using the estimates (156) and (157) instead of (166) and (168).

Let $0\leq p\leq\varsigma-1$ such that $U_{\mathfrak{d}_p}\cap U_{\mathfrak{d}_{p+1}}=\emptyset$. We first consider the following.

Lemma 7 *There exist two constants $K_p^{\mathcal{A}},M_p^{\mathcal{A}}>0$ such that*

$$\begin{aligned}
&\left|\mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_{p+1}}(\omega_{k_1}^{\mathfrak{d}_{p+1}})(\tau,m,\epsilon)-\mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_p}(\omega_{k_1}^{\mathfrak{d}_p})(\tau,m,\epsilon)\right| \\
&\quad\leq K_p^{\mathcal{A}}\exp\left(-\frac{M_p^{\mathcal{A}}}{|\tau|^{\kappa}}\right)\left(1+|m|\right)^{-\mu}e^{-\beta|m|}
\end{aligned}\tag{173}$$

for all $\epsilon\in\mathcal{E}_{p+1}\cap\mathcal{E}_p$, all $\tau\in S_{\mathfrak{d}_{p+1},\kappa,\delta_{p+1},\rho_{\nu,\kappa}}\cap S_{\mathfrak{d}_p,\kappa,\delta_p,\rho_{\nu,\kappa}}$, all $m\in\mathbb{R}$.

Proof We first notice that the functions $\tau\mapsto\omega_{k_1}^{\mathfrak{d}_p}(\tau,m,\epsilon)$ and $\tau\mapsto\omega_{k_1}^{\mathfrak{d}_{p+1}}(\tau,m,\epsilon)$ are analytic continuations of the common m_{k_1}-Borel transform $\omega_{k_1}(\tau,m,\epsilon)=\sum_{n\geq1}U_n(m,\epsilon)\tau^n/\Gamma(n/k_1)$, which defines a continuous function on $D(0,\rho)\times\mathbb{R}\times D(0,\epsilon_0)$, holomorphic w.r.t. (τ,ϵ) on $D(0,\rho)\times D(0,\epsilon_0)$ for any $m\in\mathbb{R}$ with estimates: there exists a constant $C_{\omega_{k_1}}>0$ with

$$\left|\omega_{k_1}(\tau,m,\epsilon)\right|\leq C_{\omega_{k_1}}\left(1+|m|\right)^{-\mu}e^{-\beta|m|}\frac{|\tau|}{1+|\tau|^{2k_1}}e^{\nu|\tau|^{\kappa}}\tag{174}$$

for all $\tau\in D(0,\rho)$, all $m\in\mathbb{R}$, all $\epsilon\in D(0,\epsilon_0)$. From the proof of Proposition 13, we know that the function $G(\tau,h)$ is holomorphic w.r.t. $(\tau,h)\in\mathbb{C}^2$ whenever τ/h belongs to an open unbounded sector with direction $d=0$ and aperture π/κ. As a result, the integral of the function $h\mapsto\omega_{k_1}(h,m,\epsilon)G(\tau,h)/h$, for all $(m,\epsilon)\in\mathbb{R}\times D(0,\epsilon_0)$, all $\tau\in S_{\mathfrak{d}_{p+1},\kappa,\delta_{p+1},\rho_{\nu,\kappa}}\cap S_{\mathfrak{d}_p,\kappa,\delta_p,\rho_{\nu,\kappa}}$, along the union of a segment starting from 0 to $(\rho/2)e^{i\gamma_{p+1}^1}$, an arc of circle with radius $\rho/2$ which connects $(\rho/2)e^{i\gamma_{p+1}^1}$ and $(\rho/2)e^{i\gamma_p^1}$ and a segment starting from $(\rho/2)e^{i\gamma_p^1}$ to 0, is equal to zero. Therefore, we can write the difference $\mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_{p+1}}(\omega_{k_1}^{\mathfrak{d}_{p+1}})-\mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_p}(\omega_{k_1}^{\mathfrak{d}_p})$ as a sum of three integrals

$$\begin{aligned}
&\mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_{p+1}}(\omega_{k_1}^{\mathfrak{d}_{p+1}})(\tau,m,\epsilon)-\mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_p}(\omega_{k_1}^{\mathfrak{d}_p})(\tau,m,\epsilon) \\
&\quad=\int_{L_{\rho/2,\gamma_{p+1}^1}}\omega_{k_1}^{\mathfrak{d}_{p+1}}(h,m,\epsilon)G(\tau,h)\frac{dh}{h}
\end{aligned}$$

$$
\begin{aligned}
& - \int_{L_{\rho/2,\gamma_p^1}} \omega_{k_1}^{\mathfrak{d}_p}(h,m,\epsilon) G(\tau,h) \frac{dh}{h} \\
& + \int_{C_{\rho/2,\gamma_p^1,\gamma_{p+1}^1}} \omega_{k_1}(h,m,\epsilon) G(\tau,h) \frac{dh}{h},
\end{aligned} \tag{175}
$$

where $L_{\rho/2,\gamma_{p+1}^1} = [\rho/2,+\infty)e^{i\gamma_{p+1}^1}$, $L_{\rho/2,\gamma_p^1} = [\rho/2,+\infty)e^{i\gamma_p^1}$, and $C_{\rho/2,\gamma_p^1,\gamma_{p+1}^1}$ is an arc of circle with radius $\rho/2$ connecting $(\rho/2)e^{i\gamma_p^1}$ and $(\rho/2)e^{i\gamma_{p+1}^1}$ with a well-chosen orientation.

We give estimates for the quantity

$$
I_1^A = \left| \int_{L_{\rho/2,\gamma_{p+1}^1}} \omega_{k_1}^{\mathfrak{d}_{p+1}}(h,m,\epsilon) G(\tau,h) \frac{dh}{h} \right|.
$$

From the estimates (114) and (168), we find that

$$
\begin{aligned}
I_1^A &\leq \int_{\rho/2}^{+\infty} C_{\omega_{k_1}}^{\mathfrak{d}_{p+1}} \bigl(1+|m|\bigr)^{-\mu} e^{-\beta|m|} \frac{r}{1+r^{2k_1}} e^{\nu r^\kappa} c_1 \exp\left(-c_2 \left(\frac{r}{|\tau|}\right)^\kappa\right) \frac{dr}{r} \\
&\leq c_1 C_{\omega_{k_1}}^{\mathfrak{d}_{p+1}} \bigl(1+|m|\bigr)^{-\mu} e^{-\beta|m|} \int_{\rho/2}^{+\infty} \frac{|\tau|^\kappa}{(c_2 - |\tau|^\kappa \nu)\kappa(\rho/2)^{\kappa-1}} \frac{(c_2 - |\tau|^\kappa \nu)\kappa r^{\kappa-1}}{|\tau|^\kappa} \\
&\quad \times \exp\left(-\bigl(c_2 - |\tau|^\kappa \nu\bigr)\left(\frac{r}{|\tau|}\right)^\kappa\right) dr \\
&\leq c_1 C_{\omega_{k_1}}^{\mathfrak{d}_{p+1}} \bigl(1+|m|\bigr)^{-\mu} e^{-\beta|m|} \frac{|\tau|^\kappa}{(c_2 - |\tau|^\kappa \nu)\kappa(\rho/2)^{\kappa-1}} \exp\left(-\bigl(c_2 - |\tau|^\kappa \nu\bigr)\left(\frac{\rho/2}{|\tau|}\right)^\kappa\right) \\
&\leq c_1 C_{\omega_{k_1}}^{\mathfrak{d}_{p+1}} \bigl(1+|m|\bigr)^{-\mu} e^{-\beta|m|} \frac{|\tau|^\kappa}{c_2(1-\frac{1}{2^\kappa})\kappa(\rho/2)^{\kappa-1}} \\
&\quad \times \exp\left(-\left(c_2\left(1-\frac{1}{2^\kappa}\right)\left(\frac{\rho/2}{|\tau|}\right)^\kappa\right)\right)
\end{aligned} \tag{176}
$$

for all $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$, all $\tau \in S_{\mathfrak{d}_{p+1},\kappa,\delta_{p+1},\rho_{\nu,\kappa}} \cap S_{\mathfrak{d}_p,\kappa,\delta_p,\rho_{\nu,\kappa}}$, all $m \in \mathbb{R}$.

In the same way, we also give estimates for the integral

$$
I_2^A = \left| \int_{L_{\rho/2,\gamma_p^1}} \omega_{k_1}^{\mathfrak{d}_p}(h,m,\epsilon) G(\tau,h) \frac{dh}{h} \right|.
$$

Namely, from the estimates (114) and (168), following the same steps as above in (176), we find that

$$
\begin{aligned}
I_2^A &\leq c_1 C_{\omega_{k_1}}^{\mathfrak{d}_p} \bigl(1+|m|\bigr)^{-\mu} e^{-\beta|m|} \frac{|\tau|^\kappa}{c_2(1-\frac{1}{2^\kappa})\kappa(\rho/2)^{\kappa-1}} \\
&\quad \times \exp\left(-\left(c_2\left(1-\frac{1}{2^\kappa}\right)\left(\frac{\rho/2}{|\tau|}\right)^\kappa\right)\right)
\end{aligned} \tag{177}
$$

for all $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$, all $\tau \in S_{\mathfrak{d}_{p+1},\kappa,\delta_{p+1},\rho_{\nu,\kappa}} \cap S_{\mathfrak{d}_p,\kappa,\delta_p,\rho_{\nu,\kappa}}$, all $m \in \mathbb{R}$.

Finally, we give upper bound estimates for the integral

$$
I_3^A = \left| \int_{C_{\rho/2,\gamma_p^1,\gamma_{p+1}^1}} \omega_{k_1}(h,m,\epsilon) G(\tau,h) \frac{dh}{h} \right|.
$$

Bearing in mind (114) and (174), we find that

$$
\begin{aligned}
I_3^{\mathcal{A}} &\leq \left| \int_{\gamma_p^1}^{\gamma_{p+1}^1} C_{\omega_{k_1}} \big(1+|m|\big)^{-\mu} e^{-\beta|m|} \frac{\rho/2}{1+(\rho/2)^{2k_1}} e^{\nu(\rho/2)^\kappa} c_1 \exp\left(-c_2\left(\frac{\rho/2}{|\tau|}\right)^\kappa\right) d\theta \right| \\
&\leq c_1 C_{\omega_{k_1}} \frac{\rho}{2} \big|\gamma_p^1 - \gamma_{p+1}^1\big| \big(1+|m|\big)^{-\mu} e^{-\beta|m|} \exp\left(-\big(c_2 - |\tau|^\kappa \nu\big)\left(\frac{\rho/2}{|\tau|}\right)^\kappa\right) \\
&\leq c_1 C_{\omega_{k_1}} \frac{\rho}{2} \big|\gamma_p^1 - \gamma_{p+1}^1\big| \big(1+|m|\big)^{-\mu} e^{-\beta|m|} \exp\left(-\left(c_2\left(1-\frac{1}{2^\kappa}\right)\right)\left(\frac{\rho/2}{|\tau|}\right)^\kappa\right)
\end{aligned}
\tag{178}
$$

for all $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$, all $\tau \in S_{\mathfrak{d}_{p+1},\kappa,\delta_{p+1},\rho_{\nu,\kappa}} \cap S_{\mathfrak{d}_p,\kappa,\delta_p,\rho_{\nu,\kappa}}$, all $m \in \mathbb{R}$.

Finally, gathering the above inequalities, (176), (177), and (178), we deduce from the decomposition (175) that

$$
\begin{aligned}
&\big|\mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_{p+1}}\big(\omega_{k_1}^{\mathfrak{d}_{p+1}}\big)(\tau,m,\epsilon) - \mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_p}\big(\omega_{k_1}^{\mathfrak{d}_p}\big)(\tau,m,\epsilon)\big| \\
&\quad\leq c_1\big(C_{\omega_{k_1}}^{\mathfrak{d}_{p+1}} + C_{\omega_{k_1}}^{\mathfrak{d}_p}\big)\big(1+|m|\big)^{-\mu} e^{-\beta|m|} \frac{\rho_{\nu,\kappa}^\kappa}{c_2(1-\frac{1}{2^\kappa})\kappa(\rho/2)^{\kappa-1}} \\
&\qquad\times \exp\left(-\left(c_2\left(1-\frac{1}{2^\kappa}\right)\left(\frac{\rho/2}{|\tau|}\right)^\kappa\right)\right) \\
&\qquad+ c_1 C_{\omega_{k_1}} \frac{\rho}{2}\big|\gamma_p^1 - \gamma_{p+1}^1\big|\big(1+|m|\big)^{-\mu} e^{-\beta|m|} \exp\left(-\left(c_2\left(1-\frac{1}{2^\kappa}\right)\right)\left(\frac{\rho/2}{|\tau|}\right)^\kappa\right)
\end{aligned}
\tag{179}
$$

for all $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$, all $\tau \in S_{\mathfrak{d}_{p+1},\kappa,\delta_{p+1},\rho_{\nu,\kappa}} \cap S_{\mathfrak{d}_p,\kappa,\delta_p,\rho_{\nu,\kappa}}$, all $m \in \mathbb{R}$. We conclude that the inequality (173) holds. □

Using the analytic continuation property (167) and the fact that the functions $u \mapsto \omega_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon)\exp(-(\frac{u}{\epsilon t})^{k_2})/u$ (resp. $u \mapsto \omega_{k_2}^{\mathfrak{d}_{p+1}}(u,m,\epsilon)\exp(-(\frac{u}{\epsilon t})^{k_2})/u$) are holomorphic on $S_{\mathfrak{d}_p}^b \cup S_{\mathfrak{d}_p}$ (resp. on $S_{\mathfrak{d}_{p+1}}^b \cup S_{\mathfrak{d}_p}$), we can deform the straight lines of integration L_{γ_p} (resp. $L_{\gamma_{p+1}}$) in such a way that

$$
\begin{aligned}
&u^{\mathfrak{d}_{p+1}}(t,z,\epsilon) - u^{\mathfrak{d}_p}(t,z,\epsilon) \\
&\quad= \frac{k_2}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\rho_{\nu,\kappa}/2,\gamma_{p+1}}} \omega_{k_2}^{\mathfrak{d}_{p+1}}(u,m,\epsilon) \exp\left(-\left(\frac{u}{\epsilon t}\right)^{k_2}\right) e^{izm} \frac{du}{u}\, dm \\
&\qquad- \frac{k_2}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\rho_{\nu,\kappa}/2,\gamma_p}} \omega_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon) \exp\left(-\left(\frac{u}{\epsilon t}\right)^{k_2}\right) e^{izm} \frac{du}{u}\, dm \\
&\qquad+ \frac{k_2}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{C_{\rho_{\nu,\kappa}/2,\theta_{p,p+1},\gamma_{p+1}}} \omega_{k_2}^{\mathfrak{d}_{p+1}}(u,m,\epsilon) \exp\left(-\left(\frac{u}{\epsilon t}\right)^{k_2}\right) e^{izm} \frac{du}{u}\, dm \\
&\qquad- \frac{k_2}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{C_{\rho_{\nu,\kappa}/2,\theta_{p,p+1},\gamma_p}} \omega_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon) \exp\left(-\left(\frac{u}{\epsilon t}\right)^{k_2}\right) e^{izm} \frac{du}{u}\, dm \\
&\qquad+ \frac{k_2}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{0,\rho_{\nu,\kappa}/2,\theta_{p,p+1}}} \big(\mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_{p+1}}\big(\omega_{k_1}^{\mathfrak{d}_{p+1}}\big)(u,m,\epsilon) - \mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_p}\big(\omega_{k_1}^{\mathfrak{d}_p}\big)(u,m,\epsilon)\big) \\
&\qquad\times \exp\left(-\left(\frac{u}{\epsilon t}\right)^{k_2}\right) e^{izm} \frac{du}{u}\, dm,
\end{aligned}
\tag{180}
$$

where $L_{\rho_{\nu,\kappa}/2,\gamma_{p+1}} = [\rho_{\nu,\kappa}/2,+\infty)e^{\sqrt{-1}\gamma_{p+1}}$, $L_{\rho_{\nu,\kappa}/2,\gamma_p} = [\rho_{\nu,\kappa}/2,+\infty)e^{\sqrt{-1}\gamma_p}$, $C_{\rho_{\nu,\kappa}/2,\theta_{p,p+1},\gamma_{p+1}}$ is an arc of circle with radius $\rho_{\nu,\kappa}/2$, connecting $(\rho_{\nu,\kappa}/2)e^{\sqrt{-1}\theta_{p,p+1}}$ and $(\rho_{\nu,\kappa}/2)e^{\sqrt{-1}\gamma_{p+1}}$ with a well-chosen orientation, where $\theta_{p,p+1}$ denotes the bisecting direction of the sector $S_{\mathfrak{d}_{p+1},\kappa,\delta_{p+1},\rho_{\nu,\kappa}} \cap S_{\mathfrak{d}_p,\kappa,\delta_p,\rho_{\nu,\kappa}}$ and likewise $C_{\rho_{\nu,\kappa}/2,\theta_{p,p+1},\gamma_p}$ is an arc of circle with radius $\rho_{\nu,\kappa}/2$, connecting the points $(\rho_{\nu,\kappa}/2)e^{\sqrt{-1}\theta_{p,p+1}}$ and $(\rho_{\nu,\kappa}/2)e^{\sqrt{-1}\gamma_p}$ with a well chosen orientation and finally $L_{0,\rho_{\nu,\kappa}/2,\theta_{p,p+1}} = [0,\rho_{\nu,\kappa}/2]e^{\sqrt{-1}\theta_{p,p+1}}$.

Following the same lines of arguments as in the estimates (170) and (172), we get the inequalities

$$\begin{aligned}
J_1 &= \left|\frac{k_2}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_{\rho_{\nu,\kappa}/2,\gamma_{p+1}}} \omega_{k_2}^{\mathfrak{d}_{p+1}}(u,m,\epsilon)\exp\left(-\left(\frac{u}{\epsilon t}\right)^{k_2}\right)e^{izm}\frac{du}{u}\,dm\right| \\
&\leq \frac{2k_2 C_{\omega_{k_2}^{\mathfrak{d}_{p+1}}}}{(2\pi)^{1/2}}\frac{|\epsilon|^{k_2}}{(\beta-\beta')\delta_2 k_2(\frac{\rho_{\nu,\kappa}}{2})^{k_2-1}}\exp\left(-\delta_2\frac{(\rho_{\nu,\kappa}/2)^{k_2}}{|\epsilon|^{k_2}}\right), \\
J_2 &= \left|\frac{k_2}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_{\rho_{\nu,\kappa}/2,\gamma_p}} \omega_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon)\exp\left(-\left(\frac{u}{\epsilon t}\right)^{k_2}\right)e^{izm}\frac{du}{u}\,dm\right| \\
&\leq \frac{2k_2 C_{\omega_{k_2}^{\mathfrak{d}_p}}}{(2\pi)^{1/2}}\frac{|\epsilon|^{k_2}}{(\beta-\beta')\delta_2 k_2(\frac{\rho_{\nu,\kappa}}{2})^{k_2-1}}\exp\left(-\delta_2\frac{(\rho_{\nu,\kappa}/2)^{k_2}}{|\epsilon|^{k_2}}\right), \\
J_3 &= \left|\frac{k_2}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{C_{\rho_{\nu,\kappa}/2,\theta_{p,p+1},\gamma_{p+1}}} \omega_{k_2}^{\mathfrak{d}_{p+1}}(u,m,\epsilon)\exp\left(-\left(\frac{u}{\epsilon t}\right)^{k_2}\right)e^{izm}\frac{du}{u}\,dm\right| \\
&\leq \frac{2k_2 C_{\omega_{k_2}^{\mathfrak{d}_{p+1}}}}{(2\pi)^{1/2}(\beta-\beta')}|\gamma_{p+1}-\theta_{p,p+1}|\frac{\rho_{\nu,\kappa}}{2}\exp\left(-\delta_2\left(\frac{\rho_{\nu,\kappa}/2}{|\epsilon|}\right)^{k_2}\right), \\
J_4 &= \left|\frac{k_2}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{C_{\rho_{\nu,\kappa}/2,\theta_{p,p+1},\gamma_p}} \omega_{k_2}^{\mathfrak{d}_p}(u,m,\epsilon)\exp\left(-\left(\frac{u}{\epsilon t}\right)^{k_2}\right)e^{izm}\frac{du}{u}\,dm\right| \\
&\leq \frac{2k_2 C_{\omega_{k_2}^{\mathfrak{d}_p}}}{(2\pi)^{1/2}(\beta-\beta')}|\gamma_p-\theta_{p,p+1}|\frac{\rho_{\nu,\kappa}}{2}\exp\left(-\delta_2\left(\frac{\rho_{\nu,\kappa}/2}{|\epsilon|}\right)^{k_2}\right)
\end{aligned} \tag{181}$$

for all $t \in \mathcal{T}$ and $|\mathrm{Im}(z)| \leq \beta'$ with $|t| < (\frac{\delta_1}{\delta_2+\nu'\epsilon_0^{k_2}})^{1/k_2}$, for some $\delta_1,\delta_2 > 0$, for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$.

In the last part of the proof, it remains to give upper bounds for the integral

$$\begin{aligned}
J_5 = \Bigg|\frac{k_2}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_{0,\rho_{\nu,\kappa}/2,\theta_{p,p+1}}} &\left(\mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_{p+1}}(\omega_{k_1}^{\mathfrak{d}_{p+1}})(u,m,\epsilon) - \mathrm{Acc}_{k_2,k_1}^{\mathfrak{d}_p}(\omega_{k_1}^{\mathfrak{d}_p})(u,m,\epsilon)\right) \\
&\times \exp\left(-\left(\frac{u}{\epsilon t}\right)^{k_2}\right)e^{izm}\frac{du}{u}\,dm\Bigg|.
\end{aligned}$$

By construction, there exists $\delta_1 > 0$ such that $\cos(k_2(\theta_{p,p+1} - \arg(\epsilon t))) \geq \delta_1$ for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, all $t \in \mathcal{T}$. From Lemma 7, we find that

$$\begin{aligned}
J_5 &\leq \frac{k_2}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_0^{\rho_{\nu,\kappa}/2} K_p^{\mathcal{A}}\big(1+|m|\big)^{-\mu}e^{-\beta|m|}\exp\left(-\frac{M_p^{\mathcal{A}}}{r^\kappa}\right) \\
&\quad\times \exp\left(-\frac{\cos(k_2(\theta_{p,p+1}-\arg(\epsilon t)))}{|\epsilon t|^{k_2}}r^{k_2}\right)e^{-m\,\mathrm{Im}(z)}\frac{dr}{r}\,dm \\
&\leq \frac{k_2 K_p^{\mathcal{A}}}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|}\,dm \times J_5(\epsilon t),
\end{aligned} \tag{182}$$

where

$$J_5(\epsilon t) = \int_0^{\rho_{\nu,\kappa}/2} \exp\left(-\frac{M_p^{\mathcal{A}}}{r^{\kappa}}\right) \exp\left(-\frac{\delta_1}{|\epsilon t|^{k_2}} r^{k_2}\right) \frac{dr}{r}. \tag{183}$$

The study of estimates for $J_5(\epsilon t)$ as ϵ tends to zero rests on the following two lemmas.

Lemma 8 (Watson's lemma. Exercise 4, p.16 in [2]) *Let $b > 0$ and $f : [0,b] \to \mathbb{C}$ be a continuous function having the formal expansion $\sum_{n\geq 0} a_n t^n \in \mathbb{C}[[t]]$ as its asymptotic expansion of Gevrey order $\kappa > 0$ at 0, meaning there exist $C, M > 0$ such that*

$$\left| f(t) - \sum_{n=0}^{N-1} a_n t^n \right| \leq C M^N N!^{\kappa} |t|^N,$$

for every $N \geq 1$ and $t \in [0,\delta]$, for some $0 < \delta < b$. Then the function

$$I(x) = \int_0^b f(s) e^{-\frac{s}{x}}\, ds$$

admits the formal power series $\sum_{n\geq 0} a_n n! x^{n+1} \in \mathbb{C}[[x]]$ as its asymptotic expansion of Gevrey order $\kappa + 1$ at 0, it is to say, there exist $\tilde{C}, \tilde{K} > 0$ such that

$$\left| I(x) - \sum_{n=0}^{N-1} a_n n! x^{n+1} \right| \leq \tilde{C} \tilde{K}^{N+1} (N+1)!^{1+\kappa} |x|^{N+1},$$

for every $N \geq 0$ and $x \in [0,\delta']$ for some $0 < \delta' < b$.

Lemma 9 (Exercise 3, p.18 in [2]) *Let $\delta, q > 0$, and $\psi : [0,\delta] \to \mathbb{C}$ be a continuous function. The following assertions are equivalent*:

1. *There exist $C, M > 0$ such that $|\psi(x)| \leq C M^n n!^q |x|^n$, for every $n \in \mathbb{N}$, $n \geq 0$, and $x \in [0,\delta]$.*
2. *There exist $C', M' > 0$ such that $|\psi(x)| \leq C' e^{-M'/x^{\frac{1}{q}}}$, for every $x \in (0,\delta]$.*

We make the change of variable $r^{k_2} = s$ in the integral (183) and we get

$$J_5(\epsilon t) = \frac{1}{k_2} \int_0^{(\rho_{\nu,\kappa}/2)^{k_2}} \exp\left(-\frac{M_p^{\mathcal{A}}}{s^{\kappa/k_2}}\right) \exp\left(-\frac{\delta_1}{|\epsilon t|^{k_2}} s\right) \frac{ds}{s}.$$

We put $\psi_{\mathcal{A},p}(s) = \exp(-\frac{M_p^{\mathcal{A}}}{s^{\kappa/k_2}})/s$. From Lemma 9, there exist constants $C, M > 0$ such that

$$\left|\psi_{\mathcal{A},p}(s)\right| \leq C M^n (n!)^{\frac{k_2}{\kappa}} |s|^n$$

for all $n \geq 0$, all $s \in [0, (\rho_{\nu,\kappa}/2)^{k_2}]$. In other words, $\psi_{\mathcal{A},p}(s)$ admits the null formal series $\hat{0} \in \mathbb{C}[[s]]$ as asymptotic expansion of Gevrey order k_2/κ on $[0, (\rho_{\nu,\kappa}/2)^{k_2}]$. By Lemma 8, we deduce that the function

$$I_{\mathcal{A},p}(x) = \int_0^{(\rho_{\nu,\kappa}/2)^{k_2}} \psi_{\mathcal{A},p}(s) e^{-\frac{s}{x}}\, ds$$

has the formal series $\hat{0} \in \mathbb{C}[[x]]$ as asymptotic expansion of Gevrey order $\frac{k_2}{\kappa} + 1 = \frac{k_2}{k_1}$ on some segment $[0,\delta']$ with $0 < \delta' < (\rho_{\nu,\kappa}/2)^{k_2}$. Hence, using again Lemma 9, we get two constants $C', M' > 0$ with

$$I_{\mathcal{A},p}(x) \leq C' \exp\left(-\frac{M'}{x^{k_1/k_2}}\right)$$

for $x \in [0,\delta']$. We deduce the existence of two constants $C_{J_5} > 0$, $M_{J_5} > 0$ with

$$J_5(\epsilon t) \leq C_{J_5} \exp\left(-\frac{M_{J_5}}{|\epsilon t|^{k_1}}\right) \tag{184}$$

for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, all $t \in \mathcal{T} \cap D(0,h_{\mathcal{A},p})$, for some $h_{\mathcal{A},p} > 0$. Gathering the last inequality, (184), and (182) yields

$$J_5 \leq \frac{2C_{J_5}k_2K_p^{\mathcal{A}}}{(2\pi)^{1/2}(\beta-\beta')} \exp\left(-\frac{M_{J_5}}{h_{\mathcal{A},p}^{k_1}|\epsilon|^{k_1}}\right) \tag{185}$$

for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, all $t \in \mathcal{T} \cap D(0,h_{\mathcal{A},p})$.

In conclusion, taking into account the above inequalities (181) and (185), we deduce from the decomposition (180) that

$$\begin{aligned}
&\left|u^{\mathfrak{d}_{p+1}}(t,z,\epsilon) - u^{\mathfrak{d}_p}(t,z,\epsilon)\right| \\
&\quad\leq \frac{2k_2(C_{\omega_{k_2}^{\mathfrak{d}_{p+1}}} + C_{\omega_{k_2}^{\mathfrak{d}_p}})}{(2\pi)^{1/2}} \frac{|\epsilon|^{k_2}}{(\beta-\beta')\delta_2 k_2 (\frac{\rho_{\nu,\kappa}}{2})^{k_2-1}} \exp\left(-\delta_2 \frac{(\rho_{\nu,\kappa}/2)^{k_2}}{|\epsilon|^{k_2}}\right) \\
&\qquad + \frac{2k_2}{(2\pi)^{1/2}(\beta-\beta')}\left(C_{\omega_{k_2}^{\mathfrak{d}_{p+1}}}|\gamma_{p+1}-\theta_{p,p+1}| + C_{\omega_{k_2}^{\mathfrak{d}_p}}|\gamma_p - \theta_{p,p+1}|\right)\frac{\rho_{\nu,\kappa}}{2} \\
&\qquad \times \exp\left(-\delta_2\left(\frac{\rho_{\nu,\kappa}/2}{|\epsilon|}\right)^{k_2}\right) \\
&\qquad + \frac{2C_{J_5}k_2K_p^{\mathcal{A}}}{(2\pi)^{1/2}(\beta-\beta')} \exp\left(-\frac{M_{J_5}}{h_{\mathcal{A},p}^{k_1}|\epsilon|^{k_1}}\right)
\end{aligned}$$

for all $t \in \mathcal{T}$ with $|t| < (\frac{\delta_1}{\delta_2+\nu'\epsilon_0^{k_2}})^{1/k_2}$ and $|t| \leq h_{\mathcal{A},p}$ for some constants $\delta_1, \delta_2, h_{\mathcal{A},p} > 0$, $|\mathrm{Im}(z)| \leq \beta'$, for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$. Therefore the inequality (163) holds. □

6 Existence of formal series solutions in the complex parameter and asymptotic expansion in two levels

6.1 Summable and multisummable formal series and a Ramis-Sibuya theorem with two levels

In the next definitions we recall the meaning of Gevrey asymptotic expansions for holomorphic functions and k-summability. We also give the signification of (k_2,k_1)-summability for power series in a Banach space, as described in [2].

Definition 9 Let $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ be a complex Banach space and let $\mathcal{E}$ be a bounded open sector centered at 0. Let $k > 0$ be a positive real number. We say that a holomorphic function $f : \mathcal{E} \to \mathbb{E}$ admits a formal power series $\hat{f}(\epsilon) = \sum_{n\geq 0} a_n\epsilon^n \in \mathbb{E}[[\epsilon]]$ as its asymptotic expansion of Gevrey order $1/k$ if, for any closed proper subsector $\mathcal{W} \subset \mathcal{E}$ centered at 0, there exist

$C, M > 0$ with

$$\left\| f(\epsilon) - \sum_{n=0}^{N-1} a_n \epsilon^n \right\|_{\mathbb{E}} \leq CM^N (N!)^{1/k} |\epsilon|^N \tag{186}$$

for all $N \geq 1$, all $\epsilon \in \mathcal{W}$.

If, moreover, the aperture of $\mathcal{E}$ is larger than $\frac{\pi}{k} + \delta$ for some $\delta > 0$, then the function f is the unique holomorphic function on $\mathcal{E}$ satisfying (186). In that case, we say that $\hat{f}$ is k-summable on $\mathcal{E}$ and that f defines its k-sum on $\mathcal{E}$. In addition, the function f can be reconstructed from the analytic continuation of the k_1-Borel transform

$$\hat{\mathcal{B}}_{k_1} \hat{f}(\tau) = \sum_{n \geq 0} a_n \frac{\tau^n}{\Gamma(1 + \frac{n}{k_1})}$$

on an unbounded sector and by applying a k_1-Laplace transform to it; see Section 3.2 from [2].

Definition 10 Let $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ be a complex Banach space and let $0 < k_1 < k_2$ be two positive real numbers. Let $\mathcal{E}$ be a bounded open sector centered at 0 with aperture $\frac{\pi}{k_2} + \delta_2$ for some $\delta_2 > 0$ and let $\mathcal{F}$ be a bounded open sector centered at 0 with aperture $\frac{\pi}{k_1} + \delta_1$ for some $\delta_1 > 0$ such that the inclusion $\mathcal{E} \subset \mathcal{F}$ holds.

A formal power series $\hat{f}(\epsilon) = \sum_{n \geq 0} a_n \epsilon^n \in \mathbb{E}[[\epsilon]]$ is said to be (k_2, k_1)-summable on $\mathcal{E}$ if there exist a formal series $\hat{f}_2(\epsilon) \in \mathbb{E}[[\epsilon]]$ which is k_2-summable on $\mathcal{E}$ with k_2-sum $f_2 : \mathcal{E} \to \mathbb{E}$ and a second formal series $\hat{f}_1(\epsilon) \in \mathbb{E}[[\epsilon]]$ which is k_1-summable on $\mathcal{F}$ with k_1-sum $f_1 : \mathcal{F} \to \mathbb{E}$ such that $\hat{f} = \hat{f}_1 + \hat{f}_2$. Furthermore, the holomorphic function $f(\epsilon) = f_1(\epsilon) + f_2(\epsilon)$ defined on $\mathcal{E}$ is called the (k_2, k_1)-sum of $\hat{f}$ on $\mathcal{E}$. In that case, the function $f(\epsilon)$ can be reconstructed from the analytic continuation of the k_1-Borel transform of $\hat{f}$ by applying successively some acceleration operator and Laplace transform of order k_2; see Section 6.1 from [2].

In this section, we state a version of the classical Ramis-Sibuya theorem (see [28], Theorem XI-2-3) with two different Gevrey levels which describes also the case when multisummability holds on some sector. We mention that a similar multi-level version of the Ramis-Sibuya theorem has already been stated in [22] and also in a previous work of the authors; see [29].

Theorem (RS) *Let $0 < k_1 < k_2$ be positive real numbers. Let $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ be a Banach space over $\mathbb{C}$ and $\{\mathcal{E}_i\}_{0 \leq i \leq \nu-1}$ be a good covering in $\mathbb{C}^*$; see Definition 7. For all $0 \leq i \leq \nu - 1$, let G_i be a holomorphic function from $\mathcal{E}_i$ into the Banach space $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ and let the cocycle $\Delta_i(\epsilon) = G_{i+1}(\epsilon) - G_i(\epsilon)$ be a holomorphic function from the sector $Z_i = \mathcal{E}_{i+1} \cap \mathcal{E}_i$ into $\mathbb{E}$ (with the convention that $\mathcal{E}_\nu = \mathcal{E}_0$ and $G_\nu = G_0$). We make the following assumptions.*

(1) *The functions $G_i(\epsilon)$ are bounded as $\epsilon \in \mathcal{E}_i$ tends to the origin in $\mathbb{C}$, for all $0 \leq i \leq \nu - 1$.*

(2) *For some finite subset $I_1 \subset \{0, \dots, \nu - 1\}$ and for all $i \in I_1$, the functions $\Delta_i(\epsilon)$ are exponentially flat on Z_i of order k_1, for all $0 \leq i \leq \nu - 1$. This means that there exist constants $K_i, M_i > 0$ such that*

$$\|\Delta_i(\epsilon)\|_{\mathbb{E}} \leq K_i \exp\left(-\frac{M_i}{|\epsilon|^{k_1}}\right) \tag{187}$$

for all $\epsilon \in Z_i$.

(3) *For all* $i \in I_2 = \{0, \dots, \nu - 1\} \setminus I_1$, *the functions* $\Delta_i(\epsilon)$ *are exponentially flat of order* k_2 *on* Z_i, *for all* $0 \leq i \leq \nu - 1$. *This means that there exist constants* $K_i, M_i > 0$ *such that*

$$\|\Delta_i(\epsilon)\|_{\mathbb{E}} \leq K_i \exp\left(-\frac{M_i}{|\epsilon|^{k_2}}\right) \tag{188}$$

for all $\epsilon \in Z_i$.

Then there exist a convergent power series $a(\epsilon) \in \mathbb{E}\{\epsilon\}$ *near* $\epsilon = 0$ *and two formal series* $\hat{G}^1(\epsilon), \hat{G}^2(\epsilon) \in \mathbb{E}[[\epsilon]]$ *such that* $G_i(\epsilon)$ *obeys the following decomposition*:

$$G_i(\epsilon) = a(\epsilon) + G_i^1(\epsilon) + G_i^2(\epsilon), \tag{189}$$

where $G_i^1(\epsilon)$ *is holomorphic on* $\mathcal{E}_i$ *and has* $\hat{G}^1(\epsilon)$ *as asymptotic expansion of Gevrey order* $1/k_1$ *on* $\mathcal{E}_i$, $G_i^2(\epsilon)$ *is holomorphic on* $\mathcal{E}_i$ *and carries* $\hat{G}^2(\epsilon)$ *as asymptotic expansion of Gevrey order* $1/k_2$ *on* $\mathcal{E}_i$, *for all* $0 \leq i \leq \nu - 1$.

Assume, moreover, that some integer $i_0 \in I_2$ *is such that* $I_{\delta_1,i_0,\delta_2} = \{i_0 - \delta_1, \dots, i_0, \dots, i_0 + \delta_2\} \subset I_2$ *for some integers* $\delta_1, \delta_2 \geq 0$ *and with the property that*

$$\mathcal{E}_{i_0} \subset S_{\pi/k_1} \subset \bigcup_{h \in I_{\delta_1,i_0,\delta_2}} \mathcal{E}_h, \tag{190}$$

where S_{π/k_1} *is a sector centered at* 0 *with aperture a bit larger than* π/k_1. *Then the formal series* $\hat{G}(\epsilon)$ *is* (k_2, k_1)*-summable on* $\mathcal{E}_{i_0}$ *and its* (k_2, k_1)*-sum is* $G_{i_0}(\epsilon)$ *on* $\mathcal{E}_{i_0}$.

Proof We consider two holomorphic cocycles $\Delta_i^1(\epsilon)$ and $\Delta_i^2(\epsilon)$ defined on the sectors Z_i in the following way:

$$\Delta_i^1(\epsilon) = \begin{cases} \Delta_i(\epsilon) & \text{if } i \in I_1, \\ 0 & \text{if } i \in I_2, \end{cases} \qquad \Delta_i^2(\epsilon) = \begin{cases} 0 & \text{if } i \in I_1, \\ \Delta_i(\epsilon) & \text{if } i \in I_2 \end{cases}$$

for all $\epsilon \in Z_i$, all $0 \leq i \leq \nu - 1$. We need the following lemma.

Lemma 10 (1) *For all* $0 \leq i \leq \nu - 1$, *there exist bounded holomorphic functions* $\Psi_i^1 : \mathcal{E}_i \to \mathbb{C}$ *such that*

$$\Delta_i^1(\epsilon) = \Psi_{i+1}^1(\epsilon) - \Psi_i^1(\epsilon) \tag{191}$$

for all $\epsilon \in Z_i$, *where by convention* $\Psi_\nu^1(\epsilon) = \Psi_0^1(\epsilon)$. *Moreover, there exist coefficients* $\varphi_m^1 \in \mathbb{E}$, $m \geq 0$, *such that, for each* $0 \leq l \leq \nu - 1$ *and any closed proper subsector* $\mathcal{W} \subset \mathcal{E}_l$, *centered at* 0, *there exist two constants* $\check{K}_l, \check{M}_l > 0$ *with*

$$\left\| \Psi_l^1(\epsilon) - \sum_{m=0}^{M-1} \varphi_m^1 \epsilon^m \right\|_{\mathbb{E}} \leq \check{K}_l (\check{M}_l)^M (M!)^{1/k_1} |\epsilon|^M \tag{192}$$

for all $\epsilon \in \mathcal{W}$, *all* $M \geq 1$.

(2) *For all* $0 \leq i \leq \nu - 1$, *there exist bounded holomorphic functions* $\Psi_i^2 : \mathcal{E}_i \to \mathbb{C}$ *such that*

$$\Delta_i^2(\epsilon) = \Psi_{i+1}^2(\epsilon) - \Psi_i^2(\epsilon) \tag{193}$$

for all $\epsilon \in Z_i$, *where by convention* $\Psi_\nu^2(\epsilon) = \Psi_0^2(\epsilon)$. *Moreover, there exist coefficients* $\varphi_m^2 \in \mathbb{E}$, $m \geq 0$, *such that, for each* $0 \leq l \leq \nu - 1$ *and any closed proper subsector* $\mathcal{W} \subset \mathcal{E}_l$, *centered at* 0, *there exist two constants* $\hat{K}_l, \hat{M}_l > 0$ *with*

$$\left\| \Psi_l^2(\epsilon) - \sum_{m=0}^{M-1} \varphi_m^2 \epsilon^m \right\|_{\mathbb{E}} \leq \hat{K}_l (\hat{M}_l)^M (M!)^{1/k_2} |\epsilon|^M \tag{194}$$

for all $\epsilon \in \mathcal{W}$, *all* $M \geq 1$.

Proof The proof is a consequence of Lemma XI-2-6 from [28], which provides the so-called classical Ramis-Sibuya theorem in Gevrey classes. □

We consider now the bounded holomorphic functions

$$a_i(\epsilon) = G_i(\epsilon) - \Psi_i^1(\epsilon) - \Psi_i^2(\epsilon)$$

for all $0 \leq i \leq \nu - 1$, all $\epsilon \in \mathcal{E}_i$. By definition, for $i \in I_1$ or $i \in I_2$, we have

$$a_{i+1}(\epsilon) - a_i(\epsilon) = G_{i+1}(\epsilon) - G_i(\epsilon) - \Delta_i^1(\epsilon) - \Delta_i^2(\epsilon) = G_{i+1}(\epsilon) - G_i(\epsilon) - \Delta_i(\epsilon) = 0$$

for all $\epsilon \in Z_i$. Therefore, each $a_i(\epsilon)$ is the restriction on $\mathcal{E}_i$ of a holomorphic function $a(\epsilon)$ on $D(0,r) \setminus \{0\}$. Since $a(\epsilon)$ is, moreover, bounded on $D(0,r) \setminus \{0\}$, the origin turns out to be a removable singularity for $a(\epsilon)$, which, as a consequence, defines a convergent power series on $D(0,r)$.

Finally, one can write the following decomposition:

$$G_i(\epsilon) = a(\epsilon) + \Psi_i^1(\epsilon) + \Psi_i^2(\epsilon)$$

for all $\epsilon \in \mathcal{E}_i$, all $0 \leq i \leq \nu - 1$. Moreover, $a(\epsilon)$ is a convergent power series and from (192) we know that $\Psi_i^1(\epsilon)$ has the series $\hat{G}^1(\epsilon) = \sum_{m\geq 0} \varphi_m^1 \epsilon^m$ as asymptotic expansion of Gevrey order $1/k_1$ on $\mathcal{E}_i$ and due to (194) $\Psi_i^2(\epsilon)$ carries the series $\hat{G}^2(\epsilon) = \sum_{m\geq 0} \varphi_m^2 \epsilon^m$ as asymptotic expansion of Gevrey order $1/k_2$ on $\mathcal{E}_i$, for all $0 \leq i \leq \nu - 1$. Therefore, the decomposition (189) holds.

Assume now that some integer $i_0 \in I_2$ is such that $I_{\delta_1,i_0,\delta_2} = \{i_0 - \delta_1, \ldots, i_0, \ldots, i_0 + \delta_2\} \subset I_2$ for some integers $\delta_1, \delta_2 \geq 0$ and with the property (190). Then, in the decomposition (189), we observe from the construction above that the function $G_{i_0}^1(\epsilon)$ can be analytically continued on the sector S_{π/k_1} and has the formal series $\hat{G}^1(\epsilon)$ as asymptotic expansion of Gevrey order $1/k_1$ on S_{π/k_1} (this is the consequence of the fact that $\Delta_h^1(\epsilon) = 0$ for $h \in I_{\delta_1,i_0,\delta_2}$). Hence, $G_{i_0}^1(\epsilon)$ is the k_1-sum of $\hat{G}^1(\epsilon)$ on S_{π/k_1} in the sense of Definition 9. Moreover, we already know that the function $G_{i_0}^2(\epsilon)$ has $\hat{G}^2(\epsilon)$ as an asymptotic expansion of Gevrey order $1/k_2$ on $\mathcal{E}_{i_0}$, meaning that $G_{i_0}^2(\epsilon)$ is the k_2-sum of $\hat{G}^2(\epsilon)$ on $\mathcal{E}_{i_0}$. In other words, by Definition 10, the formal series $\hat{G}(\epsilon)$ is (k_2,k_1)-summable on $\mathcal{E}_{i_0}$ and its (k_2,k_1)-sum is the function $G_{i_0}(\epsilon) = a(\epsilon) + G_{i_0}^1(\epsilon) + G_{i_0}^2(\epsilon)$ on $\mathcal{E}_{i_0}$. □

6.2 Construction of formal power series solutions in the complex parameter with two levels of asymptotics

In this subsection, we establish the second main result of our work, namely the existence of a formal power series $\hat{u}(t,z,\epsilon)$ in the parameter ϵ whose coefficients are bounded holomorphic functions on the product of a sector with small radius centered at 0 and a strip in $\mathbb{C}^2$ that is a solution of (195) and which is the common Gevrey asymptotic expansion of order $1/k_1$ of the actual solutions $u^{\mathfrak{d}_p}(t,z,\epsilon)$ of (154) constructed in Theorem 1. Furthermore, this formal series $\hat{u}$ and the corresponding functions $u^{\mathfrak{d}_p}$ have a fine structure which involves two levels of Gevrey asymptotics.

We first start by showing that the forcing terms $f^{\mathfrak{d}_p}(t,z,\epsilon)$ share a common formal power series $\hat{f}(t,z,\epsilon)$ in ϵ as asymptotic expansion of Gevrey order $1/k_1$ on $\mathcal{E}_p$.

Lemma 11 *Let us assume that the hypotheses of Theorem* 1 *hold. Then there exists a formal power series*

$$\hat{f}(t,z,\epsilon) = \sum_{m\geq 0} f_m(t,z)\epsilon^m/m!$$

whose coefficients $f_m(t,z)$ belong to the Banach space $\mathbb{F}$ of bounded holomorphic functions on $(\mathcal{T}\cap D(0,h''))\times H_{\beta'}$ equipped with supremum norm, where $h''>0$ is constructed in Theorem 1, *which is the common asymptotic expansion of Gevrey order $1/k_1$ on $\mathcal{E}_p$ of the functions $f^{\mathfrak{d}_p}$, seen as holomorphic functions from $\mathcal{E}_p$ into $\mathbb{F}$, for all $0\leq p\leq \varsigma-1$.*

Proof We consider the family of functions $f^{\mathfrak{d}_p}(t,z,\epsilon)$, $0\leq p\leq\varsigma-1$ constructed in (159). For all $0\leq p\leq\varsigma-1$, we define $G_p^f(\epsilon) := (t,z)\mapsto f^{\mathfrak{d}_p}(t,z,\epsilon)$, which is by construction a holomorphic and bounded function from $\mathcal{E}_p$ into the Banach space $\mathbb{F}$ of bounded holomorphic functions on $(\mathcal{T}\cap D(0,h''))\times H_{\beta'}$ equipped with the supremum norm, where $\mathcal{T}$ is introduced in Definition 8 and $h''>0$ is set in Theorem 1.

Bearing in mind the estimates (162) and (163) and from the fact that $k_2>k_1$, we see in particular that the cocycle $\Theta_p^f(\epsilon) = G_{p+1}^f(\epsilon) - G_p^f(\epsilon)$ is exponentially flat of order k_1 on $Z_p = \mathcal{E}_p\cap\mathcal{E}_{p+1}$, for all $0\leq p\leq\varsigma-1$.

From Theorem (RS) stated above in Section 6.1, we deduce the existence of a convergent power series $a^f(\epsilon)\in\mathbb{F}\{\epsilon\}$ and a formal series $\hat{G}^{1,f}(\epsilon)\in\mathbb{F}[[\epsilon]]$ such that $G_p^f(\epsilon)$ obeys the following decomposition:

$$G_p^f(\epsilon) = a^f(\epsilon) + G_p^{1,f}(\epsilon),$$

where $G_p^{1,f}(\epsilon)$ is holomorphic on $\mathcal{E}_p$ and has $\hat{G}^{1,f}(\epsilon)$ as its asymptotic expansion of Gevrey order $1/k_1$ on $\mathcal{E}_p$, We define

$$\hat{f}(t,z,\epsilon) = \sum_{m\geq 0} f_m(t,z)\epsilon^m/m! := a^f(\epsilon) + \hat{G}^{1,f}(\epsilon).$$

□

The second main result of this work can be stated as follows.

Theorem 2 (a) *Let us assume that the hypotheses of Theorem* 1 *hold. Then there exists a formal power series*

$$\hat{u}(t,z,\epsilon) = \sum_{m\geq 0} h_m(t,z)\epsilon^m/m!,$$

a solution of the equation

$$
\begin{aligned}
&Q(\partial_z)\big(\partial_t \hat{u}(t,z,\epsilon)\big) \\
&\quad = c_{1,2}(\epsilon)\big(Q_1(\partial_z)\hat{u}(t,z,\epsilon)\big)\big(Q_2(\partial_z)\hat{u}(t,z,\epsilon)\big) \\
&\qquad + \epsilon^{(\delta_D-1)(k_2+1)-\delta_D+1} t^{(\delta_D-1)(k_2+1)} \partial_t^{\delta_D} R_D(\partial_z)\hat{u}(t,z,\epsilon) + \sum_{l=1}^{D-1} \epsilon^{\Delta_l} t^{d_l} \partial_t^{\delta_l} R_l(\partial_z)\hat{u}(t,z,\epsilon) \\
&\qquad + c_0(t,z,\epsilon)R_0(\partial_z)\hat{u}(t,z,\epsilon) + c_F(\epsilon)\hat{f}(t,z,\epsilon)
\end{aligned}
\tag{195}
$$

whose coefficients $h_m(t,z)$ belong to the Banach space $\mathbb{F}$ of bounded holomorphic functions on $(\mathcal{T} \cap D(0,h'')) \times H_{\beta'}$ equipped with supremum norm, where $h'' > 0$ is constructed in Theorem 1, *which is the common asymptotic expansion of Gevrey order $1/k_1$ on $\mathcal{E}_p$ of the functions $u^{\mathfrak{d}_p}$, seen as holomorphic functions from $\mathcal{E}_p$ into $\mathbb{F}$, for all $0 \le p \le \varsigma - 1$. Additionally, the formal series can be decomposed into a sum of three terms,*

$$\hat{u}(t,z,\epsilon) = a(t,z,\epsilon) + \hat{u}_1(t,z,\epsilon) + \hat{u}_2(t,z,\epsilon),$$

where $a(t,z,\epsilon) \in \mathbb{F}\{\epsilon\}$ is a convergent series near $\epsilon = 0$ and $\hat{u}_1(t,z,\epsilon)$, $\hat{u}_2(t,z,\epsilon)$ belong to $\mathbb{F}[[\epsilon]]$ with the property that, accordingly, the function $u^{\mathfrak{d}_p}$ shares a similar decomposition:

$$u^{\mathfrak{d}_p}(t,z,\epsilon) = a(t,z,\epsilon) + u_1^{\mathfrak{d}_p}(t,z,\epsilon) + u_2^{\mathfrak{d}_p}(t,z,\epsilon),$$

where $\epsilon \mapsto u_1^{\mathfrak{d}_p}(t,z,\epsilon)$ is a $\mathbb{F}$-valued function having $\hat{u}_1(t,z,\epsilon)$ as asymptotic expansion of Gevrey order $1/k_1$ on $\mathcal{E}_p$ and where $\epsilon \mapsto u_2^{\mathfrak{d}_p}(t,z,\epsilon)$ is a $\mathbb{F}$-valued function having $\hat{u}_2(t,z,\epsilon)$ as asymptotic expansion of Gevrey order $1/k_2$ on $\mathcal{E}_p$, for all $0 \le p \le \varsigma - 1$.

(b) *We make now the further assumption completing the four properties described in Definition* 8 *that the good covering $\{\mathcal{E}_p\}_{0\le p\le\varsigma-1}$ and that the family of unbounded sectors $\{U_{\mathfrak{d}_p}\}_{0\le p\le\varsigma-1}$ satisfy the following property*:

(5) *There exist $0 \le p_0 \le \varsigma - 1$ and two integers $\delta_1, \delta_2 \ge 0$ such that, for all $p \in I_{\delta_1,p_0,\delta_2} = \{p_0 - \delta_1, \dots, p_0, \dots, p_0 + \delta_2\}$, the unbounded sectors $U_{\mathfrak{d}_p}$ are such that the intersection $U_{\mathfrak{d}_p} \cap U_{\mathfrak{d}_{p+1}}$ contains the sector $U_{\mathfrak{d}_p,\mathfrak{d}_{p+1}} = \{\tau \in \mathbb{C}^*/\arg(\tau) \in [\mathfrak{d}_p, \mathfrak{d}_{p+1}]\}$ and such that*

$$\mathcal{E}_{p_0} \subset S_{\pi/k_1} \subset \bigcup_{h \in I_{\delta_1,p_0,\delta_2}} \mathcal{E}_h,$$

where S_{π/k_1} is a sector centered at 0 *with aperture slightly larger than π/k_1.*

Then the formal series $\hat{u}(t,z,\epsilon)$ is (k_2,k_1)-summable on $\mathcal{E}_{p_0}$ and its (k_2,k_1)-sum is given by $u^{\mathfrak{d}_{p_0}}(t,z,\epsilon)$.

Proof We consider the family of functions $u^{\mathfrak{d}_p}(t,z,\epsilon)$, $0 \le p \le \varsigma - 1$ constructed in Theorem 1. For all $0 \le p \le \varsigma - 1$, we define $G_p(\epsilon) := (t,z) \mapsto u^{\mathfrak{d}_p}(t,z,\epsilon)$, which is by construction a holomorphic and bounded function from $\mathcal{E}_p$ into the Banach space $\mathbb{F}$ of bounded holomorphic functions on $(\mathcal{T} \cap D(0,h'')) \times H_{\beta'}$ equipped with the supremum norm, where $\mathcal{T}$ is introduced in Definition 8, $h'' > 0$ is set in Theorem 1 and $\beta' > 0$ is the width of the strip $H_{\beta'}$ on which the coefficient $c_0(t,z,\epsilon)$ and the forcing term $f^{\mathfrak{d}_p}(t,z,\epsilon)$ are defined with respect to z; see (155) and (159).

Bearing in mind the estimates (162) and (163) we see that the cocycle $\Theta_p(\epsilon) = G_{p+1}(\epsilon) - G_p(\epsilon)$ is exponentially flat of order k_2 on $Z_p = \mathcal{E}_p \cap \mathcal{E}_{p+1}$, for all $p \in I_2 \subset \{0,\ldots,\varsigma-1\}$ such that the intersection $U_{\mathfrak{d}_p} \cap U_{\mathfrak{d}_{p+1}}$ contains the sector $U_{\mathfrak{d}_p,\mathfrak{d}_{p+1}}$ and is exponentially flat of order k_1 on $Z_p = \mathcal{E}_p \cap \mathcal{E}_{p+1}$, for all $p \in I_1 \subset \{0,\ldots,\varsigma-1\}$ such that the intersection $U_{\mathfrak{d}_p} \cap U_{\mathfrak{d}_{p+1}}$ is empty.

From Theorem (RS) stated above in Section 6.1, we deduce the existence of a convergent power series $a(\epsilon) \in \mathbb{F}\{\epsilon\}$ and two formal series $\hat{G}^1(\epsilon), \hat{G}^2(\epsilon) \in \mathbb{F}[[\epsilon]]$ such that $G_p(\epsilon)$ obeys the following decomposition:

$$G_p(\epsilon) = a(\epsilon) + G_p^1(\epsilon) + G_p^2(\epsilon),$$

where $G_p^1(\epsilon)$ is holomorphic on $\mathcal{E}_p$ and has $\hat{G}^1(\epsilon)$ as its asymptotic expansion of Gevrey order $1/k_1$ on $\mathcal{E}_p$, $G_p^2(\epsilon)$ is holomorphic on $\mathcal{E}_p$ and carries $\hat{G}^2(\epsilon)$ as its asymptotic expansion of Gevrey order $1/k_2$ on $\mathcal{E}_p$, for all $0 \le p \le \nu - 1$. We set

$$\hat{u}(t,z,\epsilon) = \sum_{m\ge 0} h_m(t,z)\epsilon^m/m! := a(\epsilon) + \hat{G}^1(\epsilon) + \hat{G}^2(\epsilon).$$

This yields the first part (a) of Theorem 2.

Furthermore, under the assumption (b) (5) described above, Theorem (RS) claims that the formal series $\hat{G}(\epsilon) = a(\epsilon) + \hat{G}^1(\epsilon) + \hat{G}^2(\epsilon)$ is (k_2,k_1)-summable on $\mathcal{E}_{p_0}$ and that its (k_2,k_1)-sum is given by $G_{p_0}(\epsilon)$.

It remains to show that the formal series $\hat{u}(t,z,\epsilon)$ solves the main equation (195). Since $u^{\mathfrak{d}_p}(t,z,\epsilon)$ (resp. $f^{\mathfrak{d}_p}(t,z,\epsilon)$) has $\hat{u}(t,z,\epsilon)$ (resp. $\hat{f}(t,z,\epsilon)$) as its asymptotic expansion of Gevrey order $1/k_1$ on $\mathcal{E}_p$, we have in particular

$$\begin{aligned} &\lim_{\epsilon\to 0,\epsilon\in\mathcal{E}_p} \sup_{t\in\mathcal{T}\cap D(0,h''),z\in H_{\beta'}} \left|\partial_\epsilon^m u^{\mathfrak{d}_p}(t,z,\epsilon) - h_m(t,z)\right| = 0, \\ &\lim_{\epsilon\to 0,\epsilon\in\mathcal{E}_p} \sup_{t\in\mathcal{T}\cap D(0,h''),z\in H_{\beta'}} \left|\partial_\epsilon^m f^{\mathfrak{d}_p}(t,z,\epsilon) - f_m(t,z)\right| = 0 \end{aligned} \tag{196}$$

for all $0 \le p \le \varsigma - 1$, all $m \ge 0$. Now, we choose some $p \in \{0,\ldots,\varsigma-1\}$. By construction, the function $u^{\mathfrak{d}_p}(t,z,\epsilon)$ is a solution of (154). We take the derivative of order $m \ge 0$ w.r.t. ϵ on the left- and right-hand side of (154). From the Leibniz rule, we deduce that $\partial_\epsilon^m u^{\mathfrak{d}_p}(t,z,\epsilon)$ verifies the following equation:

$$\begin{aligned} &Q(\partial_z)\partial_t\, \partial_\epsilon^m u^{\mathfrak{d}_p}(t,z,\epsilon) \\ &\quad= \sum_{m_1+m_2+m_3=m} \frac{m!}{m_1!m_2!m_3!} \partial_\epsilon^{m_1} c_{1,2}(\epsilon)\big(Q_1(\partial_z)\partial_\epsilon^{m_2} u^{\mathfrak{d}_p}(t,z,\epsilon)\big) \\ &\quad\quad\times \big(Q_2(\partial_z)\partial_\epsilon^{m_3} u^{\mathfrak{d}_p}(t,z,\epsilon)\big) + \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \partial_\epsilon^{m_1}\big(\epsilon^{(\delta_D-1)(k_2+1)-\delta_D+1}\big) t^{(\delta_D-1)(k_2+1)} \\ &\quad\quad\times \partial_t^{\delta_D} R_D(\partial_z)\partial_\epsilon^{m_2} u^{\mathfrak{d}_p}(t,z,\epsilon) \\ &\quad\quad+ \sum_{l=1}^{D-1}\Bigg(\sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \partial_\epsilon^{m_1}\big(\epsilon^{\Delta_l}\big) t^{d_l} \partial_t^{\delta_l} R_l(\partial_z)\partial_\epsilon^{m_2} u^{\mathfrak{d}_p}(t,z,\epsilon)\Bigg) \end{aligned}$$

$$+ \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \partial_\epsilon^{m_1} c_0(t,z,\epsilon) R_0(\partial_z) \partial_\epsilon^{m_2} u^{\mathfrak{d}_p}(t,z,\epsilon)$$
$$+ \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \partial_\epsilon^{m_1} c_F(\epsilon) \partial_\epsilon^{m_2} f^{\mathfrak{d}_p}(t,z,\epsilon) \tag{197}$$

for all $m \geq 0$, all $(t,z,\epsilon) \in (\mathcal{T} \cap D(0,h'')) \times H_{\beta'} \times \mathcal{E}_p$. If we let ϵ tend to zero in (197) and if we use (196), we get the recursion

$$\begin{aligned}
&Q(\partial_z)\partial_t h_m(t,z) \\
&\quad= \sum_{m_1+m_2+m_3=m} \frac{m!}{m_1!m_2!m_3!} \left(\partial_\epsilon^{m_1} c_{1,2}\right)(0) \left(Q_1(\partial_z)h_{m_2}(t,z)\right)\left(Q_2(\partial_z)h_{m_3}(t,z)\right) \\
&\qquad+ \frac{m!}{(m-((\delta_D-1)(k_2+1)-\delta_D+1))!} t^{(\delta_D-1)(k_2+1)} \partial_t^{\delta_D} R_D(\partial_z) h_{m-((\delta_D-1)(k_2+1)-\delta_D+1)}(t,z) \\
&\qquad+ \sum_{l=1}^{D-1} \frac{m!}{(m-\Delta_l)!} t^{d_l} \partial_t^{\delta_l} R_l(\partial_z) h_{m-\Delta_l}(t,z) \\
&\qquad+ \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \left(\partial_\epsilon^{m_1} c_0\right)(t,z,0) R_0(\partial_z) h_{m_2}(t,z) \\
&\qquad+ \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \left(\partial_\epsilon^{m_1} c_F\right)(0) f_{m_2}(t,z)
\end{aligned} \tag{198}$$

for all $m \geq \max_{1\leq l\leq D-1}\{\Delta_l, (\delta_D-1)(k_2+1)-\delta_D+1\}$, all $(t,z) \in (\mathcal{T} \cap D(0,h'')) \times H_{\beta'}$. Since the functions $c_{1,2}(\epsilon)$, $c_0(t,z,\epsilon)$, and $c_F(\epsilon)$ are analytic w.r.t. ϵ at 0, we know that

$$\begin{aligned}
&c_{1,2}(\epsilon) = \sum_{m\geq 0} \frac{(\partial_\epsilon^m c_{1,2})(0)}{m!}\epsilon^m, \qquad c_0(t,z,\epsilon) = \sum_{m\geq 0} \frac{(\partial_\epsilon^m c_0)(t,z,0)}{m!}\epsilon^m, \\
&c_F(\epsilon) = \sum_{m\geq 0} \frac{(\partial_\epsilon^m c_F)(0)}{m!}\epsilon^m
\end{aligned} \tag{199}$$

for all $\epsilon \in D(0,\epsilon_0)$, all $z \in H_{\beta'}$. On the other hand, one can check by direct inspection from the recursion (198) and the expansions (199) that the series $\hat{u}(t,z,\epsilon) = \sum_{m\geq 0} h_m(t,z)\epsilon^m/m!$ formally solves (195). □

7 Application. Construction of analytic and formal solutions in a complex parameter of a nonlinear initial value Cauchy problem with analytic coefficients and forcing term near the origin in $\mathbb{C}^3$

In this section, we give sufficient conditions on the forcing term $F(T,m,\epsilon)$ for the functions $u^{\mathfrak{d}_p}(t,z,\epsilon)$ and its corresponding formal power series expansion $\hat{u}(t,z,\epsilon)$ w.r.t. ϵ constructed in Theorem 1 and Theorem 2 to solve a nonlinear problem with holomorphic coefficients and forcing term near the origin given by (224).

7.1 A linear convolution initial value problem satisfied by the formal forcing term $F(T,m,\epsilon)$

Let $k_1 \geq 1$ be the integer defined above in Section 5 and let $\mathbf{D} \geq 2$ be an integer. For $1 \leq l \leq \mathbf{D}$, let $\mathbf{d}_l, \boldsymbol{\delta}_l, \boldsymbol{\Delta}_l \geq 0$, be nonnegative integers. We assume that

$$1 = \boldsymbol{\delta}_1, \qquad \boldsymbol{\delta}_l < \boldsymbol{\delta}_{l+1}, \tag{200}$$

for all $1 \leq l \leq \mathbf{D} - 1$. We make also the assumption that

$$\begin{aligned} &\mathbf{d}_{\mathbf{D}} = (\boldsymbol{\delta}_{\mathbf{D}} - 1)(k_1 + 1), \qquad \mathbf{d}_l > (\boldsymbol{\delta}_l - 1)(k_1 + 1), \\ &\boldsymbol{\Delta}_l - \mathbf{d}_l + \boldsymbol{\delta}_l - 1 \geq 0, \qquad \boldsymbol{\Delta}_{\mathbf{D}} = \mathbf{d}_{\mathbf{D}} - \boldsymbol{\delta}_{\mathbf{D}} + 1 \end{aligned} \tag{201}$$

for all $1 \leq l \leq \mathbf{D} - 1$. Let $\mathbf{Q}(X), \mathbf{R}_l(X) \in \mathbb{C}[X]$, $0 \leq l \leq \mathbf{D}$, be polynomials such that

$$\deg(\mathbf{Q}) \geq \deg(\mathbf{R}_{\mathbf{D}}) \geq \deg(\mathbf{R}_l), \qquad \mathbf{Q}(im) \neq 0, \qquad \mathbf{R}_{\mathbf{D}}(im) \neq 0 \tag{202}$$

for all $m \in \mathbb{R}$, all $0 \leq l \leq \mathbf{D} - 1$. Let $\beta, \mu > 0$ be the integers defined above in Section 5. We consider sequences of functions $m \mapsto \mathbf{C}_{0,n}(m,\epsilon)$, for all $n \geq 0$, and $m \mapsto \mathbf{F}_n(m,\epsilon)$, for all $n \geq 1$, that belong to the Banach space $E_{(\beta,\mu)}$ and which depend holomorphically on $\epsilon \in D(0,\epsilon_0)$. We assume that there exist constants $\mathbf{K}_0, \mathbf{T}_0 > 0$ such that

$$\left\|\mathbf{C}_{0,n}(m,\epsilon)\right\|_{(\beta,\mu)} \leq \mathbf{K}_0\left(\frac{1}{\mathbf{T}_0}\right)^n, \qquad \left\|\mathbf{F}_n(m,\epsilon)\right\|_{(\beta,\mu)} \leq \mathbf{K}_0\left(\frac{1}{\mathbf{T}_0}\right)^n \tag{203}$$

for all $n \geq 1$, for all $\epsilon \in D(0,\epsilon_0)$. We define

$$\mathbf{C}_0(T,m,\epsilon) = \sum_{n\geq 1} \mathbf{C}_{0,n}(m,\epsilon)T^n, \qquad \mathbf{F}(T,m,\epsilon) = \sum_{n\geq 1} \mathbf{F}_n(m,\epsilon)T^n$$

which are convergent series on $D(0,\mathbf{T}_0/2)$ with values in $E_{(\beta,\mu)}$. Let $\mathbf{c}_0(\epsilon)$, $\mathbf{c}_{0,0}(\epsilon)$ and $\mathbf{c}_{\mathbf{F}}(\epsilon)$ be bounded holomorphic functions on $D(0,\epsilon_0)$ which vanish at the origin $\epsilon = 0$.

We make the assumption that the formal series $F(T,m,\epsilon) = \sum_{n\geq 1} F_n(m,\epsilon)T^n$, where the coefficients $F_n(m,\epsilon)$ are defined after the problem (154) in Section 5 satisfies the linear initial value problem

$$\begin{aligned} &\mathbf{Q}(im)\big(\partial_T F(T,m,\epsilon)\big) \\ &\quad = \sum_{l=1}^{\mathbf{D}} \mathbf{R}_l(im)\epsilon^{\boldsymbol{\Delta}_l - \mathbf{d}_l + \boldsymbol{\delta}_l - 1} T^{\mathbf{d}_l} \partial_T^{\boldsymbol{\delta}_l} F(T,m,\epsilon) \\ &\qquad + \epsilon^{-1}\frac{\mathbf{c}_0(\epsilon)}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \mathbf{C}_0(T,m-m_1,\epsilon)\mathbf{R}_0(im_1)F(T,m_1,\epsilon)\,dm_1 \\ &\qquad + \epsilon^{-1}\frac{\mathbf{c}_{0,0}(\epsilon)}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \mathbf{C}_{0,0}(m-m_1,\epsilon)\mathbf{R}_0(im_1)F(T,m_1,\epsilon)\,dm_1 \\ &\qquad + \epsilon^{-1}\mathbf{c}_{\mathbf{F}}(\epsilon)\mathbf{F}(T,m,\epsilon) \end{aligned} \tag{204}$$

for given initial data $F(0,m,\epsilon) = 0$.

The existence and uniqueness of the formal power series solution of (204) is ensured by the following.

Proposition 17 *There exists a unique formal series*

$$F(T,m,\epsilon) = \sum_{n\geq 1} F_n(m,\epsilon)T^n,$$

a solution of (204) *with initial data* $F(0,m,\epsilon) \equiv 0$, *where the coefficients* $m \mapsto F_n(m,\epsilon)$ *belong to* $E_{(\beta,\mu)}$ *for* $\beta,\mu > 0$ *given above and depend holomorphically on* ϵ *in* $D(0,\epsilon_0)$.

Proof From Proposition 4, we find that the coefficients $F_n(m,\epsilon)$ of $F(T,m,\epsilon)$ are well defined, belong to $E_{(\beta,\mu)}$ for all $\epsilon \in D(0,\epsilon_0)$, all $n \geq 1$, and satisfy the following recursion relation:

$$\begin{aligned}
&(n+1)F_{n+1}(m,\epsilon) \\
&\quad = \sum_{l=1}^{\mathbf{D}} \frac{\mathbf{R}_l(im)}{\mathbf{Q}(im)} \left(\epsilon^{\boldsymbol{\Delta}_l - \mathbf{d}_l + \boldsymbol{\delta}_l - 1} \prod_{j=0}^{\boldsymbol{\delta}_l - 1} (n + \boldsymbol{\delta}_l - \mathbf{d}_l - j) \right) F_{n+\boldsymbol{\delta}_l - \mathbf{d}_l}(m,\epsilon) \\
&\qquad + \frac{\epsilon^{-1}\mathbf{c}_0(\epsilon)}{\mathbf{Q}(im)} \sum_{n_1+n_2=n, n_1\geq 1, n_2 \geq 1} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \mathbf{C}_{0,n_1}(m-m_1,\epsilon)\mathbf{R}_0(im_1)F_{n_2}(m_1,\epsilon)\,dm_1 \\
&\qquad + \frac{\epsilon^{-1}\mathbf{c}_{0,0}(\epsilon)}{(2\pi)^{1/2}\mathbf{Q}(im)} \int_{-\infty}^{+\infty} \mathbf{C}_{0,0}(m-m_1,\epsilon)\mathbf{R}_0(im_1)F_n(m_1,\epsilon)\,dm_1 \\
&\qquad + \frac{\epsilon^{-1}\mathbf{c}_{\mathbf{F}}(\epsilon)}{\mathbf{Q}(im)} \mathbf{F}_n(m,\epsilon)
\end{aligned} \tag{205}$$

for all $n \geq \max_{1\leq l \leq \mathbf{D}} \mathbf{d}_l$. □

7.2 Analytic solutions for an auxiliary linear convolution problem resulting from a m_{k_1}-Borel transform applied to the linear initial value convolution problem

Using (8.7) from [9], p.3630, we can expand the operators $T^{\boldsymbol{\delta}_l(k_1+1)}\partial_T^{\boldsymbol{\delta}_l}$ in the form

$$T^{\boldsymbol{\delta}_l(k_1+1)}\partial_T^{\boldsymbol{\delta}_l} = \left(T^{k_1+1}\partial_T\right)^{\boldsymbol{\delta}_l} + \sum_{1\leq p\leq \boldsymbol{\delta}_l - 1} A_{\boldsymbol{\delta}_l,p} T^{k_1(\boldsymbol{\delta}_l - p)} \left(T^{k_1+1}\partial_T\right)^p, \tag{206}$$

where $A_{\boldsymbol{\delta}_l,p}$, $p = 1,\dots,\boldsymbol{\delta}_l - 1$ are real numbers, for all $1 \leq l \leq \mathbf{D}$. We define integers $\mathbf{d}_{l,k_1} \geq 0$ to satisfy

$$\mathbf{d}_l + k_1 + 1 = \boldsymbol{\delta}_l(k_1+1) + \mathbf{d}_{l,k_1} \tag{207}$$

for all $1 \leq l \leq \mathbf{D}$. Multiplying (204) by T^{k_1+1} and using (206), (207) we can rewrite (204) in the form

$$\begin{aligned}
&\mathbf{Q}(im)\left(T^{k_1+1}\partial_T F(T,m,\epsilon)\right) \\
&\quad = \sum_{l=1}^{\mathbf{D}} \mathbf{R}_l(im) \Bigg(\epsilon^{\boldsymbol{\Delta}_l - \mathbf{d}_l + \boldsymbol{\delta}_l - 1} T^{\mathbf{d}_{l,k_1}} \left(T^{k_1+1}\partial_T\right)^{\boldsymbol{\delta}_l} F(T,m,\epsilon) \\
&\qquad + \sum_{1\leq p\leq \boldsymbol{\delta}_l - 1} A_{\boldsymbol{\delta}_l,p} \epsilon^{\boldsymbol{\Delta}_l - \mathbf{d}_l + \boldsymbol{\delta}_l - 1} T^{k_1(\boldsymbol{\delta}_l - p) + \mathbf{d}_{l,k_1}} \left(T^{k_1+1}\partial_T\right)^p F(T,m,\epsilon) \Bigg) \\
&\qquad + \epsilon^{-1} T^{k_1+1} \frac{\mathbf{c}_0(\epsilon)}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \mathbf{C}_0(T,m-m_1,\epsilon)\mathbf{R}_0(im_1)F(T,m_1,\epsilon)\,dm_1 \\
&\qquad + \epsilon^{-1} T^{k_1+1} \frac{\mathbf{c}_{0,0}(\epsilon)}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \mathbf{C}_{0,0}(m-m_1,\epsilon)\mathbf{R}_0(im_1)F(T,m_1,\epsilon)\,dm_1 \\
&\qquad + \epsilon^{-1}\mathbf{c}_{\mathbf{F}}(\epsilon) T^{k_1+1}\mathbf{F}(T,m,\epsilon).
\end{aligned} \tag{208}$$

As above, we denote $\psi_{k_1}(\tau,m,\epsilon)$ the formal m_{k_1}-Borel transform of $F(T,m,\epsilon)$ w.r.t. T and $\boldsymbol{\varphi}_{k_1}(\tau,m,\epsilon)$ the formal m_{k_1}-Borel transform of $\mathbf{C}_0(T,m,\epsilon)$ with respect to T and $\boldsymbol{\psi}_{k_1}(\tau,m,\epsilon)$ the formal m_{k_1}-Borel transform of $\mathbf{F}(T,m,\epsilon)$ w.r.t. T,

$$\psi_{k_1}(\tau,m,\epsilon)=\sum_{n\geq 1}F_n(m,\epsilon)\frac{\tau^n}{\Gamma(\frac{n}{k_1})},\qquad \boldsymbol{\varphi}_{k_1}(\tau,m,\epsilon)=\sum_{n\geq 1}\mathbf{C}_{0,n}(m,\epsilon)\frac{\tau^n}{\Gamma(\frac{n}{k_1})},$$
$$\boldsymbol{\psi}_{k_1}(\tau,m,\epsilon)=\sum_{n\geq 1}\mathbf{F}_n(m,\epsilon)\frac{\tau^n}{\Gamma(\frac{n}{k_1})}.$$

Following a similar reasoning as in the steps (76), (77), (78), and (79), using (203) we find that $\boldsymbol{\varphi}_{k_1}(\tau,m,\epsilon)\in F^{\mathfrak{d}_p}_{(\nu,\beta,\mu,k_1,k_1)}$ and $\boldsymbol{\psi}_{k_1}(\tau,m,\epsilon)\in F^{\mathfrak{d}_p}_{(\nu,\beta,\mu,k_1,k_1)}$, for all $\epsilon\in D(0,\epsilon_0)$, for all the unbounded sectors $U_{\mathfrak{d}_p}$ centered at 0 and bisecting direction $\mathfrak{d}_p\in\mathbb{R}$ introduced in Definition 8, for some $\nu>0$.

Observe that $\mathbf{d}_{\mathbf{D},k_1}=0$. Using the computation rules for the formal m_{k_1}-Borel transform in Proposition 8, we deduce the following equation satisfied by $\psi_{k_1}(\tau,m,\epsilon)$:

$$\begin{aligned}
&\mathbf{Q}(im)\bigl(k_1\tau^{k_1}\psi_{k_1}(\tau,m,\epsilon)\bigr)\\
&\quad=\mathbf{R}_{\mathbf{D}}(im)\Biggl(k_1^{\boldsymbol{\delta}_{\mathbf{D}}}\tau^{\boldsymbol{\delta}_{\mathbf{D}}k_1}\psi_{k_1}(\tau,m,\epsilon)\\
&\qquad+\sum_{1\leq p\leq\boldsymbol{\delta}_{\mathbf{D}}-1}A_{\boldsymbol{\delta}_{\mathbf{D}},p}\frac{\tau^{k_1}}{\Gamma(\boldsymbol{\delta}_{\mathbf{D}}-p)}\int_0^{\tau^{k_1}}\bigl(\tau^{k_1}-s\bigr)^{\boldsymbol{\delta}_{\mathbf{D}}-p-1}\bigl(k_1^p s^p\psi_{k_1}\bigl(s^{1/k_1},m,\epsilon\bigr)\bigr)\frac{ds}{s}\Biggr)\\
&\qquad+\sum_{l=1}^{\mathbf{D}-1}\mathbf{R}_l(im)\Biggl(\epsilon^{\boldsymbol{\Delta}_l-\mathbf{d}_l+\boldsymbol{\delta}_l-1}\frac{\tau^{k_1}}{\Gamma(\frac{\mathbf{d}_{l,k_1}}{k_1})}\int_0^{\tau^{k_1}}\bigl(\tau^{k_1}-s\bigr)^{\frac{\mathbf{d}_{l,k_1}}{k_1}-1}\bigl(k_1^{\boldsymbol{\delta}_l}s^{\boldsymbol{\delta}_l}\psi_{k_1}\bigl(s^{1/k_1},m,\epsilon\bigr)\bigr)\frac{ds}{s}\\
&\qquad+\sum_{1\leq p\leq\boldsymbol{\delta}_l-1}A_{\boldsymbol{\delta}_l,p}\epsilon^{\boldsymbol{\Delta}_l-\mathbf{d}_l+\boldsymbol{\delta}_l-1}\frac{\tau^{k_1}}{\Gamma(\frac{\mathbf{d}_{l,k_1}}{k_1}+\boldsymbol{\delta}_l-p)}\\
&\qquad\times\int_0^{\tau^{k_1}}\bigl(\tau^{k_1}-s\bigr)^{\frac{\mathbf{d}_{l,k_1}}{k_1}+\boldsymbol{\delta}_l-p-1}\bigl(k_1^p s^p\psi_{k_1}\bigl(s^{1/k_1},m,\epsilon\bigr)\bigr)\frac{ds}{s}\Biggr)\\
&\qquad+\epsilon^{-1}\frac{\tau^{k_1}}{\Gamma(1+\frac{1}{k_1})}\int_0^{\tau^{k_1}}\bigl(\tau^{k_1}-s\bigr)^{1/k_1}\Biggl(\frac{\mathbf{c}_0(\epsilon)}{(2\pi)^{1/2}}s\int_0^s\int_{-\infty}^{+\infty}\boldsymbol{\varphi}_{k_1}\bigl((s-x)^{1/k_1},m-m_1,\epsilon\bigr)\\
&\qquad\times\mathbf{R}_0(im_1)\psi_{k_1}\bigl(x^{1/k_1},m_1,\epsilon\bigr)\frac{1}{(s-x)x}\,dx\,dm_1\Biggr)\frac{ds}{s}\\
&\qquad+\epsilon^{-1}\frac{\tau^{k_1}}{\Gamma(1+\frac{1}{k_1})}\int_0^{\tau^{k_1}}\bigl(\tau^{k_1}-s\bigr)^{1/k_1}\frac{\mathbf{c}_{0,0}(\epsilon)}{(2\pi)^{1/2}}\\
&\qquad\times\Biggl(\int_{-\infty}^{+\infty}\mathbf{C}_{0,0}(m-m_1,\epsilon)\mathbf{R}_0(im_1)\psi_{k_1}\bigl(s^{1/k_1},m_1,\epsilon\bigr)\,dm_1\Biggr)\frac{ds}{s}\\
&\qquad+\epsilon^{-1}\mathbf{c}_{\mathbf{F}}(\epsilon)\frac{\tau^{k_1}}{\Gamma(1+\frac{1}{k_1})}\int_0^{\tau^{k_1}}\bigl(\tau^{k_1}-s\bigr)^{1/k_1}\boldsymbol{\psi}_{k_1}\bigl(s^{1/k_1},m,\epsilon\bigr)\frac{ds}{s}.
\end{aligned}\tag{209}$$

We make the additional assumption that there exists an unbounded sector

$$S_{\mathbf{Q},\mathbf{R}_{\mathbf{D}}}=\bigl\{z\in\mathbb{C}/|z|\geq r_{\mathbf{Q},\mathbf{R}_{\mathbf{D}}},\bigl|\arg(z)-d_{\mathbf{Q},\mathbf{R}_{\mathbf{D}}}\bigr|\leq\boldsymbol{\eta}_{\mathbf{Q},\mathbf{R}_{\mathbf{D}}}\bigr\}$$

with direction $d_{\mathbf{Q},\mathbf{R_D}} \in \mathbb{R}$, aperture $\eta_{\mathbf{Q},\mathbf{R_D}} > 0$ for some radius $r_{\mathbf{Q},\mathbf{R_D}} > 0$, such that

$$\frac{\mathbf{Q}(im)}{\mathbf{R_D}(im)} \in S_{\mathbf{Q},\mathbf{R_D}} \tag{210}$$

for all $m \in \mathbb{R}$. We factorize the polynomial $\mathbf{P}_m(\tau) = \mathbf{Q}(im)k_1 - \mathbf{R_D}(im)k_1^{\delta_\mathbf{D}}\tau^{(\delta_\mathbf{D}-1)k_1}$ in the form

$$\mathbf{P}_m(\tau) = -\mathbf{R_D}(im)k_1^{\delta_\mathbf{D}} \prod_{l=0}^{(\delta_\mathbf{D}-1)k_1-1} \big(\tau - \mathbf{q}_l(m)\big), \tag{211}$$

where

$$\begin{aligned}\mathbf{q}_l(m) = {}& \left(\frac{|\mathbf{Q}(im)|}{|\mathbf{R_D}(im)|k_1^{\delta_\mathbf{D}-1}}\right)^{\frac{1}{(\delta_\mathbf{D}-1)k_1}} \\ &\times \exp\left(\sqrt{-1}\left(\arg\left(\frac{\mathbf{Q}(im)}{\mathbf{R_D}(im)k_1^{\delta_\mathbf{D}-1}}\right)\frac{1}{(\delta_\mathbf{D}-1)k_1} + \frac{2\pi l}{(\delta_\mathbf{D}-1)k_1}\right)\right)\end{aligned} \tag{212}$$

for all $0 \le l \le (\delta_\mathbf{D}-1)k_1 - 1$, all $m \in \mathbb{R}$.

We choose the family of unbounded sectors $U_{\mathfrak{d}_p}$ centered at 0, a small closed disc $\bar{D}(0,\rho)$ (introduced in Definition 8) and we prescribe the sector $S_{\mathbf{Q},\mathbf{R_D}}$ in such a way that the following conditions hold.

(1) There exists a constant $\mathbf{M}_1 > 0$ such that

$$\big|\tau - \mathbf{q}_l(m)\big| \ge \mathbf{M}_1\big(1 + |\tau|\big) \tag{213}$$

for all $0 \le l \le (\delta_\mathbf{D}-1)k_1 - 1$, all $m \in \mathbb{R}$, all $\tau \in U_{\mathfrak{d}_p} \cup \bar{D}(0,\rho)$, for all $0 \le p \le \varsigma - 1$.

(2) There exists a constant $\mathbf{M}_2 > 0$ such that

$$\big|\tau - \mathbf{q}_{l_0}(m)\big| \ge \mathbf{M}_2\big|\mathbf{q}_{l_0}(m)\big| \tag{214}$$

for some $l_0 \in \{0,\dots,(\delta_\mathbf{D}-1)k_1 - 1\}$, all $m \in \mathbb{R}$, all $\tau \in U_{\mathfrak{d}_p} \cup \bar{D}(0,\rho)$, for all $0 \le p \le \varsigma - 1$.

By construction of the roots (212) in the factorization (211) and using the lower bound estimates (213), (214), we get a constant $C_\mathbf{P} > 0$ such that

$$\begin{aligned}\big|\mathbf{P}_m(\tau)\big| &\ge \mathbf{M}_1^{(\delta_\mathbf{D}-1)k_1-1}\mathbf{M}_2\big|\mathbf{R_D}(im)\big|k_1^{\delta_\mathbf{D}}\left(\frac{|\mathbf{Q}(im)|}{|\mathbf{R_D}(im)|k_1^{\delta_\mathbf{D}-1}}\right)^{\frac{1}{(\delta_\mathbf{D}-1)k_1}}\big(1+|\tau|\big)^{(\delta_\mathbf{D}-1)k_1-1} \\ &\ge \mathbf{M}_1^{(\delta_\mathbf{D}-1)k_1-1}\mathbf{M}_2\frac{k_1^{\delta_\mathbf{D}}}{(k_1^{\delta_\mathbf{D}-1})^{\frac{1}{(\delta_\mathbf{D}-1)k_1}}}(r_{\mathbf{Q},\mathbf{R_D}})^{\frac{1}{(\delta_\mathbf{D}-1)k_1}}\big|\mathbf{R_D}(im)\big| \\ &\quad\times\left(\min_{x\ge 0}\frac{(1+x)^{(\delta_\mathbf{D}-1)k_1-1}}{(1+x^{k_1})^{(\delta_\mathbf{D}-1)-\frac{1}{k_1}}}\right)\big(1+|\tau|^{k_1}\big)^{(\delta_\mathbf{D}-1)-\frac{1}{k_1}} \\ &= C_\mathbf{P}(r_{\mathbf{Q},\mathbf{R_D}})^{\frac{1}{(\delta_\mathbf{D}-1)k_1}}\big|\mathbf{R_D}(im)\big|\big(1+|\tau|^{k_1}\big)^{(\delta_\mathbf{D}-1)-\frac{1}{k_1}}\end{aligned} \tag{215}$$

for all $\tau \in U_{\mathfrak{d}_p} \cup \bar{D}(0,\rho)$, all $m \in \mathbb{R}$, all $0 \le p \le \varsigma - 1$.

In the next proposition, we give sufficient conditions under which (209) has a solution $\psi_{k_1}^{\mathfrak{d}_p}(\tau,m,\epsilon)$ in the Banach space $F_{(\nu,\beta,\mu,k_1,k_1)}^{\mathfrak{d}_p}$ where β, μ are defined above.

Proposition 18 *Under the assumption that*

$$\boldsymbol{\delta}_{\mathbf{D}} \geq \boldsymbol{\delta}_l + \frac{1}{k_1} \tag{216}$$

for all $1 \leq l \leq \mathbf{D} - 1$, *there exist a radius* $r_{\mathbf{Q},\mathbf{R}_{\mathbf{D}}} > 0$, *a constant* $\boldsymbol{\upsilon} > 0$, *and constants* $\varsigma_{0,0}, \varsigma_0, \varsigma_1, \varsigma_{1,0}, \varsigma_F, \varsigma_2 > 0$ (*depending on* k_1, $C_{\mathbf{P}}$, μ, ν, ϵ_0, $\mathbf{R}_l$, $\boldsymbol{\Delta}_l$, $\boldsymbol{\delta}_l$, $\mathbf{d}_l$ *for* $0 \leq l \leq \mathbf{D}$) *such that if*

$$\begin{aligned}
&\sup_{\epsilon \in D(0,\epsilon_0)} \left| \frac{\mathbf{c}_0(\epsilon)}{\epsilon} \right| \leq \varsigma_{1,0}, \qquad \left\| \boldsymbol{\varphi}_{k_1}(\tau, m, \epsilon) \right\|_{(\nu,\beta,\mu,k_1,k_1)} \leq \varsigma_1, \\
&\sup_{\epsilon \in D(0,\epsilon_0)} \left| \frac{\mathbf{c}_{0,0}(\epsilon)}{\epsilon} \right| \leq \varsigma_{0,0}, \qquad \left\| \mathbf{C}_{0,0}(m, \epsilon) \right\|_{(\beta,\mu)} \leq \varsigma_0, \\
&\sup_{\epsilon \in D(0,\epsilon_0)} \left| \frac{\mathbf{c}_{\mathbf{F}}(\epsilon)}{\epsilon} \right| \leq \varsigma_F, \qquad \left\| \boldsymbol{\psi}_{k_1}(\tau, m, \epsilon) \right\|_{(\nu,\beta,\mu,k_1,k_1)} \leq \varsigma_2
\end{aligned} \tag{217}$$

for all $\epsilon \in D(0,\epsilon_0)$, (209) *has a unique solution* $\psi_{k_1}^{\mathfrak{d}_p}(\tau, m, \epsilon)$ *in the space* $F_{(\nu,\beta,\mu,k_1,k_1)}^{\mathfrak{d}_p}$ *with the property that* $\|\psi_{k_1}^{\mathfrak{d}_p}(\tau, m, \epsilon)\|_{(\nu,\beta,\mu,k_1,k_1)} \leq \boldsymbol{\upsilon}$, *for all* $\epsilon \in D(0,\epsilon_0)$, *where* $\beta, \mu > 0$ *are defined above, for any unbounded sector* $U_{\mathfrak{d}_p}$ *and disc* $\bar{D}(0,\rho)$ *that satisfy the constraints* (213), (214), *for all* $0 \leq p \leq \varsigma - 1$.

The proof of Proposition 18 follows exactly the same steps as the corresponding one of Proposition 14, therefore we skip completely the details.

As a result, we find that the m_{k_1}-Borel transform $\psi_{k_1}(\tau, m, \epsilon)$ of the formal series $F(T, m, \epsilon)$ solution of (204) is convergent w.r.t. τ on $D(0,\rho)$ as series in coefficients in $E_{(\beta,\mu)}$, for all $\epsilon \in D(0,\epsilon_0)$, and can be analytically continued on each unbounded sector $U_{\mathfrak{d}_p}$ as a function $\tau \mapsto \psi_{k_1}^{\mathfrak{d}_p}(\tau, m, \epsilon)$ which belongs to the space $F_{(\nu,\beta,\mu,k_1,k_1)}^{\mathfrak{d}_p}$. In other words, the assumed constraints (156) are fulfilled.

7.3 A linear initial value Cauchy problem satisfied by the analytic forcing terms $f^{\mathfrak{d}_p}(t,z,\epsilon)$

We keep the notations and the assumptions made in the previous subsection. From the assumption (203), we deduce that the functions

$$\begin{aligned}
\check{\mathbf{C}}_0(T, z, \epsilon) &= \mathbf{c}_{0,0}(\epsilon)\mathcal{F}^{-1}\big(m \mapsto \mathbf{C}_{0,0}(m, \epsilon)\big)(z) \\
&\quad + \sum_{n \geq 1} \mathbf{c}_0(\epsilon)\mathcal{F}^{-1}\big(m \mapsto \mathbf{C}_{0,n}(m, \epsilon)\big)(z) T^n, \\
\check{\mathbf{F}}(T, z, \epsilon) &= \sum_{n \geq 1} \mathcal{F}^{-1}\big(m \mapsto \mathbf{F}_n(m, \epsilon)\big)(z) T^n
\end{aligned} \tag{218}$$

represent bounded holomorphic functions on $D(0, \mathbf{T}_0/2) \times H_{\beta'} \times D(0,\epsilon_0)$ for any $0 < \beta' < \beta$ (where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform defined in Proposition 9). We define the coefficients

$$\mathbf{c}_0(t, z, \epsilon) = \check{\mathbf{C}}_0(\epsilon t, z, \epsilon), \qquad \mathbf{f}(t, z, \epsilon) = \check{\mathbf{F}}(\epsilon t, z, \epsilon), \tag{219}$$

which are holomorphic and bounded on $D(0,r) \times H_{\beta'} \times D(0,\epsilon_0)$ where $r\epsilon_0 \leq \mathbf{T}_0/2$.

Proposition 19 *Under the constraints* (200), (201), (202), (203) *and the assumptions* (210), (213), (214), (216), (217), *the forcing term* $f^{\mathfrak{d}_p}(t,z,\epsilon)$ *represented by* (159) *solves the following linear Cauchy problem*:

$$\begin{aligned}&\mathbf{Q}(\partial_z)\big(\partial_t f^{\mathfrak{d}_p}(t,z,\epsilon)\big)\\&\quad=\epsilon^{(\boldsymbol{\delta}_{\mathbf{D}}-1)(k_1+1)-\boldsymbol{\delta}_{\mathbf{D}}+1}t^{(\boldsymbol{\delta}_{\mathbf{D}}-1)(k_1+1)}\partial_t^{\boldsymbol{\delta}_{\mathbf{D}}}\mathbf{R}_{\mathbf{D}}(\partial_z)f^{\mathfrak{d}_p}(t,z,\epsilon)\\&\qquad+\sum_{l=1}^{\mathbf{D}-1}\epsilon^{\boldsymbol{\Delta}_l}t^{\mathbf{d}_l}\partial_t^{\delta_l}\mathbf{R}_l(\partial_z)f^{\mathfrak{d}_p}(t,z,\epsilon)+\mathbf{c}_0(t,z,\epsilon)\mathbf{R}_0(\partial_z)f^{\mathfrak{d}_p}(t,z,\epsilon)+\mathbf{c}_{\mathbf{F}}(\epsilon)\mathbf{f}(t,z,\epsilon)\end{aligned}\tag{220}$$

for given initial data $f^{\mathfrak{d}_p}(0,z,\epsilon)\equiv 0$, *for all* $t\in\mathcal{T}$, $z\in H_{\beta'}$, *and* $\epsilon\in\mathcal{E}_p$ (*provided that the radius* $r_{\mathcal{T}}$ *of* $\mathcal{T}$ *fulfills the restriction* $\epsilon_0 r_{\mathcal{T}}\le\min(h',T_0/2,\mathbf{T}_0/2)$).

Proof From Proposition 18, we know that the formal series $F(T,m,\epsilon)=\sum_{n\ge1}F_n(m,\epsilon)T^n$ is m_{k_1}-summable w.r.t. T in all directions $\mathfrak{d}_p$, $0\le p\le\varsigma-1$ (in the sense of Definition 4). Therefore, from the estimates (156), we deduce that the m_{k_1}-Laplace transform

$$\mathcal{L}^{\mathfrak{d}_p}_{m_{k_1}}\big(\tau\mapsto\psi^{\mathfrak{d}_p}_{k_1}(\tau,m,\epsilon)\big)(T)=k_1\int_{L_{\mathfrak{d}_p}}\psi^{\mathfrak{d}_p}_{k_1}(u,m,\epsilon)e^{-(\frac{u}{T})^{k_1}}\frac{du}{u}$$

defines a bounded and holomorphic function on any sector $S_{\mathfrak{d}_p,\theta_{k_1},h'_{k_1}}$ w.r.t. T, for all $m\in\mathbb{R}$, all $\epsilon\in D(0,\epsilon_0)$, where $S_{\mathfrak{d}_p,\theta_{k_1},h'_{k_1}}$ is a sector with bisecting direction $\mathfrak{d}_p$, aperture $\frac{\pi}{k_1}<\theta_{k_1}<\frac{\pi}{k_1}+\mathrm{ap}(U_{\mathfrak{d}_p})$, and some radius $h'_{k_1}>0$. Moreover, using the algebraic properties of the m_{k_1}-sums we deduce that $\mathcal{L}^{\mathfrak{d}_p}_{m_{k_1}}(\tau\mapsto\psi^{\mathfrak{d}_p}_{k_1}(\tau,m,\epsilon))(T)$ solves (208) and then (204) for all $T\in S_{\mathfrak{d}_p,\theta_{k_1},h'_{k_1}}$, all $m\in\mathbb{R}$, all $\epsilon\in D(0,\epsilon_0)$, vanishes at $T=0$. Now, let $F^{\mathfrak{d}_p}(T,m,\epsilon)$ be as defined in (158).

Lemma 12 *The identity*

$$F^{\mathfrak{d}_p}(T,m,\epsilon)=\mathcal{L}^{\mathfrak{d}_p}_{m_{k_1}}\big(\tau\mapsto\psi^{\mathfrak{d}_p}_{k_1}(\tau,m,\epsilon)\big)(T)$$

holds, *for all* $T\in S_{\mathfrak{d}_p,\theta,h'}$, $m\in\mathbb{R}$, $\epsilon\in D(0,\epsilon_0)$, *as defined just after the definition* (158), *for* $\frac{\pi}{k_2}<\theta<\frac{\pi}{k_2}+\mathrm{ap}(S_{\mathfrak{d}_p})$, *and some radius* $h'>0$.

Proof By construction, we can write

$$\begin{aligned}F^{\mathfrak{d}_p}(T,m,\epsilon)&=k_2\int_{L_{\mathfrak{d}_p}}\bigg(\int_{L_{\mathfrak{d}_p}}\psi^{\mathfrak{d}_p}_{k_1}(h,m,\epsilon)\\&\qquad\times\bigg(-\frac{k_2k_1}{2i\pi}u^{k_2}\int_{V_{\mathfrak{d}_p,k_2,\delta'}}\exp\bigg(-\Big(\frac{h}{\nu}\Big)^{k_1}+\Big(\frac{u}{\nu}\Big)^{k_2}\bigg)\frac{d\nu}{\nu^{k_2+1}}\bigg)\frac{dh}{h}\bigg)e^{-(\frac{u}{T})^{k_2}}\frac{du}{u}\end{aligned}$$

for some $0<\delta'<\frac{\pi}{\kappa}$, where $V_{\mathfrak{d}_p,k_2,\delta'}$ is defined in Proposition 13. Using Fubini's theorem yields

$$F^{\mathfrak{d}_p}(T,m,\epsilon)=k_1\int_{L_{\mathfrak{d}_p}}\psi^{\mathfrak{d}_p}_{k_1}(h,m,\epsilon)A(T,h)\frac{dh}{h},\tag{221}$$

where

$$A(T,h) = k_2 \int_{L_{\mathfrak{d}_p}} \frac{-k_2}{2i\pi} u^{k_2} \left(\int_{V_{\mathfrak{d}_p,k_2,\delta'}} \exp\left(-\left(\frac{h}{v}\right)^{k_1} + \left(\frac{u}{v}\right)^{k_2} \right) \frac{dv}{v^{k_2+1}} \right) e^{-(\frac{u}{T})^{k_2}} \frac{du}{u}$$
$$= \mathcal{L}^{\mathfrak{d}_p}_{m_{k_2}} \left(u \mapsto \left(\mathcal{B}^{\mathfrak{d}_p}_{m_{k_2}} \left(v \mapsto e^{-(\frac{h}{v})^{k_1}} \right) \right)(u) \right)(T) \tag{222}$$

for all $T \in S_{\mathfrak{d}_p,\theta,h'}$, $m \in \mathbb{R}$, $\epsilon \in D(0,\epsilon_0)$. But we observe from the inversion formula (110) that $A(T,h) = \exp(-(h/T)^{k_1})$. Gathering (221) and (222) yields Lemma 12. □

From Lemma 12, we deduce that $F^{\mathfrak{d}_p}(T,m,\epsilon)$ solves (204) for all $T \in S_{\mathfrak{d}_p,\theta,h'}$, all $m \in \mathbb{R}$, and all $\epsilon \in D(0,\epsilon_0)$. Hence, using the properties of the Fourier inverse transform from Proposition 9, we deduce that the analytic forcing term $f^{\mathfrak{d}_p}(t,z,\epsilon) = \mathcal{F}^{-1}(m \mapsto F^{\mathfrak{d}_p}(\epsilon t,m,\epsilon))(z)$ solves the linear Cauchy problem (220), for all $t \in \mathcal{T}$, all $z \in H_{\beta'}$, and all $\epsilon \in \mathcal{E}_p$. □

We are in a position to state the main result of this section.

Theorem 3 *We take for granted that the assumptions of Theorem* 1 *hold. We also make the hypothesis that the constraints* (200), (201), (202), (203) *and the assumptions* (210), (213), (214), (216), (217) *hold. We denote* $P(t,z,\epsilon,\partial_t,\partial_z)$ *and* $\mathbf{P}(t,z,\epsilon,\partial_t,\partial_z)$ *the linear differential operators*

$$\begin{aligned}
P(t,z,\epsilon,\partial_t,\partial_z) &= Q(\partial_z)\partial_t - \epsilon^{(\delta_D-1)(k_2+1)-\delta_D+1} t^{(\delta_D-1)(k_2+1)} \partial_t^{\delta_D} R_D(\partial_z) \\
&\quad - \sum_{l=1}^{D-1} \epsilon^{\Delta_l} t^{d_l} \partial_t^{\delta_l} R_l(\partial_z) - c_0(t,z,\epsilon) R_0(\partial_z), \\
\mathbf{P}(t,z,\epsilon,\partial_t,\partial_z) &= \mathbf{Q}(\partial_z)\partial_t - \epsilon^{(\boldsymbol{\delta}_{\mathbf{D}}-1)(k_1+1)-\boldsymbol{\delta}_{\mathbf{D}}+1} t^{(\boldsymbol{\delta}_{\mathbf{D}}-1)(k_1+1)} \partial_t^{\boldsymbol{\delta}_{\mathbf{D}}} \mathbf{R}_{\mathbf{D}}(\partial_z) \\
&\quad - \sum_{l=1}^{\mathbf{D}-1} \epsilon^{\boldsymbol{\Delta}_l} t^{\mathbf{d}_l} \partial_t^{\boldsymbol{\delta}_l} \mathbf{R}_l(\partial_z) - \mathbf{c}_0(t,z,\epsilon) \mathbf{R}_0(\partial_z).
\end{aligned} \tag{223}$$

Then the functions $u^{\mathfrak{d}_p}(t,z,\epsilon)$ *constructed in Theorem* 1 *solve the following nonlinear PDE*:

$$\begin{aligned}
&\mathbf{P}(t,z,\epsilon,\partial_t,\partial_z) P(t,z,\epsilon,\partial_t,\partial_z) u^{\mathfrak{d}_p}(t,z,\epsilon) \\
&\quad = c_{1,2}(\epsilon) \mathbf{P}(t,z,\epsilon,\partial_t,\partial_z) \left(Q_1(\partial_z) u^{\mathfrak{d}_p}(t,z,\epsilon) Q_2(\partial_z) u^{\mathfrak{d}_p}(t,z,\epsilon) \right) \\
&\qquad + c_F(\epsilon) \mathbf{c}_{\mathbf{F}}(\epsilon) \mathbf{f}(t,z,\epsilon)
\end{aligned} \tag{224}$$

whose coefficients and forcing term $\mathbf{f}$ *are analytic functions on* $D(0,r_{\mathcal{T}}) \times H_{\beta'} \times D(0,\epsilon_0)$, *with vanishing initial data* $u^{\mathfrak{d}_p}(0,z,\epsilon) \equiv 0$, *for all* $t \in \mathcal{T}$, *all* $z \in H_{\beta'}$ *and all* $\epsilon \in \mathcal{E}_p$. *Moreover, the formal power series* $\hat{u}(t,z,\epsilon) = \sum_{m\geq 0} h_m(t,z)\epsilon^m/m!$ *constructed in Theorem* 2 *formally solves the same equation* (224).

Proof The reason why $u^{\mathfrak{d}_p}(t,z,\epsilon)$ solves (224) follows directly from the fact that $u^{\mathfrak{d}_p}(t,z,\epsilon)$ solves the nonlinear equation

$$\begin{aligned}
&P(t,z,\epsilon,\partial_t,\partial_z) u^{\mathfrak{d}_p}(t,z,\epsilon) \\
&\quad = c_{1,2}(\epsilon) \left(Q_1(\partial_z) u^{\mathfrak{d}_p}(t,z,\epsilon) Q_2(\partial_z) u^{\mathfrak{d}_p}(t,z,\epsilon) \right) + c_F(\epsilon) f^{\mathfrak{d}_p}(t,z,\epsilon)
\end{aligned}$$

according to Theorem 1 and from the additional feature that $f^{\mathfrak{d}_p}(t,z,\epsilon)$ solves the linear equation

$$\mathbf{P}(t,z,\epsilon,\partial_t,\partial_z)f^{\mathfrak{d}_p}(t,z,\epsilon) = \mathbf{c_F}(\epsilon)\mathbf{f}(t,z,\epsilon),$$

as shown in Proposition 19. Finally in order to show that $\hat{u}(t,z,\epsilon)$ formally solves (224) we see that with the help of the second equality in (196) and following exactly the same lines of arguments as in the last part of Theorem 2, one can show that the power series $\hat{f}(t,z,\epsilon) = \sum_{m\geq 0} f_m(t,z)\epsilon^m/m!$ constructed in Lemma 11 formally solves the linear equation

$$\mathbf{P}(t,z,\epsilon,\partial_t,\partial_z)\hat{f}(t,z,\epsilon) = \mathbf{c_F}(\epsilon)\mathbf{f}(t,z,\epsilon). \tag{225}$$

Combining (195) and (225) yields the result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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