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Covering Rational Ruled Surfaces

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Abstract

We present an algorithm that covers any given rational ruled surface with two rational parametrizations. In addition, we present an algorithm that transforms any rational surface parametrization into a new rational surface parametrization without affine base points and such that the degree of the corresponding maps is preserved.

1 Introduction

One of the most important features of rational varieties, at least in practice, is the possibility to choose between parametric or implicit representations depending on the nature of the problem one is dealing with; examples are the computation of intersections, plotting figures, line and surface integrals, etc. However, when using the parametric representations additional difficulties may appear, and the feasibility of the strategy is affected. In particular, if the parametrization is not surjective, some solving strategies may fail.

In [SSV14b], Example 1 illustrates a situation where the computation of the intersection of two surfaces fails when a non surjective parametrization is used. Let us see another motivating example.

Example 1.1. The Hausdorff distance appears naturally in applications in computer aided design, pattern matching and pattern recognition (see e.g. [BYLLM11], [CMXP10], [KOYKE10]), when measuring the resemblance between two geometric objects. The computation or estimation of the

Hausdorff distance implies, in particular, measuring the distance of a point to a set. Let us assume that we want to measure the distance of the point $A = (4/5, 6/5, 1)$ to the surface S defined by $f(x, y, z) = xy - 2yz + z^2$. Applying Lagrange multipliers one gets that the distance of A to S is $\sqrt{2}/5 = 0.283\ldots$ and it is reachable at $B = (1, 1, 1) \in S$. Nevertheless, in general, approaching this problem using implicit equations turns to be computationally intractable. Instead, one can try to use a parametrization of the surface so that the problem reduces to a optimization problem without constrains. In our case, S is rational, indeed it is rational ruled surface, and can be parametrized as

$$\mathcal{P}(s, t) = \left((s^2 - 1)t, s^2 t, t(s^2 + s) \right).$$

However, if we try to optimize the function $\|A - \mathcal{P}(s, t)\|^2$ we find that the best critical value is obtained at $(0.889\ldots, -0.042\ldots, 0.155\ldots)$ and the distance is then estimated as $1.504\ldots$. The problem is that $B \in S \smallsetminus \mathrm{Image}(\mathcal{P})$, so it cannot be found with the parametrization. Nevertheless, $\mathcal{P}$ satisfies the hypothesis in Theorem 2.6, and therefore we can determine that the $S \smallsetminus \mathrm{Image}(\mathcal{P})$ is included in the line (t, t, t). Thus, we now optimize the $\|A - (t, t, t)\|^2$ to get B as solution.

In the case of curves, non-surjectivity is not so important since every rational proper parametrization of a curve may miss at most one point that can be easily computed (see e.g. [AR07], [Sen02]). The situation changes when working with rational parametrizations of algebraic surfaces: the missing subset can be of dimension 1.

Some authors have addressed the problem of finding surjective parametrizations of rational surfaces; see [BR95], [GC91] for the case of quadrics or [SSV14b] for certain particular types of rational surfaces. Alternatively, one can compute finitely many rational parametrizations such that the union of their images covers the whole surface. This was done for the real general case, in [BR95], by computing a cover with 2^n parametrizations, where n is the dimension of the rational variety; i.e. in the surface case, with four pieces. In [SSV14a] we show that, if a surface admits a rational parametrization without projective base points, then it can be covered with at most three pieces. Continuing with this research, in this paper we analyze the problem of covering rational ruled surfaces. The next example shows that for the same surface, changing the parametrization, can make the missing subset bigger.

Example 1.2. We consider the ruled surface S given by $x^2 y - 2xy^2 + 2y^3 - 3y^2 z + 3yz^2 - z^3 = 0$. It can be parametrized in *ruled form* as

$$\mathcal{P}(s, t) = \left(\frac{(s^3 - 1)t}{s(s + 1)}, \frac{s^2 t}{s + 1}, \frac{(s^2 + 1)t}{s + 1} \right)$$

Applying the algorithm in [SSV14b] for computing the critical sets, we obtain that $\mathcal{P}$ covers all the surface but the three lines $\{x = 2y = z\}, \{x = y = z\}, \{y = z = 0\}$. However, the reparametrization

$$\mathcal{P}(s, ts(s + 1)) = \left((s^2 - 1)t, ts^2, (s^2 + s)t \right),$$

that is also in *ruled form*, only misses the line $\{x = y = z\}$ (see Theorem 2.6).

In this paper we prove that a rational ruled surface can always be covered with two rational surface parametrizations in *ruled form*. More precisely, we prove that there always exists a rational parametrization that, at most, misses a line on the surface; then the second parametrization covers that line. In order to compute the first parametrization we need parametrizations without affine base points. Later we consider this problem in general, and we present an algorithm that transforms any rational surface parametrization into a new parametrization without affine base points.

2 Covering Ruled Surfaces: Main Results

In the sequel, we show that every rational ruled surface can be covered by means of, at most, two rational parametrizations.

Definition 2.1. Let k be an algebraically closed field of characteristic zero. A *standardized ruled surface parametrization* of a ruled surface S is a triple of rational functions that determines a dominant rational map

$$
\begin{array}{rcc}
\mathcal{P}: & k^2 & \dashrightarrow & S \\
\\
& (s,t) & \mapsto & \left(\dfrac{r_1(s) + t \cdot p_1(s)}{q(s)}, \dfrac{r_2(s) + t \cdot p_2(s)}{q(s)}, \dfrac{r_3(s) + t \cdot p_3(s)}{q(s)} \right)
\end{array}
$$

such that those p_i that are nonzero have the same degree and do not have any common root (note that not all three of them are zero).

Remark 2.2. In a standardized ruled surface parametrization, if two of the polynomials p_i are zero, then the third has to be a nonzero constant. In addition, we observe that, in that case, say $p_1 = p_2 = 0$, then $\mathcal{P}(s, (-r_3 + qt)/p_3) = (r_1/q, r_2/q, t)$. So, S is a cylinder over the plane curve $(r_1/q, r_2/q, 0)$ and hence, applying the results in [Sen02], S can be parametrized surjectively.

Lemma 2.3. *Every rational ruled surface admits a standardized ruled surface parametrization.*

Proof. By [SPD14], every rational ruled surface admits a rational parametrization of the form

$$
\mathcal{P}(s,t) = \left(\frac{\alpha_1(s) + t\beta_1(s)}{\gamma(s)}, \frac{\alpha_2(s) + t\beta_2(s)}{\gamma(s)}, \frac{\alpha_3(s) + t\beta_3(s)}{\gamma(s)} \right).
$$

If two β_i are zero, say $\beta_1 = \beta_2 = 0$, then $\mathcal{P}(s, t/\beta_3(t))$ is standardized. Let us suppose that at least two β_i are nonzero. Then, we can assume that those components of $\mathcal{P}(s,t)$ depending on t also do depend on s; if this is not the case a suitable change of the form $(s, as + bt)$ provides a parametrization with this property. Furthermore, applying a transformation of the form $(\frac{as+b}{cs+d}, t)$, we can assume that all nonzero β_i have the same degree. It only remains to ensure that the gcd of the polynomial coefficients of t are coprime. But this can be achieved by performing the transformation $(s, t/\Delta(s))$, where Δ is the gcd of the nonzero β_i. □

Associated to the standardized ruled surface parametrization $\mathcal{P}(s,t)$, we consider the polynomials

$$
\begin{aligned}
H_1 &= r_1(s) + t \cdot p_1(s) - x \cdot q(s), \\
H_2 &= r_2(s) + t \cdot p_2(s) - y \cdot q(s), \\
H_3 &= r_3(s) + t \cdot p_3(s) - z \cdot q(s),
\end{aligned}
\tag{1}
$$

as well as the polynomials $A_{ij} = p_i H_j - p_j H_i \in k[x,y,z,s]$ for $i \neq j$. We express A_{ij} as

$$
\begin{aligned}
A_{12} &= qp_2 x - qp_1 y - \alpha_{12}, \\
A_{13} &= qp_3 x - qp_1 z - \alpha_{13}, \\
A_{23} &= qp_3 y - qp_2 z - \alpha_{23}, \\
\alpha_{ij} &= -p_i r_j + p_j r_i.
\end{aligned}
\tag{2}
$$

We have the following lemma.

Lemma 2.4. *Let $\mathcal{P}$ be a standardized ruled surface parametrization without affine base points of a surface S. Then $S \setminus \mathrm{Image}(\mathcal{P})$ is contained in the variety $\mathcal{W}$ defined by $\{\mathrm{LC}_s(A_{ij})\}_{i \neq j}$, where LC_s denotes the leading coefficient w.r.t. s.*

Proof. In the ring $k[x, y, z, s, t, w]$ we consider the ideal

$$I = (H_1(s, t, x), \ H_2(s, t, y), \ H_3(s, t, z), \ w \cdot q(s) - 1).$$

Then $\text{Image}(\mathcal{P}) = \pi(V(I))$ where $\pi(x, y, z, s, t, w) = (x, y, z)$. We will use the extension theorem (see e.g. Ch.3, Th 3, p. 115 in [CLO07]) to determine which points $(x, y, z) \in S$ can be lifted to $V(I)$. To this end we define

$$I_1 = I \cap k[x, y, z, s, t], \quad I_2 = I \cap k[x, y, z, s], \quad I_3 = I \cap k[x, y, z].$$

- Extension from I_1 to I: a point $(x_0, y_0, z_0, s_0, t_0)$ has an extension provided $q(s_0) \neq 0$. But if $q(s_0) = 0$ we see from the equations that $r_i(s_0) + t_0 p_i(s_0) = 0$ for all i, and (s_0, t_0) would be a base point, contrary to the hypotheses.

- Extension from I_2 to I_1: in order to extend a point (x_0, y_0, z_0, s_0) to the coordinate t it suffices that p_1, p_2, p_3 do not simultaneously vanish at s_0. This always holds since by definition they have no common root. Note that if two of the p_i are zero, the other is a nonzero constant, and the extension is possible.

- Extension from I_3 to I_2: a point (x, y, z) can be extended to the coordinate s if for at least one of the polynomials A_{ij} the leading coefficient in s does not vanish at the point.

$\square$

Lemma 2.5. *The variety $\mathcal{W}$ introduced in Lemma 2.4 is either empty or a line. Furthermore, $\mathcal{W} = \emptyset$ if and only if $\deg(\alpha_{ij}) > \deg(p_k q)$, for some different $i, j \in \{1, 2, 3\}$ and nonzero p_k.*

Proof. Let us assume $p_1 \neq 0$. If $\deg(\alpha_{ij}) > \deg(p_1 q)$, for some different $i, j \in \{1, 2, 3\}$, then $\text{LC}_s(A_{ij})$ is a nonzero constant and $\mathcal{W} = \emptyset$.

If $\deg(\alpha_{ij}) \leq \deg(p_1 q)$ for all $i \neq j$, we distinguish two cases. If $p_2 = p_3 = 0$ then $A_{12} = -p_1 q y - \alpha_{12}, A_{13} = -p_1 q z - \alpha_{13}, A_{23} = 0$. Then, $\mathcal{W}$ is defined by two linear polynomials, one depending on y and the other on z. So $\mathcal{W}$ is a line. In the second case, let us assume that at least two p_i are nonzero. Since $p_3 A_{12} - p_2 A_{13} + p_1 A_{23} = 0$, then

$$\text{LC}_s(p_1 A_{23}) = \text{LC}_s(p_1) \text{LC}_s(A_{23}) = \text{LC}_s(-p_3 A_{12} + p_2 A_{13}).$$

Let us see that

$$\text{LC}_s(p_1) \text{LC}_s(A_{23}) = -\text{LC}_s(p_3) \text{LC}_s(A_{12}) + \text{LC}_s(p_2) \text{LC}_s(A_{13}).$$

If either p_2 or p_3 is zero, the result is clear. So, let none of them be zero. Then, $\deg_s(p_3 A_{12}) = \deg_s(p_2 A_{13})$. Since the leading coefficient of $p_3 A_{12}$ does depend on $\{x, y\}$ and the leading coefficient of $p_2 A_{13}$ does depend on $\{y, z\}$, $\deg_s(-p_3 A_{12} + p_2 A_{13}) = \deg_s(p_3 A_{12}) = \deg_s(p_2 A_{13})$, from where the above equality on the leading coefficients follows. In this situation we get that $\mathcal{W}$ is defined by $\text{LC}_s(A_{12}), \text{LC}_s(A_{13})$. Now, the result follows by taking into account that the rank of the linear system $\{\text{LC}_s(A_{12}) = 0 = \text{LC}_s(A_{13})\}$ is 2. $\square$

Using the previous results one gets the following theorem.

Theorem 2.6. *Let $\mathcal{P}$ be a standardized ruled surface parametrization without affine base points of a surface S. Then $S \smallsetminus \text{Image}(\mathcal{P})$ is contained in a line. Furthermore,*

1. *if there exists $i, j \in \{1, 2, 3\}$ such that $i \neq j$ and $\deg(\alpha_{ij}) > \deg(p_k q)$ for nonzero p_k, then $\mathcal{P}(s, t)$ is normal.*

2. *if for all $i, j \in \{1, 2, 3\}$, with $i \neq j$, $\deg(\alpha_{ij}) \leq \deg(p_k q)$ for nonzero p_k, then $S \smallsetminus \mathrm{Image}(\mathcal{P})$ is included in the line $V(\mathrm{LC}_s(A_{12}), \mathrm{LC}_s(A_{13}), \mathrm{LC}_s(A_{23}))$.*

In Example 3.4, one can see that the parametrization covers all the line $\mathcal{W}$ but a point, while in Example 3.5, the parametrization only covers two points on the line $\mathcal{W}$.

In the previous theorem we have imposed the condition of not having affine base points. Let us see that this is always achievable.

Lemma 2.7. *Every standardized ruled surface parametrization can be reparametrized into another one without affine base points and where the degree of the induced map is preserved.*

Proof. Let us assume first that all the p_i are nonzero. Let $f(s)$ be a polynomial such that $f(s_1) = t_1$ for some base point (s_1, t_1). With the change $(s, 1/t + f(s))$ the resulting parametrization is

$$\left(\frac{Q_i(s) \cdot t + p_i(s)}{t \cdot q(s)} \right)_{i=1,2,3} \qquad \text{where } Q_i(s) = r_i(s) + f(s) p_i(s).$$

Since s_1 is a common root of q and the Q_i, if we define $\widetilde{Q} = \gcd(Q_1, Q_2, Q_3, q)$, we have $\deg(\widetilde{Q}) \geq 1$. Now with the change $(s, 1/(\widetilde{Q}t))$ we obtain the new parametrization

$$\left(\frac{Q_i/\widetilde{Q} + p_i \cdot t}{q/\widetilde{Q}} \right)_{i=1,2,3}.$$

Note that this is standardized as well, but the degree of the denominator is strictly smaller than the original. Therefore repeating this procedure finitely many times we obtain a standardized parametrization without affine base points (since that is the case when the denominator is a constant). Finally, note that all transformations considered are birational, and hence the degrees of the maps are preserved.

If any $p_i = 0$, the corresponding component of the parametrization does not change after the first reparametrization, resulting in $Q_i = r_i$. The second reparametrization does not change the component as well, but the common factor $\widetilde{Q}$ of r_i and q can be directly simplified in that fraction. $\qquad \square$

Remark 2.8. In the previous result we can have some control on the removal of base points that occurs effectively in each iteration. Namely, suppose that the zeros of q are $s_1, \ldots, s_l$ where $s_1, \ldots, s_k$, $k \leq l$, are the first coordinates of the base points of $\mathcal{P}$. Note that for each of $s_1, \ldots, s_k$ there is exactly one base point (s_i, t_i).

Let $f(s)$ be an interpolating polynomial of

$$(s_1, t_1), \ldots, (s_k, t_k), (s_{k+1}, 0), \ldots, (s_l, 0).$$

As before, we define $Q_i(s) = r_i(s) + f(s) p_i(s)$ and $\widetilde{Q} = \gcd(Q_1, Q_2, Q_3, q)$, and make the change $(s, \widetilde{Q}t + f(s))$ to obtain

$$\mathcal{P}^{(1)} = \left(\frac{r_i^{(1)} + p_i \cdot t}{q^{(1)}} \right)_{i=1,2,3} \qquad \text{where } r_i^{(1)} = Q_i/\widetilde{Q}, \; q^{(1)} = q/\widetilde{Q}.$$

Note that the roots of $\widetilde{Q}$ are precisely $s_1, \ldots, s_k$. We will show that $\mathcal{P}^{(1)}$ has at most as many base points as the number of multiple roots of q among $s_1, \ldots, s_k$. To this end let (α, β) be a base point of $\mathcal{P}^{(1)}$. Since α is a root of $q^{(1)}$, it must be a multiple root of q, say $\alpha = s_i$. If $i > k$ then $Q_i(\alpha) = r_i(\alpha)$. Now, by definition of β, we have $\beta = -r_i^{(1)}(\alpha)/p_i(\alpha)$ for some i. But then the point $(\alpha, \beta \widetilde{Q}(\alpha))$ is a base point of $\mathcal{P}$, contradiction.

Corollary 2.9. *Every rational ruled surface can be parametrized in an standardized way that misses at most one line.*

Theorem 2.10. *Every rational ruled surface can be covered with at most two surface parametrizations.*

Proof. By Lemma 2.7 we can assume that we are given an standardized parametrization $\mathcal{P}$ without affine base points. We use for $\mathcal{P}$ the notation in Definition 2.1 and in the previous results. By Theorem 2.6, we can also assume that $\max\{\deg(\alpha_{12}), \deg(\alpha_{13}), \deg(\alpha_{23})\} \leq \deg(p_k q)$ for nonzero p_k.

First we assume that all $p_i(s)$ are nonzero. Consider the reparametrizations

$$\mathcal{Q}(s,t) = \mathcal{P}\left(s, \frac{qt - r_3}{p_3}\right) = \left(\frac{qp_1 t + \alpha_{13}}{p_3 q}, \frac{qp_2 t + \alpha_{23}}{p_3 q}, t\right)$$

and

$$\mathcal{H}(s,t) = \mathcal{Q}\left(\frac{1}{s}, t\right).$$

Because of our above degree assumptions, we know that the degrees in s of the numerator and denominator of each (first and second) component of $\mathcal{Q}$ are equal. Therefore, s is not a factor of the denominators in $\mathcal{H}(s,t)$. So, $\mathcal{H}(0,t)$ is well defined and, indeed,

$$\mathcal{H}(0,t) = \left(\frac{\text{LC}_s(qp_1 t + \alpha_{13})}{\text{LC}_s(p_3 q)}, \frac{\text{LC}_s(qp_2 t + \alpha_{23})}{\text{LC}_s(p_3 q)}, t\right)$$

that parametrizes the line $\mathcal{W}$.

A similar argument with obvious modifications works in the case when some p_i are zero. $\square$

3 Covering Ruled Surfaces: Algorithm and Examples

In order to derive an algorithm from the previous results, we need to algorithmically show how to remove the affine base points of an standardized ruled parametrization. This, essentially, requires to compute interpolation polynomials (see proof of Lemma 2.7 and Remark 2.8). In the following lemma we see how to actually compute the interpolation polynomial without explicitly determining the coordinates of the base points; i.e. without approximating roots.

Lemma 3.1. *Let $\mathcal{P}(s,t)$ be an standardized ruled parametrization as in Def. 2.1 with affine base points. Let I be the ideal generated by $\{p_1 t + r_1, p_2 t + r_2, p_3 t + r_3, q\}$ in $k[s,t]$. Then, there exists a polynomial of the form $t - f(s)$ in $\sqrt{I}$ where $f(s)$ interpolates the affine base points of $\mathcal{P}(s,t)$.*

Proof. As observed in Remark 2.8, all affine base points of $\mathcal{P}$ have different s-coordinate. Thus, there exists an interpolating polynomial $f(s)$ passing through all base points. So, $t - f(s)$ vanishes on all the points in the variety of I. So, $t - f(s) \in \sqrt{I}$. $\square$

Now, we are ready to outline our algorithm.

Algorithm 3.2. Given a rational parametrization $\mathcal{P}(s,t)$ of a ruled surface $\mathcal{S}$, the algorithm computes a covering of $\mathcal{S}$.

1. If $\mathcal{P}$ is not of the form $((r_i(s) + p_i(s)t)/q(s))_{i=1,2,3}$ apply the algorithm in [SPD14] and replace $\mathcal{P}$.

2. If $\mathcal{P}$ is not in standardized form (see Def. 2.1) do the following

   (a) If some of the numerators of $\mathcal{P}$ does not depend on s, replace $\mathcal{P}$ by $\mathcal{P}(s, as + bt)$ with $a, b \in k$.

   (b) If the polynomials p_1, p_2, p_3 do not have the same degree, replace $\mathcal{P}$ by $\mathcal{P}((as+b)/(cs+d), t)$ where $a, b, c, d \in k$ and $ad - bc \neq 0$.

   (c) Replace $\mathcal{P}$ by reparametrization $\mathcal{P}(s, t/\Delta(s))$ where Δ is the gcd of the nonzero p_i.

3. Calculate $\sqrt{I}$, where I is the ideal generated by $\{p_1 t + r_1, p_2 t + r_2, p_3 t + r_3, q\}$ in $k[s, t]$. This can be done with a relatively inexpensive Gröbner basis computation (see e.g. Ex 2.3.23 and 24 in [AL94] and [Sei74]).

4. Calculate a Gröbner basis of $\sqrt{I}$ with respect to the lexicographical ordering $t > s$.

   (a) If the basis does not contain a polynomial of the form $t - f(s)$, by elementary properties of Gröbner basis it follows that there is no polynomial of that form in $\sqrt{I}$, so by Lemma 3.1 we know that $\mathcal{P}$ does not have affine base points.

   (b) In the other case, let $t - f(s)$ belong to the basis, do

      i. Replace $\mathcal{P}$ by $\mathcal{P}(s, 1/t + f(s))$.

      ii. Let $\widetilde{Q}$ be the gcd of the coefficients of t of the numerators of $\mathcal{P}$ and q, then replace $\mathcal{P}$ by $\mathcal{P}(s, 1/(\widetilde{Q}t))$.

      iii. Repeat Steps 3 and 4 while $\sqrt{I}$ has an element of the form $s - f(t)$.

5. Compute the polynomials α_{ij} (see (2)).

6. If there exist $i, j \in \{1, 2, 3\}$ such that $i \neq j$ and $\deg(\alpha_{ij}) > \deg(p_k q)$ for nonzero p_k, RETURN $\mathcal{P}(s, t)$.

7. Assume that $p_k \neq 0$, compute

$$\mathcal{H}(s, t) = \mathcal{P}\left(\frac{1}{s}, \frac{q(1/s)t - r_k(1/s)}{p_k(1/s)}\right)$$

and RETURN $[\mathcal{P}(s, t), \mathcal{H}(s, t)]$.

Remark 3.3. Remark 2.8 shows that, in general, the number of iterations of the loop in Step 4 is small. Indeed it is bounded by the maximum multiplicity of the roots of the denominator $q(t)$ of $\mathcal{P}$.

In addition we observe that all parametrizations in the output of the algorithm are of ruled form, that is, of the form $(\alpha_1(s), \alpha_2(s), \alpha_3(s)) + t(\beta_1(s), \beta_2(s), \beta_3(s))$.

Let us illustrate Algorithm 3.2 by some examples.

Example 3.4. We consider the parametrization

$$\mathcal{P}(s, t) = \left(\frac{r_1(s) + tp_1(s)}{q(s)}, \frac{r_2(s) + tp_2(s)}{q(s)}, \frac{r_3(s) + tp_3(s)}{q(s)}\right) =$$

$$\left(\frac{t\left(s^2 + s + 1\right) + s}{s\left(s - 1\right)}, \frac{t\left(s^2 + 2s\right) + s}{s\left(s - 1\right)}, \frac{t\left(s^2 + 1\right) + s}{s\left(s - 1\right)}\right).$$

It parametrizes the degree 3 ruled surface defined by

$$F(x,y,z) = 5\,x^3 - 9\,x^2y - 8\,x^2z + 5\,xy^2 + 11\,xyz + 3\,xz^2 - y^3 - 3\,y^2z - 3\,yz^2 - 4\,x^2 + 4\,xy + 4\,xz - y^2 - 2\,yz - z^2.$$

We observe that $\mathcal{P}(s,t)$ is in standardized form. So, we go to Step 3 in Algorithm 3.2. I is the ideal generated by $\{p_1t + r_1, p_2t + r_2, p_3t + r_3, q\}$ in $k[s,t]$. We get $\sqrt{I} = I$, and a Gröbner basis w.r.t. the lexicographic ordering $t > s$ (Step 4) is $\{s,t\}$. So, in Step 4 (b) we get that $f(s) = 0$; note that the origin is the only affine base point. In Step 4 (b, i), we replace $\mathcal{P}(s,t)$ by $\mathcal{P}(s, 1/t + 0)$, namely

$$\mathcal{P}(s,t) = \left(\frac{s^2 + st + s + 1}{ts\,(s-1)}, \frac{s(s+2+t)}{ts\,(s-1)}, \frac{s^2 + st + 1}{ts\,(s-1)} \right).$$

In Step 4 (b, ii), $\widetilde{Q} = \gcd(s, s, s, s(s-1)) = s$. So, we replace $\mathcal{P}$ by $\mathcal{P}(s, 1/(st))$, namely

$$\mathcal{P}(s,t) = \left(\frac{s^2t + st + t + 1}{s - 1}, \frac{s^2t + 2\,st + 1}{s - 1}, \frac{s^2t + t + 1}{s - 1} \right). \tag{3}$$

Now, the lexicographic order Gröbner basis of $\sqrt{I}$ is $\{1\}$, hence $\mathcal{P}$ does not have base points. In Step 5 we get

$$\alpha_{12} = s - 1, \alpha_{13} = -s, \alpha_{23} = 2\,s - 1.$$

In Step 6 the boolean conditions do not hold. In Step 7 we calculate the parametrization

$$\mathcal{H} = \mathcal{P}\left(\frac{1}{s}, \frac{q(1/s)t - r_3(1/s)}{p_3(1/s)} \right) =$$

$$\left(\frac{s^3t - 2\,s^3 - s^2 - 2\,s - t}{(s^2 + 1)\,(s - 1)}, -\frac{s^3 - 2\,s^2t + 2\,s^2 + st + 2\,s + t}{(s^2 + 1)\,(s - 1)}, \frac{st - 2\,s - t}{s - 1} \right).$$

The algorithm returns the covering $[\mathcal{P}(s,t), \mathcal{H}(s,t)]$ where $\mathcal{P}$ is the parametrization in (3). Continuing
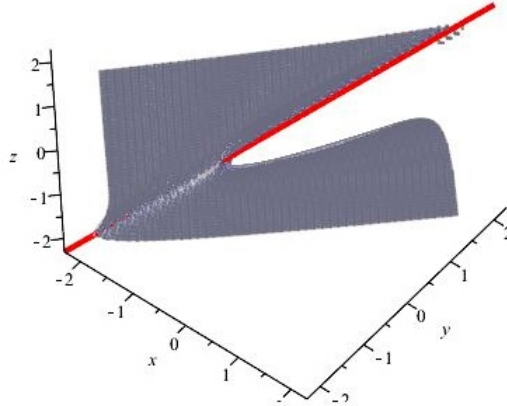

Figure 1: The surface in Example 3.4 and line (t,t,t).

with the example, since $\mathcal{P}$ in (3) is an standardized ruled parametrization without affine base points, by Theorem 2.6, the possible missing points of $\mathcal{P}$ are included in the line defined by $\{x - y = 0, x - z = 0, y - z = 0\}$, that is, the line (t,t,t); see Fig. 1. In fact, $\mathcal{P}$ covers all the line except the origin, by taking $\mathcal{P}(s,0)$. Nevertheless, $\mathcal{H}$ covers the whole line by taking $\mathcal{H}(0,t)$.

Example 3.5. We consider the parametrization

$$\mathcal{P}(s,t) = \left(\frac{r_1(s) + tp_1(s)}{q(s)}, \frac{r_2(s) + tp_2(s)}{q(s)}, \frac{r_3(s) + tp_3(s)}{q(s)} \right) =$$

$$\left(\frac{ts^3 + 2\,s^2 + 1}{s^2 - 1}, \frac{t\left(s^3 + 2\right) + s + 1}{s^2 - 1}, \frac{t\left(s^3 + s + 1\right) + 1}{s^2 - 1}\right).$$

It parametrizes the degree 5 ruled surface defined by

$F(x, y, z) = 9\,x^5 - 45\,x^4y - 24\,x^4z + 77\,x^3y^2 + 80\,x^3yz - 15\,x^3z^2 - 83\,x^2y^3 + 34\,x^2y^2z - 147\,x^2yz^2 + 78\,x^2z^3 + 66\,xy^4 - 60\,xy^3z - 189\,xy^2z^2 + 460\,xyz^3 - 236\,xz^4 + 16\,y^5 - 86\,y^4z + 111\,y^3z^2 + 118\,y^2z^3 - 332\,yz^4 + 168\,z^5 - 104\,x^4 + 319\,x^3y + 108\,x^3z - 207\,x^2y^2 - 621\,x^2yz + 452\,x^2z^2 + 147\,xy^3 - 382\,xy^2z + 1034\,xyz^2 - 848\,xz^3 + 297\,y^4 - 1549\,y^3z + 3390\,y^2z^2 - 3380\,yz^3 + 1344\,z^4 + 304\,x^3 - 741\,x^2y + 389\,x^2z - 267\,xy^2 + 1761\,xyz - 2314\,xz^2 - 4\,y^3 + 922\,y^2z - 1930\,z^2y + 1816\,z^3 - 70\,x^2 + 597\,yx - 2060\,zx + 748\,y^2 - 1703\,zy + 2940\,z^2 - 761\,x - 62\,y + 2085\,z + 746.$

We observe that $\mathcal{P}(s, t)$ is in standardized form, so we go to Step 3 in Algorithm 3.2. I is the ideal generated by $\{p_1t + r_1, p_2t + r_2, p_3t + r_3, q\}$ in $k[s, t]$. We get $\sqrt{I} = I$, and a Gröbner basis w.r.t. the lexicographic ordering $t > s$ (Step 4) is $\{1\}$. Thus $\mathcal{P}(s, t)$ does not have affine base points and we go to Step 5 to get

$$\alpha_{12} = 2s^5 - s^4 + 4s^2 + 2, \alpha_{13} = 2s^5 + 2s^3 + 2s^2 + s + 1, \alpha_{23} = -s^4 - s^2 - 2s + 1.$$

In Step 6 the boolean conditions do not hold. In Step 7 we get the parametrization

$$\mathcal{H} = \mathcal{P}\left(\frac{1}{s}, \frac{q(1/s)t - r_3(1/s)}{p_3(1/s)}\right) =$$

$$\left(-\frac{s^5 + s^4 + 2\,s^3 - ts^2 + 4\,s^2 + t + 2}{\left(s^3 + s^2 + 1\right)\left(s^2 - 1\right)},\right.$$

$$\left.\frac{2\,s^5t - 3\,s^5 - 2\,s^4 - 2\,ts^3 - s^3 + ts^2 - 2\,s^2 - s - t}{\left(s^3 + s^2 + 1\right)\left(s^2 - 1\right)}, \frac{ts^2 - 2\,s^2 - t}{s^2 - 1}\right).$$

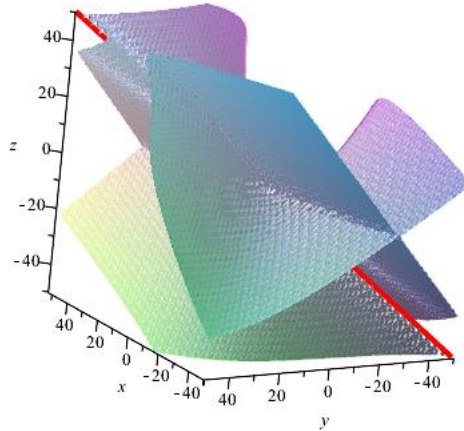The algorithm returns the covering $[\mathcal{P}(s, t), \mathcal{H}(s, t)]$.

Figure 2: The surface in Example 3.5 and line $(t + 2, t, t)$.

Next, since the input parametrization $\mathcal{P}$ is an standardized ruled parametrization without affine base points, by Theorem 2.6, the possible missing points of $\mathcal{P}$ are included in the line defined by

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$\{x - y - 2, x - z - 2, y - z\}$, that is, the line $(t + 2, t, t)$; see Fig. 2. In fact, on the line, $\mathcal{P}$ reaches only the points

$$\mathcal{P}\left(\frac{1}{2}(1 - i\sqrt{7}), \frac{1}{4}(3 + i\sqrt{7})\right) = \left(\frac{3}{32}i\sqrt{7} + \frac{65}{32}, \frac{3}{32}i\sqrt{7} + \frac{1}{32}, \frac{3}{32}i\sqrt{7} + \frac{1}{32}\right),$$

$$\mathcal{P}\left(\frac{1}{2}(1 + i\sqrt{7}), \frac{1}{4}(3 - i\sqrt{7})\right) = \left(-\frac{3}{32}i\sqrt{7} + \frac{65}{32}, -\frac{3}{32}i\sqrt{7} + \frac{1}{32}, -\frac{3}{32}i\sqrt{7} + \frac{1}{32}\right)$$

Nevertheless, $\mathcal{H}$ covers the whole line by taking $\mathcal{H}(0, t)$.

Example 3.6. We consider the parametrization

$$\mathcal{P}(s, t) = \left(\frac{r_1(s) + tp_1(s)}{q(s)}, \frac{r_2(s) + tp_2(s)}{q(s)}, \frac{r_3(s) + tp_3(s)}{q(s)}\right) =$$

$$\left(\frac{ts + (-3s + 2)s^5}{(s - 1)s^2}, \frac{t(s + 1) + (-5s + 3)s^2}{(s - 1)s^2}, \frac{t(s + 2) + (-8s + 5)s^2}{(s - 1)s^2}\right).$$

It parametrizes the degree 4 ruled surface defined by

$F(x, y, z) = xy^3 - 3y^2zx + 3xyz^2 - xz^3 - 2y^4 + 7zy^3 - 9y^2z^2 + 5yz^3 - z^4 - 9y^2x + 18zyx - 9z^2x - 5y^3 - 9y^2z + 16z^2y - 5z^3 + 27yx - 27zx - 78y^2 + 89zy - 23z^2 - 27x - 14y + 17z + 12.$

We observe that $\mathcal{P}(s, t)$ is in standardized form, so we go to Step 3 in Algorithm 3.2. I is the ideal generated by $\{p_1t + r_1, p_2t + r_2, p_3t + r_3, q\}$ in $k[s, t]$. We get $\sqrt{I} \neq I$, and a Gröbner basis of $\sqrt{I}$ w.r.t. the lexicographic ordering $t > s$ (Step 4) is $\{t^2 - t, -t + s\}$. So, in Step 4 (b) we get that $f(s) = s$. Note that the affine base points are $(0, 0)$ and $(1, 1)$ and $t = s$ is the interpolating line; observe that the corresponding Gröbner basis of I, $\{t^2 - t, st - t, s^2 - t\}$, that does not read the interpolating polynomial of minimal degree, although it contains the parabola $t = s^2$ that passes through the base points. In Step 4 (b, i), we replace $\mathcal{P}(s, t)$ by $\mathcal{P}(s, 1/t + s)$, namely

$$\mathcal{P}(s, t) = \left(-\frac{3s^5t - 2s^4t - ts - 1}{st(s - 1)}, -\frac{5s^3t - 4ts^2 - ts - s - 1}{t(s - 1)s^2},\right.$$

$$\left.-\frac{8s^3t - 6ts^2 - 2ts - s - 2}{t(s - 1)s^2}\right).$$

In Step 4 (b, ii), $\widetilde{Q} = s^2 - s$. So, we replace $\mathcal{P}$ by $\mathcal{P}(s, 1/((s^2 - s)t))$, namely

$$\mathcal{P}(s, t) = \left(\frac{-3s^4 - s^3 - s^2 + ts - s}{s}, \frac{ts - 5s + t - 1}{s}, \frac{ts - 8s + 2t - 2}{s}\right).$$

Now, the lexicographic order Gröbner basis of $\sqrt{I}$ is $\{s, t - 1\}$, and hence $\mathcal{P}$ still have one base point, namely $(0, 1)$. Now, the interpolation polynomial is $f(s) = 1$ and $\widetilde{Q} = s$. Repeating the steps as above we reach at the end of Step 4

$$\mathcal{P}(s, t) = (-(3s^2 + s - t + 1)s, ts + t - 4, ts + 2t - 7) \tag{4}$$

In Step 5 we get

$$\alpha_{12} = -3s^5 - 4s^4 - 2s^3 + 3s^2, \alpha_{13} = -3s^5 - 7s^4 - 3s^3 + 5s^2, \alpha_{23} = -3s^2 + s$$

In Step 6 the boolean conditions do hold, and the output is the parametrization $\mathcal{P}$ in (4) which is normal.

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4 Removal of Base Points: the General Case

In Lemma 2.7 we have seen that, for the special case of standardized ruled surfaces, one can always find a reparametrization such that the new parametrization does not have affine base points. In this section we see that the ideas applied in the proof of that lemma can be generalized to any rational parametrization. More precisely we have the following result.

Theorem 4.1. *Let* $\mathcal{P}\colon k^2 \dashrightarrow k^3$ *be an affine rational parametrization, with nonconstant components, of a surface. Then there exists a rational reparametrization* $\mathcal{P} \circ \psi$ *without affine base points. Moreover,* $\deg(\mathcal{P}) = \deg(\mathcal{P} \circ \psi)$ *as rational maps; in particular, properness is preserved.*

Proof. If $\mathcal{P}$ has no affine base points, take as ψ the identity.

We can assume without loss of generality that, after a suitable linear birational change,

$$\mathcal{P}(s,t) = \left(\frac{p_1(s,t)}{q(s,t)}, \frac{p_2(s,t)}{q(s,t)}, \frac{p_3(s,t)}{q(s,t)} \right)$$

where $\deg(p_1) = \deg(p_2) = \deg(p_3) = \deg(q)$, $\gcd(p_1, p_2, p_3, q) = 1$, and the projective point $(0 : 1 : 0)$ does not belong to any of the projectivizations of the four curves determined by numerators and denominator. We also assume that there are no two affine base points with the same s-coordinate, since this can be achieved by composition with $(s,t) \to (s + \lambda t, t)$ for generic λ without losing the previous assumptions.

By the last assumption, there exists an interpolation polynomial $f(s)$ for the affine base points, i.e. for every base point (s_i, t_i) we have $t_i = f(s_i)$; note that the gcd condition implies finiteness of the base point set. We define the birational reparametrization

$$\psi(s,t) = \left(s, \frac{1}{t} + f(s) \right)$$

and $\widetilde{\mathcal{P}} = \mathcal{P} \circ \psi$. We will prove that $\widetilde{\mathcal{P}}$ has no affine base points. To this end we write

$$\mathcal{P} = \left(\frac{a_n t^n + a_{n-1}(s) t^{n-1} + \cdots + a_0(s)}{b_n t^n + b_{n-1}(s) t^{n-1} + \cdots + b_0(s)}, \ldots, \ldots \right)$$

with $a_n, b_n \neq 0$ and $\deg(a_i), \deg(b_i) \leq n - i$. This is possible by the hypothesis on the degrees of p_1, p_2, p_3, q, and the fact that t^n appears in all of them with nonzero coefficient (equivalent to the hypothesis on $(0 : 1 : 0)$.) Then

$$\widetilde{\mathcal{P}} = \left(\frac{a_n(1 + tf(s))^n + ta_{n-1}(s)(1 + tf(s))^{n-1} + \cdots + t^n a_0(s)}{b_n(1 + tf(s))^n + tb_{n-1}(s)(1 + tf(s))^{n-1} + \cdots + t^n b_0(s)}, \ldots, \ldots \right).$$

This new parametrization cannot have any base points of the form $(s_0, 0)$, since $\widetilde{\mathcal{P}}(s_0, 0) = (a_n/b_n, \ldots, \ldots)$. On the other hand, if (s_0, t_0) is a base point of $\widetilde{\mathcal{P}}$ with $t_0 \neq 0$, then $\psi(s_0, t_0)$ is a base point (s_i, t_i) of $\mathcal{P}$. But this is impossible: if $\psi(s_0, t_0) = (s_i, t_i)$ then $s_0 = s_i$ and $1/t_0 + f(s_0) = t_i$ which imply $1/t_0 = 0$, contradiction. Finally, note that the previous transformations are birational, and ψ is a birational map from k^2 on k^2, and hence the degree of the parametrization maps is preserved. $\square$

The reasoning in the previous proof leads to an algorithmic process to remove the affine base points of a surface parametrization. To be more precise, let

$$\mathcal{P}(s,t) = \left(\frac{p_1(s,t)}{q(s,t)}, \frac{p_2(s,t)}{q(s,t)}, \frac{p_3(s,t)}{q(s,t)} \right)$$

be the surface parametrization. First, we observe that some assumptions on the parametrization are done, namely

1. [degree and gcd condition] $\deg(p_1) = \deg(p_2) = \deg(p_3) = \deg(q)$, and $\gcd(p_1, p_2, p_3, q) = 1$,

2. [condition on $(0 : 1 : 0)$] the projective point $(0 : 1 : 0)$ does not belong to any of the projectivizations of the four curves determined by numerators and denominator,

3. [general position of the base points] there are no two affine base points with the same s-coordinate.

Observe that, in the rational ruled case, condition 3 is satisfied while, in general, conditions 1 and 2 fail because of the particular structure of standardized form, that we wanted to be preserved. So, in Section 2, we have developed an *ad hoc* proof for the ruled case.

Once the parametrization satisfies these conditions, one computes the interpolation polynomial $f(s)$ passing through the affine base points. Then, $\psi(s, t) = (s, \frac{1}{t} + f(s))$. We observe that condition 1 can always be achieved by a birational change of the form

$$\left(\frac{a_1 t + b_1 s + c_1}{d_1 t + e_1 s + h_1}, \frac{a_2 t + b_2 s + c_2}{d_2 t + e_2 s + h_2} \right),$$

and condition 2 with a linear change $(s + \lambda t, t)$. In the following lemma we see how to check the third condition and how to actually compute the interpolation polynomial $f(s)$ without approximating roots. This result extends Lemma 3.1 to the general case.

Lemma 4.2. *Let I be the ideal generated by $\{p_1, p_2, p_3, q\}$ in $k[s, t]$.*

1. *Condition 3 is satisfied if and only if there exists a polynomial of the form $t - g(s)$ in $\sqrt{I}$.*

2. *If $t - g(s) \in \sqrt{I}$, then $g(s)$ interpolates the affine base points.*

Proof. If condition 3 holds, then $t - f(s)$ vanishes on all the points in the variety of I. So, $t - f(s) \in \sqrt{I}$. The converse is trivial, and (2) follows from (1). □

Algorithm 4.3. The input is a rational surface parametrization with affine base points, and the output is a parametrization of the same surface without base points.

1. Reparametrize the input to satisfy conditions 1 and 2.

2. Calculate $\sqrt{I}$; see Step 3 in Algorithm 3.2.

3. Calculate a Gröbner basis of $\sqrt{I}$ with respect to the lexicographical ordering $t > s$.

   (a) If the basis contains a polynomial of the form $t - f(s)$, then by the previous Lemma condition 3 is satisfied and we can apply the reparametrization of Theorem 4.1 to RETURN $\mathcal{P}(s, 1/t + f(s))$.

   (b) In the negative case, by elementary properties of Gröbner bases it follows that there is no polynomial of that form in $\sqrt{I}$. Again by Lemma 4.2, condition 3 is not satisfied. Apply a transformation $(s + \lambda t, t)$ for random λ in the ground field and go to step 2.

As a consequence of Theorem 4.1 and Algorithm 4.3, the following corollaries hold.

Corollary 4.4. *Every rational surface can be parametrized without affine base points.*

Corollary 4.5. *Every rational surface parametrization can be reparametrized, without affine base points, without extending the field of coefficients and the degree as rational maps.*

We illustrate the ideas of this section by an example.

Example 4.6. We consider the rational parametrization

$$\mathcal{P}(s,t) = \left(\frac{p_1(s,t)}{q(s,t)}, \frac{p_2(s,t)}{q(s,t)}, \frac{p_3(s,t)}{q(s,t)} \right) = \left(\frac{4\,s^2 - 4\,st + t^2 - 6\,s + 3\,t}{2\,s^2 + 8\,st + 3\,t^2 - 8\,s - 11\,t}, \right.$$

$$\left. \frac{s^2 - 6\,st - t^2 + s + 7\,t}{2\,s^2 + 8\,st + 3\,t^2 - 8\,s - 11\,t}, \frac{-3\,s^2 + 22\,st + 4\,t^2 - 5\,s - 26\,t}{2\,s^2 + 8\,st + 3\,t^2 - 8\,s - 11\,t} \right)$$

Its base points are $\{(0,0),(2,1),(1,2),(1,-1)\}$. We observe that $\mathcal{P}(s,t)$ satisfies conditions 1 and 2. Let I be the ideal generated by $\{p_1, p_2, p_3, q\}$. A Gröbner basis of $\sqrt{I}$ w.r.t. the lexicographic order with $t > s$ is

$$\{s^3 - 3s^2 + 2s, -s^2 + 2st + s - 2t, 2s^2 + t^2 - 4s - t\}.$$

Since there is no polynomial of the form $t - f(s)$ in the basis, condition 3 fails, and we perform a change of parameters. For example $\mathcal{P}$ is replaced by $\mathcal{P}(s+t,t)$ (note that in this example a simple coordinate swap would work as well, because the points have different t-coordinates). Applying again the Gröbner basis computation to $\sqrt{I}$ for the new $\mathcal{P}$, we obtain the basis

$$\{s^4 - 2s^3 - s^2 + 2s, 2s^3 - 3s^2 - s + 2t\}.$$

The second polynomial implies that $t - (-s^3 + (3/2)s^2 + (1/2)s) \in \sqrt{J}$. So condition 3 is now satisfied and $f(s) = -s^3 + (3/2)s^2 + (1/2)s$. Therefore, performing the transformation $\mathcal{P}(s, 1/t + f(s))$ we get a new parametrization without affine base points, namely

$$\left(\frac{4s^6t^2 - 12s^5t^2 - 11s^4t^2 + 42s^3t^2 - 8s^3t + 7s^2t^2 + 12s^2t - 30t^2s + 20st - 12t + 4}{52s^6t^2 - 156s^5t^2 + 17s^4t^2 + 226s^3t^2 - 104s^3t - 69s^2t^2 + 156s^2t - 70t^2s + 100st - 76t + 52}, \right.$$

$$-2\frac{12s^6t^2 - 36s^5t^2 + 7s^4t^2 + 46s^3t^2 - 24s^3t - 19s^2t^2 + 36s^2t - 10t^2s + 20st - 16t + 12}{52s^6t^2 - 156s^5t^2 + 17s^4t^2 + 226s^3t^2 - 104s^3t - 69s^2t^2 + 156s^2t - 70t^2s + 100st - 76t + 52},$$

$$\left. \frac{92s^6t^2 - 276s^5t^2 + 51s^4t^2 + 358s^3t^2 - 184s^3t - 143s^2t^2 + 276s^2t - 82t^2s + 156st - 124t + 92}{52s^6t^2 - 156s^5t^2 + 17s^4t^2 + 226s^3t^2 - 104s^3t - 69s^2t^2 + 156s^2t - 70t^2s + 100st - 76t + 52} \right).$$

Example 4.7. In [Wan04], section 4.5, the author tests his implicitization algorithm with a family of rational surface parametrizations collected from different papers. For those having affine points, we apply Algorithm 4.3:

1. Example 1 in [Wan04]. The parametrization is

$$\mathcal{P} = \left(\frac{st^2 - t^3 - t}{t^2 - 2\,t + 1}, \frac{t^3 - st - t^2 + t + 1}{t^2 - 2\,t + 1}, \frac{st - 2\,t}{t^2 - 2\,t + 1} \right).$$

The Gröbner basis of $\sqrt{I}$ w.r.t. the lexicographical ordering $t > s$ is $\{s - 2, t - 1\}$; indeed $\mathcal{P}$ has the affine base point $(2,1)$. So the interpolating polynomial is $f(s) = 1$. Therefore, $\mathcal{P}(s, 1/t + 1)$ does not have affine base points.

2. Example 6 in [Wan04]. The parametrization is

$$\mathcal{P} = \left(\frac{s\,(s+t-1)}{s^2 + st + t^2 - 1}, \frac{t\,(s+t-1)}{s^2 + st + t^2 - 1}, \frac{s+t-1}{s^2 + st + t^2 - 1} \right).$$

The Gröbner basis of $\sqrt{I}$ w.r.t. the lexicographical ordering $t > s$ is $\{s^2 - s, s + t - 1\}$; indeed $\mathcal{P}$ has the affine base points $(0,1),(1,0)$. So the interpolating polynomial is $f(s) = 1 - s$. Therefore, $\mathcal{P}(s, 1/t + (1 - s))$ does not have affine base points.

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3. Example 9 in [Wan04]. The parametrization is

$$\mathcal{P} = \left(\frac{s^2 t + 2\,t^3 + s^2 + 4\,st + 4\,t^2 + 3\,s + 2\,t + 2}{s^3 + s^2 t + t^3 + s^2 + t^2 - s - t - 1}, \right.$$

$$\frac{-s^3 - 2\,st^2 - 2\,s^2 - st + s - 2\,t + 2}{s^3 + s^2 t + t^3 + s^2 + t^2 - s - t - 1},$$

$$\left. \frac{-s^3 - 2\,s^2 t - 3\,st^2 - 3\,s^2 - 3\,st + 2\,t^2 - 2\,s - 2\,t}{s^3 + s^2 t + t^3 + s^2 + t^2 - s - t - 1} \right).$$

The Gröbner basis of $\sqrt{I}$ w.r.t. the lexicographical ordering $t > s$ is

$$\{9\,s^6 + 8\,s^5 - 12\,s^4 + 27\,s^3 + 34\,s^2 - 44\,s - 40,$$
$$1665\,s^5 + 382\,s^4 - 2152\,s^3 + 4939\,s^2 + 1540\,s + 3288\,t - 4268\}.$$

So $\mathcal{P}$ has 6 affine base points, and the interpolation polynomial is

$$f(s) = -\frac{555}{1096}\,s^5 - \frac{191}{1644}\,s^4 + \frac{269}{411}\,s^3 - \frac{4939}{3288}\,s^2 - \frac{385}{822}\,s + \frac{1067}{822}.$$

Therefore, $\mathcal{P}(s, 1/t + f(s))$ does not have affine base points.

4. Example 10 in [Wan04]. The parametrization is

$$\mathcal{P} = \left(\frac{-s^4 + 4\,s^3 t - 2\,s^2 t^2 + st^3 + s^2 t - 2\,t^3}{-s^3 t + 6\,s^2 t^2 - 3\,st^3 + t^4 + s^3 - 2\,st^2}, \right.$$

$$\frac{-s^3 t - 2\,s^3 + s^2 t + 3\,st^2 - t^3}{-s^3 t + 6\,s^2 t^2 - 3\,st^3 + t^4 + s^3 - 2\,st^2},$$

$$\left. \frac{-st^3 + s^3 - 4\,s^2 t - st^2 + 6\,t^3}{-s^3 t + 6\,s^2 t^2 - 3\,st^3 + t^4 + s^3 - 2\,st^2} \right).$$

The Gröbner basis of $\sqrt{I}$ w.r.t. the lexicographical ordering $t > s$ is

$$\{s^6 - 7\,s^5 - 20\,s^4 + 173\,s^3 - 27\,s^2 + s,$$
$$-176\,s^5 + 1205\,s^4 + 3605\,s^3 - 29867\,s^2 + 2371\,s + 703\,t\}.$$

So $\mathcal{P}$ has 6 affine base points, and the interpolation polynomial is

$$f(s) = \frac{176}{703}\,s^5 - \frac{1205}{703}\,s^4 - \frac{3605}{703}\,s^3 + \frac{29867}{703}\,s^2 - \frac{2371}{703}\,s.$$

Therefore, $\mathcal{P}(s, 1/t + f(s))$ does not have affine base points.

5   Acknowledgements

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