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Detecting when an implicit equation or a rational parametrization defines a conical or cylindrical surface, or a surface of revolution

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Abstract—Given an implicit polynomial equation or a rational parametrization, we develop algorithms to determine whether the set of real and complex points defined by the equation, i.e. the *surface* defined by the equation, in the sense of Algebraic Geometry, is a cylindrical surface, a conical surface, or a surface of revolution. The algorithms are directly applicable to, and formulated in terms of, the implicit equation or the rational parametrization. When the surface is cylindrical, we show how to compute the direction of its rulings; when the surface is conical, we show how to compute its vertex; and when the surface is a surface of revolution, we show how to compute its axis of rotation directly from the defining equations.

equations of a surface, cylindrical surface, conical surface, surface of revolution, shape recognition.

1 Introduction

Surfaces invariant under either scaling or rotation are common both in the natural world and in human creations. Cylindrical surfaces are the surfaces invariant under scaling in a fixed direction; conical surfaces are the surfaces invariant under scaling from a fixed point; and surfaces of revolution are the surfaces invariant under rotation around a fixed axis. These invariance properties make it easy to generate these surfaces in a computer graphics package and then to use these surfaces for industrial design.

However, the information regarding how surfaces are generated can be lost when geometric data are exchanged between different solid modeling systems. This problem is extensively addressed by Elber et al. in the Introduction to [5], where the authors compare the data file formats employed by several modeling systems. If the history corresponding to the construction of a model is not preserved when moving from one modeling system to another, some operations can become much more difficult. Elber et. al cite two such examples: automatic generation of toolpaths for NC machining in the case of surfaces of revolution, and surface-surface intersections. We can identify several more examples. For instance, if a surface is recognized as a cylindrical surface, its offset can easily be computed as another cylindrical surface whose rulings are parallel to the rulings of the original surface, and whose directrix is the offset of a normal section of the original surface. Similarly, the offset of a surface of revolution is also a surface of revolution [14] that can be generated by rotating the offset of the directrix curve about the axis of revolution of the original surface. The symmetries of a cylindrical surface follow from the symmetries of a section of the surface normal to the direction of the rulings [2]. Furthermore, the symmetries of a surface of revolution also follow from the symmetries of the directrix curve [2]. Therefore, if information on the intrinsic geometry of an object is lost when transferring the object to a new modeling system, it is often essential to be able to recover that information.

Detecting the shape of a surface is also important in the field of *reverse engineering*. In reverse engineering the challenge is to build a physical model, either from a computer program or from a mathematical model. Identifying characteristic properties of a virtual object is necessary in order to build an accurate physical model. For this reason, methods for recognizing surfaces such as planes, spheres, circular cylinders, circular cones, cylindrical surfaces, conical surfaces, ruled surfaces, developable surfaces and surfaces of revolution have been investigated in a number of papers, including [3], [6], [8], [11], [13], where the input is a point cloud, and approximations to these surfaces are detected by resorting to least-squares techniques.

However, in these papers the input is assumed to be noisy. In our case, we start with with an exact object, either a polynomial in the variables x, y, z, or a rational parametrization $\boldsymbol{x}(t,s)$, therefore defining a *surface* in the sense of Algebraic Geometry, i.e. a 3D object consisting of a real part and a complex part. When we have an implicit form, the zero set of the polynomial provides all the real and complex points of the surface. When we have a rational parametrization, the real and complex points of the surface are obtained by evaluating $\boldsymbol{x}(t,s)$ for real and complex values of t, s. We emphasize that the input to our algorithms is an equation, not just a set of points in space. In particular, throughout this paper, whenever we speak of *surfaces*, the readers should understand that we mean surfaces in the sense of Algebraic Geometry. In Algebraic Geometry, the

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surfaces we deal with are called *algebraic surfaces*, since these surfaces can be defined as the zero set of a polynomial in x, y, z. When this polynomial is provided, we say that we know the *implicit equation* of the surface, or that the surface is given implicitly. When a rational parametrization is given, we say that the surface is given in *rational* form. In this case, there are algorithms to find the implicit equation of the surface. However, such algorithms have a certain cost in terms of computation time, so avoiding the computation of the implicit form of the surface, and therefore working directly in rational form, is preferred.

Given such an implicit or rational representation of an algebraic surface, we develop algorithms to determine whether this surface is cylindrical, conical, or a surface of revolution. When the surface is cylindrical, we show how to compute the direction of its rulings; when the surface is conical, we show how to compute its vertex; and when the surface is a surface of revolution, we show how to compute its axis of rotation. In the case of implicit surfaces, the characterizations we provide for cylindrical surfaces, conical surfaces and surfaces of revolution are polynomial identities in terms of the polynomial defining the surface and its first derivatives. In the case of rational surfaces, we get polynomial identities in terms of the components of the parametrization, and the normal of the parametrization. In all these cases, the expressions we derive are quite simple and easy to use in practice, even for potential users with a minimal background in mathematical methods.

Similar problems are treated in [5], where the authors show how to recognize spheres, circular cones, circular cylinders and surfaces of revolution. Recognizing cylindrical surfaces and surfaces of revolution is also addressed in [11], [12], and the results in these papers are extended in [7] to conical surfaces and other types of surfaces not treated in this paper, like helical surfaces and spiral surfaces. Recognizing surfaces of revolution is also investigated in [1], [16] and some aspects regarding the recognition of cylindrical and conical surfaces are studied in [9]. In Section 4 of this paper, we compare our results with the results in these papers.

Some of the results we include in this paper are already known, and we properly acknowledge this prior work in the places of this paper where this happens. In particular, some of these results are covered in [7], [11], [12], although formulated in terms of *Plücker coordinates* and *linear complexes* (see also Section 2.3 of the paper). In contrast, in this paper we only use Plücker coordinates to prove some preliminary facts about surfaces of revolution, but our results are ultimately formulated in terms of the implicit equation of the algebraic surface, or the components of the parametrization of the rational surface, we wish to analyze.

Additionally, although we studied surfaces of revolution in one of our previous papers [1] the results on surfaces of revolution in this paper are qualitatively different. Our previous paper [1] assumes that the surface is given implicitly by a polynomial, and focuses more on deriving certain algebraic properties of the implicit equation of a surface of revolution. These algebraic properties are then exploited to provide an algorithm for computing the axis of revolution, if any, which in turn allows us to recognize whether or not an implicitly defined surface is a surface of revolution. In contrast, the ideas in the current paper are applicable not only to implicit algebraic surfaces, but also to surfaces defined by a rational parametrization, and lead to a much more efficient procedure for recognizing a surface as a surface of revolution. In the implicit case, we provide a simple relationship between the first derivatives of the polynomial defining the surface and the variables on which the polynomial depends (see Theorem 13) that must be satisfied by certain parameters in order to guarantee that the surface is a surface of revolution. In the rational case, we provide a simple relationship between the parametrization of the surface and the normal vector of the parametrization (see Theorem 16) that must also be satisfied by certain parameters for the surface to be a surface of revolution. These two conditions lead to linear systems whose consistency is equivalent to the surface being a surface of revolution.

Recently, recognizing other types of surfaces has also been addressed. Detecting canal surfaces is considered in [17]. Moreover, detecting and canonically reparametrizing rational ruled surfaces is investigated in [15], and similar questions for translational rational surfaces are explored in [10].

2 The implicit case.

In this section we show how to detect algebraic cylindrical surfaces, conical surfaces and surfaces of revolution given in implicit algebraic form. Recall that a surface S is *algebraic* if it is the zero set of a polynomial $p(x, y, z)$. Furthermore, $p(x, y, z) = 0$ is the *implicit equation* of the surface S. In this section, we will assume that $p(x, y, z)$ has real coefficients.

Cylindrical and *conical* surfaces are special cases of ruled surfaces. A surface is *cylindrical* if through each point on the surface there is a straight line on the surface parallel to a fixed direction; this line is called a *generatrix* or a *ruling* of the surface. A surface is *conical* if through each point on the surface there exists a straight line on the surface passing through a fixed point, the *vertex* of the surface. Notice that cylindrical surfaces are invariant under the 1-parameter group $\mathcal{T}_{\mathbf{v}}$ of translations parallel to the direction $\mathbf{v}$ of the surface, and conical surfaces are invariant under the 1-parameter group $\mathcal{N}_{\mathbf{p}_0}$ of uniform scaling from the vertex $\mathbf{p}_0$ of the surface. Let $\mathbf{v} = (v_1, v_2, v_3)$. Then every element of $\mathcal{T}_{\mathbf{v}}$ can be written as $T_\lambda(x, y, z) = (x+\lambda v_1, y+\lambda v_2, z+\lambda v_3)$, with λ a (real or complex) constant. Similarly, let $\mathbf{p}_0 = (x_0, y_0, z_0)$. Then every element of $\mathcal{N}_{\mathbf{p}_0}$ can be written as $T_\lambda(x, y, z) = (x_0+\lambda(x-x_0), y_0+\lambda(y-y_0), z_0+\lambda(z-z_0))$.

A surface S is a *surface of revolution* if there exists a line $\mathcal{A}$, called the *axis* of S, such that S is invariant under the 1-parameter group $\mathcal{R}_{\mathcal{A}}$ of rotations about $\mathcal{A}$. The elements of $\mathcal{R}_{\mathcal{A}}$ are parametrized by the rotation angle θ, or equivalently by λ, where $\cos(\theta) = \frac{1-\lambda^2}{1+\lambda^2}$, $\sin(\theta) = \frac{2\lambda}{1+\lambda^2}$. The only cylindrical surfaces that are also surfaces of revolution, are the unions of circular cylinders with the same axis. Similarly, the only conical surfaces that are also surfaces of revolution, are the unions of circular cones with the same axis and same vertex.

Thus, the surfaces we are interested in are invariant under a 1-parameter group of transformations $\mathcal{G}_\lambda$, such that a generic $T_\lambda \in \mathcal{G}_\lambda$ is: (1) affine; (2) non-singular; and (3) the components of T_λ are linear mappings in x, y, z whose coefficients are rational functions of λ. In particular, $\mathcal{T}_{\mathbf{v}}$, $\mathcal{N}_{\mathbf{p}_0}$ or $\mathcal{R}_{\mathcal{A}}$ are this type. The algebraic surface defined by $p(x,y,z)$ is invariant under $\mathcal{G}_\lambda$ if for all *real or complex* points (x,y,z) such that $p(x,y,z)=0$, and for all real or complex λ where T_λ is well-defined, the zero sets of $p(T_\lambda(x,y,z))$ and $p(x,y,z)$ coincide.

Now we want to relate the invariance of an algebraic surface under $\mathcal{G}_\lambda$, where $\mathcal{G}_\lambda$ satisfies properties (1), (2), (3) listed above, with the invariance of the defining implicit polynomial. A first observation is that since the well-defined transformations in $\mathcal{G}_\lambda$ are affine and non-singular, these transformations are invertible, and their inverse is also affine. Hence these transformations preserve irreducibility, i.e. if $f(x,y,z)$ is irreducible, then $f(T_{\lambda_i}(x,y,z))$ is also irreducible for any λ_i. Furthermore, using this fact, one can prove the following result.

Lemma 1. *Let $f(x,y,z)$ be irreducible, and let T_λ be a generic transformation of $\mathcal{G}_\lambda$. Then $f(T_\lambda(x,y,z))=\mu(\lambda)\cdot q(x,y,z,\lambda)$, where $\mu(\lambda)$ is a rational function, and q is an irreducible polynomial in x,y,z,λ explicitly depending on at least one of the variables x,y,z.*

Proof. Since the coefficients of the components of each $T_\lambda \in \mathcal{G}_\lambda$ are rational functions in λ, $f(T_\lambda(x,y,z)) = \mu(\lambda)\cdot q(x,y,z,\lambda)$, where $\mu(\lambda)$ is a rational function in λ, and q is a polynomial explicitly depending on at least one of the variables x,y,z. Now suppose that there exist q_1,q_2, explicitly depending on at least one of the variables x,y,z, such that $q=q_1\cdot q_2$. Since there are at most finitely many values $\lambda=\lambda_i$ such that $q_j(x,y,z,\lambda_i)$ is a constant for $j=1,2$, we deduce that for almost all λ_i, $q(T_{\lambda_i})$ is reducible. Therefore $f(T_{\lambda_i}(x,y,z))$ is also reducible for almost all λ_i, which is a contradiction. □

Proposition 2. *Let $p(x,y,z)$ be a polynomial with real coefficients. Then the surface S defined by $p(x,y,z)$ is invariant under $\mathcal{G}_\lambda$ if and only if for a generic transformation $T_\lambda\in\mathcal{G}_\lambda$, we have $p(T_\lambda(x,y,z))=\widehat{\mu}(\lambda)\cdot p(x,y,z)$, where $\widehat{\mu}(\lambda)$ is a rational function of λ.*

Proof. The implication ($\Leftarrow$) is straightforward, so let us prove ($\Rightarrow$). Let $p=p_1^{r_1}\cdots p_k^{r_k}$, where p_i is irreducible for $i=1,\ldots,k$, and $p_i\neq p_j$ for $i\neq j$. Then $p(T_\lambda)=p_1^{r_1}(T_\lambda)\cdots p_k^{r_k}(T_\lambda)$. Now notice that since p_i is irreducible as a polynomial in x,y,z, it follows that p_i is also irreducible as a polynomial in x,y,z,λ. Additionally by Lemma 1, for $i=1,\ldots,k$ we have $p_i(T_\lambda)=\mu_i(\lambda)\cdot q_i$, where q_i is irreducible as a polynomial in x,y,z,λ and $\mu_i(\lambda)$ is a rational function in λ. Since S is invariant under T_λ for all λ, the zero set (in $\mathbb{C}^4$) of the polynomial $\tilde{p}=p_1\cdots p_k$ is included in the zero set of $\tilde{q}=q_1\cdots q_k$, where both $\tilde{p}$ and $\tilde{q}$ are seen as polynomials in x,y,z,λ. Therefore, by Study's Lemma (see Theorem 2.1.4 in [4]) each p_j divides some q_i. Since p_j,q_i are irreducible, $p_i(T_\lambda)=\widehat{\mu}_i(\lambda)\cdot p_j$. Suppose that $i\neq j$, and let $\lambda=\lambda_0$ so that T_{λ_0} is the identity (for translations and rotations, this happens for $\lambda_0=0$; for uniform scaling, this happens for $\lambda_0=1$). Then we have $p_i(T_{\lambda_0}(x,y,z))=p_i(x,y,z)=\widehat{\mu}_i(\lambda_0)\cdot p_j(x,y,z)$. But this is a contradiction, because p_i,p_j are different irreducible factors of p. So $i=j$, i.e. for each $i=1,\ldots,k$ there exists $\widehat{\mu}_i(\lambda)$ such that $p_i(T_\lambda)=\widehat{\mu}_i(\lambda)\cdot p_i$. Therefore, we have

$$p(T_\lambda)=p_1^{r_1}(T_\lambda)\cdots p_k^{r_k}(T_\lambda)=\widehat{\mu}_1(\lambda)^{r_1}\cdots\widehat{\mu}_k(\lambda)^{r_k}\cdot p_1^{r_1}\cdots p_k^{r_k},$$

and calling $\widehat{\mu}(\lambda)=\widehat{\mu}_1(\lambda)^{r_1}\cdots\widehat{\mu}_k(\lambda)^{r_k}$, we get $p(T_\lambda)=\widehat{\mu}(\lambda)\cdot p$. □

Notice that the fact that all the *real* components of $p(x,y,z)=0$ are mapped to themselves under a transformation T does not guarantee that the surface, in the sense of Algebraic Geometry, defined by $p(x,y,z)$ is invariant under T; the complex components must also be mapped onto themselves. For instance, while the surface defined by $p_1(x,y,z)=x^2+y^2-1$ is certainly cylindrical, because this surface is invariant under any translation

$$X=x,\ Y=y,\ Z=z+\lambda,$$

the surface defined by $p_2(x,y,z)=(x^2+y^2-1)(x^2+y^2+z^2+1)$ is not cylindrical, since this surface is not invariant under these translations. Indeed, when we plug the above translation into $p_2(x,y,z)$ we do not get a scalar multiple of $p_2(x,y,z)$, which is the invariance condition. Certainly the real part of the surface defined by $p_2(x,y,z)$ is cylindrical, and in fact shares a whole component with the surface defined by $p_1(x,y,z)$. But one cannot consider the surface defined by $p_2(x,y,z)$ a cylindrical surface in the sense of Algebraic Geometry. Similarly and in the same sense, $(x^2+y^2-z^2)(x^2+y^2+z^2+1)=0$ is not a conical surface, and $(x^2+y^2-1)(x^2+y^4+1)=0$ is not a surface of revolution about the z-axis.

Remark 1. *Observe that if $p(x,y,z)$ is irreducible and has at least one non-singular real point, then the surface $p(x,y,z)=0$ is invariant under a certain transformation T (for instance, a translation or a rotation) if and only if the real part of the surface is invariant under T. This result is essentially a consequence of Study's Lemma as well. Therefore, whenever $p(x,y,z)$ is irreducible and has one non-singular real point, the algebraic surface defined by $p(x,y,z)$ is cylindrical, conical, or a surface of revolution if and only if the set of real points of the surface is invariant under a translation, under a perspective transformation from a certain point, or under rotations about a certain line. Notice that both conditions, i.e. irreducibility and the existence of a non-singular real point, are necessary. Indeed, as we observed before, $p_2(x,y,z)=(x^2+y^2-1)(x^2+y^2+z^2+1)$, which is reducible, does not define a cylindrical surface in the sense of Algebraic Geometry, although the real points correspond to a circular cylinder. Also, $p(x,y,z)=x^2+2y^2$, which is irreducible but whose real part, the z-axis, consists entirely of singular points, does not define a surface of revolution, although certainly its real part is invariant under any rotation about the z-axis.*

Next we present our main results, which characterize algebraic cylindrical surfaces, conical surfaces and surfaces of revolution. The sufficiency of these conditions could be deduced from the results in [7] and [12], where similar problems are investigated, and results are given in terms of Plücker coordinates and linear complexes. However, the necessity of our results cannot be deduced in a straightforward way from the results in [7] and [12]. Additionally, in the case of cylindrical and conical surfaces we do not use Plücker coordinates and linear complexes at all. In the case of surfaces of revolution, we use Plücker coordinates and linear complexes only to prove certain preliminary facts about surfaces of revolution. Nevertheless, in all these cases, our results are ultimately formulated in terms of the implicit equation of the surface rather than in terms of linear complexes.

2.1 Detecting Cylindrical surfaces.

In this subsection we consider cylindrical algebraic surfaces. Recall that the *gradient* vector $\nabla p = (\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z})$ provides the direction of the line normal to S at each point of S.

Theorem 3. *Let $p(x, y, z) = 0$ define an algebraic surface S. Then S is cylindrical if and only if there exists a vector $\boldsymbol{v} \neq 0$ such that*

$$\boldsymbol{v} \cdot \nabla p = 0. \tag{1}$$

Furthermore, all the rulings of S are parallel to $\boldsymbol{v}$.

Proof. ($\Rightarrow$) Let $\boldsymbol{v}$ be parallel to the rulings of S, and consider an affine change of coordinates so that the new z-axis is parallel to $\boldsymbol{v}$. In this new system of coordinates, the equation of S must be $\tilde{p}(x, y) = 0$. Therefore, $\nabla\tilde{p}\cdot(0, 0, 1) = 0$. Hence, the normal vector to S at every point of S is perpendicular to the vector $(0, 0, 1)$, which is parallel to the rulings of S. Inverting the change of coordinates, we deduce that in the original coordinates, the normal vector to S at every point of S is perpendicular to the vector $\boldsymbol{v}$, which is parallel to the rulings of S. But the normal vector to S is ∇p. ($\Leftarrow$) Since $\boldsymbol{v} \cdot \nabla p = 0$, the normal line to S at every point of S is perpendicular to $\boldsymbol{v}$. Let us apply a change of coordinates so that the new z-axis is parallel to $\boldsymbol{v}$, and let $\tilde{p}(x, y, z) = 0$ be the implicit equation of S in the new system of coordinates. Since the normal line to S at each point is perpendicular to $\boldsymbol{v}$, in the new setting $\nabla\tilde{p}$ is perpendicular to $(0, 0, 1)$. Therefore, $\tilde{p}_z = 0$, so $\tilde{p}(x, y, z) = \tilde{p}(x, y)$. But then S is cylindrical. □

From the computational point of view, for an algebraic surface S implicitly defined by $p(x, y, z) = 0$, the condition $\boldsymbol{v} \cdot \nabla p = 0$ leads to a linear system in the coordinates a, b, c of $\boldsymbol{v}$, since the coefficient of each monomial $x^i y^j z^k$ in $\boldsymbol{v} \cdot \nabla p$ must vanish. Hence, the surface S is a cylinder if and only if this linear system is consistent. Furthermore when a solution of the system exists, the solution corresponds to the direction of the rulings of S. Additionally, from the preceding analysis the following result holds.

Corollary 4. *If $p(x, y, z)$ represents a cylindrical surface S and $p(x, y, z)$ has coefficients in $\mathbb{Q}$, then there exist $a, b, c \in \mathbb{Q}$ such that (a, b, c) defines the direction of the rulings of S.*

Finally, notice that once the direction of the rulings is known, one can compute a directrix curve by intersecting the surface with a plane normal to the direction of the rulings.

2.2 Detecting conical surfaces.

Let us consider now implicitly defined algebraic conical surfaces. Let $p(x, y, z)$ be a polynomial of degree N implicitly defining a surface S. We shall write

$$p(x, y, z) = p_N(x, y, z) + p_{N-1}(x, y, z) + \cdots + p_0(x, y, z),$$

where $p_k(x, y, z)$ denotes the homogeneous form of $p(x, y, z)$ of degree k. We will need the following straightforward lemma.

Lemma 5. *Let $\{H_k(x, y, z)\}$ be a collection of homogeneous polynomials of degree k, where $k = 0, \ldots, n$. Then*

$$\sum_{k=0}^{n} c_k \cdot H_k(x, y, z)$$

is the zero polynomial if and only if $c_k = 0$ for all $k = 0, \ldots, n$.

We will begin by considering conical surfaces with vertex at the origin.

Theorem 6. *Let $p(x, y, z)$ be a polynomial of degree N. The following statements are equivalent.*

(1) *$p(x, y, z)$ represents a conical surface with vertex at the origin.*
(2) *For every constant $s \neq 0$, there is a scalar $\lambda(s) \neq 0$, not depending on x, y, z, such that $p(sx, sy, sz) = \lambda(s) \cdot p(x, y, z)$.*
(3) *$p(x, y, z) = p_N(x, y, z)$, i.e. $p(x, y, z)$ is a homogeneous polynomial of degree N.*
(4) *For every constant $s \neq 0$, $p(sx, sy, sz) = s^N \cdot p(x, y, z)$.*
(5)
$$x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y} + z\frac{\partial p}{\partial z} = N \cdot p(x, y, z).$$

Proof. (1) $\Rightarrow$ (2). Let $p(x, y, z)$ represent a cone S with vertex at the origin. Then for any point $(x, y, z) \in S$, the line connecting the origin and the point (x, y, z) is contained in S. Thus for any constant $s \neq 0$, we have $(x, y, z) \in S \Leftrightarrow (sx, sy, sz) \in S$. Therefore, $p(x, y, z) = 0 \Leftrightarrow p(sx, sy, sz) = 0$. Hence, by Proposition 2, there is a scalar $\lambda(s) \neq 0$ such that $p(sx, sy, sz) = \lambda(s) \cdot p(x, y, z)$.

(2) $\Rightarrow$ (1). Suppose that for every constant $s \neq 0$, there is a scalar $\lambda(s) \neq 0$ such that $p(sx, sy, sz) = \lambda(s) \cdot p(x, y, z)$. Let S be the surface represented by the equation $p(x, y, z) = 0$. Then for any constant $s \neq 0$, we have $(x, y, z) \in S \Leftrightarrow (sx, sy, sz) \in S$. Therefore, the line connecting every point of S with the origin lies in S. Hence, S is a conical surface with vertex at the origin.

(2) $\Rightarrow$ (3) Suppose that for every constant $s \neq 0$, there is a scalar $\lambda(s) \neq 0$ such that

$$p(sx, sy, sz) = \lambda(s) \cdot p(x, y, z). \tag{2}$$

Since each term $p_k(x, y, z)$ is a homogeneous polynomial of degree k, we have

$$p(sx, sy, sz) = \sum_{k=0}^{N} s^k p_k(x, y, z). \tag{3}$$

From (2) and (3), we get

$$\sum_{k=0}^{N} (s^k - \lambda(s)) p_k(x, y, z) \equiv 0.$$

Since by assumption $p_N(x, y, z)$ is not identically zero, by Lemma 5 $\lambda(s) = s^N$ and $p_k(x, y, z) \equiv 0$ for $k = 0, 1, \ldots, N - 1$.

(3) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (2) Obvious.

(3) ⇒ (5) The equation in statement (5) is Euler's identity for homogeneous polynomials (see Theorem 10.2, page 27 of [18]).

(5) ⇒ (3) Suppose that

$$x\frac{\partial p}{\partial x}+y\frac{\partial p}{\partial y}+z\frac{\partial p}{\partial z}=N\cdot p(x,y,z).$$

By Euler's identity for homogeneous polynomials, we have

$$x\frac{\partial p_k}{\partial x}+y\frac{\partial p_k}{\partial y}+z\frac{\partial p_k}{\partial z}=k\cdot p_k(x,y,z)$$

for each homogeneous form p_k. Summing these equations over k, and then equating the result with the previous expression for $x\frac{\partial p}{\partial x}+y\frac{\partial p}{\partial y}+z\frac{\partial p}{\partial z}$ yields

$$\sum_{k=0}^{N}(k-N)p_k(x,y,z)\equiv 0$$

Since by assumption $p_N(x,y,z)$ is not identically zero, it follows from Lemma 5 that $p_k(x,y,z)\equiv 0$ for $k=0,\ldots,N-1$. □

Now we can move to conical surfaces with vertex at a point $P=(x_0,y_0,z_0)$, not necessarily the origin.

Theorem 7. *Let $p(x,y,z)$ be a polynomial of degree N. Then the following statements are equivalent:*

(1) *$p(x,y,z)$ represents a conical surface with vertex at (x_0,y_0,z_0).*
(2) *$p(x-x_0,y-y_0,z-z_0)$ is a homogeneous polynomial.*
(3) *$p(x-x_0,y-y_0,z-z_0)=p_N(x,y,z)$.*

Proof. (1) ⇔ (2) follows from Theorem 6, and (2) ⇔ (3) follows because $p_N(x,y,z)$ is invariant under translation. □

Theorem 8. *Let $p(x,y,z)$ be a polynomial of degree N. Then $p(x,y,z)$ represents a conical surface with vertex at (x_0,y_0,z_0) if and only if:*

$$(x-x_0)\frac{\partial p}{\partial x}+(y-y_0)\frac{\partial p}{\partial y}+(z-z_0)\frac{\partial p}{\partial z}=N\cdot p(x,y,z). \quad (4)$$

Proof. This theorem follows from Theorem 6 by translating the vertex from the origin to (x_0,y_0,z_0). □

Computationally, Equation (4) provides an efficient condition to check if an algebraic surface is a cone, and to find the vertex when the surface is a cone. One just needs to find a solution (x_0,y_0,z_0) of the linear system generated by this equality. Additionally, the following result holds.

Corollary 9. *If $p(x,y,z)$ represents a cone S and $p(x,y,z)$ has coefficients in $\mathbb{Q}$, then the coordinates of the vertex of S also belong to $\mathbb{Q}$.*

Proof. If $p(x,y,z)$ has rational coefficients, the partial derivatives $\frac{\partial p}{\partial x}$, $\frac{\partial p}{\partial y}$, $\frac{\partial p}{\partial z}$ also have rational coefficients. Therefore the linear system (in x_0,y_0,z_0) corresponding to Equation (4) has coefficients in $\mathbb{Q}$. By applying Cramer's rule, if this system is consistent, the solutions x_0,y_0,z_0 are rational numbers. □

2.3 Detecting surfaces of revolution.

In this subsection we investigate algebraic surfaces of revolution. In order to do so, we recall *Plücker coordinates* [11]. Plücker coordinates provide an alternative way to represent straight lines. A line $L\subset\mathbb{R}^3$ is completely determined when we know a point $P\in L$ and a vector $\boldsymbol{w}$ parallel to L. Therefore we often write $L=(P,\boldsymbol{w})$. Now let $\bar{\boldsymbol{w}}=\boldsymbol{P}\times\boldsymbol{w}$, where $\boldsymbol{P}$ here denotes the vector connecting the point P with the origin of the coordinate system. Then the *Plücker coordinates* of L are the coordinates of $(\bar{\boldsymbol{w}},\boldsymbol{w})\in\mathbb{R}^6$. Notice that by construction $\bar{\boldsymbol{w}}\cdot\boldsymbol{w}=0$.

Plücker coordinates are unique up to multiplication by a constant. Moreover $\bar{\boldsymbol{w}}$ is independent of the choice of the point $P\in L$, since if $Q\in L$, then $(\boldsymbol{Q}-\boldsymbol{P})\times\boldsymbol{w}=0$. Furthermore, given the Plücker coordinates $(\bar{\boldsymbol{w}},\boldsymbol{w})$ of L, we can recover a point P on L from the relationship

$$\boldsymbol{P}\times\boldsymbol{w}=\bar{\boldsymbol{w}}, \quad (5)$$

by writing $\boldsymbol{P}=(x,y,z)$ and solving the linear system (5) for x,y,z.

Let (β,α) be the Plücker coordinates for a line in $\mathbb{R}^3$, and consider all the lines $(\bar{\boldsymbol{w}},\boldsymbol{w})$, written in Plücker coordinates, such that

$$\alpha\cdot\bar{\boldsymbol{w}}+\beta\cdot\boldsymbol{w}=0. \quad (6)$$

This equation (see [11], [16]) expresses the condition that the line (β,α) intersects the line $(\bar{\boldsymbol{w}},\boldsymbol{w})$. Equation (6) corresponds to a hyperplane of $\mathbb{R}^6$, which in [11] is called a *linear complex*. We say that the hyperplane is *nontrivial* if either $\alpha\neq 0$ or $\beta\neq 0$.

Now we have the following result from [11] concerning nontrivial hyperplanes and the normal lines to cylindrical surfaces, surfaces of revolution, and helical surfaces. Recall that a surface is *helical* if it is invariant under a *screw motion*, i.e. the composition of a rotation about a line $\mathcal{A}$, and a translation parallel to $\mathcal{A}$. It is easy to see [2] that a helical algebraic surface must be cylindrical.

Theorem 10. (1) *The normal lines of a $\mathcal{C}^1$ surface S lie in a nontrivial hyperplane* (6) *if and only if the surface is contained in either a cylindrical surface, a surface of revolution, or a helical surface.*

(2) *If S is a $\mathcal{C}^1$ surface and the normal lines of S lie in a nontrivial hyperplane* (6), *then:*

- *If $\alpha=0$ but $\beta\neq 0$ then S is contained in a cylindrical surface whose rulings are parallel to β.*
- *If $\alpha\neq 0$ and $\alpha\cdot\beta=0$ then S is a surface of revolution about an axis $\mathcal{A}$. Expressed in Plücker coordinates, $\mathcal{A}=(\beta,\alpha)$.*
- *If $\alpha\neq 0$ and $\alpha\cdot\beta\neq 0$ then S is contained in a non-cylindrical, helical surface.*

Observe that Theorem 10 requires only that S is a $\mathcal{C}^1$ surface. Notice that every algebraic surface implicitly defined is $\mathcal{C}^1$ except at singular points, i.e. points where ∇p vanish.

Since the gradient vector ∇p provides the direction of the line normal to the algebraic surface S implicitly defined by $p(x,y,z)=0$, the normal line to S at each point $P=(x,y,z)\in S$, written in Plücker coordinates, is

$$\begin{aligned}\mathbf{n}&=(\boldsymbol{P}\times\nabla p,\nabla p)=\\&=\left(y\tfrac{\partial p}{\partial z}-z\tfrac{\partial p}{\partial y},-x\tfrac{\partial p}{\partial z}+z\tfrac{\partial p}{\partial x},x\tfrac{\partial p}{\partial y}-y\tfrac{\partial p}{\partial x},\tfrac{\partial p}{\partial x},\tfrac{\partial p}{\partial y},\tfrac{\partial p}{\partial z}\right).\end{aligned} \quad (7)$$

Plücker coordinates are also used in [16] to detect algebraic surfaces of revolution. The following observation, which essentially corresponds to Theorem 2.4 in [16], is crucial.

Theorem 11. *The surface $p(x,y,z)=0$ is a surface of revolution if and only if $p(x,y,z)=k$ defines a surface of revolution for all $k\in\mathbb{R}$. Furthermore, the axis of revolution is the same for all $k\in\mathbb{R}$.*

Theorem 11 leads to the following corollary. Here we denote the surface defined by $\widehat{p}(x,y,z,k)=p(x,y,z)-k=0$ by S_k, $k\in\mathbb{R}$.

Corollary 12. *The surface $p(x,y,z)=0$ is a surface of revolution if and only if all the normals of the surfaces $\{S_k\}_{k\in\mathbb{R}}$ belong to a same hyperplane* (6).

Proof. By Theorem 11, $p(x,y,z)=0$ is a surface of revolution if and only if all the S_k are surfaces of revolution with the same axis, which is also the axis of S. By Theorem 10, the normals of each S_k belong to a hyperplane (6). However, this hyperplane is completely defined by the Plücker coordinates of the axis of revolution, which, according to Theorem 10, are (β,α). Since this axis of revolution is the same for all the S_k, the result follows. □

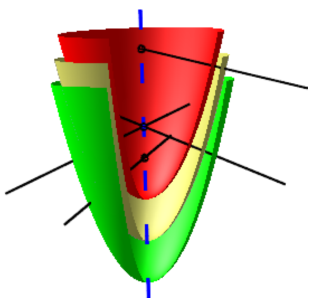

Fig. 1: Illustrating Corollary 12.

Fig. 1 illustrates Corollary 12. Here we show the paraboloids $z=x^2+y^2$, $z=x^2+y^2+1$, $z=x^2+y^2+2$, all of which are surfaces of revolution about the z-axis (plotted as a blue dashed line). We also show (as black lines) some normal lines to these paraboloids. One can see that all these lines are normal to all the paraboloids. Furthermore, all these lines intersect the axis of revolution. Hence, all of these lines belong to the same hyperplane (6).

Now we have the following very simple characterization of surfaces of revolution.

Theorem 13. *Let $p(x,y,z)$ be a polynomial, and let $\boldsymbol{P}=(x,y,z)$. Then $p(x,y,z)=0$ defines a surface of revolution S if and only if there exist $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and $\beta=(\beta_1,\beta_2,\beta_3)$ such that $\alpha\neq 0$, $\alpha\cdot\beta=0$, and*

$$\begin{aligned}&\alpha\cdot(\boldsymbol{P}\times\nabla p)+\beta\cdot\nabla p=\\&\alpha_1(y\tfrac{\partial p}{\partial z}-z\tfrac{\partial p}{\partial y})+\alpha_2(-x\tfrac{\partial p}{\partial z}+z\tfrac{\partial p}{\partial x})+\alpha_3(x\tfrac{\partial p}{\partial y}-y\tfrac{\partial p}{\partial x})\\&+\beta_1\tfrac{\partial p}{\partial x}+\beta_2\tfrac{\partial p}{\partial y}+\beta_3\tfrac{\partial p}{\partial z}=0\end{aligned}\tag{8}$$

Furthermore, the axis of revolution $\mathcal{A}$ of S written in Plücker coordinates is $\mathcal{A}=(\beta,\alpha)$.

Proof. (⇒) If $p(x,y,z)=0$ defines a surface of revolution, then by Theorem 10 the set of normal lines to S belongs to a linear complex with $\alpha\neq 0$ and $\alpha\cdot\beta=0$. Furthermore by Theorem 10, the axis of revolution $\mathcal{A}$ of S is given by $\mathcal{A}=(\beta,\alpha)$. In addition, since the Plücker coordinates of a generic normal line are given by (7), the polynomial on the left hand-side of (8) vanishes at every point of S. Moreover, by Corollary 12 the polynomial on the left hand-side of (8) vanishes at every point of S_k for every value of k. Hence, the polynomial in Equation 8 vanishes everywhere, since each point $(x,y,z)\in\mathbb{R}^3$ lies on S_k for some value of k. (⇐) This result follows directly from Theorem 10, taking (7) into account. □

Computationally, (8) leads to a linear system from which the Plücker coordinates of the axis of revolution can be computed.

Example 1. *Let S be the surface defined by $p(x,y,z)=z^4+z^3+x^2+y^2+z-1$. Consider the polynomial*

$$\begin{aligned}&\alpha_1\cdot(y\tfrac{\partial p}{\partial z}-z\tfrac{\partial p}{\partial y})+\alpha_2\cdot(-x\tfrac{\partial p}{\partial z}+z\tfrac{\partial p}{\partial x})+\alpha_3\cdot(x\tfrac{\partial p}{\partial y}-y\tfrac{\partial p}{\partial x})\\&+\beta_1\cdot\tfrac{\partial p}{\partial x}+\beta_2\cdot\tfrac{\partial p}{\partial y}+\beta_3\cdot\tfrac{\partial p}{\partial z}=-4\alpha_2xz^3-3\alpha_2xz^2+2\alpha_2xz\\&+(-\alpha_2+2\beta_1)x+4\alpha_1yz^3+3\alpha_1yz^2-2\alpha_1yz+(\alpha_1+2\beta_2)y\\&+4\beta_3z^3+3\beta_3z^2+\beta_3\end{aligned}$$

By setting to zero the coefficients of each monomial in this polynomial, we get a linear system that yields the solution

$$\alpha_1=\alpha_2=\beta_1=\beta_2=\beta_3=0$$

Hence, a solution is given by $\alpha=(0,0,\gamma)$, $\beta=0$. Since $\alpha\neq 0$ and $\alpha\cdot\beta=0$, we deduce from Theorem 10 that S is a surface of revolution, and the axis of revolution is the z-axis.

3 The rational case.

In this section we characterize when a rational parametrization

$$\boldsymbol{x}(t,s)=(x(t,s),y(t,s),z(t,s)).$$

represents a cylindrical surface, a conical surface, or a surface of revolution. We will represent the partial derivatives of $\boldsymbol{x}(t,s)$ with respect to the parameters t,s as $\boldsymbol{x}_t$ and $\boldsymbol{x}_s$. Furthermore, we recall that whenever $\boldsymbol{x}_t\times\boldsymbol{x}_s\neq 0$, the normal vector $\mathbf{N}$ to S at each point can be computed as

$$\mathbf{N}=\mathbf{N}(t,s)=\frac{\boldsymbol{x}_t\times\boldsymbol{x}_s}{\|\boldsymbol{x}_t\times\boldsymbol{x}_s\|}.$$

Whenever necessary we will denote the implicit equation of S by $p(x,y,z)$.

Now we begin with cylindrical surfaces. The following result follows in a straightforward way from Theorem 3.

Theorem 14. *The surface S rationally parametrized by $\boldsymbol{x}(t,s)$ is cylindrical if and only if there exists a constant vector $\boldsymbol{v}\neq 0$ such that*

$$\boldsymbol{v}\cdot\mathbf{N}=0.\tag{9}$$

The vector $\boldsymbol{v}$ in Theorem 14 gives the direction of the rulings of the cylindrical surface. Furthermore, from a computational point of view, for a rational parametrization $\boldsymbol{x}(t,s)$ the condition $\boldsymbol{v}\cdot\mathbf{N}=0$ provides a linear system in the coordinates of $\boldsymbol{v}$, whose consistency is equivalent to S being a cylindrical surface.

Now let us consider conical surfaces.

Theorem 15. *The surface S rationally parametrized by $\boldsymbol{x}(t,s)$ is conical if and only if there exists a point $P_0 = (x_0, y_0, z_0)$ such that*

$$(\boldsymbol{x} - P_0) \cdot \mathbf{N} = 0. \tag{10}$$

Furthermore, P_0 is the vertex of the conical surface.

Proof. ($\Rightarrow$) If S is conical then from Theorem 8 $P_0 = (x_0, y_0, z_0)$ satisfies Eq. (4). Since $\mathbf{N}$ and ∇p are parallel at the points of S, substituting the components of $\boldsymbol{x}(t,s)$ for x, y, z in Eq. (4) we get that $(\boldsymbol{x} - P_0) \cdot \mathbf{N}$ is proportional to $p(\boldsymbol{x}(t,s))$. Since $\boldsymbol{x}(t,s)$ parametrizes S, $p(\boldsymbol{x}(t,s)) = 0$ and (10) holds.

($\Leftarrow$) If (10) holds, then since $\mathbf{N}$ and ∇p are parallel at the points of S we have that the dot product of $(x - x_0, y - y_0, z - z_0)$ and ∇p vanishes at the points of S. However, from Euler's identity it follows that the degree of this dot product is equal to the degree of p. Therefore, this dot product is the result of multiplying p by a constant. But then the result follows from Theorem 6 and Theorem 8. □

Again, from a computational point of view the condition $(\boldsymbol{x} - P_0) \cdot \mathbf{N} = 0$ gives rise to a linear system in the coordinates of P_0, whose consistency is equivalent to S being a cone.

Now let us address surfaces of revolution.

Theorem 16. *The surface S rationally parametrized by $\boldsymbol{x}(t,s)$ is a surface of revolution if and only if there exist two constant vectors α, β, where $\alpha \neq 0$ and $\alpha \cdot \beta = 0$, such that*

$$\alpha \cdot (\boldsymbol{x} \times \mathbf{N}) + \beta \cdot \mathbf{N} = 0. \tag{11}$$

Furthermore, the axis of revolution $\mathcal{A}$ of S is the line written in Plücker coordinates as $\mathcal{A} = (\beta, \alpha)$.

Proof. ($\Rightarrow$) If S is a surface of revolution then from Theorem 13 there exist two constant vectors α, β such that $\alpha \neq 0$, $\alpha \cdot \beta = 0$ and (8) holds. In particular, this equation holds for all the points of S. Since ∇p and $\mathbf{N}$ are parallel at the points of S, (11) follows.

($\Leftarrow$) This condition implies that the normals to S belong to a linear complex. Since $\alpha \neq 0$ and $\alpha \cdot \beta = 0$, we conclude from Theorem 10 that S is a surface of revolution. □

Again, Equation (11) provides a linear system in the coordinates of α, β. Whenever this linear system has a solution with $\alpha \neq 0$, $\alpha \cdot \beta = 0$ then S is a surface of revolution, and the axis of revolution can be found by solving a system of linear equations.

We conclude the section with two corollaries, where we provide conditions for detecting rational circular cylinders, and rational circular cones. The circular cylinders are cylindrical surfaces of revolution, while the circular cones are conical surfaces of revolution. Note that circular cylinders and circular cones are easy to detect in implicit form. Indeed, circular cylinders and circular cones are special types of quadrics, and any quadric given implicitly can be classified by studying the symmetric 4×4 matrix associated with the implicit equation of the quadric. However, when circular cylinders and circular cones are presented in rational form, it is not so obvious how to detect them. Alternative conditions to identify circular cylinders and circular cones in parametric form can be found in [5].

Corollary 17. *The surface S rationally parametrized by $\boldsymbol{x}(t,s)$ is a circular cylinder if and only if there exist a constant vector $\boldsymbol{v} \neq 0$ satisfying Eq. (9), and two constant vectors α, β, where $\alpha \neq 0$ and $\alpha \cdot \beta = 0$, satisfying Eq. (11).*

Corollary 18. *The surface S rationally parametrized by $\boldsymbol{x}(t,s)$ is a circular cone if and only if there exist a point $P_0 = (x_0, y_0, z_0)$ satisfying Eq. (10), and two constant vectors α, β, where $\alpha \neq 0$ and $\alpha \cdot \beta = 0$, satisfying Eq. (11).*

We can speed up the computation of the vector $\boldsymbol{v}$ in Theorem 14, the point P_0 in Theorem 15, or the vectors α, β in Theorem 16 by randomly picking at least three points $(t_i, s_i) \in \mathbb{Q}^2$, in the case of Theorem 14 and Theorem 15, or at least six points $(t_i, s_i) \in \mathbb{Q}^2$, in the case of Theorem 16, and then plugging these values into Eq. (9), Eq. (10), Eq. (11). By proceeding this way, we get linear systems for $\boldsymbol{v}$, P_0 or α, β. Then we can substitute $\boldsymbol{v}$, P_0 or α, β in Eq. (9), Eq. (10), Eq. (11) to check if Eq. (9), Eq. (10), Eq. (11) hold identically. This procedure has the advantage of needing fewer symbolic operations. Additionally, by proceeding this way we can also solve the problem of checking whether or not an algebraic surface defined by a non-rational parametrization[1] is cylindrical, conical or a surface of revolution, since the proofs of Theorem 14, Theorem 15 and Theorem 16 do not really need the parametrization to be rational. The difference, in the non-rational setting, is that symbolic computations to directly extract a linear system from Eq. (9), Eq. (10), Eq. (11) are certainly more complicated and difficult to implement.

4 Comparison with other approaches.

In this section we compare our results with the results in [5], [9], [7], [12], [1] and [16], where similar problems to ours, or somehow connected to ours, are addressed.

We begin with [5]. This paper investigates methods for recognizing several types of surfaces given in rational form. In particular, the authors show how to recognize circular cylinders and circular cones by using the *mean evolute surface*, defined as

$$\mathbf{E}(t,s) = \boldsymbol{x}(t,s) + \frac{\mathbf{N}(t,s)}{2H(t,s)},$$

where $H(t,s)$ is the *mean curvature* of $\boldsymbol{x}(t,s)$; recall that $H(t,s)$ involves second order derivatives of $\boldsymbol{x}(t,s)$. The authors prove that $\boldsymbol{x}(t,s)$ defines a circular cylinder or a circular cone if and only if $\mathbf{E}(t,s)$ degenerates into a line. This condition can be used efficiently, and is fast to compute. Nevertheless, the conditions in Corollary 17 and Corollary 18 of this paper are even faster. This is to be expected, since the conditions in Corollary 17 and Corollary 18 involve only first order derivatives of $\boldsymbol{x}(t,s)$, and lead to linear systems, which can be rapidly solved.

For instance, consider the circular cone parametrized by:

$$\boldsymbol{x}(t,s) = \left(\frac{8st + 9t^2 - 10t + 5}{9t^2 - 12t + 5}, \frac{36st^2 - 32st + 4s + 4t - 4}{9t^2 - 12t + 5}, \right. \\ \left. -\frac{36st^2 - 64st - 9t^2 + 28s + 8t - 3}{9t^2 - 12t + 5} \right).$$

1. For instance, a parametrization with radical functions, or trigonometric functions.

Using the computer algebra system Maple 18, running on a personal computer revving up to 2.90 GHz, with 8 Gb of RAM, one can check that $\mathbf{E}(t,s)$ degenerates into the line parametrized as

$$\left(\frac{16}{3}s+\frac{7}{3},\frac{8}{3}s-\frac{1}{3},-\frac{16}{3}s+\frac{2}{3}\right)$$

in 0.156 seconds. After this computation, one still has to check if $\boldsymbol{x}(t,s)$ defines a cylinder or a cone, and, in the second case, find the vertex of the cone. In contrast, the conditions in Corollary 18 can be used to show that $\boldsymbol{x}(t,s)$ defines a circular cone with vertex $P_0=(1,-1,2)$ in 0.016 seconds. The difference between these two approaches can be more dramatic when the parametrization defining the surface has suffered some reparametrization, since in that case recognizing that $\mathbf{E}(t,s)$ degenerates into a line can be more complicated.

Surfaces of revolution are also treated in [5]. The authors prove that if $\boldsymbol{x}(t,s)$ defines a surface of revolution, then another surface $\mathbf{F}(t,s)$ constructed from the origin surface, called the *pseudo-focal surface in the t-isoparametric direction*, degenerates into a line, which is the axis of revolution of the surface. The axis of revolution of the surface is extracted by exploiting a least squares fit of a line to $\mathbf{F}(t,s)$. However, this property is not proved to be sufficient, so in general if the condition holds, one still needs to check that the surface is really a surface of revolution. Furthermore, in general, computing the pseudo-focal surface of an implicit surface is cumbersome. Therefore the idea cannot be efficiently generalized to the implicit setting.

In [9] the authors treat a different problem, namely recognizing and (re)parametrizing rational developable surfaces. However, the problem of identifying cylindrical and conical surfaces, both in implicit and in rational form, also comes up in this paper. First, the results in [9] require recognizing that the surface is developable. This is done by checking whether certain determinants depending on some derivatives of the polynomial implicitly defining the surface (see Theorem 3.1 in [9]) or the parametrization of the surface (see Theorem 4.1 in [9]) are identically zero. Once the surface has been identified as developable, algorithms for detecting whether the surface is cylindrical or conical are provided. In the case of cylindrical surfaces, the idea in Theorem 3 or Theorem 14 is used, but under the assumption that the surface is developable. Conical surfaces are detected by analyzing certain polynomial systems, which leads to an algorithm more complicated than ours. Surfaces of revolution are not considered.

In [12] recognizing several types of surfaces using Plücker coordinates and linear complexes is considered; the main results in [12] are summarized in Theorem 10 of this paper. The paper [7] extends the results of [12] to conical surfaces, helical surfaces and spiral surfaces. The results in both [12] and [7] are formulated in terms of Plücker coordinates and linear complexes: essentially, the idea is that the normals of cylindrical surfaces, conical surfaces, surfaces of revolution, helical surfaces and spiral surfaces always belong to a linear complex. Furthermore, the type of the equation of the linear complex determines the nature of the surface. From a practical point of view, one can find these linear complexes by computing the normals of the surface at several points. The minimum number of points depends on the number of parameters on which the linear complex depends. In general, we need at least the normals at 7 different points. In the rational case this approach is very efficient, since we can just pick rational pairs of the parameters of the surface. In the implicit case, however, this approach is less efficient, since computing points with rational coordinates on an implicit surface is difficult. This disadvantage is overcome in [7] by working in a floating-point environment, and using least squares methods. This difficulty, however, is absent in our methods, since we do not need to compute any point on the surface. In contrast, we work directly with the implicit equation of the surface, or with the parametrization of the surface, in the rational setting.

Surfaces of revolution are also addressed in [1] and [16]. In both papers, the surface is assumed to be given in implicit form, and the rational setting is not investigated. Nevertheless, the algorithm in [16] could also be adapted to the rational case. In [1] the results are more focused on exploiting the information provided by the form of highest degree of the implicit equation, and on describing the structure and properties of this form, and its relationship with the axis of revolution of the surface. Although [1] gives some necessary conditions for an implicit equation to represent a surface of revolution, a complete characterization is not provided; however, an algorithm is provided for computing the axis of revolution, when the surface is known to be a surface of revolution. This algorithm has advantages in certain cases, but the authors (see Section 4 of [1]) acknowledge that in a generic case, the algorithm provided in [16] is more efficient.

The approach in [16] is quite similar to the approach of [12] and [7]. The authors of [16] also use Plücker coordinates, and the main idea of [16] is, as in [12] and [7], to determine the linear complex containing the normals of the surface by solving a linear system. However, instead of picking points on the surface to be analyzed, the authors in [16] show that one can pick points P_i on surfaces $p(x,y,z)=k_i$, where $p(x,y,z)$ is the implicit equation of the surface to be analyzed and k_i is *any* real number. Since there is total freedom to choose k_i, we can use any point P_i; in particular we can always choose points P_i with rational coordinates, thereby making the method exact and efficient. For each P_i, the Plücker coordinates $(\bar{\boldsymbol{w}}_i,\boldsymbol{w}_i)$ of the normal line L_i to the surface $p(x,y,z)=k_i$ are computed. If the surface is a surface of revolution, each $(\bar{\boldsymbol{w}}_i,\boldsymbol{w}_i)$ satisfies Eq. (6). Proceeding in this way, a linear system in the Plücker coordinates of the axis of revolution is generated.

Our method is certainly inspired in the ideas of [16]. We make use of these ideas to directly provide a condition for an algebraic surface to be a surface of revolution in terms of the implicit equation, or the parametrization of the surface. In particular, we do not need to find points P_i or carry out computations with the coordinates of these points: we reach a linear system directly from the implicit equation of the surface, or the rational parametrization of the surface, as one can see in Theorem 13 and Theorem 16. These two theorems shed extra light on the properties that a polynomial must satisfy to be the implicit equation of a surface of revolution, or the properties that a rational parametrization must have to correspond to a rational surface of revolution. Certainly the ideas in [16] are at the core of Theorem 13 and Theorem 16. But, as one can see from the proofs of these theorems, some elaboration is required.

The timings provided by the methods in this paper are generally better than those in [1], but not better than those in [16], although often comparable. Additionally, both the methods in this paper and the method in [16] are more efficient than the method in [5].

Summarizing, our contribution in the case of surfaces of revolution is more theoretical than practical. In particular, we provide a characterization of surfaces of revolution directly in terms of the implicit equation, in the case of implicit surfaces, and in terms of the components of the parametrization and the components of the normal vector of the parametrization, in the case of rational surfaces.

5 Conclusions and Further Work.

A polynomial equation or a rational parametrization define a surface in the sense of Algebraic Geometry, i.e. a 3D object consisting of a real and a complex part. Given such an equation, we have presented algorithms that allow us to check, in a very simple and efficient way, if the surface defined by the equation is cylindrical, conical, or a surface of revolution. In practice, all our results lead to linear systems, whose consistency is equivalent to the surface being cylindrical, conical, or a surface of revolution. All these results can also be understood as characterizations for the implicit polynomials and rational parametrizations defining cylindrical surfaces, conical surfaces or surfaces of revolution. Additionally, we also provide fast tests to check whether a given rational surface is a circular cylinder or a circular cone.

Circular cylinders and circular cones are special cases of quadrics. Any quadric surface can be parametrized by a rational quadratic parametrization, i.e. a parametrization $\boldsymbol{x}(t,s)$ where the numerators and denominators of the coordinate functions $x(t,s)$, $y(t,s)$ and $z(t,s)$ are polynomials of degree two in t,s. However, apparently no direct algorithm is known for finding the type of quadric corresponding to a given rational quadratic parametrization without converting to implicit form. Certainly, our results can be applied to recognize cylindrical quadrics and conical quadrics. But our results cannot be applied to recognize ellipsoids, paraboloids, hyperboloids of one sheet or hyperboloids of two sheets. Hence, identifying the type of a quadric given in rational form, and computing the main features of the quadric (center, if any, symmetries, axes) is another interesting open question that we would like to address in the future.

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References

[1] Alcázar J.G., Goldman R. (2016), *Finding the axis of revolution of a surface of revolution*, to appear in IEEE Transactions on Visualization and Computer Graphics.

[2] Alcázar J.G., Hermoso C. (2016), *Involutions of polynomially parametrized surfaces*, Journal of Computational and Applied Mathematics Vol. 294, pp. 23–38.

[3] Chen H.Y., Lee I.K., Leopolodseder S., Pottmann H., Randrup T., Wallner J. (1999), *On surface approximation using developable surfaces*, Graphical Models and Image Processing, Vol. 61, Issue 2, pp. 110–124.

[4] De Jong T., Pfister G. (2000), *Local Analytic Geometry*, Advanced Lectures in Mathematics, Ed. Vieweg, Germany.

[5] Elber G., Kim M.S. (1997), *Geometric shape recognition of freeform curves and surfaces*, Graphical Models and Image Processing Vol. 59, No. 6, pp. 417-433.

[6] Elsässer B., Hoschek J. (1996), *Approximation of digitized points by surfaces of revolution*, Comput. Graphics, Vol. 20, No. 1, pp. 85-94.

[7] Hofer M., Odehnal B., Pottmann H., Steiner T., Wallner J. (2005), *3D shape recognition and reconstruction based on line element geometry*, Proceedings ECCV 2005 conference, Vol. 2, pp. 1532–1538.

[8] Lukacs G., Martin R., Marshall D. (1998), *Faithful least-squares fitting of spheres, cylinders, cones and tori for reliable segmentation*, Proceedings of the 5-th European Conference on Computer Vision, Vol. I, pp. 671-686, Springer-Verlag London, UK.

[9] Pérez-Díaz S., Shen L.Y. (2015), *Determination and (re)parametrization of rational developable surfaces*, Journal of Systems Science and Complexity, DOI: 10.1007/s11424-015-3119-z, pp. 1-14.

[10] Pérez-Díaz S., Shen L.Y. (2014), *Parametrization of translational surfaces*, Proceedings of the 2014 Symposium on Symbolic-Numeric Computation, pp. 128-129, ACM New York, NY, USA.

[11] Pottmann H., Randrup T. (1998), *Rotational and helical surface approximation for reverse engineering*, Computing Vol. 60, pp. 307–322.

[12] Pottann H., Hofer M., Odehnal B., Wallner J. (2004), *Line geometry for 3D shape understanding and reconstruction*, Proceedings ECCV 2004 conference, Part I, Vol. 3021 of Lecture Notes in Computer Science, pp. 297–309.

[13] Randrup T. (1998), *Approximation of surfaces by cylinders*, Computer-Aided Design, Vol. 30, No. 10, pp. 807812.

[14] San Segundo F., Sendra J.R. (2008), *Offsetting revolution surfaces*, Automated Deduction in Geometry, Vol. 6301 of the series Lecture Notes in Computer Science, pp 179-188.

[15] Shen L-Y., Pérez-Díaz S. (2014), *Characterization of rational ruled surfaces*, Journal of Symbolic Computation Vol. 63, pp. 21-45.

[16] Vršek J., Làvička M. (2015), *Determining surfaces of revolution from their implicit equations.*, Journal of Computational and Applied Mathematics Vol. 290, pp. 125–135.

[17] Vršek J., Làvička M. (2016), *Recognizing implicitly given rational canal surfaces*, Journal of Symbolic Computation Vol. 74, pp. 367377.

[18] Walker R.J. (1978), *Algebraic Curves*, Springer-Verlag.

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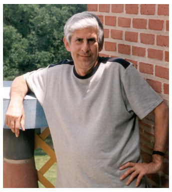

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Professor Goldman's current research interests lie in the mathematical representation, manipulation, and analysis of shape using computers. His work includes research in computer aided geometric design, solid modeling, computer graphics, polynomials and splines. He is particularly interested in algorithms for polynomial and piecewise polynomial curves and surfaces, as well as in applications of algebraic and differential geometry to geometric modeling. His most recent focus is on the uses of quaternions and Clifford algebras in computer graphics.

Dr. Goldman has published over a hundred articles in journals, books, and conference proceedings on these and related topics. He has also published three books on Computer Graphics and Geometric Modeling: *Pyramid Algorithms: A Dynamic Programming Approach to Curves and Surfaces for Geometric Modeling*, *An Integrated Introduction to Computer Graphics and Geometric Modeling*, and *Rethinking Quaternions: Theory and Computation*. Dr. Goldman is currently an Associate Editor of Computer Aided Geometric Design.

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