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On Symbolic Solutions of Algebraic Partial Differential Equations

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In this paper we present a general procedure for solving first-order autonomous algebraic partial differential equations in two independent variables. The method uses proper rational parametrizations of algebraic surfaces and generalizes a similar procedure for first-order autonomous ordinary differential equations. We will demonstrate in examples that, depending on certain steps in the procedure, rational, radical or even non-algebraic solutions can be found. Solutions computed by the procedure will depend on two arbitrary independent constants.

Keywords: Partial differential equations · algebraic surfaces · rational parametrizations · radical parametrizations

1 Introduction

Recently algebraic-geometric solution methods for first-order algebraic ordinary differential equations (AODEs) were investigated. A first result on computing solutions of AODEs using Gröbner bases was presented in [10]. Later in [3] a degree bound of rational solutions of a given AODE is computed. From this one might find a solution by

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solving algebraic equations. The starting point for algebraic-geometric methods was an algorithm by Feng and Gao [4, 5] which decides whether or not an autonomous AODE, $F(y, y') = 0$ has a rational solution and in the affirmative case computes it. In the algorithm a proper rational parametrization of an algebraic curve is used. By means of a special property of this parametrization the existence of a rational solution can be decided. From a rational solution a rational general solution can be deduced.

This result was then generalized by Ngô and Winkler [14, 15, 16] to the non-autonomous case $F(x, y, y') = 0$. Here, parametrizations of surfaces play an important role. On the basis of a proper parametrization, the algorithm builds a so called associated system of first-order linear ODEs for which solution methods exist. With the solution of the associated system, a rational general solution of the differential equation is computed.

First results on higher order AODEs can be found in [7, 8, 9]. Ngô, Sendra and Winkler [13] also classified AODEs in terms of rational solvability by considering affine linear transformations. Classes of AODEs are investigated which contain an autonomous equation. A generalization to birational transformations can be found in [12].

In [6] a solution method for autonomous AODEs is presented which generalizes the method of Feng and Gao to the computation of radical and also non-radical solutions. Again a crucial tool is the parametrization involved in the process. To the contrary of the previous algorithms also radical parametrizations can be used in this method. However, this method is not complete, for if it does not yield a solution, no conclusion on the solvability of the initial AODE can be drawn.

In this paper we present a generalization of the procedure in [6] to algebraic partial differential equations (APDEs). We restrict to first-order autonomous APDEs in two variables. Solutions computed by the procedure will depend on two arbitrary independent constants. However, the class of functions which may appear in the solution of the procedure is only defined implicitly since the procedure depends on the solution of certain ODEs.

In Sect. 2 we will recall and introduce the necessary definitions and concepts. Then we will present the general procedure for solving APDEs in Sect. 3 and show some examples.

2 Preliminaries

We consider the field of rational functions $\mathbb{K}(x, y)$ for some field $\mathbb{K}$ of characteristic zero. By $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ we denote the usual derivative by x and y respectively. Sometimes we might use the abbreviations $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$. The ring of differential polynomials is denoted as $\mathbb{K}(x, y)\{u\}$. It consists of all polynomials in u and its derivatives, i. e.

$$\mathbb{K}(x, y)\{u\} = \mathbb{K}(x, y)[u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots] \ .$$

Let $\mathbb{K}[x, y]\{u\} \subseteq \mathbb{K}(x, y)\{u\}$ be the elements which are polynomial in the variables x and y. An algebraic partial differential equation (APDE) is given by

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots) = 0$$

where $F \in \mathbb{K}[x, y]\{u\}$. In this paper we restrict to the first-order autonomous case, i. e. $F(u, u_x, u_y) = 0$.

Let $\overline{\mathbb{K}}$ be the algebraic closure of $\mathbb{K}$, and $A(\overline{\mathbb{K}})^3$ be the 3-dimensional affine space. An *algebraic surface* $\mathcal{S}$ in $A(\overline{\mathbb{K}})^3$ is a two-dimensional algebraic variety, i.e. $\mathcal{S}$ is the zero set of a squarefree non-constant polynomial $f \in \mathbb{K}[x, y, z]$, $\mathcal{S} = \{(a, b, c) \in A(\bar{\mathbb{K}})^3 \mid f(a, b, c) = 0\}$. We call the polynomial f the *defining polynomial*. An important aspect of algebraic surfaces is their rational parametrizability. We consider an algebraic surface defined by an irreducible polynomial f. A triple of rational functions $\mathcal{P}(s, t) = (p_1(s, t), p_2(s, t), p_3(s, t))$ is called a *rational parametrization* of the surface if $f(p_1(s, t), p_2(s, t), p_3(s, t)) = 0$ for all s and t and the jacobian of $\mathcal{P}$ has generic rank 2. We observe that this condition is fundamental since, otherwise, we are parametrizing a point (if the rank is 0) or a curve on the surface (if the rank is 1). A parametrization can be considered as a dominant map $\mathcal{P}(s, t) : \mathbb{K}^2 \to \mathcal{S}$. By abuse of notation we also call this map a parametrization. We call a parametrization $\mathcal{P}(s, t)$ *proper* if it is a birational map or in other words if for almost every point (a, b, c) on the surface we find exactly one pair (s, t) such that $\mathcal{P}(s, t) = (a, b, c)$ or equivalently if $\mathbb{K}(\mathcal{P}(s, t)) = \mathbb{K}(s, t)$.

Above we have considered rational parametrizations of a surface. However, we might want to deal with more general parametrizations. If so, we will say that a triple of differentiable functions $\mathcal{Q}(s, t) = (q_1(s, t), q_2(s, t), q_3(s, t))$ is a parametrization of the surface if $f(\mathcal{Q}(s, t))$ is identically zero and the jacobian of $\mathcal{Q}(s, t)$ has generic rank 2.

Let $F(u, u_x, u_y) = 0$ be an autonomous APDE. We consider the corresponding algebraic surface by replacing the derivatives by independent transcendental variables, $F(z, p, q) = 0$. Given any differentiable function $u(x, y)$ with $F(u, u_x, u_y) = 0$, then $(u(s, t), u_x(s, t), u_y(s, t))$ is a parametrization. We call this parametrization the *corresponding parametrization of the solution*. We observe that the corresponding parametrization of a solution is not necessarily a parametrization of the associated surface. For instance, let us consider the APDE $u_x = 0$. A solution would be of the form $u(x, y) = g(y)$, with g differentiable. However, this solution generates $(g(t), 0, g'(t))$ that is a curve in the surface; namely the plane $p = 0$. Now, consider the APDE $u_x = \lambda$, with λ a nonzero constant. Hence, the solutions are of the form $u(x, y) = \lambda x + g(y)$. Then, $u(x, y) = \lambda x + y$ generates the line $(\lambda s + t, \lambda, 1)$ while $u(x, y) = \lambda x + y^2$ generates the parametrization $(\lambda s + t^2, \lambda, 2t)$ of the associated plane $p = \lambda$. Clearly a solution of an APDE is a function $u(x, y)$ such that $F(u, u_x, u_y) = 0$. The examples above motivate the following definition.

Definition 2.1.
We say that a solution $u(x, y)$ of an APDE is rational *if $u(x, y)$ is a rational function over an algebraic extension of $\mathbb{K}$.*
We say that a rational solution of an APDE is proper *if the corresponding parametrization is proper.*

In the case of autonomous ordinary differential equations, every non-constant solution induces a proper parametrization of the associated curve (see [4]). However, this is not true in general for autonomous APDEs. For instance, the solution $x + y^3$ of $u_x = 1$, induces the parametrization $(s+t^3, 1, 3t^2)$ which is not proper, although its jacobian has rank 2.

Reark 2.2.
By the definition of a surface parametrization we know that the jacobian of a proper parametrization has generic rank 2*.*

3 A Method for Solving First-Order Autonomous APDEs

Let $F(u, u_x, u_y) = 0$ be an algebraic partial differential equation. We consider the surface $F(z, p, q) = 0$ and assume it admits a proper (rational) surface parametrization

$$\mathcal{Q}(s,t) = (q_1(s,t), q_2(s,t), q_3(s,t)) \ .$$

An algorithm for computing a proper rational parametrization of a surface can be found for instance in [17]. Here, we will stick to rational parametrizations, but the procedure which we present will work as well with other kinds of parametrizations, for instance radical ones. First results on radical parametrizations of surfaces can be found in [18]. Assume that $\mathcal{L}(s,t) = (p_1(s,t), p_2(s,t), p_3(s,t))$ is the corresponding parametrization of a solution of the APDE. Furthermore we assume that the parametrization $\mathcal{Q}$ can be expressed as

$$\mathcal{Q}(s,t) = \mathcal{L}(g(s,t))$$

for some invertible function $g(s,t) = (g_1(s,t), g_2(s,t))$. This assumption is motivated by the fact that in case of rational algebraic curves every non-constant rational solution of an AODE yields a proper rational parametrization of the associated algebraic curve and each proper rational parametrization can be obtained from any other proper one by a rational transformation. However, in the case of APDEs, not all rational solutions provide a proper parametrization, as mentioned in the remark after Definition 2.1. Now, using the assumption, if we can compute g^{-1} we have a solution $\mathcal{Q}(g^{-1}(s,t))$.

Let $\mathcal{J}$ be the jacobian matrix. Then we have

$$\mathcal{J}_{\mathcal{Q}}(s,t) = \mathcal{J}_{\mathcal{L}}(g(s,t)) \cdot \mathcal{J}_g(s,t) \ .$$

Taking a look at the first row we get that

$$\begin{aligned} \frac{\partial q_1}{\partial s} &= \frac{\partial p_1}{\partial s}(g)\frac{\partial g_1}{\partial s} + \frac{\partial p_1}{\partial t}(g)\frac{\partial g_2}{\partial s} = q_2(s,t)\frac{\partial g_1}{\partial s} + q_3(s,t)\frac{\partial g_2}{\partial s} \ , \\ \frac{\partial q_1}{\partial t} &= \frac{\partial p_1}{\partial s}(g)\frac{\partial g_1}{\partial t} + \frac{\partial p_1}{\partial t}(g)\frac{\partial g_2}{\partial t} = q_2(s,t)\frac{\partial g_1}{\partial t} + q_3(s,t)\frac{\partial g_2}{\partial t} \ . \end{aligned} \tag{1}$$

This is a system of quasilinear equations in the unknown functions g_1 and g_2. In case q_2 or q_3 is zero the problem reduces to ordinary differential equations. Hence, from now on we assume that $q_2 \neq 0$ and $q_3 \neq 0$. First we divide by q_2:

$$\begin{aligned} a_1 &= \frac{\partial g_1}{\partial s} + b\frac{\partial g_2}{\partial s} \quad , \\ a_2 &= \frac{\partial g_1}{\partial t} + b\frac{\partial g_2}{\partial t} \end{aligned} \tag{2}$$

with

$$a_1 = \frac{\frac{\partial q_1}{\partial s}}{q_2} \quad , \qquad a_2 = \frac{\frac{\partial q_1}{\partial t}}{q_2} \quad , \qquad b = \frac{q_3}{q_2} \quad . \tag{3}$$

By taking derivatives we get

$$\begin{aligned} \frac{\partial a_1}{\partial t} &= \frac{\partial^2 g_1}{\partial s \partial t} + \frac{\partial b}{\partial t}\frac{\partial g_2}{\partial s} + b\frac{\partial^2 g_2}{\partial s \partial t} \quad , \\ \frac{\partial a_2}{\partial s} &= \frac{\partial^2 g_1}{\partial t \partial s} + \frac{\partial b}{\partial s}\frac{\partial g_2}{\partial t} + b\frac{\partial^2 g_2}{\partial t \partial s} \quad . \end{aligned} \tag{4}$$

Subtraction of the two equations yields

$$\frac{\partial b}{\partial t}\frac{\partial g_2}{\partial s} - \frac{\partial b}{\partial s}\frac{\partial g_2}{\partial t} = \frac{\partial a_1}{\partial t} - \frac{\partial a_2}{\partial s} \quad . \tag{5}$$

This is a single quasilinear differential equation which can be solved by the method of characteristics (see for instance [19]). In case $\frac{\partial b}{\partial t} = 0$ or $\frac{\partial b}{\partial s} = 0$ equation (5) reduces to a simple ordinary differential equation.

Reark 3.1.
If both derivatives of b are zero then b is a constant. Hence, the left hand side of (5) is zero. In case the right hand side is non-zero we get a contradiction, and hence there is no solution. In case the right hand side is zero as well we get from (5) that

$$\begin{aligned} 0 = \frac{\partial a_1}{\partial t} - \frac{\partial a_2}{\partial s} &= \frac{\partial}{\partial t}\left(\frac{\frac{\partial q_1}{\partial s}}{q_2}\right) - \frac{\partial}{\partial s}\left(\frac{\frac{\partial q_1}{\partial t}}{q_2}\right) \\ &= \frac{\frac{\partial^2 q_1}{\partial t \partial s} q_2 - \frac{\partial q_1}{\partial s}\frac{\partial q_2}{\partial t}}{q_2^2} - \frac{\frac{\partial^2 q_1}{\partial s \partial t} q_2 - \frac{\partial q_1}{\partial t}\frac{\partial q_2}{\partial s}}{q_2^2} \\ &= -\frac{\frac{\partial q_1}{\partial s}\frac{\partial q_2}{\partial t} - \frac{\partial q_1}{\partial t}\frac{\partial q_2}{\partial s}}{q_2^2} \quad , \end{aligned}$$

hence,

$$0 = \frac{\partial q_1}{\partial s}\frac{\partial q_2}{\partial t} - \frac{\partial q_1}{\partial t}\frac{\partial q_2}{\partial s} \quad .$$

Moreover, since b is constant, $q_2 = kq_3$ for some constant k. But this means that the rank of the jacobian of $\mathcal{Q}$ is 1, a contradiction to $\mathcal{Q}$ being proper.

Therefore we assume from now on, that the derivatives of b are non-zero. According to the method of characteristics, we need to solve the following system of first-order ordinary differential equations

$$\frac{ds(t)}{dt} = -\frac{\frac{\partial b}{\partial t}(s(t),t)}{\frac{\partial b}{\partial s}(s(t),t)} ,$$
$$\frac{dv(t)}{dt} = \frac{\frac{\partial a_1}{\partial t}(s(t),t) - \frac{\partial a_2}{\partial s}(s(t),t)}{-\frac{\partial b}{\partial s}(s(t),t)} .$$

The second equation is linear and separable but depends on the solution of the first. The first ODE can be solved independently. Its solution $s(t) = \eta(t,k)$ will depend on an arbitrary constant k. Hence, also the solutions of the second ODE depends on k. Finally, the function g_2 we are looking for is $g_2(s,t) = v(t,\mu(s,t)) + \nu(\mu(s,t))$ where μ is computed such that $s = \eta(t,\mu(s,t))$ and ν is an arbitrary function. In case we are only looking for rational solutions we can use the algorithm of Ngô and Winkler [14, 15, 16] for solving these ODEs.

Knowing g_2 we can compute g_1 by using (1) which now reduces to a separable ODE in g_1. The remaining task is to compute h_1 and h_2 such that $g(h_1(s,t),h_2(s,t)) = (s,t)$. Then $q_1(h_1,h_2)$ is a solution of the original PDE.

Finally the method reads as

Procedure 3.2.
Given an autonomous APDE, $F(u,u_x,u_y) = 0$, where F is irreducible and $F(z,p,q) = 0$ is a rational surface with a proper rational parametrization $\mathcal{Q} = (q_1,q_2,q_3)$.

1. *Compute the coefficients b and a_i as in* (3).
2. *If $\frac{\partial b}{\partial s} = 0$ and $\frac{\partial b}{\partial t} \neq 0$ compute $g_2 = \int \frac{\frac{\partial a_1}{\partial t} - \frac{\partial a_2}{\partial s}}{\frac{\partial b}{\partial t}} ds + \kappa(t)$ and go to step 6 otherwise continue.*
   If $\frac{\partial b}{\partial s} = \frac{\partial b}{\partial t} = 0$ return "No proper solution".
3. *Solve the ODE $\frac{ds(t)}{dt} = -\frac{\frac{\partial b}{\partial t}(s(t),t)}{\frac{\partial b}{\partial s}(s(t),t)}$ for $s(t) = \eta(t,k)$ with arbitrary constant k.*
4. *Solve the ODE $\frac{dv(t)}{dt} = \frac{\frac{\partial a_1}{\partial t}(\eta(t,k),t) - \frac{\partial a_2}{\partial s}(\eta(t,k),t)}{-\frac{\partial b}{\partial s}(\eta(t,k),t)}$*
   by $v(t) = v(t,k) = \int \frac{\frac{\partial a_1}{\partial t}(\eta(t,k),t) - \frac{\partial a_2}{\partial s}(\eta(t,k),t)}{-\frac{\partial b}{\partial s}(\eta(t,k),t)} dt + \nu(k)$.
5. *Compute μ such that $s = \eta(t,\mu(s,t))$ and then $g_2(s,t) = v(t,\mu(s,t))$.*
6. *Use the second equation of* (2) *to compute $g_1(s,t) = m(s) + \int a_2 - b\frac{\partial g_2}{\partial t} dt$.*
7. *Determine $m(s)$ by using the first equation of* (2).
8. *Compute h_1, h_2 such that $g(h_1(s,t),h_2(s,t)) = (s,t)$.*
9. *Return the solution $q_1(h_1,h_2)$.*

Observe that the proper rational parametrization $\mathcal{Q}$ can be computed applying Schicho's algorithm (see [17]). In addition, we also observe that the procedure can be extended to the non-rational algebraic case, if one has an injective parametrization, in that case non-rational, of the surface defined by $F(z,p,q)=0$.

In general ν will depend on a constant c_2 and m on a constant c_1. As a special case of the procedure we will fix $\nu = c_2$. This choice is done for simplicity reasons but we may sometimes refer to cases with other choices which are a subject of further research.

Furthermore, the procedure can be considered symmetrically in step 2 for the case that $\frac{\partial b}{\partial t} = 0$ and $\frac{\partial b}{\partial s} \neq 0$. In such a case the rest of the procedure has to be changed symmetrically as well. We will not go into further details.

Theorem 3.3.
Let $F(u,u_x,u_y)=0$ be an autonomous APDE. If Procedure 3.2 returns a function $v(x,y)$ for input F, then v is a solution of F.

Proof. By the procedure we know that $v(x,y)=q_1(h_1(x,y),h_2(x,y))$ with h_i such that $g(h_1(s,t),h_2(s,t))=(s,t)$. Since g is a solution of system (1) it fulfills the assumption that $u(g_1,g_2)=q_1$ for a solution u . Hence, v is a solution. We have seen a more detailed description at the beginning of this section. □ □

Reark 3.4.
In step 3 and 4 ODEs have to be solved. Depending on the class of functions to which the requested solution should belong, these ODEs do not necessarily have a solution. Furthermore, an explicit inverse (step 8) does not necessarily exist.
It will be a subject of further research, to investigate conditions on cases were the procedure does definitely not fail.

Now, we will show that the result of Procedure 3.2 does not change if we postpone the introduction of c_1 and c_2 to the end of the procedure. It is easy to show that if $u(x,y)$ is a solution of an autonomous APDE then so is $u(x+c,y+d)$ for any constants c and d. From the procedure we see that in the computation of g_1 we use the derivative of g_2 only (and hence c_2 disappears). We can write

$$g_2 = \bar{g}_2 + c_2 \ , \qquad\qquad g_1 = \bar{g}_1 + c_1$$

for some functions $\bar{g}_1, \bar{g}_2$ which do not depend on c_1 and c_2. Let $g=(g_1,g_2)$ and $\bar{g}=(\bar{g}_1,\bar{g}_2)$. In the step 8 we are looking for a function h such that $g \circ h = \mathrm{id}$. Now $g \circ h = \bar{g} \circ h + (c_1,c_2)$. Take $\bar{h}$ such that $\bar{g} \circ \bar{h} = \mathrm{id}$. Then $g \circ \bar{h}(s-c_1, t-c_2)) = \mathrm{id}$. Hence, we can introduce the constants at the end.

In case the original APDE is in fact an AODE, the ODE in step 4 turns out to be trivial and the integral in step 7 is exactly the one which appears in the procedure for AODEs [6]. Of course then g is univariate and so is its inverse. In this sense, this new procedure generalizes the procedure in [6]. We do not specify Procedure 3.2 to handle this case.

In the following we will show some examples which can be solved by Procedure 3.2. Note, that the examples have more solutions than the one computed below. In Example 3.5 for instance, other solutions can be found by choosing different ν, e. g. $\nu(x) =$

$c_2 + x^2$. However, the results might not be rational solutions then. In general the procedure, as stated in this paper, will yield only one solution containing two arbitrary independent constants. Hence, it will not be a general solution in the sense of depending on an arbitrary function (compare [11]).

We start with a simple well known APDE which has a rational solution.

Example 3.5. (Inviscid Burgers' Equation [1, p. 7])
We consider the autonomous APDE

$$F(u, u_x, u_y) = uu_x + u_y = 0 \ .$$

Since F is of degree one in each of the derivatives, it is easy to compute a parametrization $\mathcal{Q} = \left(-\frac{t}{s}, s, t\right)$. We compute the coefficients

$$a_1 = \frac{t}{s^3} \ , \qquad a_2 = -\frac{1}{s^2} \ , \qquad b = \frac{t}{s} \ .$$

In step 3 we find $s(t) = kt$ and in step 4 we compute $v(t) = \frac{1}{kt} + \nu(k)$. Then $\mu(s, t) = \frac{s}{t}$ and hence (with $\nu = c_2$),

$$\begin{aligned} g_2 &= \frac{1}{s} + c_2 \ , \\ g_1 &= -\frac{t}{s^2} + m(s) \ . \end{aligned}$$

Using step 7 we find out that $m(s) = c_1$. Computing the inverse of g we find

$$\begin{aligned} h_1 &= \frac{1}{t - c_2} \ , \\ h_2 &= \frac{-s + c_1}{(t - c_2)^2} \ . \end{aligned}$$

Finally, we get the solution $\frac{x-c_1}{y-c_2}$.

Procedure 3.2 can also handle more complicated APDEs.

Example 3.6.
We consider the APDE

$$\begin{aligned} 0 &= F(u, u_x, u_y) \\ &= uu_x^4 + u_x^3 u_y - uu_x^3 u_y - u_x^2 u_y^2 + uu_x^2 u_y^2 + u_x u_y^3 - uu_x u_y^3 + uu_y^4 \ . \end{aligned}$$

Then

$$\mathcal{Q} = \left(-\frac{t\,(1 - t + t^2)}{1 - t + t^2 - t^3 + t^4}, t\gamma(s, t), \gamma(s, t)\right) \ ,$$

with $\gamma(s, t) = \frac{t(-10+7t)\left(-9+t^2\right)\left(-1+2t-3t^2+3t^4-2t^5+t^6\right)}{2s(45-63t+5t^2)(1-t+t^2-t^3+t^4)^2}$, is a proper parametrization of the corresponding algebraic surface. This parametrization is not easy to find. It is computed *by first using parametrization by lines and then applying a linear transformation in s. Alternatively one could use this parametrization by lines directly. Procedure 3.2 will find the same solution, but the intermediate steps need more writing space. Using the procedure with the parametrization $\mathcal{Q}$ we get*

$$g_1 = s\left(\frac{7}{10-7t} - \frac{1}{t} + \frac{2t}{-9+t^2}\right) \ , \qquad g_2 = \frac{2s\left(45-63t+5t^2\right)}{(-10+7t)\left(-9+t^2\right)} \ ,$$
$$h_1 = -\frac{t\left(-90s^3-63s^2t+10st^2+7t^3\right)}{2s\left(45s^2+63st+5t^2\right)} \ , \qquad h_2 = \frac{-t}{s} \ ,$$

and finally the solution $u(x,y) = \frac{xy\left(x^2+xy+y^2\right)}{x^4+x^3y+x^2y^2+xy^3+y^4}$. As mentioned before, $u(x+c_1, y+c_2)$ with constants c_1 and c_2 is also a solution.

The procedure presented in this paper is, however, not restricted to rational solutions nor to rational parametrizations as we will see in the following examples. We start with an example which has a radical solution.

Example 3.7. (Eikonal Equation [2, p. 2])
We consider the APDE

$$F(u, u_x, u_y) = u_x^2 + u_y^2 - 1 = 0 \ .$$

From the rational parametrization of a circle it is easy to see that

$$\mathcal{Q} = \left(s, \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

is a parametrization of the corresponding surface. Using the procedure we get some rational g_1 and g_2 which yield

$$h_2 = \frac{-s+c_1 \pm \sqrt{s^2+t^2-2sc_1+c_1^2-2tc_2+c_2^2}}{t-c_2} \ ,$$
$$h_1 = \pm\sqrt{s^2+t^2-2sc_1+c_1^2-2tc_2+c_2^2} \ .$$

Finally, we get the radical solution

$$u(x,y) = \pm\sqrt{x^2+y^2-2xc_1+c_1^2-2yc_2+c_2^2} \ .$$

In a further example we compute an exponential solution of an APDE.

Example 3.8. (Convection-Reaction Equation [1, p. 7])
We consider the APDE

$$F(u, u_x, u_y) = u_x + cu_y - du = 0 \ ,$$

where $d \neq 0$. We compute a parametrization $\mathcal{Q} = \left(\frac{s+ct}{d}, s, t\right)$ and the coefficients

$$a_1 = \frac{1}{ds} \ , \qquad a_2 = \frac{c}{ds} \ , \qquad b = \frac{t}{s} \ .$$

Solving the ODEs of steps 3–6 we get

$$g_2 = \frac{c\log(t)}{d} + c_2 \ , \qquad g_1 = c_1 + \frac{\log(s)}{d} \ .$$

Computing the inverse of g we find

$$h_1 = \mathrm{e}^{ds-dc_1} \ , \qquad h_2 = \mathrm{e}^{\frac{dt}{c}-\frac{dc_2}{c}} \ .$$

Finally, we get the solution $\frac{\mathrm{e}^{d(x-c_1)}+c\mathrm{e}^{\frac{d(y-c_2)}{c}}}{d}$.

4 Conclusion

We have introduced a procedure which, in case all steps are computable, yields a solution of the input APDE. In case one step of the procedure is not computable (in a certain class of functions) we cannot give any answer to the question of solvability of the APDE. We have shown examples of APDEs solvable by the procedure. These include rational, radical and exponential solutions. The investigation of rational solutions as well as a possible extension to an arbitrary number of variables is currently subject to further research.

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