

Radical parametrization of algebraic curves and surfaces

J. Rafael Sendra

Dept. of Physics and Mathematics, University of Alcalá de Henares (Spain)

David Sevilla

Dept. of Mathematics, University of Extremadura (Spain)

sevillad@unex.es

Abstract

Parametrization of algebraic curves and surfaces is a fundamental topic in CAGD (intersections; offsets and conchoids; etc.) There are many results on rational parametrization, in particular in the curve case, but the class of such objects is relatively small. If we allow root extraction, the class of parametrizable objects is greatly enlarged (for example, elliptic curves can be parametrized with one square root). We will describe the basics and the state of the art of the problem of parametrization of curves and surfaces by radicals.

Keywords

Radical parametrization, parametrization of curves, parametrization of surfaces

1 Introduction

It is well known that the only algebraic curves that are rationally parametrizable are those of genus zero, and there are algorithms for that purpose (Sendra et al., 2008). However, in many applications, this is a strong limitation because either the curves appearing in the process are not rational (i.e. genus zero curves) or the algebraic manipulation of the geometric object does not preserve the genus; this happens, for instance, when applying offsetting constructions (Arrondo et al., 1997) or performing conchoidal transformations (Sendra and Sendra, 2010).

On the other hand, allowing radicals rather than just rational functions greatly enlarges the class of parametrizable functions. For example, one class of curves which are clearly parametrizable by radicals is that of hyperelliptic curves. Every such curve can be written as $y^2 = P(x)$ for some polynomial $P(x)$, and we can quickly write the parametrization $x = t, y = \sqrt{P(t)}$ where the root is meant to be taking in a strictly algebraic sense, that is, as an element of an algebraic extension of the field $\mathbb{F}(t)$ where \mathbb{F} is the coefficient field of the curve. Essentially, a radical parametrization is given by rational functions whose numerators and denominators are radicals expressions of polynomials.

The roots of univariate polynomials of degree ≤ 4 can be written in terms of radicals. Therefore, curves which can be expressed as $f(x, y) = 0$ where one of the variables occurs with degree ≤ 4 can also be parametrized by radicals. In relation to this, the minimum degree of a map from the curve to \mathbb{P}^1 is called the *gonality* of the curve. Hyperelliptic curves are precisely those of gonality two and, as in the example above, can be parametrized using one square root. It is thus interesting to characterize the curves of gonality three (*trigonal*) and four (*trigonal*), and further to produce algorithms that detect these situations and compute a radical parametrization.

In relation to this, the following facts are relevant. In Zariski (1926), Zariski proved that the general complex projective curve of genus $g > 6$ is not parametrizable by radicals. Moreover, as remarked in Pirola and Schlesinger (2005), Zariski's result is sharp. Indeed, a result within Brill-Noether theory (see Brill and Noether (1873), or (Arbarello et al., 1985, Chapter V) for a more modern account) states that a curve of genus g has a linear system of dimension 1 and degree $\lceil \frac{g}{2} + 1 \rceil$ (Arbarello et al., 1985, p. 206), thus a map of that degree to \mathbb{P}^1 . The previous expression is thus an upper bound for the gonality in terms of the genus. It follows that for $g = 3, 4$ there exists generically a $3 : 1$ map whose inversion would provide a radical parametrization with cubic

roots, and for $g = 5, 6$ the inversion of the existing $4 : 1$ would provide a radical parametrization with quartic roots. These are instances of trigonal and tetragonal curves.

We do not wish to enter the thorny realm of evaluation of radical functions, so we consider them as elements in an certain algebraic field extension, not as functions. For simplicity, let us restrict the discussion to the situations where the coefficient field is algebraically closed of characteristic zero.

2 The trigonal case

An algorithm for the trigonal case is described in Schicho and Sevilla (2012). The solution is based on the *Lie algebra method* introduced in de Graaf et al. (2006) (see also de Graaf et al. (2009)). There Lie algebra computations (which mostly amount to linear algebra) are used to decide if a certain algebraic variety associated to the input curve is a rational normal scroll, which is the case precisely when the curve is trigonal. Further, one can compute an isomorphism between that variety and the scroll when it exists.

Let C be an non-hyperelliptic algebraic curve of genus $g \geq 4$, so that it is isomorphic to its image by the canonical map $\varphi: C \rightarrow \mathbb{P}^{g-1}$. In Enriques (1919) and Babbage (1939) it is proven that $\varphi(C)$ is the intersection of the quadrics that contain it, except when C is trigonal (that is, it has a g_3^1) or isomorphic to a plane quintic ($g = 6$). In those cases, the corresponding varieties are minimal degree surfaces, see (Griffiths and Harris, 1978, p. 522 and onwards).

From this situation we exclude the curves with genus lower than 3 since they are hyperelliptic, thus they have a g_2^1 which can be made into a g_3^1 by adding a base point; the problem is then to find a point in the curve over the field of definition. Also, if the curve is non-hyperelliptic of genus 3, it is isomorphic to its canonical image which is a quartic in \mathbb{P}^2 , and the system of lines through any point of the curve cuts out a g_3^1 .

The following theorem summarizes the classification of canonical curves according to the intersection of the quadric hypersurfaces that contain them.

Theorem 1 (Griffiths and Harris (1978, p. 535)) *For any canonical curve $C \subset \mathbb{P}^{g-1}$ over an algebraically closed field, either*

1. C is entirely cut out by quadric hypersurfaces; or
2. C is trigonal, in which case the intersection of all quadrics containing C is isomorphic to the rational normal scroll swept out by the trichords of C ; or
3. C is isomorphic to a plane quintic, in which case the intersection of the quadrics containing C is isomorphic to the Veronese surface in \mathbb{P}^5 , swept out by the conic curves through five coplanar points of C .

There exist efficient algorithms for the computation of the canonical map, determination of hyperellipticity, and calculation of the space of forms of a given degree containing a curve, for example in Magma (Bosma et al., 1997) and at least partially in Maple. Thus the problem lies in *recognizing* which of the previous types is that of the intersection of the quadrics containing C .

Definition 2 *Every finite-dimensional Lie algebra L can be written as a semidirect sum of two parts called a solvable part and a semisimple part. The latter is called a Levi subalgebra of L , and it is unique up to conjugation, so we will speak of “the” Levi subalgebra of L and denote it as $LSA(L)$. For a variety X , we will denote $LSA(L(X))$ simply by $LSA(X)$.*

The Lie algebra of a curve of genus 2 or higher is zero since its automorphism group is finite. The rest of the cases that arise in Theorem 1 are studied in the next result.

Theorem 3 (Oda (1988, Section 3.4)) *Let k be an algebraically closed field of characteristic zero. As above, let $S_{m,n}$ be the the rational normal scroll with parameters m, n , and let V be the image of the Veronese map $\mathbb{P}^2 \rightarrow \mathbb{P}^5$.*

1. $LSA(S_{m,n}) \cong \mathfrak{sl}_2$ if $m \neq n$.
2. $LSA(S_{m,m}) \cong \mathfrak{sl}_2 + \mathfrak{sl}_2$ (a direct sum of two Lie algebras)
3. $LSA(V) \cong \mathfrak{sl}_3$.

Therefore, just by looking at the dimension of the Levi subalgebra we can discard the two cases where the curve is not trigonal. In other words, we can recognize a trigonal curve by the dimension of its Levi subalgebra.

Corollary 4 *Let k be any field of characteristic zero, let C be a canonical curve and X be the intersection of the quadrics that contain it. Then one of the following occurs:*

- *If $\dim LSA(X) = 0$ then $X = C$ and C is not trigonal.*
- *If $\dim LSA(X) = 3$ then X is a twist of $S_{m,n}$ with $m \neq n$ and C is trigonal.*
- *If $\dim LSA(X) = 6$ then X is a twist of $S_{m,m}$ and C is trigonal.*
- *If $\dim LSA(X) = 8$ then $X \cong V$ and C is not trigonal.*

3 The tetragonal case and higher gonality

For the cases of genus 5 and 6, a solution was presented in Harrison (2013) which involves the study of minimal free resolutions of certain geometric constructions. Unfortunately there has been no extension to tetragonal curves of arbitrary genus.

Very recently, in Schicho et al. (2013) the authors have published a deterministic algorithm that calculates the gonality of a given curve and a map to \mathbb{P}^1 that realizes the gonality. The methods they use are based on syzygies, and are quite limited in practical computations. On the other hand, in (Schicho et al., 2013, Theorem 1.3) an algorithm for the case of gonality up to 4 for curves in characteristic $\neq 2, 3$.

Although these results allow us to find lowest degree maps, their invertibility by radicals is generally not possible outside the cases discussed above.

4 Parametrization by lines and adjoints

A more direct approach for particular cases is shown in Sendra and Sevilla (2011). First, it is established that the construction of offsets and conchoids, two common constructions in CAGD, is closed under parametrizability by radicals. That is, an offset or conchoid constructed over a curve that is parametrizable by radicals will also be a curve of such type. Another class of curves that can be quickly parametrized by radicals are those of degree d and possessing a point of multiplicity $d - r$ for some $r \leq 4$; in this situation one can produce the parametrization by considering a pencil of lines through the point. Finally, by employing adjoint curves as it is done in the rational case, it is possible to parametrize by radicals curves of genus up to 4.

The caveat is that this method produces $g : 1$ maps where g is the genus. This means that trigonal curves of genus 4 are parametrized by quartic roots, although they can be parametrized by cubic roots; analogously, for curves of genus 5 or 6 a quintic or sextic polynomial in one variable needs to be resolved by radicals in order to produce a parametrization, whereas we know that they can be parametrized by quartic roots.

In any case, these methods provide efficient radical parametrizations for curves that are of practical interest.

5 Radical parametrization of surfaces

It is possible to exploit the results outlined in the previous section for the case of surfaces. As in the curve case, only a narrow class of algebraic surfaces can be parametrized rationally. Namely, the two genera must be zero. What follows is taken from Sendra and Sevilla (2013).

However, by using resolution of univariate polynomials by radicals, it is clear that one can parametrize several new classes of surfaces by radicals. For example, if a surface is given as the zeros of $F(x,y,z)$ where the degree of any of the variables is less or equal than 4, we can parametrize by solving F as a univariate polynomial.

In a more geometric fashion, a surface of degree d that possess a point of multiplicity $d - r$ for some $r \leq 4$ can be parametrized by the pencil of lines through the point. Therefore, every surface of degree 5 is parametrizable by radicals, and so is every singular surface of degree 6.

If we regard $F(x, y, z)$ as a polynomial in x, y with coefficients in $\mathbb{F}(z)$, we can use the parametrization by adjoints methods of the previous section. The caveat here is that, if the relevant construction employs a point in the curve case (not problematic since our coefficient field is algebraically closed), in the surface case it is necessary that there exist such a point satisfying that its coordinates are radical functions. Otherwise, the parametrization we obtain would be radical in x, y but not in z . This produces several cases depending on the genus of $F(x, y)(z)$ and the existence of points with the property just mentioned.

Finally, as in the curve case, certain geometric constructions are proven to be closed under parametrizability by radicals.

Acknowledgements

This contribution is partially supported by the Ministerio de Economía y Competitividad under the project MTM2011-25816-C02-01, by the Austrian Science Fund (FWF) P22766-N18, and by Junta de Extremadura and FEDER funds. The first author is a member of the of the Research Group ASYNACS (Ref. CCEE2011/R34).

References

- Arbarello, E., Cornalba, M., Griffiths, P.A., Harris, J., 1985. Geometry of algebraic curves. Vol. I. volume 267 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York.
- Arrondo, E., Sendra, J., Sendra, J.R., 1997. Parametric generalized offsets to hypersurfaces. *J. Symbolic Comput.* 23, 267–285. Parametric algebraic curves and applications (Albuquerque, NM, 1995).
- Babbage, D.W., 1939. A note on the quadrics through a canonical curve. *J. London Math. Soc.* 14, 310–315.
- Brill, A., Noether, M., 1873. ber die algebraischen functionen und ihre anwendungen in der geometrie. *Math. Ann.* 7, 269–310.
- Bosma, W., Cannon, J., Playoust, C., 1997. The Magma algebra system. I. The user language. *J. Symbolic Comput.* 24, 235–265. Computational algebra and number theory (London, 1993).
- de Graaf, W.A., Harrison, M., Pílníková, J., Schicho, J., 2006. A Lie algebra method for rational parametrization of Severi-Brauer surfaces. *J. Algebra* 303, 514–529.
- de Graaf, W.A., Pílníková, J., Schicho, J., 2009. Parametrizing del Pezzo surfaces of degree 8 using Lie algebras. *J. Symbolic Comput.* 44, 1–14.
- Enriques, F., 1919. Sulle curve canoniche di genere p dello spazio a $p - 1$ dimensioni. *Rend. Accad. Sci. Ist. Bologna* , 80–82.
- Griffiths, P., Harris, J., 1978. Principles of algebraic geometry. Wiley-Interscience [John Wiley & Sons], New York. Pure and Applied Mathematics.
- Harrison, M., 2013. Explicit solution by radicals, gonal maps and plane models of algebraic curves of genus 5 or 6. *J. Symbolic Comput.* 51, 3–21.
- Oda, T., 1988. Convex bodies and algebraic geometry. volume 15 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin. An introduction to the theory of toric varieties, Translated from the Japanese.
- Pirola, G.P., Schlesinger, E., 2005. A curve algebraically but not rationally uniformized by radicals. *J. Algebra* 289, 412–420.
- Schicho, J., Schreyer, F.O., Weimann, M., 2013. Computational aspects of gonal maps and radical parametrization of curves. ArXiv: math.AG/1304.2551v1.

- Schicho, J., Sevilla, D., 2012. Effective radical parametrization of trigonal curves, in: Computational algebraic and analytic geometry. Amer. Math. Soc., Providence, RI. volume 572 of *Contemp. Math.*, pp. 221–231.
- Sendra, J.R., Sendra, J., 2010. Rational parametrization of conchoids to algebraic curves. *Appl. Algebra Engrg. Comm. Comput.* 21, 285–308.
- Sendra, J.R., Sevilla, D., 2011. Radical parametrizations of algebraic curves by adjoint curves. *J. Symbolic Comput.* 46, 1030–1038.
- Sendra, J.R., Sevilla, D., 2013. First steps towards radical parametrization of algebraic surfaces. *Comput. Aided Geom. Design* 30, 374–388.
- Sendra, J.R., Winkler, F., Pérez-Díaz, S., 2008. Rational algebraic curves. volume 22 of *Algorithms and Computation in Mathematics*. Springer, Berlin. A computer algebra approach.
- Zariski, O., 1926. Sull'impossibilità di risolvere parametricamente per radicali un'equazione algebrica $f(x, y) = 0$ di genere $p > 6$ a moduli generali. *Atti Accad. Naz. Lincei Rend., Cl. Sc. fis. Mat. Natur., serie VI* 3, 660–666.