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Bounding and Estimating the Hausdorff distance between real space algebraic curves.*

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Abstract

In this paper, given two real space algebraic curves, not necessarily bounded, whose Hausdorff distance is finite, we provide bounds of their distance. These bounds are related to the distance between the projections of the space curves onto a plane (say, $z = 0$), and the distance between the z-coordinates of points in the original curves. Using these bounds we provide an estimation method for a bound of the Hausdorff distance between two such curves and we check in applications that the method is accurate and fast.

Keywords: Hausdorff distance, space curve, projection, implicit representation, rational parametrization.

1 Introduction

The Hausdorff distance has proven to be an appropriate tool for measuring the resemblance between two geometric objects, becoming in consequence a widely used tool in computer aided design, pattern matching and pattern recognition (see for instance [2], [5], [10] and [12]). Several variants of the Haussdorff distance have been developed to

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match specific patterns of objects, in this paper, we study the computation of the Hausdorff distance between two real space algebraic curves $\mathcal{E}_1$ and $\mathcal{E}_2$. We briefly recall the notion of Hausdorff distance; for further details we refer to [1]. In a metric space (X, d), for $\emptyset \neq B \subset X$ and $a \in X$ we define $\mathrm{d}(a, B) = \inf_{b\in B}\{\mathrm{d}(a, b)\}$. Moreover, for $A, B \subset X \setminus \emptyset$ we define

$$\mathrm{H_d}(A, B) = \max\{\sup_{a\in A}\{\mathrm{d}(a, B)\}, \sup_{b\in B}\{\mathrm{d}(b, A)\}\}.$$

By convention $\mathrm{H_d}(\emptyset, \emptyset) = 0$ and, for $\emptyset \neq A \subset X$, $\mathrm{H_d}(A, \emptyset) = \infty$. The function $\mathrm{H_d}$ is called the Hausdorff distance induced by d. In our case, since we will be working in $(\mathbb{C}^3, \mathrm{d})$ or $(\mathbb{R}^3, \mathrm{d})$, d being the usual unitary or Euclidean distance, we simplify the notation writing $\mathrm{H}(A, B)$.

The problem of computing the Hausdorff distance has proven not to be an easy one. We should note in the first place that there is no effective algorithm for the exact computation of the Hausdorff distance between algebraic varieties. Some recent works that approach special cases are [5], [9] and [12]. There exist theoretical results, as Lojasiewicz inequality for the compact case (see [8]), that relate by means of a constant the Hausdorff distance to the evaluation of the implicit equation(s) of one of the varieties at a parametrization of the other variety. However, this constant is hard to compute. Furthermore, if both varieties are given in implicit form, the computation of the Hausdorff distance is even harder. Also, for the compact case, there are techniques to approximate the distance into a fixed size frame (see [4]), or by using biarcs (see [10]), or polylines (see [2]) as well as under the phenomenon where the point sets are given imprecisely (see [11]). Additionally, for plane curves, in [9] it is shown how to bound the Hausdorff distance using foot-point distance. Altogether shows that bounding and estimating the Hausdorff distance is an active research area.

Among all these different variants of the problem we, here, deal with the algebraic global case for space curves. That is, we are given two real space algebraic curves and we want to provide bounds of the distance of the two algebraic sets and not of certain parts (subsets) included within a bounded frame. Afterwards, these bounds can be used to obtain estimations of the Hausdorff distance in a chosen bounded frame. Having such bounds would be useful for measuring the performance of approximate parametrization and approximate implicitization methods, see [6], [13], [14], [15], in other words to decide how much the input and output of such methods resemble each other. In [13] we provided bounds for the Hausdorff distance between two algebraic plane curves and, in practice, we used estimations of these bounds in [14]. In [15], we gave a method to estimate bounds of $\sup_{Q\in\mathcal{E}_1^{\mathbb{R}}}\{\mathrm{d}(Q, \mathcal{E}_2)\}$ for space curves, considering the intersection of normal planes through regular points of $\mathcal{E}_1^{\mathbb{R}}$ with $\mathcal{E}_2$, and conversely; the curves, although real, are considered over the field of complex numbers and the super-index $\mathbb{R}$ means the real part of the curve.

In this paper, considering planes orthogonal to a plane of projection ($z = 0$), we will be able to relate bounds of $\sup_{P\in\mathcal{E}_1^{\mathbb{R}}}\{\mathrm{d}(P, \mathcal{E}_2)\}$ with bounds for the distance between the real parts of the projected curves, as well as with bounds for the distance between the z–coordinates of points in $\mathcal{E}_1^{\mathbb{R}}$ and $\mathcal{E}_2$. To derive this relation a Gröbner basis of the ideal of $\mathcal{E}_2$ w.r.t. the pure lexicographic order with $z > x > y$ is a fundamental tool, [3]. The relation between bounds of the space curves and bounds of the plane curves, allows the use every method developed thus far to estimate the Hausdorff distance between plane curves to achieve estimations of the distance for space curves.

In Section 2, we present a situation in which the Hausdorff distance between the real part of two algebraic curves is finite. In such situation, an estimation method for the supremum of the distances between points in a given curve and the other curve, is given in Section 3. Examples of application of such method are provided in Section 4 and conclusions are derived in Section 5.

2 Notation and General Assumptions

In this section we fix the **notation** that will be used throughout the paper. We consider a computable subfield $\mathbb{K}$ of the field $\mathbb{R}$ of real numbers, as well as its algebraic closure $\mathbb{F}$; in practice, we may think that $\mathbb{K}$ is the field $\mathbb{Q}$ of rational numbers. We denote by $\mathbb{F}^2$ and $\mathbb{F}^3$ the affine plane and the affine space over $\mathbb{F}$, respectively. Similarly, we denote by $\mathbb{P}^2(\mathbb{F})$ and $\mathbb{P}^3(\mathbb{F})$ the projective plane and the projective space over $\mathbb{F}$, respectively. Furthermore, if $\mathcal{A} \subset \mathbb{F}^3$ (similarly if $\mathcal{A} \subset \mathbb{F}^2$) we denote by $\mathcal{A}^* \subset \mathbb{F}^3$ its Zariski closure, and by $\mathcal{A}^h \subset \mathbb{P}^3(\mathbb{F})$ the projective closure of $\mathcal{A}^*$. We will consider (x, y, z) as affine coordinates and $(x : y : z : w)$ as projective coordinates. Also, we denote by $\mathcal{A}^\infty$ the points at infinity of $\mathcal{A}$, that is, the intersection of $\mathcal{A}^h$ with the projective plane (line in the planar case) of equation $w = 0$.

In addition, we will consider two irreducible real space curves $\mathcal{E}_1, \mathcal{E}_2 \subset \mathbb{C}^3$ satisfying the following assumptions:

A_1. $\mathcal{E}_1^\infty = \mathcal{E}_2^\infty$,

A_2. $\operatorname{card}(\mathcal{E}_1^\infty) = \operatorname{card}(\mathcal{E}_2^\infty) = \deg(\mathcal{E}_1) = \deg(\mathcal{E}_2)$,

A_3. $\mathcal{E}_1, \mathcal{E}_2$ are not included in a plane of the form $ax + by = c$; note that this is not a loss of generality.

We denote by $\mathcal{E}_i^{\mathbb{R}}$ the real part of $\mathcal{E}_i$, that is, $\mathcal{E}_i^{\mathbb{R}} = \mathcal{E}_i \cap \mathbb{R}^3$. Then, in [15] Theorem 6.4., it is shown that if assumptions A_1 and A_2 are satisfied then $\mathrm{H}(\mathcal{E}_1^{\mathbb{R}}, \mathcal{E}_2^{\mathbb{R}}) < \infty$. Alternatively, one may replace assumptions A_1 and A_2 by the requirement that $\mathcal{E}_1^{\mathbb{R}}$ and $\mathcal{E}_2^{\mathbb{R}}$ are both compact, in which case, $\mathrm{H}(\mathcal{E}_1^{\mathbb{R}}, \mathcal{E}_2^{\mathbb{R}}) < \infty$. In addition, note that the compactness of $\mathcal{E}_i^{\mathbb{R}}$ can be deduced from the structure of the curve at infinity.

In addition to the assumptions above, we consider that the projection of each $\mathcal{E}_i$ over the plane $z = 0$ is birational. Note that for almost all projections (see e.g. [7] pp. 155), this holds and hence we can assume this w.l.o.g. We denote this projection map by π_z^i to distinguish between the projection restricted to $\mathcal{E}_1$ and to $\mathcal{E}_2$.

We consider a Gröbner basis $\mathcal{F} = \{F_0, F_1, \ldots, F_{\ell_1}\} \subset \mathbb{K}[x, y, z]$ of the ideal of $\mathcal{E}_1$ and a Gröbner basis $\mathcal{G} = \{G_0, G_1, \ldots, G_{\ell_2}\} \subset \mathbb{K}[x, y, z]$ of the ideal of $\mathcal{E}_2$, both bases w.r.t. the pure lexicographic order with $z > x > y$. Let F_0 be the smallest polynomial of

$\mathcal{F}$, then $F_0 \in \mathbb{K}[x,y]$ and it is an implicit representation of the projected curve $\pi_z^1(\mathcal{E}_1)^*$; similarly with G_0 and $\pi_z^2(\mathcal{E}_2)^*$. On the other hand, since we are assuming the projection π_z^i to be birational on $\mathcal{E}_i$, both Gröbner base contain a linear polynomial in z. Say that $F_1 = f_1(x,y)z - f_2(x,y)$ and that $G_1 = g_1(x,y)z - g_2(x,y)$. Note that this implies that the inverse of $\pi_z^1 : \mathcal{E}_1 \to \pi_z^1(\mathcal{E}_1)$ is $(x, y, f_2(x,y)/f_1(x,y)) \leftarrow (x,y)$ and that the inverse of $\pi_z^2 : \mathcal{E}_2 \to \pi_z^2(\mathcal{E}_2)$ is $(x, y, g_2(x,y)/g_1(x,y)) \leftarrow (x,y)$.

In the next, d denotes the usual unitary distance in $\mathbb{C}^3$, $\|\cdot\|$ denotes the associated norm to d, and $|\cdot|$ denotes the module in the field $\mathbb{C}$ of complex numbers.

3 Bounding $\mathrm{H}(\mathcal{E}_1^{\mathbb{R}}, \mathcal{E}_2^{\mathbb{R}})$

We are interested in bounding the Hausdorff distance between $\mathcal{E}_1^{\mathbb{R}}$ and $\mathcal{E}_2^{\mathbb{R}}$, namely

$$H(\mathcal{E}_1^{\mathbb{R}}, \mathcal{E}_2^{\mathbb{R}}) = \max\{\sup_{Q\in\mathcal{E}_1^{\mathbb{R}}}\{\mathrm{d}(Q, \mathcal{E}_2^{\mathbb{R}})\}, \sup_{P\in\mathcal{E}_2^{\mathbb{R}}}\{\mathrm{d}(P, \mathcal{E}_1^{\mathbb{R}})\}\},$$

where the distance between a point $P \in \mathbb{C}^3$ and a set $\emptyset \neq A \subset \mathbb{C}^3$ is defined as

$$\mathrm{d}(P, A) = \inf_{Q\in A}\{\mathrm{d}(P, Q)\}.$$

In [15], we gave a method to estimate

$$\sup_{P\in\mathcal{E}_1^{\mathbb{R}}}\{\mathrm{d}(P, \mathcal{E}_2)\} \text{ and } \sup_{P\in\mathcal{E}_1^{\mathbb{R}}}\{\mathrm{d}(P, \mathcal{E}_2^{\mathbb{R}})\},$$

assuming that $\mathcal{E}_1$ is rational, and considering the intersection, with $\mathcal{E}_2$, of normal planes through regular points of $\mathcal{E}_1^{\mathbb{R}}$. In this paper, considering planes orthogonal to the plane of projection ($z = 0$), we will be able to relate the given bounds with bounds for the distance between $\pi_z^1(\mathcal{E}_1^{\mathbb{R}})^* \cap \mathbb{R}^2$ and $\pi_z^2(\mathcal{E}_2)^*$, as well as with bounds for the distance between the z–coordinates of points in $\mathcal{E}_1^{\mathbb{R}}$ and $\mathcal{E}_2$.

For this purpose, in the following, let $\mathbf{P} = (\alpha, \beta, \gamma)$ be a point on $\mathcal{E}_1$. We search for an upper bound of

$$\mathrm{d}(\mathbf{P}, \mathcal{E}_2^{\mathbb{R}}) = \inf\{\mathrm{d}(\mathbf{P}, Q) \mid Q \in \mathcal{E}_2^{\mathbb{R}}\} \tag{1}$$

whose computation could be accessible. Let us consider the pencil of all orthogonal planes (in $\mathbb{C}^3$) to the plane $z = 0$, and passing through $\mathbf{P}$ (see Figure 1 below). This pencil can be parametrized as:

$$\mathcal{L}(h, \mathbf{P}, k_1, k_2) := \mathbf{P} + k_1 v(h) + k_2 v_2, \tag{2}$$

where $v_2 = (0, 0, 1)$ and $v(h)$ is an arbitrary unitary vector in the projection plane, say

$$v(h) := \left(\frac{h^2-1}{h^2+1}, \frac{2h}{h^2+1}, 0\right), \ h \in \mathbb{C}.$$

Note that $v(h)$ provides all vectors of the unit circle, on the plane $z = 0$, with the exception of $(1, 0, 0)$; see Section 6.3. in [16]. Therefore, $\mathcal{L}(h, \mathbf{P}, k_1, k_2)$ parametrizes all the planes in the pencil. For $h_0 \in \mathbb{C}$, let $\Pi(h_0, \mathbf{P})$ be the complex plane given by $\mathcal{L}(h_0, \mathbf{P}, k_1, k_2)$; that is, k_1, k_2 take values in $\mathbb{C}$. Similarly, let $\Pi^{\mathbb{R}}(h_0, \mathbf{P})$ be the real plane

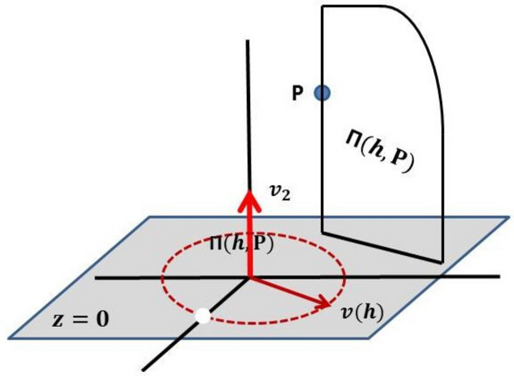

Figure 1.

given by $\mathcal{L}(h_0, \mathbf{P}, k_1, k_2)$ where h_0 is taken real and k_1, k_2 take values in $\mathbb{R}$. We observe that

$$\mathbb{C}^3 = \bigcup_{h_0 \in \mathbb{C}} \Pi(h_0, \mathbf{P}), \quad \mathbb{R}^3 = \bigcup_{h_0 \in \mathbb{R}} \Pi^{\mathbb{R}}(h_0, \mathbf{P}). \tag{3}$$

Let us see the first equality, the second is analogous. Clearly $\cup_{h_0 \in \mathbb{C}} \Pi(h_0, \mathbf{P}) \subset \mathbb{C}^3$. Now, let $P = (a, b, c) \in \mathbb{C}^3$. If $(a, b) = (\alpha, \beta)$ then P is, indeed, in all the planes, so let $v = (a - \alpha, b - \beta) \neq \overline{0}$. Then $P \in \Pi(h_0, \mathbf{P})$, where h_0 is such that $v(h_0) = v$ is $b \neq \beta$ and $h_0 = 0$ if $b = \beta$, which proves the equality.

In this situation, for every $h_0 \in \mathbb{C}$, we consider the polynomials in $\mathbb{C}[k_1, k_2]$ defined as

$$\mathcal{D}_i^{h_0}(\mathbf{P}, k_1, k_2) := G_i(\mathcal{L}(h_0, \mathbf{P}, k_1, k_2)), \quad i = 0, \ldots, \ell_2.$$

Definition 3.1. We introduce the sets

1. $K_{h_0} := \{(k_1, k_2) \in \mathbb{C}^2 \mid \mathcal{D}_i^{h_0}(\mathbf{P}, k_1, k_2) = 0, i = 0, \ldots, \ell_2\}$.
2. For $h_0 \in \mathbb{R}$, let $K_{h_0}^{\mathbb{R}} := K_{h_0} \cap \mathbb{R}^2$.

Remark 3.2. We observe that:

1. K_{h_0} consists in the parameter values generating the intersection points of $\mathcal{E}_2$ and $\Pi(h_0, \mathbf{P})$. That is

$$\mathcal{E}_2 \cap \Pi(h_0, \mathbf{P}) = \{\mathcal{L}(h_0, \mathbf{P}, k_1, k_2) \mid (k_1, k_2) \in K_{h_0}\}.$$

   Moreover, because of assumption A_3 in Section 2, $K_{h_0} \neq \emptyset$.

2. Using (3) one has

$$\mathcal{E}_2 = \mathcal{E}_2 \cap \mathbb{C}^3 = \mathcal{E}_2 \cap (\bigcup_{h_0 \in \mathbb{C}} \Pi(h_0, \mathbf{P})) = \bigcup_{h_0 \in \mathbb{C}} \{\mathcal{L}(h_0, \mathbf{P}, k_1, k_2) \mid (k_1, k_2) \in K_{h_0}\}.$$

3. For $h_0 \in \mathbb{R}$, $K^{\mathbb{R}}_{h_0}$ consists in the parameter values generating the real intersection points of $\mathcal{E}_2$ and $\Pi(h_0,\mathbf{P})$. That is
$$\mathcal{E}_2^{\mathbb{R}} \cap \Pi(h_0,\mathbf{P}) = \{\mathcal{L}(h_0,\mathbf{P},k_1,k_2) \,|\, (k_1,k_2) \in K^{\mathbb{R}}_{h_0}\}.$$
Moreover, since $\pi_z^2(\mathcal{E}_2)$ is a real curve, one has that for infinitely many $h_0 \in \mathbb{R}$ it holds that $K^{\mathbb{R}}_{h_0} \neq \emptyset$.

4. Reasoning as above, one has
$$\mathcal{E}_2^{\mathbb{R}} = \bigcup_{h_0 \in \mathbb{R}} \{\mathcal{L}(h_0,\mathbf{P},k_1,k_2) \,|\, (k_1,k_2) \in K^{\mathbb{R}}_{h_0}\}.$$

Lemma 3.3. *It holds that*

1. $\mathrm{d}(\mathbf{P},\mathcal{E}_2) = \inf \bigcup_{h_0\in\mathbb{C}} \{||k_1 v(h_0) + k_2 v_2|| \mid (k_1,k_2) \in K_{h_0}\}$.
2. $\mathrm{d}(\mathbf{P},\mathcal{E}_2^{\mathbb{R}}) = \inf \bigcup_{h_0\in\mathbb{R}} \{||k_1 v(h_0) + k_2 v_2|| \mid (k_1,k_2) \in K^{\mathbb{R}}_{h_0}\}$.

Proof. It follows from the definition of distance of a point to a set, and Remark 3.2. □

For a given $\mathbf{P}$, we consider values of $\mathbf{h}_0 \in \mathbb{R}$ such that:

- $K^{\mathbb{R}}_{\mathbf{h}_0} \neq \emptyset$ (see Remark 3.2)
- The points in $\pi_z^2(\mathcal{E}_2) \cap \Pi(\mathbf{h}_0,\mathbf{P})$ are 1:1 invertible by the map $\pi_z^2 : \mathcal{E}_2 \to \pi_z^2(\mathcal{E}_2)$.

We can obtain such $\mathbf{h}_0$ as follows. Compute the finite set Ω of points of $\pi_z^2(\mathcal{E}_2)$ that are not 1:1 invertible under π_z^2. Then, consider the pencil of lines in the plane $z=0$, passing through $\pi_z^1(\mathbf{P})$, and take one line cutting $\pi_z^2(\mathcal{E}_2^{\mathbb{R}}) \setminus \Omega$. Now, $\mathbf{h}_0$ is given by the direction vector of that line.

In this situation, it holds that (recall that, by Remark 3.2, $K_{\mathbf{h}_0} \neq \emptyset$ and by construction $K^{\mathbb{R}}_{\mathbf{h}_0} \neq \emptyset$)
$$\begin{array}{rcl} \mathrm{d}(\mathbf{P},\mathcal{E}_2) & \leq & \min\{||k_1 v(\mathbf{h}_0) + k_2 v_2|| \mid (k_1,k_2) \in K_{\mathbf{h}_0}\} \leq \\ & \leq & \min\{|k_1| + |k_2| \mid (k_1,k_2) \in K_{\mathbf{h}_0})\} \text{ and,} \\ \mathrm{d}(\mathbf{P},\mathcal{E}_2^{\mathbb{R}}) & \leq & \min\{|k_1| + |k_2| \mid (k_1,k_2) \in K^{\mathbb{R}}_{\mathbf{h}_0}\}. \end{array}$$

We observe that the polynomial $\mathcal{D}_0^{\mathbf{h}_0}$ does not depend on k_2. So we write it as $\mathcal{D}_0^{\mathbf{h}_0}(\mathbf{P},k_1)$. Let $g(x,y) := g_2(x,y)/g_1(x,y)$ (see Section 2). $\mathcal{L}(\mathbf{h}_0,\mathbf{P},k_1,k_2)$ can be expressed as $(\mathfrak{a}(k_1),\mathfrak{b}(k_1),\gamma+k_2)$ where $\mathfrak{a}$ and $\mathfrak{b}$ are linear polynomials in k_1 (see (2)). Then, by construction, the following univariate rational function in $\mathbb{R}(k_1)$ is well defined
$$\overline{g}(\mathbf{h}_0,\mathbf{P},k_1) := g(\mathfrak{a}(k_1),\mathfrak{b}(k_1)).$$

The following lemma gives a characterization of the sets $K_{\mathbf{h}_0}$ and $K^{\mathbb{R}}_{\mathbf{h}_0}$ using only G_0 and G_1.

Lemma 3.4. *It holds that*

1. $K_{\mathbf{h}_0} = \{(k_1, \overline{g}(\mathbf{h}_0, \mathbf{P}, k_1) - \gamma) \mid \mathcal{D}_0^{\mathbf{h}_0}(\mathbf{P}, k_1) = 0\}$.

2. $K_{\mathbf{h}_0}^{\mathbb{R}} = \{(k_1, \overline{g}(\mathbf{h}_0, \mathbf{P}, k_1) - \gamma) \mid \mathcal{D}_0^{\mathbf{h}_0}(\mathbf{P}, k_1) = 0, k_1 \in \mathbb{R}\}$.

Proof. We prove statement 1. A similar reasoning is valid for statement 2. Let Δ be the set on the r.h.s of the equality in statement 1. Let $(k_1, k_2) \in K_{\mathbf{h}_0}$. Then, $\mathcal{D}_i^{\mathbf{h}_0}(\mathbf{P}, k_1, k_2) = 0$ for $i = 0, \ldots, \ell_2$. In particular $\mathcal{D}_0^{\mathbf{h}_0}(\mathbf{P}, k_1) = 0$, and $\mathcal{D}_1^{\mathbf{h}_0}(\mathbf{P}, k_1, k_2) = 0$. More precisely,

$$\mathcal{D}_1^{\mathbf{h}_0}(\mathbf{P}, k_1, k_2) = G_1(\mathcal{L}(\mathbf{h}_0, \mathbf{P}, k_1, k_2)) = G_1(\mathfrak{a}, \mathfrak{b}, \gamma + k_2) = g_1(\mathfrak{a}, \mathfrak{b})(\gamma + k_2) - g_2(\mathfrak{a}, \mathfrak{b}) = 0.$$

That is, $k_2 = g(\mathfrak{a}, \mathfrak{b}) - \gamma = \overline{g}(\mathbf{h}_0, \mathbf{P}, k_1) - \gamma$. Therefore, $(k_1, k_2) \in \Delta$.

Conversely, let $(k_1, k_2) \in \Delta$. Then, $k_2 = \overline{g}(\mathbf{h}_0, \mathbf{P}, k_1) - \gamma$ and $\mathcal{D}_0^{\mathbf{h}_0}(\mathbf{P}, k_1) = 0$. Let $(\mathfrak{a}(k_1), \mathfrak{b}(k_1), \gamma + k_2) = \mathcal{L}(\mathbf{h}_0, \mathbf{P}, k_1, k_2)$, then $(\mathfrak{a}, \mathfrak{b}) \in \pi_z^2(\mathcal{E}_2)$. By construction of $\mathbf{h}_0$, $(\mathfrak{a}, \mathfrak{b})$ is invertible via π_z^2 to $(\mathfrak{a}, \mathfrak{b}, c^*) = (\pi_z^2)^{-1}(\mathfrak{a}, \mathfrak{b}) \in \mathcal{E}_2$ where $c^* = g(\mathfrak{a}, \mathfrak{b}) = \overline{g}(\mathbf{h}_0, \mathbf{P}, k_1) = k_2 + \gamma$. Therefore, $(\mathfrak{a}, \mathfrak{b}, \gamma + k_2) = \mathcal{L}(\mathbf{h}_0, \mathbf{P}, k_1, k_2) \in \mathcal{E}_2$, and hence

$$G_i(\mathfrak{a}, \mathfrak{b}, \gamma + k_2) = G_i(\mathcal{L}(\mathbf{h}_0, \mathbf{P}, k_1, k_2)) = \mathcal{D}_i^{\mathbf{h}_0}(\mathbf{P}, k_1, k_2) = 0 \text{ for } i = 0, \ldots, \ell_2.$$

So, $(k_1, k_2) \in K_{\mathbf{h}_0}$. □

Summarizing, we have proved the next result.

Theorem 3.5. *Let $\mathbf{P} = (\alpha, \beta, \gamma)$, and let $\mathbf{h}_0 \in \mathbb{R}$ be such that*

(i) $K_{\mathbf{h}_0}^{\mathbb{R}} \neq \emptyset$,

(ii) the points in $\pi_z^2(\mathcal{E}_2) \cap \Pi(\mathbf{h}_0, \mathbf{P})$ are 1:1 invertible by the map $\pi_z^2 : \mathcal{E}_2 \to \pi_z^2(\mathcal{E}_2)$.

Then, it holds that

1. $\mathrm{d}(\mathbf{P}, \mathcal{E}_2) \leq \min\{|k_1| + |\overline{g}(\mathbf{P}, k_1) - \gamma| \mid \mathcal{D}_0^{\mathbf{h}_0}(\mathbf{P}, k_1) = 0\}$,

2. $\mathrm{d}(\mathbf{P}, \mathcal{E}_2^{\mathbb{R}}) \leq \min\{|k_1| + |\overline{g}(\mathbf{P}, k_1) - \gamma| \mid \mathcal{D}_0^{\mathbf{h}_0}(\mathbf{P}, k_1) = 0, k_1 \in \mathbb{R}\}$.

Let us give a geometric interpretation of Theorem 3.5; see Figure2. We do it for statement 1; similarly for statement 2. First, observe that the solutions of $\mathcal{D}_0^{\mathbf{h}_0}(\mathbf{P}, k_1) = 0$ provide the values of the parameter k_1 reaching the intersection points of the line, passing through $(\alpha, \beta) = \pi_z^1(\mathbf{P})$ in the direction of $\pi_z(v(\mathbf{h}_0))$, and the plane curve $\pi_z^2(\mathcal{E}_2)^*$. Say that Q is one of these intersection points, then $|k_1| = \mathrm{d}((\alpha, \beta), Q)$ and $|\overline{g}(\mathbf{h}_0, \mathbf{P}, k_1) - \gamma|$ is the difference (in module) between the z-coordinate of $\mathbf{P}$ and z-coordinate of $(\pi_z^2)^{-1}(Q)$.

Applying the previous theorem one has the following results.

Corollary 3.6. *Let $\mathcal{E}_1^{\mathbb{R}}, \mathcal{E}_2^{\mathbb{R}}$ be bounded curves satisfying our general assumptions, then*

$$\mathrm{H}(\mathcal{E}_1^{\mathbb{R}}, \mathcal{E}_2^{\mathbb{R}}) \leq \mathrm{H}(\pi_z^1(\mathcal{E}_1^{\mathbb{R}}), \pi_z^2(\mathcal{E}_2^{\mathbb{R}})) + \max\{|c_{1,1} - c_{2,1}|, |c_{1,2} - c_{2,2}|\}$$

where $\mathcal{E}_i^{\mathbb{R}}$ is included in the box $[a_{i,1}, a_{i,2}] \times [b_{i,1}, b_{i,2}] \times [c_{i,1}, c_{i,2}]$.

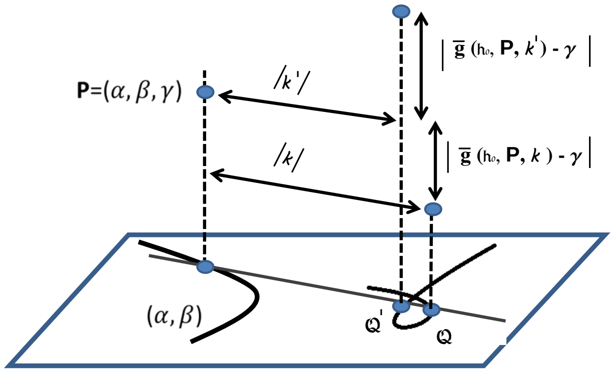

Figure 2.

Corollary 3.7. *Let $\mathcal{E}_1^{\mathbb{R}}, \mathcal{E}_2^{\mathbb{R}}$ satisfy our general assumptions, and let $\mathcal{E}_i^{\mathbb{R}}$ be included in the plane $z = h_i$ then*

$$\mathrm{H}(\mathcal{E}_1^{\mathbb{R}}, \mathcal{E}_2^{\mathbb{R}}) \leq \mathrm{H}(\pi_z^1(\mathcal{E}_1^{\mathbb{R}}), \pi_z^2(\mathcal{E}_1^{\mathbb{R}})) + |h_1 - h_2|.$$

As, a simple illustration, consider the circles $\mathcal{C}_1$ given by $\{x^2 + y^2 = r_1^2, z = h_1\}$ and $\mathcal{C}_2$ given by $\{x^2 + y^2 = r_2^2, z = h_2\}$. Then,

$$\mathrm{H}(\mathcal{E}_1^{\mathbb{R}}, \mathcal{E}_2^{\mathbb{R}}) = \sqrt{(r_2 - r_1)^2 + (h_2 - h_1)^2}$$

while our bound would be $|r_2 - r_1| + |h_2 - h_1|$.

4 Estimating $\mathrm{H}(\mathcal{E}_1^{\mathbb{R}}, \mathcal{E}_2^{\mathbb{R}})$

In this section we derive a global technique to estimate the Hausdorff distance between $\mathcal{E}_1^{\mathbb{R}}, \mathcal{E}_2^{\mathbb{R}}$ by means of the results developed in Section 3. Of course, the methods described below, at some step, need to use numerical approximations, and hence they provide approximations of the bound. For this purpose, we take different points $\mathbf{P}$ on $\mathcal{E}_1^{\mathbb{R}}$ and for each of them, we get an upper bound of $\mathrm{d}(\mathbf{P}, \mathcal{E}_2^{\mathbb{R}})$. Then, by taking the maximum of these bounds we estimate $\mathcal{D}(P, \mathcal{E}_2^{\mathbb{R}}) = \sup_{P \in \mathcal{E}_1^{\mathbb{R}}}(P, \mathcal{E}_2^{\mathbb{R}})$.

In order to estimate $\sup_{P \in \mathcal{E}_1^{\mathbb{R}}}(P, \mathcal{E}_2^{\mathbb{R}})$, similarly for $\sup_{P \in \mathcal{E}_2^{\mathbb{R}}}(P, \mathcal{E}_1^{\mathbb{R}})$ we present the following general strategy.

<u>General Strategy</u>

1. Compute a finite subset $\mathcal{A}$ of points in $\mathcal{E}_1^{\mathbb{R}}$.

2. For each $\mathbf{P} := (\alpha, \beta, \gamma) \in \mathcal{A}$, take $n_0 \in \mathbb{N}$, set $\mathcal{M}$ as the empty set, and execute the next steps:

   (a) Determine $\overline{\mathcal{B}} = \left\{-1 + \frac{2n}{n_0} \,|\, n = 1, \ldots, n_0\right\} \subset [-1, 1]$.

   (b) Determine the subset $\mathcal{B}$ of $\overline{\mathcal{B}}$ of those values $\mathbf{h}_0$ such that

   $$g_1\left(\alpha + k_1\frac{\mathbf{h}_0^2 - 1}{\mathbf{h}_0^2 + 1}, \beta + k_1\frac{2\mathbf{h}_0}{\mathbf{h}_0^2 + 1}\right) \neq 0.$$

   Recall that $G_1 = g_1(x, y)z - g_2(x, y)$. If $\mathcal{B} = \emptyset$ go to Step 2 and take a bigger $n_0 \in \mathbb{N}$.

   (c) For each $\mathbf{h}_0 \in \mathcal{B}$ compute

   $$\mathcal{D}_0^{\mathbf{h}_0}(\mathbf{P}, k_1) = G_0\left(\alpha + k_1\frac{\mathbf{h}_0^2 - 1}{\mathbf{h}_0^2 + 1}, \beta + k_1\frac{2\mathbf{h}_0}{\mathbf{h}_0^2 + 1}\right).$$

   Recall that the projected curve is defined by $G_0(x, y)$.

   (d) Compute the set $\mathcal{R}$ of real roots of $\mathcal{D}_0^{\mathbf{h}_0}(\mathbf{P}, k_1)$. If $\mathcal{R} = \emptyset$ go to Step 2 and take a bigger $n_0 \in \mathbb{N}$.

   (e) Compute

   $$\mathcal{H} = \left\{ |k_1| + \left| \gamma - \frac{g_2\left(\alpha + k_1\frac{\mathbf{h}_0^2-1}{\mathbf{h}_0^2+1}, \beta + k_1\frac{2\mathbf{h}_0}{\mathbf{h}_0^2+1}\right)}{g_1\left(\alpha + k_1\frac{\mathbf{h}_0^2-1}{\mathbf{h}_0^2+1}, \beta + k_1\frac{2\mathbf{h}_0}{\mathbf{h}_0^2+1}\right)} \right| \text{ where } k_1 \in \mathcal{R} \right\}$$

   (f) Append $\min(\mathcal{H})$ to $\mathcal{M}$.

3. Return $\max(\mathcal{M})$.

Let us comment on the steps above. In Step 1, one has to compute points on $\mathcal{E}_1^{\mathbb{R}}$. If $\mathcal{E}_1^{\mathbb{R}}$ is not rational, these points can be computed by intersecting the plane curve defined by G_0 with lines defined over $\mathbb{Q}$. If $\mathcal{E}_1$ is rational, then one can always take a proper rational parametrization, with coefficients in $\mathbb{K}$ (see [16]), of the plane curve to afterwards give rational values to the parameter. Of course, the non-rational case would need approximation methods and the points would not be really on $\pi_z^1(\mathcal{E}_1^{\mathbb{R}})^*$, while in the rational case, the points would be on the curve. Step 2 is, theoretically speaking, based on Theorem 3.5. Beside that, the most remarkable fact is that the set

$$\left\{\left(\frac{h^2 - 1}{h^2 + 1}, \frac{2h}{h^2 + 1}\right) \text{ with } h \in [-1, 1]\right\}$$

is the west-half part of the unit circle. Therefore, for a sufficiently big partition $\overline{\mathcal{B}}$ of $[-1, 1]$ the lines passing through $\mathbf{P}$, in the directions defined by the taken vectors in the circle, must intersect the real curve $\pi_z^1(\mathcal{E}_1^{\mathbb{R}})^*$.

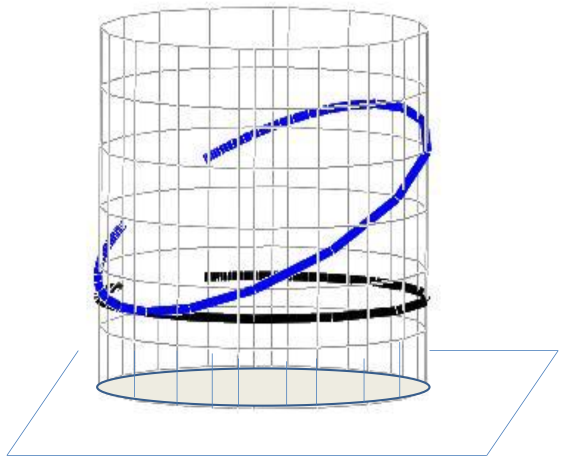

Figure 3.

Alternatively, one may use direct techniques, as for instance, given $P \in \mathcal{E}_1^{\mathbb{R}}$ optimizing the distance function from P to $\mathcal{E}_2^{\mathbb{R}}$; this, if $\mathcal{E}_2^{\mathbb{R}}$ is rational (say $\mathcal{Q}(t)$ is a real rational parametrization of $\mathcal{E}_2$) can be done optimizing the univariate rational function $\|P - \mathcal{Q}(t)\|^2$ and, if $\mathcal{E}_2^{\mathbb{R}}$ is not rational, optimizing the rational function $\|\mathbf{P} - (x, y, z)\|^2$ under the conditions $\{G_0(x, y) = 0, G_1(x, y, z) = 0\}$ (note that G_0, G_1 provides $\mathcal{E}_2^{\mathbb{R}}$ as complete intersection). Nevertheless, using these strategies also requires at some point the numerical approximation of the solution of algebraic systems, and hence they also provide estimations. We would like to emphasize that the general strategy we present can be used with curves given in both parametric and implicit form.

In the following section, we illustrate these comments.

5 Experiments and discussion

In this section, we illustrate and discuss our ideas with some examples. Computations were carried out with Maple 15.

Example 5.1. Let us consider the real circle $\mathcal{E}_1$ and the real ellipse $\mathcal{E}_2$ on the same cylinder $x^2 + y^2 = 1$ (see Figure 3), defined respectively by

$$\mathcal{F} = \{F_0 = -1 + y^2 + x^2, F_1 = z - 9, \}, \ \mathcal{G} = \{G_0 = -1 + y^2 + x^2, G_1 = z - 10 + y\}.$$

We may check easily that assumptions in Section 2 hold. On the other hand, both curves are rational and can be, respectively, parametrized by

$$\mathcal{P}_1(t) = \left(\frac{t^2 - 1}{t^2 + 1}, \frac{2t}{t^2 + 1}, 9 \right), \ \mathcal{P}_2(t) = \left(\frac{t^2 - 1}{t^2 + 1}, \frac{2t}{t^2 + 1}, 10 - \frac{2t}{t^2 + 1} \right).$$

Now, considering the sampling of points in $\mathcal{E}_1^{\mathbb{R}}$ defined by

$$\mathcal{A} = \{\mathcal{P}_1((-2)^i) \,|\, i = 1, \dots, 10\},$$

we apply the following three strategies to compute $\mathrm{d}(\mathbf{P}, \mathcal{E}_2^{\mathbb{R}})$, with $\mathbf{P} \in \mathcal{A}$:

- **Strategy-1.** Minimize the function $\|\mathbf{P} - \mathcal{P}_2(t)\|^2$.
- **General Strategy-2.** The general strategy in Section 4 (with $n_0 = 100$, see step 2 in general strategy).
- **Strategy-3.** Minimize the function $\|\mathbf{P} - (x, y, z)\|^2$, under the conditions $G_0(x, y, z) = G_1(x, y, z) = 0$.

The results obtained are shown in the following table:

i	Strategy-1	General Strategy-2	Strategy-3
1	1.8000000	1.8000000	1.8000000
2	.52941176	.52941176	.52941176
3	1.2461538	1.2461538	1.2461538
4	.87548638	.87548638	.87548638
5	1.0624390	1.0624390	1.0624390
6	.96875763	.96875763	.96875763
7	1.0156240	1.0156240	1.0156240
8	.99218762	.99218762	.99218760
9	1.0039062	1.0039062	1.0039062
10	.99804688	.99804688	.99804682

Let us now use Corollary 3.6. Clearly the distance between the projected curves is 0, and the z-coordinates of points in $\mathcal{E}_1^{\mathbb{R}}$ and $\mathcal{E}_2^{\mathbb{R}}$ are in the interval $[9, 11]$. Therefore, the bound given by the corollary is 2.

Example 5.2. In this example we see that the results obtained using our general strategy, being faster, are close to the standard optimization techniques. We consider the real curves $\mathcal{E}_1$ and $\mathcal{E}_2$ defined, respectively, by the Gröbner bases

$$\mathcal{F} = \{F_0 = y^3 - yx - x^3, F_1 = xz - y^2, F_2 = zy - y - x^2, F_3 = z^2 - z - yx\}$$

$$\mathcal{G} = \{G_0 = -9 - x + 13y + 3\,x^2 - yx - 6\,y^2 - x^3 + y^3, G_1 = -3 - z - x + 4\,y + xz - y^2,$$
$$G_2 = 3 + zy - 2\,y + 2\,x - 2\,z - x^2, G_3 = -yx + 2\,x + y - 3\,z + z^2\},$$

$\deg(\mathcal{E}_1) = \deg(\mathcal{E}_2) = 3$ and

$$\mathcal{E}_1^{\infty} = \mathcal{E}_2^{\infty} = \{(1:1:1:0)\} \cup \{(\lambda : -1 - \lambda : 0) \,|\, \lambda^2 + \lambda + 1 = 0\}.$$

Therefore, assumption in Section 2. On the other hand , both curves are rational and can be parametrized, respectively, by

$$\mathcal{P}_1(t) = \left(\frac{t}{t^3-1}, \frac{t^2}{t^3-1}, \frac{t^3}{t^3-1}\right), \ \mathcal{P}_2(t) = \left(\frac{t+t^3-1}{t^3-1}, \frac{t^2+2\,t^3-2}{t^3-1}, \frac{2\,t^3-1}{t^3-1}\right).$$

Now, considering the sampling of points in $\mathcal{E}_1^{\mathbb{R}}$ defined by

$$\mathcal{A} = \{\mathcal{P}_1((-2)^i) \mid i = 1, \ldots, 10\} \cup \{\mathcal{P}_1(1 + \frac{1}{(-2)^i}) \mid i = 1, \ldots, 10\}$$

we apply the following three strategies to compute $\mathcal{D}(\mathbf{P}, \mathcal{E}_2^{\mathbb{R}})$, with $\mathbf{P} \in \mathcal{A}$:

- **Strategy-1.** Minimize the function $\|\mathbf{P} - \mathcal{P}_2(t)\|^2$.
- **General Strategy-2.** The general strategy in Section 4 (with $n_0 = 100$, see step 2 in general strategy).
- **Strategy-3.** Minimize the function $\|\mathbf{P} - (x, y, z)\|^2$, under the conditions $G_0(x, y, z) = G_1(x, y, z) = 0$.

The results obtained are shown the following table:

$(-2)^i$ i	Appr-1	Global Strategy	Appr-3	$1 + \frac{1}{(-2)^i}$ i	Appr-1	Global Strategy	Appr-3
1	1.665	2.130	1.665	1	.7904	.8933	.7904
2	1.121	1.546	1.121	2	.1898	.2680	.1898
3	1.395	1.901	1.395	3	.8110	.9879	.8110
4	1.246	1.715	1.246	4	.8137	.9955	.8137
5	1.319	1.808	1.319	5	.8160	.9993	.8160
6	1.282	1.761	1.282	6	.8164	.9999	.8164
7	1.300	1.785	1.300	7	.8165	1.000	.8165
8	1.291	1.773	1.291	8	.8165	1.000	.8165
9	1.296	1.779	1.296	9	.8165	1.000	.8165
10	1.293	1.776	1.293	10	.8165	1.000	.8164

Example 5.3. Let $\mathcal{E}_1$ be the real affine space curve defined by the polynomials

$$\{-2\,y + y^2 - 5\,y\,z + 4\,z + 4\,z^2 + 1 + \frac{x}{100}, -y - 3\,y\,z + 4\,z + 4\,z^2 + 1 + x^2.\}$$

A Gröbner basis of the ideal of $\mathcal{E}_1$ w.r.t. the pure lexicographic order with $z > x > y$ is $\mathcal{F} = \{F_1, F_2, F_3\}$ with

$$\begin{aligned}
F_1 &= 100y - x - 100y^2 + 200yz + 100x^2, \\
F_2 &= -2x - 100y^2 + yx + 200x^2 - 4zx + 100y^3 - 300yx^2 + 400x^2z, \\
F_3 &= 200 + 100y - 3x + 800z - 300y^2 + 800z^2 + 500x^2.
\end{aligned}$$

This curve is not rational, it was introduced in [15], Example 2.2. Applying the approximate parametrization algorithm presented in [15] (Algorithm-2) to the curve $\mathcal{E}_1$, with $\epsilon = \frac{1}{100}$ we obtain a rational curve $\mathcal{E}_2$ given by the parametrization $\mathcal{P}(t) = \left(\frac{p_1(t)}{q(t)}, \frac{p_2(t)}{q(t)}, \frac{p_3(t)}{q(t)}\right)$,

$$
\begin{aligned}
p_1(t) &= \frac{-29347}{1074175} - \frac{130}{7617817}t + \frac{52047}{4296700}t^2 + \frac{7618077}{7617817}t^3 + \frac{454}{214835}t^4,\\
p_2(t) &= \frac{-24807}{1074175}t - \frac{65}{7617817}t^2 + \frac{61127}{4296700}t^3 + \frac{7618012}{7617817}t^4 + \frac{130}{7617817},\\
p_3(t) &= \frac{110741}{8593400}t^3 - \frac{209969}{8593400}t - \frac{65}{7617817}t^2 + \frac{7618142}{7617817},\\
q(t) &= -2 - t^2 + t^4 = (t^2-2)(t^2+1).
\end{aligned}
$$

It was verified in [15], Example 2.2, Part-1 that $\deg(\mathcal{E}_1) = 4$ and that

$$\mathcal{E}_1^\infty = \left\{ \left(1 : \pm\sqrt{2} : \pm\frac{\sqrt{2}}{\delta} : 0\right), \left(1 : \pm\sqrt{-1} : \pm\frac{\sqrt{-1}}{\delta} : 0\right) \right\}.$$

By construction, the output $\mathcal{E}_2$ of Algorithm-2, in [15] verifies that $\mathcal{E}_1^\infty = \mathcal{E}_2^\infty$ and $\deg(\mathcal{E}_1) = \deg(\mathcal{E}_2)$. Therefore assumptions in Section 2 hold.

Let us consider a set of points $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ in $\mathcal{E}_2^{\mathbb{R}}$, with

$$\mathcal{A}_1 = \{\mathcal{P}((-2)^i) \mid i = 1, \ldots, 50\}$$

and $\mathcal{A}_2$, $\mathcal{A}_3$ points coming from sequences of real numbers approaching the real poles of the parametrization $\overline{\mathcal{P}}(t)$, constructed as follows. For the real pole $\sqrt{2}$ of $q(t)$ (real root of t^2-2) we consider a finite sequence of isolating intervals $\{J_i\}_{i=1,\ldots,10}$ of length $1/2^{(i+5)}$, and we take the middle point ξ_i. Then $\mathcal{A}_2 = \{\mathcal{P}(\xi_i) \mid i = 1, \ldots, 10\}$. Similarly for $-\sqrt{2}$ we obtain $\mathcal{A}_3 = \{\mathcal{P}(\zeta_i) \mid i = 1, \ldots, 10\}$.

We apply the following two strategies to compute $\mathrm{d}(\mathbf{P}, \mathcal{E}_1^{\mathbb{R}})$, with $\mathbf{P} \in \mathcal{A}$:

- **General Strategy-2.** The general strategy in Section 4 (with $n_0 = 100$, see step 2 in general strategy).
- **Strategy-3.** Minimize the function $\|\mathbf{P} - (x, y, z)\|^2$, under the conditions $F_0(x, y, z) = F_1(x, y, z) = 0$.

Note that, Strategy-1 used in the previous examples cannot be applied in this case since the curve $\mathcal{E}_1$ is not rational.

The following table shows the computation of $\mathrm{d}(\mathbf{P}, \mathcal{E}_1^{\mathbb{R}})$ for the set of points $\{\mathcal{P}((-2)^i) \mid i = 1, \ldots, 10\} \subset \mathcal{A}_1$, and Figure 4 shows the results of all the points in $\mathcal{A}_1$ (with $i = 1..50$). Observe that in the figure, the values obtained with the strategy 1 are very close to zero.

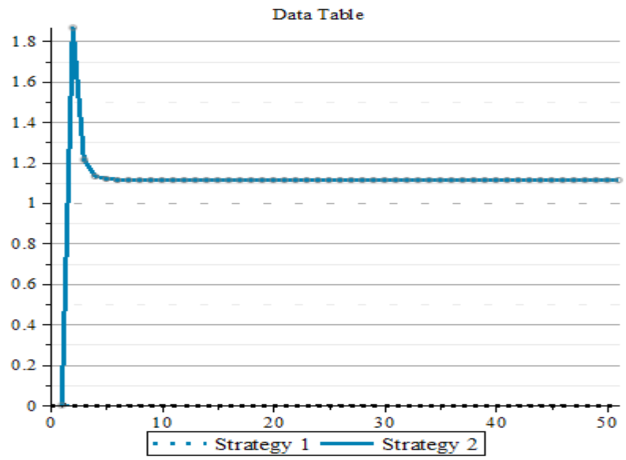

Figure 4.

i	General Strategy-1	Strategy-2
1	0.14603302e-2	1.8658154
2	0.70143524e-3	1.2138578
3	0.38179348e-3	1.1309156
4	0.15934946e-3	1.1189717
5	0.11133542e-3	1.1131302
6	0.23978835e-4	1.1135176
7	0.43685096e-4	1.1125872
8	0.98563044e-5	1.1128686
9	0.26770423e-4	1.1126820
10	0.18313527e-4	1.1127638

Figure 5 compares timing data (in seconds) for strategy 1 and 2, for $i = 1..50$.

Note that, for strategy 1, the time is controlled very close to 0.5 seconds, while for strategy 2 the time varies between 0.5 and 2.5 seconds.

The next table shows the computation of $\mathrm{d}(\mathbf{P}, \mathcal{E}_1^{\mathbb{R}})$ for the set of points in $\mathcal{A}_2$ and $\mathcal{A}_3$, respectively.

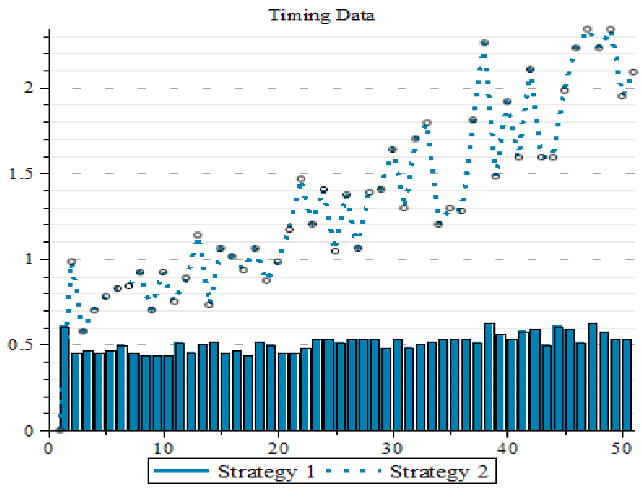

Figure 5

$\mathcal{A}_2$ i	Global Strategy-1	Strategy-2	$\mathcal{A}_3$ i	Global Strategy-1	Strategy-2
1	0.20303000e-2	3907.4207	1	0.2052300011e-2	3892.6527663229954
2	0.2025090259e-2	156.66023629709701	2	0.2046570260e-2	157.40151886971572
3	0.2027940125e-2	328.3997526712342	3	0.2049330154e-2	326.25257206712337
4	0.2029400058e-2	715.632028751385	4	0.2050600058e-2	712.324131576008
5	0.2029900024e-2	1751.3426630081156	5	0.2051200029e-2	1744.8178749540477
6	0.2031000007e-2	6342.847323098147	6	0.2050000007e-2	6319.12782165575
7	0.2027000002e-2	20362.940130115443	7	0.2050000002e-2	20285.935481082102
8	0.2034000003e-2	18425.126552372083	8	0.2050000002e-2	18354.369576100464
9	0.2010000000e-2	387007.3109	9	0.2020000000e-2	385543.8814
10	0.2029000001e-2	42988.73817	10	0.2049000001e-2	42826.16004

For the chosen set $\mathcal{A}$ the estimation of $\sup_{P\in\mathcal{E}_2^{\mathbb{R}}}(P,\mathcal{E}_1^{\mathbb{R}})$ using our general strategy equals $\max(\mathcal{M}) = 0.002052300011$, while the estimation using the strategy 2 (implemented using Lagrange multipliers) is 387007.890132416, which is clearly provoked by the numerical approximation.

6 Conclusion

Given two real algebraic space curves $\mathcal{E}_1$ and $\mathcal{E}_2$ such that the Hausdorff distance between their real parts is guaranteed to be finite, in this paper we provide upper bounds of the distance from a point of $\mathcal{E}_1$ to $\mathcal{E}_2$ and $\mathcal{E}_2^{\mathbb{R}}$, using the distance between the projection of this objects into the plane $z = 0$ and the distance between their z-coordinates. Based on this bound we propose a method to estimate $\sup_{P\in\mathcal{E}_1^{\mathbb{R}}}(P,\mathcal{E}_2^{\mathbb{R}})$, which can be used with curves being given in implicit or parametric representation. We apply this method to pairs of curves given in different representations and we compare the results with optimization methods. The answers provided by our method were similar of significantly better, specially in the cases were multivariate optimization is needed and the use of numeric approximations cannot be avoided.

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