

ON THE SINGULAR LOCUS OF RATIONAL SURFACE PARAMETRIZATIONS

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ABSTRACT. Given a rational projective parametrization of a rational surface, we decompose (with the exception of the parametrization base points) the projective plane of parameters as union of sets such that in each of these sets the multiplicity of the achieved points on the surface is invariant.

INTRODUCTION

Let $\mathcal{P}(\bar{\mathbf{m}}) = (\mathbf{p}_1(\bar{\mathbf{m}}) : \cdots : \mathbf{p}_4(\bar{\mathbf{m}}))$, with $\bar{\mathbf{m}} = (\mathbf{t} : \mathbf{s} : \mathbf{v})$, be rational projective parametrization of a rational projective surface $\mathcal{S} \subset \mathbb{P}^3(\mathbb{K})$ (\mathbb{K} is an algebraically closed field of characteristic 0). Our goal is to decompose $\mathbb{P}^2(\mathbb{K}) \setminus \mathfrak{B}$ (where \mathfrak{B} is the finite set of projective base points of \mathcal{P}) as $\cup_{k=1}^{\ell} \mathfrak{S}_k$ such that if $(\mathbf{t}_0 : \mathbf{s}_0 : \mathbf{v}_0) \in \mathfrak{S}_k$ then $\text{mult}(\mathcal{P}(\mathbf{t}_0, \mathbf{s}_0, \mathbf{v}_0), \mathcal{S}) = k$.

The elements in \mathfrak{S}_1 are called \mathcal{P} -simple points of \mathcal{S} , and the elements in \mathfrak{S}_k , with $k > 1$, \mathcal{P} -singularities of \mathcal{S} of multiplicity k . We refer to these points as affine (either \mathcal{P} -simple or \mathcal{P} -singular) points if $\mathbf{v}_0 \neq 0$ and points (either \mathcal{P} -simple or \mathcal{P} -singular) at infinity if $\mathbf{v}_0 = 0$. Moreover, we represent the multiplicity of $\bar{\mathbf{m}}_0$ as $\text{mult}(\bar{\mathbf{m}}_0) = \text{mult}(\mathcal{P}(\bar{\mathbf{m}}_0), \mathcal{S})$.

The polynomials $\mathbf{p}_j, j = 1, 2, 3, 4$, are assumed to be homogeneous of the same degree and coprime. Therefore the parametrization $\mathcal{P}(\bar{\mathbf{m}})$ induces the regular map $\mathcal{P} : \mathbb{P}^2(\mathbb{K}) \setminus \mathfrak{B} \rightarrow \mathcal{S}; \bar{\alpha} \mapsto \mathcal{P}(\bar{\alpha})$, where $\mathfrak{B} = \{\bar{\alpha} \in \mathbb{P}^2(\mathbb{K}) \mid \mathbf{p}_j(\bar{\alpha}) = 0, j = 1, 2, 3, 4\}$; we call the elements in \mathfrak{B} the (projective) base points of $\mathcal{P}(\bar{\mathbf{m}})$. We will be able to decompose, as above, $\mathbb{P}^2(\mathbb{K}) \setminus \mathfrak{B}$.

\mathfrak{B} is either zero dimensional or empty. So, we will be missing (at most) finitely many parameter values in $\mathbb{P}^2(\mathbb{K})$. If $\mathfrak{B} = \emptyset$, since \mathcal{S} is irreducible and \mathcal{P} regular, then $\mathcal{P}(\mathbb{P}^2(\mathbb{K})) = \mathcal{S}$ (see e.g. Thm. 2, page 57, in [8]). If $\mathfrak{B} \neq \emptyset$, our method will determine all singularities of \mathcal{S} . However, if $\mathfrak{B} \neq \emptyset$ the method will generate all singularities in the dense set $\mathcal{P}(\mathbb{P}^2(\mathbb{K}) \setminus \mathfrak{B}) \subset \mathcal{S}$. For avoiding this deficiency one may consider reparametrizing normally the parametrization (see [6]).

Our method is based on the generalization of the ideas in [2] in combination with the results in [3] and [4] that perform the computations without implicitizing. Intuitively speaking, the method works as follows: first we state a formula for computing the multiplicity of an affine point w.r.t. an affine surface. Then, we analyze the multiplicity of the (affine) parameter values of the form $(\mathbf{t}_0 : \mathbf{s}_0 : 1)$ to later study the parameter values (at infinity) of the form $(\mathbf{t}_0 : \mathbf{s}_0 : 0)$. In order to compute $\text{mult}((\mathbf{t}_0 : \mathbf{s}_0 : 1))$ we consider the four *canonical* affine rational parametrizations (we call them $\mathcal{P}_{x_1}, \dots, \mathcal{P}_{x_4}$) generated by $\mathcal{P}(\mathbf{t}, \mathbf{s}, \mathbf{v})$ by dehomogenizing w.r.t. the first, second, third and fourth component of the parametrization, respectively and taking $\mathbf{v} = 1$. Then, we apply the multiplicity formula to $(\mathbf{t}_0 : \mathbf{s}_0 : 1)$ via \mathcal{P}_{x_4} . This first attempt will classify all affine parameter values with the exception of a proper closed set, and hence with the exception of finitely many component of dimension either 1 or 0. By using consecutively $\mathcal{P}_{x_3}, \mathcal{P}_{x_2}$ and \mathcal{P}_{x_1} we achieve the multiplicity of all

affine parameter values not covered by \mathcal{P}_{x_4} and not being base points. Finally, we deal with the parameter values at infinity with a similar strategy but dehomogenizing with either $\mathfrak{t} = 1$ or $\mathfrak{s} = 1$.

For further details we refer the reader to [7] where an extended version of this abstract can be found.

Some further notation: In the sequel, we also use the following notation

$$\begin{aligned} \mathcal{P}_{x_1} &= \left(\frac{p_2(t, s)}{q_{1,1}(t, s)}, \frac{p_3(t, s)}{q_{1,2}(t, s)}, \frac{p_4(t, s)}{q_{1,3}(t, s)} \right), & \mathcal{P}_{x_3} &= \left(\frac{p_1(t, s)}{q_{3,1}(t, s)}, \frac{p_2(t, s)}{q_{3,2}(t, s)}, \frac{p_4(t, s)}{q_{3,3}(t, s)} \right), \\ \mathcal{P}_{x_2} &= \left(\frac{p_1(t, s)}{q_{2,1}(t, s)}, \frac{p_3(t, s)}{q_{2,2}(t, s)}, \frac{p_4(t, s)}{q_{2,3}(t, s)} \right), & \mathcal{P}_{x_4} &= \left(\frac{p_1(t, s)}{q_{4,1}(t, s)}, \frac{p_2(t, s)}{q_{4,2}(t, s)}, \frac{p_3(t, s)}{q_{4,3}(t, s)} \right). \end{aligned}$$

where all rational functions are in reduced form. Moreover, we say that $(t_0, s_0) \in \mathbb{K}^2$ is an (affine) base point of $\mathcal{P}(\overline{\mathfrak{m}})$ if $\mathfrak{p}_j(t_0, s_0, 1) = 0$, $j = 1, 2, 3, 4$. We denote by \mathfrak{B}_a the set of affine base points of $\mathcal{P}(\overline{\mathfrak{m}})$. Observe that \mathfrak{B}_a can be naturally embedded in \mathfrak{B} . One may prove that $\mathfrak{B}_a = \bigcap_{i=1}^4 \{(t_0, s_0) \in \mathbb{K}^2 \mid \text{lcm}(q_{i,1}, q_{i,2}, q_{i,3}) = 0\}$. If $\Theta : \mathbb{K}^n \rightarrow \mathbb{K}^m$ is a rational affine map, we denote by $\deg(\Theta)$ the degree of the map Θ (see e.g. [8] pp.143, or [1] pp.80). **General assumption:** We assume that for every two different polynomials $\mathfrak{p}_i, \mathfrak{p}_j$ it does not exist $\lambda \in \mathbb{K}$ such that $\mathfrak{p}_i = \lambda \mathfrak{p}_j$. Note that if $\mathfrak{p}_i = \lambda \mathfrak{p}_j$, then \mathcal{S} is the plane of equation $x_i - \lambda x_j = 0$, and the problem is trivial. Note that this requirement implies that none of the dehomogenizations \mathcal{S}_{x_i} is empty, and that \mathcal{S}_{x_i} is not a plane parallel to any of the affine coordinate plane in \mathbb{K}^3 .

1. THE MULTIPLICITY FORMULA

In this section, we state a formula for computing the multiplicity of a point in \mathbb{K}^3 w.r.t. an affine rational surface in \mathbb{K}^3 , when a rational parametrization (not necessarily proper) is provided. For that purpose, throughout this section, $\mathcal{Z} \subset \mathbb{K}^3$ is a rational affine surface and

$$\mathcal{Q}(t, s) = \left(\frac{N_1(t, s)}{D_1(t, s)}, \frac{N_2(t, s)}{D_2(t, s)}, \frac{N_3(t, s)}{D_3(t, s)} \right)$$

is a rational parametrization (in reduced form) of \mathcal{Z} ; we assume w.l.o.g. that \mathcal{Z} is not a plane parallel to the coordinate planes of \mathbb{K}^3 .

Let $f(x, y, z)$ be the defining polynomial of \mathcal{Z} and $F(x, y, z, w)$ its homogenization. For any $A = (a, b, c) \in \mathbb{K}^3$, we consider the polynomial $G(x, y, z, w) = F(x + aw, y + bw, z + cw, w)$. It is clear that $\text{mult}(A, \mathcal{Z}) = \deg(G) - \deg_w(G)$. Since $N_1/D_1 - a \neq 0$

$$\mathcal{Q}^* = \left(\frac{N_2(t, s) - bD_2(t, s)}{N_1(t, s) - aD_1(t, s)} \cdot \frac{D_1(t, s)}{D_2(t, s)}, \frac{N_3(t, s) - cD_3(t, s)}{N_1(t, s) - aD_1(t, s)} \cdot \frac{D_1(t, s)}{D_3(t, s)}, \frac{D_1(t, s)}{N_1(t, s) - aD_1(t, s)} \right)$$

parametrizes the affine surface defined by $G(1, y, z, w)$; note that, since G is homogeneous, $\deg_w(G) = \deg_w(G(1, y, z, w))$. In this situation, let

$$\Phi_{2,3}(A)(t, s) = \left(\frac{N_2(t, s) - bD_2(t, s)}{N_1(t, s) - aD_1(t, s)} \cdot \frac{D_1(t, s)}{D_2(t, s)}, \frac{N_3(t, s) - cD_3(t, s)}{N_1(t, s) - aD_1(t, s)} \cdot \frac{D_1(t, s)}{D_3(t, s)} \right)$$

and let $\Phi_{2,3}(A) : \mathbb{K}^2 \rightarrow \mathbb{K}^2$ be the induced map. Moreover, if for $i = 1, 2$, $\chi_i^A(t, s)$ denotes the i -component of $\Phi_{2,3}(A)(t, s)$, let $\mathfrak{g}_i^{\mathcal{Q}, A} = \text{Numer}(\chi_i^A(t, s) - \chi_i^A(h_1, h_2))$, $i = 1, 2$, where h_1, h_2 are new variables, and let $K = \text{gcd}(\mathfrak{g}_1^{\mathcal{Q}, A}, \mathfrak{g}_2^{\mathcal{Q}, A})$ where the gcd is computed in $\mathbb{K}[h_1, h_2][t, s]$.

Then, we define the polynomial $\mathfrak{g}^{\mathcal{Q},A}$ as $\mathfrak{g}^{\mathcal{Q},A} = K(t, s, h_1, h_2)$ if $\deg_{\{t,s\}}(K) > 0$ and $\mathfrak{g}^{\mathcal{Q},A} = 1$ if $\deg_{\{t,s\}}(K) = 0$.

In the following theorem and corollaries we assume that none of the projective curves defined by each of the non-constant polynomials in $\{N_1, N_2, N_3, D_1, D_2, D_3\}$ passes through $(0 : 1 : 0)$, and that for each $A = (a, b, c) \in \mathbb{K}^3$ (similarly A_0) none of the projective curves defined by the each of the non-constant polynomials in $\{N_2 - bD_2, N_3 - cD_3, N_1 - aD_1\}$ passes through $(0 : 1 : 0)$ (note that, if necessary, one can always perform a suitable polynomial linear change of parameters).

Theorem 1.1. (The general formula) *It holds that*

- (1) $\text{mult}(A, \mathcal{Z}) = \deg(\mathcal{Z})$ iff $\mathfrak{g}^{\mathcal{Q},A} \neq 1$.
- (2) $\text{mult}(A, \mathcal{Z}) < \deg(\mathcal{Z})$ iff $\mathfrak{g}^{\mathcal{Q},A} = 1$. Furthermore, if $\mathfrak{g}^{\mathcal{Q},A} = 1$ then

$$\deg(\mathcal{Z}) - \text{mult}(A, \mathcal{Z}) = \frac{\deg(\Phi_{23}(A))}{\deg(\mathcal{Q})}.$$

Corollary 1.2. *It holds that*

- (1) \mathcal{Z} is a plane iff \exists (a non-empty dense subset) $\Omega \subset \mathcal{Z}$ such that $\forall A \in \Omega, \mathfrak{g}^{\mathcal{Q},A} \neq 1$.
- (2) Let \mathcal{Z} not be a plane. \mathcal{Z} is a cone of vertex A if and only if $\mathfrak{g}^{\mathcal{Q},A} \neq 1$.
- (3) Let \mathcal{Z} not be a plane. There exists at most one $A \in \mathbb{K}^3$ such that $\mathfrak{g}^{\mathcal{Q},A} \neq 1$.

Corollary 1.3. (The multiplicity formula) *Let $A_0 \in \mathbb{K}^3 \setminus \mathcal{Z}$ and let $A \in \mathbb{K}^3$. Then*

- (1) if $\mathfrak{g}^{\mathcal{Q},A} = 1$, then $\text{mult}(A, \mathcal{Z}) = \frac{\deg(\Phi_{23}(A_0)) - \deg(\Phi_{23}(A))}{\deg(\mathcal{Q})}$.
- (2) if $\mathfrak{g}^{\mathcal{Q},A} \neq 1$, then $\text{mult}(A, \mathcal{Z}) = \frac{\deg(\Phi_{23}(A_0))}{\deg(\mathcal{Q})}$.

Corollary 1.4. (Criterion for simple points) *Let $A_0 \in \mathbb{K}^3 \setminus \mathcal{Z}$ and let $A \in \mathbb{K}^3$. If \mathcal{Z} is not a plane, the following statements are equivalent*

- (1) A is a simple point of \mathcal{Z} .
- (2) $\mathfrak{g}^{\mathcal{Q},A} = 1$ and $\deg(\Phi_{2,3}(A_0)) - \deg(\Phi_{23}(A)) = \deg(\mathcal{Q})$.

Observe that if we know how to compute $\deg(\mathcal{Q})$, $\deg(\Phi_{23}(A))$ for any given $A \in \mathbb{K}^3$, and if we know how to compute a point out of the surface (recall that we do not have the implicit equation of \mathcal{Z}), Corollary 1.3 provides a method for computing the multiplicity of any point in \mathbb{K}^3 , and Corollary 1.4 a method to check whether it is simple on the surface. We note that, once the parametrization is given, $\deg(\mathcal{Q})$ is fixed. However, $\deg(\Phi_{23}(A))$ will vary depending on A . Both quantities can be derived by applying elimination theory techniques as Gröbner basis. Indeed, they can be computed by means or resultants as shown in [3] without determining the implicit equation of \mathcal{Q} .

For the feasibility of the formulas above, we need to compute a point $A_0 \notin \mathcal{Z}$. For that, we assume that we know the partial degree m , w.r.t. one of the variables (say w.r.t. x), of the defining polynomial of \mathcal{Z} ; in [5] one can see how to compute m . This means that for almost all affine lines \mathcal{L} of the type $\{y = \lambda, z = \mu\}$ (recall that \mathcal{Z} is not a plane parallel to the coordinate planes) $\text{Card}(\mathcal{L} \cap \mathcal{Z}) = m$. Then, the idea is as follows. We take values for (λ, μ) till the number of different points on \mathcal{Z} generated by $\mathcal{Q}(t, s)$ is m . Once we have found a suitable (λ, μ) , every point $(\alpha, \lambda, \mu) \notin \mathcal{W}(\lambda, \mu)$ is not on \mathcal{Z} .

2. COMPUTING THE \mathcal{P} -SINGULARITIES

We start with the affine \mathcal{P} -singularities. We see \mathbb{K}^2 embedded in $\mathbb{P}^2(\mathbb{K})$ by means of the map $\mathbf{j} : \mathbb{K}^2 \rightarrow \mathbb{P}^2(\mathbb{K}), (t_0, s_0) \mapsto (t_0 : s_0 : 1)$; in this sense, we will be determining the affine \mathcal{P} -singularities of \mathcal{S} . For this purpose, let $\Delta_i := \{(t_0, s_0) \mid \text{lcm}(q_{i,1}, q_{i,2}, q_{i,3})(t_0, s_0) = 0\}$ and \mathfrak{B}_a be the set of base points of $\mathcal{P}_{x_4}(t, s)$. Note that $\mathbf{j}(\mathfrak{B}_a) \subset \mathfrak{B}$. The basic idea consists in applying the results presented in Section 1 to a generic point on \mathcal{S} . We proceed as follows:

First Level. We decompose $\Lambda_1 := \mathbb{K}^2 \setminus \Delta_4$ as $\Lambda_1 := \cup_{k=1}^{\ell_1} \mathfrak{F}_k^1$ such that if $(t_0, s_0) \in \mathfrak{F}_k^1$ then $\mathcal{P}_{x_4}(t_0, s_0)$ is a point of \mathcal{S}_{x_4} of multiplicity k .

Second Level. If $\Delta_4 \setminus \mathfrak{B}_a \neq \emptyset$ we decompose $\Lambda_2 := \Delta_4 \setminus \Delta_3$ as $\Lambda_2 := \cup_{k=1}^{\ell_2} \mathfrak{F}_k^2$ such that if $(t_0, s_0) \in \mathfrak{F}_k^2$ then $\mathcal{P}_{x_3}(t_0, s_0)$ is a point of \mathcal{S}_{x_3} of multiplicity k .

Third level. If $\Lambda_2 \setminus \mathfrak{B}_a \neq \emptyset$ we decompose $\Lambda_3 := (\Delta_4 \cap \Delta_3) \setminus \Delta_2$ as $\Lambda_3 = \cup_{k=1}^{\ell_3} \mathfrak{F}_k^3$ such that if $(t_0, s_0) \in \mathfrak{F}_k^3$ then $\mathcal{P}_{x_2}(t_0, s_0)$ is a point of \mathcal{S}_{x_2} of multiplicity k .

Fourth Level. If $\Lambda_3 \setminus \mathfrak{B}_a \neq \emptyset$ we decompose $\Lambda_4 := (\Delta_4 \cap \Delta_3 \cap \Delta_2) \setminus \Delta_1$ as $\Lambda_4 = \cup_{k=1}^{\ell_4} \mathfrak{F}_k^4$ such that if $(t_0, s_0) \in \mathfrak{F}_k^4$ then $\mathcal{P}_{x_1}(t_0, s_0)$ is a point of \mathcal{S}_{x_1} of multiplicity k .

Note that at this point, $\Lambda_4 \setminus \mathfrak{B}_a = \emptyset$. Moreover $\mathbf{j}(\cup_{i=1}^4 \mathfrak{F}_k^i) \subset \mathfrak{S}_k$.

For computing the \mathcal{P} -singularities at infinity, we proceed as follows. First we analyze whether $A = (0 : 1 : 0)$ is a \mathcal{P} -singularity. For this purpose, we check whether $A \in \mathfrak{B}$. If $A \notin \mathfrak{B}$, then we compute $\text{mult}(A)$ by applying our formula to a suitable dehomogenization of $\mathcal{P}(\bar{\mathbf{m}})$. Next, we study the points in $\mathcal{E} = \{(1 : \lambda_0 : 0) \mid \lambda_0 \in \mathbb{K}\}$. First we consider the points in $\mathcal{E}^* = \mathcal{E} \cap \mathfrak{B}$ and finally we deal with the points in $\mathcal{E} \setminus \mathcal{E}^{**}$; all of them by taking a suitable dehomogenization of the parametrization.

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