

Detecting Symmetries of Rational Plane Curves

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Abstract

Given a rational algebraic curve defined by means of a rational parametrization, we address here the problem of deterministically detecting whether the curve exhibits some kind of symmetry (central, mirror, rotation), and of computing the elements of the symmetry in the affirmative case. We provide effective methods for solving these questions without any conversion to implicit form. The underlying idea is the existing relationship between two proper parametrizations of a same curve, which in turn leads to algorithms where only univariate polynomials are involved. These methods have been implemented and tested in the computer algebra system Maple 15; evidence of their applicability, as well as a detailed theoretical analysis, is given.

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1 Introduction

The problem of detecting the symmetries of a curve has been extensively studied mainly because of its applications in Pattern Recognition and Computer Graphics. In the first case, a common problem is how to choose, from a database of curves, the one which best suits a given object, represented by means of an equation (see for example [9], [14], [24], [28], [29], [30]). For this purpose, first one must place the shape to be identified in a “canonical position” so that the comparison can be carried out, and thus the symmetries of the studied curve must be computed. Among others, in this context the computation of symmetries has been addressed in [9], using splines, in [4], [5], [32], by means of differential invariants, in [12], [13], [28], using a complex representation of the implicit equation of the curve, or in [9], [26], [27], [31], using moments. In the case of Computer Graphics, the detection of symmetries and similarities is important (also in the 3D case) to gain understanding when analyzing pictures, and also in order to perform tasks like compression, shape editing or shape completion. In this field, most of the techniques have to do with statistical methods and, in particular, *clustering* (see for example the papers [17], [3], [2], [19], where the technique of *transformation voting* is used). Other alternatives include robust auto-alignment (see [25]), spherical harmonics analysis (see [16]) or primitive fitting (see [22]), to quote a few.

In almost all the above references, the goal is to find *approximate symmetries* of the object. This is perfectly reasonable in applications because in many cases either the input is, up to a certain extent, a fuzzy object (even with some missing or occluded parts), or it is an exact object (for instance, an algebraic curve) but which is modeling (and therefore is an *approximate* instance of) a real object. However, here we will consider a different perspective. We assume that our input is exact, and we want to deterministically detect its symmetries. More precisely, our input will be a planar curve \mathcal{C} defined by means of a rational parametrization

$$\phi(t) = \left(\frac{p_1(t)}{q_1(t)}, \frac{p_2(t)}{q_2(t)} \right),$$

where $p_i(t), q_i(t)$ are polynomials with integer coefficients. Our goal, then, is to find: (1) a deterministic algorithm for detecting the symmetries of the curve (so, we want an exact “yes or no” answer for the question of whether the curve has certain symmetries); (2) in the affirmative case, an algorithm for finding the elements of the symmetry.

Up to our knowledge, the only papers where the same question (i.e. deterministic detection of symmetries) has been efficiently solved (so that the algorithms work with serious inputs in degree and norm, and not only in easy, very low-

degree cases) are those by Lebmair and Richter-Gebert ([12], [13]). However, in these papers the curve to be analyzed is assumed to be given in implicit form, with some restrictions on the form of the implicit equation. Compared to this, our method here applies directly to the parametrization, without any need to convert it into implicit form, and has no restrictions. Additionally, the methods of differential invariants can be applied to parametric curves (in fact not necessarily rational, but with more general parametrizations) and can be performed exactly, to give a deterministic answer. However, these methods are not really efficient in their “deterministic version” (they work only with very simple, low-degree curves); otherwise, they were conceived to give rise to powerful numeric algorithms (and this “conversion to numerics” is in fact the core in these papers) and are perfectly successful when applied to fuzzy objects.

More precisely, we consider here three types of symmetry: central symmetry (i.e. symmetry with respect to a point, which is the center of gravity of the curve), mirror symmetry (i.e. symmetry with respect to an axis), and rotation symmetry (which means that the curve is invariant under a non-trivial rotation around a point). Notice that while central symmetry is a particular case of rotation symmetry (namely, with $\theta = \pi$), it is interesting enough as to be addressed separately. As for the computation of the elements of the symmetry, our method provides symbolic expressions for them, but these expressions depend on the real roots of certain polynomials. As a consequence, these elements (centers of symmetry, symmetry axes, etc.) are approximated. However, this approximation is certified, in the sense that one knows in advance whether the symmetry is present or not; furthermore, a numerical test can be applied to evaluate the exactness of the computed element (see Section 7).

The main ingredient behind our method is the existing relationship between two *proper* (i.e. injective up to finitely many parameter values) parametrizations of a same curve. The rough, general idea is the following: if the curve \mathcal{C} has a certain symmetry, then by applying this symmetry we can obtain another parametrization $\tilde{\phi}(t)$ from the original $\phi(t)$; whenever $\phi(t)$ is proper, then $\tilde{\phi}(t)$ will be also proper, and both parametrizations will be related by means of a certain transformation whose general form is well-known (it is a *Möebius transformation*), depending on 4 real parameters. This argument can be reverted as to characterize the existence of the symmetry. Hence, we detect symmetry iff we can find real values for these parameters. For each kind of symmetry, we can show that these 4 parameters can be expressed as rational functions of just one of them. Thus, in the end the problem boils down to computing greatest common divisors of univariate polynomials, and checking whether certain univariate polynomials have some real root, or not (in fact, the number of real roots provides also information on the number of symmetry elements, since different solutions correspond to different elements). In this sense, we have implemented our algorithms in the computer

algebra system Maple 15, and we have tested them over many examples. We report on this in the last section of the paper. Additionally, we also present an improvement of our method for curves which admit a parametrization of the type $\left(\frac{p(t)}{(t^2+1)^r}, \frac{q(t)}{(t^2+1)^s}\right)$ where $p(t), q(t)$ are polynomials, $r, s \in \mathbb{N}$ and either $r > 0$ or $s > 0$. This family is important from the point of view of applications because it contains most of the *trigonometric curves*, i.e. curves which are parametrized by truncated Fourier Series (see for example [1], [8], [20], [21]).

The structure of the paper is the following. In Section 2 we present some preliminary results to be used throughout the paper. The detection of central, mirror and rotation symmetry is addressed in Section 3, Section 4 and Section 5, respectively. The special type of parametrizations mentioned above is addressed in Section 6. Finally, implementation issues are considered in Section 7. An appendix contains the (long) proof of a result which is needed in Section 4.

2 Preliminary Results

Along the paper we consider a plane algebraic curve \mathcal{C} defined by means of a rational parametrization $\phi(t) = (x(t), y(t))$ where $x(t) = \frac{p_1(t)}{q_1(t)}$, $y(t) = \frac{p_2(t)}{q_2(t)}$, with $p_i, q_i \in \mathbb{R}[t]$ for $i = 1, 2$, and $\gcd(p_i, q_i) = 1$ for $i = 1, 2$. Furthermore, we will assume that $\phi(t)$ is *proper*, i.e. that it is injective for almost all (complex) values of t . Algorithms for checking properness can be found for example in [23]. Also, from the algorithm in page 193 of [23] it follows that every rational curve can be properly reparametrized without extending the ground field. Now the following result will be crucial for us.

Theorem 1 *Let $\phi_1(t), \phi_2(t) \in \mathbb{R}[t]$ be two proper rational parametrizations of a same curve. Then there exists a unique function $\varphi(t) = \frac{\alpha t + \beta}{\gamma t + \delta}$, with $\alpha\delta - \beta\gamma \neq 0$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, fulfilling that $\phi_2(t) = \phi_1(\varphi(t))$.*

Proof. The existence of $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ fulfilling $\phi_2(t) = \phi_1(\varphi(t))$ is guaranteed by Lemma 4.17 in [23]. So, we just have to prove that $\alpha, \beta, \gamma, \delta$ can be assumed to be real. For this purpose, observe first that $\varphi(t)$, if it exists, is unique. Indeed, if $\varphi_1(t), \varphi_2(t)$ both fulfill the conditions in the statement, we get that $\phi_2(t) = \phi_1(\varphi_1(t)) = \phi_1(\varphi_2(t))$. Since $\phi_1(t)$ is proper, we deduce that $\varphi_1(t) = \varphi_2(t)$ for almost all complex values of t . Furthermore, since $\varphi_1(t), \varphi_2(t)$ are analytic, $\varphi_1(t) = \varphi_2(t)$ follows from the Identity Theorem (see e.g. page 81 in [10]) Thus, in order to prove the statement, we just need to find one $\varphi(t)$

with real coefficients. Now from page 97 in [23] one may see that $\varphi(t) = \phi_1^{-1} \circ \phi_2$ fulfills the conditions of the statement. Also, it has real coefficients because $\phi_1(t), \phi_2(t)$ are parametrizations with real coefficients, and the computation of the inverse ϕ_1^{-1} does not extend the ground field (see page 107 in [23]). \square

The functions $\varphi(t) = \frac{\alpha t + \beta}{\gamma t + \delta}$ are called *Möbius transformations*. In the rest of the paper we will maintain the notation $\varphi(t)$ for this type of functions, and we will assume, according to the above result, that $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Also, in the sequel we will often use the complex notation $z(t) = x(t) + \mathbf{i}y(t)$ for the points of \mathcal{C} , therefore seen as elements of \mathbb{C} . Furthermore, in the rest of the paper we will denote as $\|\cdot\|$ the Euclidean norm over \mathbb{C} . Additionally, along the paper we will speak of the *degree* of the parametrization $\phi(t)$ to denote the maximum power in the numerators and denominators of its components.

In the sequel, we will assume that \mathcal{C} is neither a line nor a circle, which can be considered as trivial cases from the point of view of symmetry analysis.

3 Central Symmetry

In [12], [13] it is proven that the center of symmetry of an algebraic curve, if it exists, is unique. Furthermore, the curve \mathcal{C} has central symmetry with respect to a point z_0 (in complex notation) iff $z_0 - (z(t) - z_0) = 2z_0 - z(t)$ is also a point of \mathcal{C} for every value of t (notice that $2z_0 - z(t)$ corresponds to the symmetric of a generic point $z(t)$ of \mathcal{C} with respect to z_0). This happens iff $\tilde{z}(t) = 2z_0 - z(t)$ is also a parametrization of \mathcal{C} . Furthermore, if $z(t)$ is proper (which holds by hypothesis) then $\tilde{z}(t)$ is also proper. So, the following result follows from Theorem 1.

Theorem 2 *The curve \mathcal{C} is symmetric with respect to a point z_0 iff there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\alpha\delta - \beta\gamma \neq 0$, such that*

$$z(\varphi(t)) = 2z_0 - z(t) \tag{1}$$

Hence, detecting central symmetry is equivalent to check whether there exist $\varphi(t), z_0$ in the conditions of Theorem 2. In turn, these conditions lead to the analysis of a polynomial system (with six unknowns: the four parameters involved in $\varphi(t)$ and the two coordinates of z_0). In general this system is too complicated. So, let us find extra conditions on the parameters of $\varphi(t)$. We start with the following lemma. Here, we denote $\varphi^2(t) = (\varphi \circ \varphi)(t)$.

Lemma 3 *Assume that \mathcal{C} is symmetric with respect to z_0 . Then the function $\varphi(t)$ satisfies that $\varphi^2(t) = t$.*

Proof. From $z(\varphi(t)) = 2z_0 - z(t)$, we get that $z(\varphi^2(t)) = 2z_0 - z(\varphi(t)) = 2z_0 - (2z_0 - z(t))$; so, $z(\varphi^2(t)) = z(t)$. Now since $z(t)$ is a proper parametrization, we get that $\varphi^2(t) = t$ for almost all (complex) values of t , i.e. $\varphi^2(t) - t = 0$ for infinitely many (complex) values. Since $\varphi^2(t) - t$ is analytic, from the Identity Theorem (see page 81 in [10]) it follows that it must be identically zero. \square

By explicitly computing $\varphi^2(t)$, we deduce the following corollary from the above lemma.

Corollary 4 *If \mathcal{C} is symmetric with respect to z_0 , then $(\alpha+\delta)\beta = 0$, $(\alpha+\delta)\gamma = 0$, $\alpha^2 = \delta^2$.*

Let us distinguish now two cases for $\varphi(t)$, namely $\delta = 0$ and $\delta \neq 0$, which are analyzed in the following subsections. Also, for technical reasons in the sequel we will assume that $z(t)$ (and therefore its derivatives) is well defined for $t = 0$; notice that this can always be achieved by applying a linear change of parameter (which does not affect properness of the curve).

3.1 Case $\delta = 0$

From Corollary 4, $\alpha^2 = \delta^2$. So, $\alpha = 0$ and $\varphi(t) = k/t$, $k \in \mathbb{R}$. Now if \mathcal{C} has central symmetry and $\varphi(t)$ has this form, by differentiating the equality (1) we get

$$z'(k/t) \cdot (-k)/t^2 = -z'(t) \tag{2}$$

Lemma 5 *The equation (2) cannot be an identity $\forall k$.*

Proof. Substituting $k = 0$ in the above equation, we deduce that $z'(t)$ is identically 0, which implies that $z(t)$ is constant. So, \mathcal{C} would not be a curve, which is absurd. \square

Let $\xi(k)$ be the gcd of the numerator in (2) (considered as a polynomial in t). From the above lemma, we have that $\xi(k)$ is not identically 0. Hence, we deduce the following result.

Theorem 6 *If \mathcal{C} has central symmetry and $\varphi(t) = k/t$, then $\xi(k) = 0$. Conversely, if $k_0 \in \mathbb{R}$ is a real root of $\xi(k)$ and the denominator in (1) does not vanish, then \mathcal{C} has central symmetry with $\varphi(t) = k_0/t$, and the symmetry center can be computed from (1).*

3.2 Case $\delta \neq 0$

Since $\alpha^2 = \delta^2$, we can distinguish the subcases $\alpha = \delta$, and $\alpha = -\delta$, respectively. In the first case, since we are assuming $\delta \neq 0$ we have $\alpha + \delta \neq 0$, and from Corollary 4 we get $\beta = \gamma = 0$. So, $\varphi(t) = t$. However, in this situation from Theorem 2 we get that $z(t)$ is a constant, i.e. \mathcal{C} is not a curve. So, the only possibility is $\alpha = -\delta \neq 0$. Now since $\delta \neq 0$, by dividing if necessary the numerator and denominator of $\varphi(t)$ by δ , we can assume that $\delta = 1$; thus, $\alpha = -1$. Hence, if \mathcal{C} has central symmetry then by differentiating (1) and evaluating at $t = 0$, we get

$$z'(\beta) \cdot (1 + \beta\gamma) = z'(0) \quad (3)$$

We distinguish two cases, depending on whether $\beta = 0$ or $\beta \neq 0$. If $\beta = 0$, then $\varphi(t) = \frac{-t}{\gamma t + 1}$. In this situation, we consider the equation

$$z'(\varphi(t))\varphi'(t) + z'(t) = 0 \quad (4)$$

which results from differentiating (1) and substituting $\varphi(t) = \frac{-t}{\gamma t + 1}$. One may argue as in Lemma 5 to derive that this equation cannot be an identity. So, the gcd of the numerator $\nu(\gamma)$ of the above expression (considered as a polynomial in t) cannot be identically 0. If this polynomial has some real root, by substituting $t = 0$ in (1) we get that \mathcal{C} has central symmetry, with symmetry center $z(0)$. Hence the following result holds.

Theorem 7 *If \mathcal{C} has central symmetry and $\varphi(t) = \frac{\alpha t + \beta}{\gamma t + \delta}$ with $\delta \neq 0$, $\beta = 0$, then (i) $\alpha = -1$, $\delta = 1$; (ii) $\nu(\gamma) = 0$. Conversely, if $\gamma_0 \in \mathbb{R}$ is a real root of $\nu(\gamma)$, such that no denominator in (1) vanishes identically, then \mathcal{C} has central symmetry with $\varphi(t) = \frac{-t}{\gamma_0 t + 1}$, and the symmetry center is $z(0)$.*

If $\beta \neq 0$, from (3) we get

$$\gamma = \left[\frac{z'(0)}{z'(\beta)} - 1 \right] \cdot \frac{1}{\beta}$$

The right hand-side of the above equality is a complex number, but from Theorem 1 we know that γ can be taken real. So, denoting $\gamma = f(\beta) + \mathbf{i}g(\beta)$ we have that $\gamma = f(\beta)$ and $g(\beta) = 0$, where $f(\beta), g(\beta)$ are rational functions of β .

Lemma 8 *Under our assumptions, $g(\beta)$ is not identically 0.*

Proof. A direct computation shows that $g(\beta) = \frac{-x'(0)y'(\beta) + y'(0)x'(\beta)}{x'^2(\beta) + y'^2(\beta)}$. So, $g(\beta)$ is identically 0 iff $x'(0)y'(\beta) = y'(0)x'(\beta)$ for all $\beta \in \mathbb{R}$. However, in such case \mathcal{C} is a line, which is excluded by hypothesis. \square

Finally, consider the equation (4) with $\varphi(t) = \frac{-t + \beta}{f(\beta)t + 1}$. Let us denote by $\xi(\beta)$ the gcd of the coefficients in t of the numerator of this expression. Also, let $\eta(\beta) = \gcd(g(\beta), \xi(\beta))$. Then the following theorem holds.

Theorem 9 *If \mathcal{C} has central symmetry and $\varphi(t) = \frac{\alpha t + \beta}{\gamma t + \delta}$ with $\delta \neq 0$, then (i) $\alpha = -1$, $\delta = 1$, $\gamma = f(\beta)$; (ii) $\eta(\beta) = 0$. Conversely, if $\beta_0 \in \mathbb{R}$ is a real root of $\eta(\beta)$, such that $f(\beta)$ is well defined and no denominator in (1), (3), vanishes identically, then \mathcal{C} has central symmetry with $\varphi(t) = \frac{-t + \beta_0}{f(\beta_0)t + 1}$, and the symmetry center can be computed from (1).*

3.3 Full Algorithm

The following algorithm follows from the preceding subsections.

ALGORITHM CENTRAL SYMMETRY: Given a curve \mathcal{C} by means of a proper parametrization $z(t)$ (in complex form), well-defined at $t = 0$, the algorithm **checks** if it has **central symmetry**, and **computes** the **symmetry center** in the affirmative case.

- (1) ($\varphi(t) = k/t$) **Check** if $\xi(k)$ has any root $k_0 \in \mathbb{R}$ in the conditions of Theorem 6. In the affirmative case, compute the symmetry center z_0 , and **return** “The symmetry center is z_0 ”.
- (2) $\left(\varphi(t) = \frac{-t + \beta}{f(\beta)t + 1}, \delta \neq 0, \beta = 0 \right)$ **Check** if $\nu(\gamma)$ has any root $\gamma_0 \in \mathbb{R}$ in the conditions of Theorem 7. In the affirmative case, **return** “The symmetry center is $z(0)$ ”.
- (3) $\left(\varphi(t) = \frac{-t + \beta}{f(\beta)t + 1}, \delta \neq 0, \beta \neq 0 \right)$ **Check** if $\eta(\beta)$ has any root $\beta_0 \in \mathbb{R}$ in the conditions of Theorem 9. In the affirmative case, compute the symmetry center z_0 , and **return** “The symmetry center is z_0 ”.
- (4) If (1), (2), (3) have not succeeded, then **return** “The curve has not central symmetry”.

Lemma 5 and Lemma 8 guarantee that the algorithm terminates. Its correctness follows from the preceding results in this section.

4 Mirror Symmetry

Let \mathcal{L} be a line passing through a point z_0 (in complex notation) and forming an angle θ with the x -axis. Then \mathcal{C} exhibits mirror symmetry with respect to \mathcal{L} iff the curve $\tilde{\mathcal{C}}$, parametrized by $(z(t) - z_0)e^{i\theta}$, obtained by applying over \mathcal{C} a translation of vector $-z_0$ followed by a rotation around the origin of angle $-\theta$, is symmetric with respect to the x -axis. This condition is equivalent to $(\overline{z(t)} - \bar{z}_0) \cdot e^{-i\theta}$ being also a point of $\tilde{\mathcal{C}}$ for every value of t . So, from Theorem 1 the following result follows.

Theorem 10 *The curve \mathcal{C} is symmetric with respect to a line \mathcal{L} passing through a point z_0 (in complex notation) and forming an angle θ with the x -axis iff there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\alpha\delta - \beta\gamma \neq 0$, such that*

$$[z(\varphi(t)) - z_0] \cdot e^{2i\theta} = \overline{z(t)} - \bar{z}_0 \quad (5)$$

Lemma 11 *Assume that \mathcal{C} is symmetric with respect to z_0 . Then the function $\varphi(t)$ satisfies that $\varphi^2(t) = t$.*

Proof. Substituting t by $\varphi(t)$ in (5), we get that $[z(\varphi^2(t)) - z_0] \cdot e^{2i\theta} = \overline{z(\varphi(t))} - \bar{z}_0$. Substituting here $z(\varphi(t))$ also from (5), we get $z(\varphi^2(t)) = z(t)$. Now one argues as in Lemma 3. \square

So, Corollary 4 holds also in this case, and we can consider the same discussion as in the section before. For this purpose, in the rest of the section we will assume that $z(0)$ (and therefore all the derivatives $z^{(p)}(0)$, for $p \in \mathbb{N}$) are well-defined. This can be achieved by almost all linear changes of parameter. Additionally, w.l.o.g. we will suppose that $z'(0) \neq 0$.

4.1 Case $\delta = 0$

Here $\varphi(t) = k/t$. Now differentiating (5) w.r.t. t , we get

$$z'(k/t) \cdot (-k) \cdot e^{2i\theta} = \overline{z'(t)} \cdot t^2 \quad (6)$$

In particular, this implies that

$$e^{2i\theta} = \frac{\overline{z'(t)} \cdot t^2}{z'(k/t) \cdot (-k)} \quad (7)$$

Differentiating (6) again,

$$z''(k/t) \cdot k^2 \cdot e^{2i\theta} = \overline{z''(t)} \cdot t^4 + \overline{z'(t)} \cdot 2t^3 \quad (8)$$

By dividing (6) and (8), and clearing denominators, we get that

$$-z'(k/t) \cdot [\overline{z''(t)} \cdot t^2 + \overline{z'(t)} \cdot 2t] = z''(k/t) \cdot k \cdot \overline{z'(t)} \quad (9)$$

In addition to this, taking modules in (7) it holds that

$$\left| \frac{\overline{z'(t)} \cdot t^2}{z'(k/t) \cdot (-k)} \right|^2 = 1 \quad (10)$$

Lemma 12 *The equations (9) and (10) cannot be identities at the same time $\forall k$.*

Proof. Assume that both expressions are simultaneously identities. In that case, (9) is a differential equation, and by integrating it we get $t^2 \overline{z'(t)} = C \cdot z'(k/t)$. By taking (10) into account, we deduce that $C = \pm k$. So, $t^2 \overline{z'(t)} = \pm k \cdot z'(k/t)$. Since this last equality holds for all t, k , making $k = 0$ (recall that $z'(0)$ is well defined) we get that $z'(t) = 0$; so, $z(t)$ is constant and therefore \mathcal{C} is not a curve, which is a contradiction. \square

Now let us represent by $\xi_1(k)$ the gcd of the coefficients in t of the polynomial in (9), and let $\xi_2(k)$ be the gcd of the numerator of the polynomial obtained when clearing denominators in (10). Also, let

$$\frac{d}{dt} \left(\frac{\overline{z'(t)} \cdot t^2}{z'(k/t) \cdot (-k)} \right) = 0 \quad (11)$$

and let $\xi_3(k)$ be the gcd of the coefficients in t of the numerator of the above expression. Notice that the condition $\xi_3(k) = 0$ expresses the property that the right-hand side of (7) is a constant complex number; this condition, together with the condition $\xi_2(k) = 0$, ensures that the right-hand side of (7) is a complex number of modulus 1, and therefore that (7) holds. Now if \mathcal{C} exhibits mirror symmetry with $\varphi(t) = k/t$, then k must be a common root of $\xi_1(k), \xi_2(k), \xi_3(k)$. On the other hand, by Lemma 12, $\eta(k) = \gcd(\xi_1(k), \xi_2(k), \xi_3(k))$ cannot be the zero polynomial.

Theorem 13 *If \mathcal{C} has mirror symmetry and $\varphi(t) = k/t$, then $\eta(k) = 0$. Conversely, if $k_0 \in \mathbb{R}$ fulfills $\eta(k_0) = 0$, and the denominators in (5), (7), (9), do not vanish, then \mathcal{C} has mirror symmetry and the symmetry axis can be computed from (5).*

The equation of the symmetry axis is obtained after performing the substitution of $\varphi(t)$ and $e^{2i\theta}$ (in terms of k, t) on (5). Indeed, this way we reach a complex expression

$$Az_0 + B\overline{z_0} + C = 0$$

with A, B, C being complex numbers. Regarding z_0 as a complex variable, and substituting $z_0 = x + iy$, the equation of the real line is the gcd of the real and imaginary parts.

4.2 Case $\delta \neq 0$

Arguing as in Section 3, we distinguish the subcases $\alpha = \delta$, and $\alpha = -\delta$. In the first case, we conclude that $\varphi(t) = t$. However, let us see that in this case \mathcal{C} exhibits mirror symmetry only if it is a line. Indeed, if $\varphi(t) = t$ and \mathcal{C} has this type of symmetry, then from Theorem 10 we have that $(z(t) - z_0)e^{i\theta} = \overline{(z(t) - z_0)e^{-i\theta}}$. So, $(z(t) - z_0)e^{i\theta} = \overline{(z(t) - z_0)e^{i\theta}}$, and hence $(z(t) - z_0)e^{i\theta} \in \mathbb{R}$. One may check then that the imaginary part of $(z(t) - z_0)e^{i\theta}$ is $(x(t) - x_0)\sin(\theta) + (y(t) - y_0)\cos(\theta)$, and this expression must be identically 0. So, we get that

$$\frac{y(t) - y_0}{x(t) - x_0} = -\text{tg}(\theta) = \text{constant}$$

which means that \mathcal{C} is a line.

Hence, we only need to consider the case $\alpha = -\delta$. In this case, since $\delta \neq 0$, we can assume $\delta = 1$, and so $\alpha = -1$. Now differentiating (5) with respect to t , we get

$$z'(\varphi(t)) \cdot \varphi'(t) \cdot e^{2i\theta} = \overline{z'(t)} \quad (12)$$

where $\varphi'(t) = \frac{\Delta}{(t\gamma + \delta)^2}$, and $\Delta = \alpha\delta - \beta\gamma = -1 - \beta\gamma$. So, evaluating (12) at $t = 0$ we get that

$$z'(\beta) \cdot \Delta \cdot e^{2i\theta} = \overline{z'(0)} \quad (13)$$

Also, from here

$$e^{2i\theta} = \frac{\overline{z'(0)}}{z'(\beta) \cdot \Delta} \quad (14)$$

Differentiating again (12) and evaluating at $t = 0$, we get

$$z''(\beta) \cdot \Delta^2 \cdot e^{2i\theta} = \overline{z''(0)} + 2\gamma\overline{z'(0)} \quad (15)$$

By combining the above two equations, we deduce that

$$\gamma = \left[z'(\beta) \cdot \frac{\overline{z''(0)}}{z'(0)} + z''(\beta) \right] \cdot \frac{-1}{\beta z''(\beta) + 2z'(\beta)} \quad (16)$$

The above expression does not make sense when $tz''(t) + 2z'(t)$ is identically 0. However, this corresponds to a differential equation that can be integrated, whose solution is $z(t) = C_1 + \frac{C_2}{t}$. And it is easy to see that this corresponds to a line, which is a case excluded by hypothesis. Now let us write $\gamma = \gamma(\beta) = f(\beta) + \mathbf{i}\xi_1(\beta)$. Since γ can be taken real, we have that $\gamma = f(\beta)$, and $\xi_1(\beta) = 0$. Thus $\Delta = -1 - \beta f(\beta)$; by taking modules in (14) we get that

$$|z'(\beta)|^2 - \frac{|z'(0)|^2}{(1 - \beta f(\beta))^2} = 0 \quad (17)$$

Now let us represent by $\xi_2(\beta)$ the numerator in (17). Then the following result holds. The (quite long) proof of this result is provided in Appendix I, so as not to stop the flow of the paper.

Lemma 14 *Under our hypotheses, if \mathcal{C} exhibits mirror symmetry, then the polynomials $\xi_1(\beta)$ and $\xi_2(\beta)$ cannot be identically 0 at the same time.*

Additionally, let us consider the equation

$$\frac{d}{dt} \left(\frac{\overline{z'(t)}}{z'(\varphi(t)) \cdot \varphi'(t)} \right) = 0 \quad (18)$$

which is obtained by solving (12) for $e^{2i\theta}$, and differentiating the resulting expression. This condition, together with (17), guarantees that (12) holds. Let $\xi_3(\beta)$ be the gcd of the numerator of (18), and let $\eta(\beta) = \gcd(\xi_1(\beta), \xi_2(\beta), \xi_3(\beta))$, which, from Lemma 14, is not identically 0. Then we have the following theorem on the existence of mirror symmetry.

Theorem 15 *If \mathcal{C} has mirror symmetry and $\varphi(t) = \frac{\alpha t + \beta}{\gamma t + \delta}$ with $\delta \neq 0$, then it holds that: (i) $\alpha = -1$, $\delta = 1$, $\gamma = f(\beta)$; (ii) $\eta(\beta) = 0$. Conversely, if: (i) $\beta_0 \in \mathbb{R}$ fulfills $\eta(\beta_0) = 0$; (ii) no denominator in (5), (14), (15), (16), (18) vanishes identically, then \mathcal{C} has mirror symmetry with $\varphi(t) = \frac{-t + \beta_0}{f(\beta_0)t + 1}$ and the symmetry axis can be computed from (5).*

4.3 Full Algorithm

The following algorithm follows from the preceding subsections.

ALGORITHM MIRROR SYMMETRY: Given a curve \mathcal{C} by means of a proper parametrization $z(t) = x(t) + iy(t)$ (in complex form), well-defined at $t = 0$, such that $z'(0) \neq 0$, the algorithm **checks** if it has **mirror symmetry**, and **computes** the **symmetry axes** in the affirmative case.

- (1) ($\varphi(t) = k/t$) **Check** if $\eta(k)$ has any real roots $k_1, \dots, k_r \in \mathbb{R}$ in the conditions of Theorem 13. In the affirmative case, let $\mathcal{L}_1, \dots, \mathcal{L}_r$ be the symmetry axes they correspond to.
- 2) $\left(\varphi(t) = \frac{-t + \beta}{f(\beta)t + 1}, \delta \neq 0 \right)$ **Check** if $\eta(\beta)$ has any real roots β_1, \dots, β_s in the conditions of Theorem 15. In the affirmative case, let $\tilde{\mathcal{L}}_1, \dots, \tilde{\mathcal{L}}_s$ be the symmetry axes they correspond to,.
- (3) If (1) and (2) have not succeeded, then **return** “The curve has not central symmetry”. Otherwise, **return** the list of \mathcal{L}_i 's and $\tilde{\mathcal{L}}_j$'s.

Lemma 12 and Lemma 14 guarantee that this algorithm terminates. The correctness follows from the results in this section.

5 Rotation Symmetry

In [12], [13] it is proven that the center of rotation of an algebraic curve, if it exists, is unique. Furthermore, the curve \mathcal{C} has rotational symmetry iff there exists a point z_0 and an angle θ , such that for every value of t , $\tilde{z}(t) = z_0 + e^{i\theta} \cdot (z(t) - z_0)$ is also a point of \mathcal{C} (notice that this expression describes a rotation of $z(t)$ around z_0). Hence, from Theorem 1 we get the following result.

Theorem 16 *The curve \mathcal{C} has rotational symmetry with center z_0 and angle θ iff there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\alpha\delta - \beta\gamma \neq 0$, such that*

$$z(\varphi(t)) = z_0 + e^{i\theta} \cdot (z(t) - z_0) \quad (19)$$

In [12] it is proven that θ has the form $\theta = \frac{2\pi}{n}$, where $n \in \mathbb{N}$. Moreover, from Bezout's Theorem one may see that $n \leq 2d$, where d is the degree of \mathcal{C} . Now in the sequel let us assume that \mathcal{C} has rotational symmetry, and let us find conditions on the parameters of $\varphi(t)$. We will suppose that $z(0)$ is well-defined, and $z'(0) \neq 0$. As in the other sections, we distinguish the cases $\delta = 0$ and $\delta \neq 0$, respectively.

5.1 Case $\delta = 0$

We consider first the following lemma, which can be proven in a similar way to Lemma 3 or Lemma 11.

Lemma 17 *If \mathcal{C} has rotation symmetry, then there exists $n \in \mathbb{N}$ such that $\varphi^n(t) = t$.*

Now we can identify the parameters of $\varphi(t)$ with the elements of the matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

and one can check that the parameters in $\varphi^n(t) = (\varphi \circ \dots \circ \varphi)(t)$ can be identified with the coefficients of A^n .

Lemma 18 *If \mathcal{C} has rotation symmetry and $\delta = 0$, the matrix A has two real and distinct eigenvalues (as a consequence, it is diagonalizable over the reals).*

Proof. When $\delta = 0$, the characteristic polynomial of A is $\lambda^2 - \alpha\lambda - \beta\gamma$. The roots of this polynomial are

$$\lambda = \frac{\alpha \pm \sqrt{\alpha^2 + 4\beta^2\gamma^2}}{2}$$

Since $\beta \cdot \gamma \neq 0$ (because otherwise $\alpha\delta - \beta\gamma = 0$), the discriminant of the equation is strictly positive. \square

Lemma 19 *If \mathcal{C} has rotation symmetry and $\delta = 0$, then the eigenvalues λ_1, λ_2 of A satisfy $\lambda_1 = -\lambda_2$.*

Proof. From Lemma 17, one may see that for some $n \in \mathbb{N}$, it holds that $A^n = \mu \cdot I$, with $\mu \in \mathbb{R}$. So, A^n has obviously just one eigenvalue, μ . However, since the eigenvalues of A^n are the n -th powers of λ_1, λ_2 , we deduce that $\lambda_1^n = \lambda_2^n$. So, $\lambda_1 = \xi \cdot \lambda_2$ where ξ is an n -th root of the unity. But since from Lemma 18 both λ_1, λ_2 are real and distinct, we have that $\xi = -1$. \square

Proposition 20 *If \mathcal{C} has rotation symmetry and $\delta = 0$, then $\alpha = 0$. As a result, $\varphi(t) = k/t$.*

Proof. The trace of A is $\alpha + \delta$. Since the trace is similarity-invariant, and from Lemma 19 we have that $\lambda_1 = -\lambda_2$, we get that $\alpha + \delta = 0$. Since $\delta = 0$, $\alpha = 0$, too. \square

Hence, in this case we have

$$z(k/t) = z_0 + e^{i\theta}(z(t) - z_0) \quad (20)$$

and differentiating with respect to t , and solving for $e^{i\theta}$,

$$e^{i\theta} = \frac{z'(k/t) \cdot (-k/t^2)}{z'(t)} \quad (21)$$

Taking modules, we get

$$\left| \frac{z'(k/t) \cdot (-k/t^2)}{z'(t)} \right|^2 - 1 = 0 \quad (22)$$

Lemma 21 *Under our assumptions, the equation (22) cannot be an identity.*

Proof. If (22) is an identity then by substituting $k = 0$ we get $|z'(t)| = 0$, in which case \mathcal{C} is not a curve. \square

So, let $\xi_1(k)$ be the gcd of the coefficients of the numerator of (22) (considered as a polynomial in t , with coefficients in k). From Lemma 21 we know that it is not identically 0. Also, let $\xi_2(k)$ be the gcd of the coefficients of the numerator of the derivative with respect to t of the right-hand side of (21). As in other cases, the conditions $\xi_1(k) = 0$ and $\xi_2(k) = 0$, taken together, guarantee that (21) holds. Also, let $\eta(k) = \gcd(\xi_1(k), \xi_2(k))$. Then we have the following result.

Theorem 22 *If \mathcal{C} has rotation symmetry and $\varphi(t) = k/t$, then $\eta(k) = 0$. Conversely, if $k_0 \in \mathbb{R}$ fulfills $\eta(k_0) = 0$ and no denominator in (19), (21), vanishes identically, then \mathcal{C} has rotation symmetry with $\varphi(t) = k_0/t$, and the elements of this symmetry can be computed from (20) and (21).*

5.2 Case $\delta \neq 0$

In this case we can assume $\delta = 1$. Now differentiating (19) with respect to t , and taking into account that $\varphi'(t) = \frac{\Delta}{(\gamma t + \delta)^2}$, we get

$$z'(\varphi(t)) \cdot \Delta = e^{i\theta} \cdot z'(t)(\gamma t + \delta)^2 \quad (23)$$

(where $\delta = 1$). Evaluating at $t = 0$, we deduce

$$z'(\beta) \cdot \Delta = e^{i\theta} \cdot z'(0) \quad (24)$$

Since $\Delta \neq 0$ and because of the hypotheses, one may check that $z'(\beta) \neq 0$. Now by differentiating (23) again, substituting $t = 0$ and taking (24) into account, we have that

$$\gamma = \frac{1}{2z'(0)} \cdot \left[\frac{z''(\beta)z'(0)\Delta}{z'(\beta)} - z''(0) \right] \quad (25)$$

Let us consider the real and imaginary parts of γ :

$$\begin{aligned} \operatorname{Re}(\gamma) &= \operatorname{Re} \left(\frac{z''(\beta)}{2z'(\beta)} \right) \cdot \Delta - \operatorname{Re} \left(\frac{z''(0)}{2z'(0)} \right) \\ \operatorname{Im}(\gamma) &= \operatorname{Im} \left(\frac{z''(\beta)}{2z'(\beta)} \right) \cdot \Delta - \operatorname{Im} \left(\frac{z''(0)}{2z'(0)} \right), \end{aligned}$$

and let us write $f(\beta) = \operatorname{Re}(\gamma)$, $g(\beta) = \operatorname{Im}(\gamma)$. Since γ can always be taken real, we have $\gamma = f(\beta)$, and $g(\beta) = 0$. Now let us denote $h_1(\beta) = \operatorname{Im} \left(\frac{z''(\beta)}{2z'(\beta)} \right)$, $w = \operatorname{Im} \left(\frac{z''(0)}{2z'(0)} \right)$. Observe that $h_1(\beta)$, considered as a polynomial in β , cannot be identically 0. Indeed, one may see that this happens iff $x'(t)y''(t) = x''(t)y'(t)$, i.e.

$$\frac{x''(t)}{x'(t)} = \frac{y''(t)}{y'(t)}$$

for every t . Integrating twice, this implies that \mathcal{C} is a line, which is excluded by hypothesis. As a result, $\Delta = h(\beta) = \frac{w}{h_1(\beta)}$. Hence, plugging this into $f(\beta)$, we get that γ can be written as a rational function of β , and since $\Delta = \alpha - \beta\gamma$, so is α , i.e. $\alpha = \Delta + \beta f(\beta)$. Furthermore, from (24) we obtain

$$\Delta^2 = \frac{|z'(0)|^2}{|z'(\beta)|^2}$$

So, we deduce that:

$$(h(\beta))^2 - \frac{|z'(0)|^2}{|z'(\beta)|^2} = 0 \quad (26)$$

Lemma 23 *Under our hypotheses, the equation (26) cannot be an identity.*

Proof. Since $z' = x' + iy'$, then (26) can be written as

$$\frac{4(x'^2 + y'^2)^3}{(-x''y' + x'y'')^2} = C$$

with C a constant. If $C = 0$ then \mathcal{C} cannot be a curve. So, assume that $C \neq 0$, and let $k = \frac{4}{C}$. Then we have that

$$k = \frac{(-x''y' + x'y'')^2}{(x'^2 + y'^2)^3} = \dots = \frac{\left[\left(\frac{y'}{x'}\right)'\right]^2}{x'^2 \left(1 + \left(\frac{y'}{x'}\right)^2\right)^3}$$

Writing $u = \frac{y'}{x'}$, $k_1 = \sqrt{k}$ (notice that $k > 0$), we get

$$k_1 x' = \frac{u'}{(1 + u^2)^{3/2}}$$

Integrating, we have that

$$k_1 x + k_2 = \frac{u}{\sqrt{1 + u^2}}$$

Now since $u = \frac{y'}{x'}$, after substituting this and taking squares, we get (after some calculations),

$$\frac{(k_1 x + k_2)^2 x'^2}{1 - (k_1 x + k_2)^2} = y'^2$$

Taking square-roots, and integrating, it holds that

$$\frac{1}{k_1} \sqrt{1 - (k_1 x + k_2)^2} = y + k_3$$

From here one may easily get that the implicit equation of \mathcal{C} is that of a circumference, which is excluded by hypothesis. \square

So, let $\xi_1(\beta)$ be the numerator of (26), which, from the above lemma, is not identically 0. We also consider the equation

$$\frac{d}{dt} \left(\frac{z'(\varphi(t)) \cdot \varphi'(t)}{z'(t)} \right) = 0 \tag{27}$$

which is obtained after solving for $e^{i\theta}$ in (23), and differentiating with respect to t . Let $\xi_2(\beta)$ be the gcd of the coefficients of the numerator of $\frac{d}{dt} \left(\frac{z'(\varphi(t)) \cdot \varphi'(t)}{z'(t)} \right)$.

As in other cases, the conditions $\xi_1(\beta) = 0$, $\xi_2(\beta) = 0$, taken together, guarantee that (23) holds. Finally, let $\eta(\beta) = \text{gcd}(\xi_1(\beta), \xi_2(\beta))$, which is not identically 0 because of Lemma 23.

Theorem 24 *If \mathcal{C} has rotation symmetry and $\varphi(t) = \frac{\alpha t + \beta}{\gamma t + \delta}$ with $\delta \neq 0$, then it holds that: (i) $\delta = 1$, $\gamma = f(\beta)$, $\alpha = h(\beta) + \beta f(\beta)$; (ii) $\eta(\beta) = 0$. Conversely, if $\beta_0 \in \mathbb{R}$ fulfills: (i) $\eta(\beta_0) = 0$; (ii) no denominator in (19), (25), (26), (27), vanishes identically, then \mathcal{C} has rotation symmetry with the considered $\varphi(t)$, and the symmetry elements can be obtained from (19) and (24).*

5.3 Full Algorithm

The following algorithm follows from the preceding subsections.

ALGORITHM ROTATION SYMMETRY: Given a curve \mathcal{C} by means of a proper parametrization $z(t) = x(t) + iy(t)$ (in complex form), well-defined at $t = 0$, such that $z'(0) \neq 0$, the algorithm **checks** if it has **rotation symmetry**, and computes the **rotation center** and the **angles** in the affirmative case.

- (1) ($\varphi(t) = k/t$) **Check** if $\eta(k)$ has any real roots $k_1, \dots, k_r \in \mathbb{R}$ in the conditions of Theorem 22. In the affirmative case, let z_0 be the rotation center and let $\theta_1, \dots, \theta_r$ be the angles they correspond to.
- 2) $\left(\varphi(t) = \frac{\alpha t + \beta}{\gamma t + 1}, \delta \neq 0 \right)$ **Check** if $\eta(\beta)$ has any real roots β_1, \dots, β_s in the conditions of Theorem 24. In the affirmative case, let z_0 be the rotation center and let $\tilde{\theta}_1, \dots, \tilde{\theta}_s$ be the angles they correspond to.
- (3) If (1) and (2) have not succeeded, then **return** “The curve has not rotation symmetry”. Otherwise, **return** the rotation center, and the list of angles.

Lemma 21 and Lemma 23 guarantee that this algorithm terminates. The correctness follows from the preceding subsections in this section.

6 A Special Case

In this section we consider a curve \mathcal{C} parametrized by a proper parametrization

$$(x(t), y(t)) = \left(\frac{p(t)}{(t^2 + 1)^r}, \frac{q(t)}{(t^2 + 1)^s} \right)$$

where $p(t), q(t)$ are polynomials, $r, s \in \mathbb{N}$ with either $r > 0$ or $s > 0$ (in the sequel, without loss of generality we will assume that $r > 0$), and $\gcd(p(t), t^2 + 1) = 1$, $\gcd(q(t), t^2 + 1) = 1$. This subclass of rational curves is important because it contains most of the so-called *trigonometric curves*. These are the

real plane curves where each coordinate can be given parametrically by a trigonometric polynomial, that is, a truncated Fourier series:

$$\begin{aligned}x &= \sum_{k=0}^m (a_k \cos(k\theta) + b_k \sin(k\theta)) \\y &= \sum_{k=0}^n (c_k \cos(k\theta) + d_k \sin(k\theta))\end{aligned}$$

with $a_i, b_i, c_i, d_i \in \mathbb{R}$. These curves appear in various areas of mathematics, physics and engineering (one may see the introduction to [8] for a list of different contexts where they arise). They have been considered for example in [1], [8], [20], [33], [34], [21]. From the results in [8] (see Theorem 2.1 therein), it follows that every trigonometric curve has either a *simple* trigonometric representation (i.e. a parametrization of the above kind, which is injective for almost every point when restricted to $\theta \in [0, 2\pi]$), or a *polynomial simplification*, i.e. a polynomial parametrization which coincides with the curve when the parameter is restricted to a certain real interval $[a, b]$. For the first ones, De Moivre's formula together with the usual parametrization

$$\cos(\theta) = \frac{1 - t^2}{1 + t^2}, \quad \sin(\theta) = \frac{2t}{1 + t^2}$$

applied on the simple representation provides a parametrization of the kind considered in this section.

In the sequel, we will denote $m = \deg(p(t))$, $n = \deg(q(t))$. Now for these curves, and specially for central symmetry and mirror symmetry, we can provide a sharper class of tentative $\varphi(t)$'s, which shortens (in some cases, dramatically, as it is shown in the next section) the computations.

6.1 Central Symmetry

Following the ideas in Section 3, if \mathcal{C} has central symmetry then $z(\varphi(t)) = 2z_0 - z(t)$. If we consider only the first coordinate, we get that $x(\varphi(t)) = 2x_0 - x(t)$; substituting $\varphi(t)$ and performing some calculations, we have that

$$\frac{P(t)}{(\gamma t + \delta)^{m-2r} \cdot [(\alpha t + \beta)^2 + (\gamma t + \delta)^2]^r} = 2x_0 - \frac{p(t)}{(t^2 + 1)^r}$$

where $P(t)$ is a polynomial which can be written as $P(t) = a_m(\alpha t + \beta)^m + (\gamma t + \delta)\tilde{P}(t)$. Notice that since $\gcd(p(t), t^2 + 1) = 1$ and $t^2 + 1$ is irreducible then $(\alpha t + \beta)^2 + (\gamma t + \delta)^2$ cannot divide $P(t)$. Additionally, from the irreducibility

of $t^2 + 1$ we deduce that:

- (a) If $m > 2r$, then $\gamma t + \delta$ must divide $t^2 + 1$. Hence, the only possibility is $\gamma = 0$. Furthermore, $(\alpha t + \beta)^2 + (\gamma t + \delta)^2 = \{\gamma = 0\} = \alpha^2 t^2 + 2\alpha\beta t + \beta^2 + \delta^2$ must also divide $t^2 + 1$. So, $\alpha\beta = 0$ and $\alpha^2 = \beta^2 + \delta^2$. Since $\alpha \neq 0$ (otherwise we would get $\alpha\delta - \beta\gamma = 0$) we get $\beta = 0$, $\alpha = \pm\delta$. Discarding the solution $\alpha = \delta$ (which implies $x(t), y(t)$ constant) we get $\alpha = -\delta$, i.e. $\varphi(t) = -t$.
- (b) If $m \leq 2r$, then $(\alpha t + \beta)^2 + (\gamma t + \delta)^2 = (\alpha^2 + \gamma^2)t^2 + 2(\alpha\beta + \gamma\delta)t + \beta^2 + \delta^2$ must divide $t^2 + 1$. So, $\alpha\beta + \gamma\delta = 0$ and $\alpha^2 + \gamma^2 = \beta^2 + \delta^2$. Now we distinguish two cases:
- If $\delta = 0$, then $\alpha\beta = 0$. If $\beta = 0$ then we get $\alpha^2 + \gamma^2 = 0$, and since α, γ are real, $\alpha = \gamma = 0$, which contradicts $\alpha\delta - \beta\gamma \neq 0$. So, in this case $\alpha = 0$, $\gamma = \pm\beta$, and hence $\varphi(t) = 1/t$ or $\varphi(t) = -1/t$.
 - If $\delta \neq 0$, then we can assume $\delta = 1$. Moreover, from Subsection 3.2 we know that $\alpha = -\delta$, and so $\alpha = -1$. Hence, $\gamma = \beta$ and $\varphi(t) = \frac{-t + \beta}{\beta t + 1}$.

As a consequence, the only possibilities for $\varphi(t)$ are: $-t, 1/t, -1/t$ and $\varphi(t) = \frac{-t + \beta}{\beta t + 1}$.

6.2 Mirror Symmetry

The same idea can be used in the case of mirror symmetry and rotation symmetry. In the first case, the condition (5) can be written as

$$\overline{z(t)} = \overline{z_0} + [z(\varphi(t)) - z_0] \cdot e^{2i\theta}$$

Decomposing $\overline{z(t)}$ into real and imaginary parts, we get

$$\begin{aligned} \frac{p(t)}{(t^2 + 1)^r} &= x_0^* + \frac{R(t) \cdot \cos(2\theta)}{(\gamma t + \delta)^{m-2r} \cdot [(\alpha t + \beta)^2 + (\gamma t + \delta)^2]^r} + \frac{S(t) \cdot \sin(2\theta)}{(\gamma t + \delta)^{n-2s} \cdot [(\alpha t + \beta)^2 + (\gamma t + \delta)^2]^s} \\ \frac{q(t)}{(t^2 + 1)^s} &= y_0^* + \frac{M(t) \cdot \sin(2\theta)}{(\gamma t + \delta)^{m-2r} \cdot [(\alpha t + \beta)^2 + (\gamma t + \delta)^2]^r} + \frac{N(t) \cdot \cos(2\theta)}{(\gamma t + \delta)^{n-2s} \cdot [(\alpha t + \beta)^2 + (\gamma t + \delta)^2]^s} \end{aligned}$$

where $R(t), S(t), M(t), N(t)$ are polynomials, $x_0^* = x_0 - x_0 \cos(2\theta) + y_0 \sin(2\theta)$, $y_0^* = -y_0 - y_0 \cos(2\theta) - x_0 \sin(2\theta)$. Now arguing as in Subsection 6.1, we have that:

- (1) If $m - 2r \leq 0$ and $n - 2s \leq 0$ simultaneously, then the only possibilities are $\varphi(t) = 1/t, \varphi(t) = -1/t$, or $\varphi(t) = \frac{-t + \beta}{\beta t + 1}$.
- (2) Otherwise there is one more possibility, namely $\varphi(t) = -t$.

So, the possible $\varphi(t)$'s are exactly the same as in the case before.

6.3 Rotation Symmetry

Arguing as in the above subsection, if $m - 2r \leq 0$ and $n - 2s \leq 0$ then in the case $\delta = 0$, the only possibilities are $\varphi(t) = 1/t$ or $\varphi(t) = -1/t$. However, when $\delta \neq 0$, since in this case it is not true in general that $\alpha = -\delta$, we cannot reach a simple form for $\varphi(t)$, similar to that in the preceding subsections.

7 Implementation and experimentation

From the algorithms in Sections 3, 4, 5, 6 we have that the detection of central, mirror or rotation symmetry reduces to checking the existence of real roots of certain univariate polynomials. As a consequence, it can be performed deterministically. However, in order to find the elements of the symmetry (i.e. symmetry centers, symmetry axes, etc.), in general a numeric approach of the real roots of the polynomials is required. In this sense, our algorithms have been implemented and tested in the computer algebra system Maple 15. As for the computation of the elements of the symmetry, we have observed that the precision needed in this computation must be increased accordingly with the size of the input; in particular, as the degree is increased, the infinity norm of the polynomials arising in intermediate steps of the algorithm grows as well. So, in our implementation we have followed the following strategy:

- (i) The real roots of the polynomials are approached numerically with a certain precision which is initially computed based on the size of some intermediate polynomials, appearing when certain steps of the algorithm are executed, which we have heuristically detected as “good indicators”.
- (ii) The symmetry elements are approached using this precision.
- (iii) Afterwards, these elements are tested to check their accuracy. For this purpose, we compute an index, which must be very close to 0: if we identify that this index is not small, the precision is increased, and the computations start again. The index is computed as follows: we generate a number of points (typically, 100) in the curve, we compute the symmetric points of these with respect to the elements that we have determined, and we “estimate” the average distance from these points to the curve. In general, we observe a very well performance of the indicators pointed out in (i). We must say that as the size of the output grows, the time consumed by this last part (the test) is bigger and tends to be even longer than the time consumed in the computation of the symmetry elements; the reason behind is the necessity, in these cases, of a big (sometimes, huge) number of digits

to guarantee an accurate computation.

Also, in our experiments we have seen that generally we can manage curves up to degree 10 in a fast way. In some cases we can go a bit further, although this seems to be more difficult in the case of mirror symmetry. We can always go further if we avoid the computation of certain polynomials, but this implies that the answer yes/no on the existence of the considered symmetry is not deterministic anymore. Still, in this case we can always determine the tentative elements of the symmetry, and check a fortiori whether they are fake or not (see later the timings for the mirror symmetry algorithm) by using the test above. Finally, let us mention also that the timings improve dramatically in the case of the curves addressed in Section 6. All the experiments have been performed on an Intel Core revving up to 2 GHz., with 8 Gb. of RAM memory.

So, let us address in more detail each of the algorithms. The following table shows some data corresponding to the algorithm for checking central symmetry. Here, we provide the timing (in seconds), which includes the computation of the symmetry elements and the test on their accuracy, as well as data about the input curves (maximum infinity norm of the numerators and denominators, degree of the parametrization).

Curve	$\deg_t(\phi(t))$	Norm	Sym. (Y/N)	Timing	Comments
Trisectrix	3	3	N	2.372	
Lemniscata	4	30	Y	2.106	
Three-leaved rose	4	3	N	2.403	
1	5	9	Y	2.262	
Astroid	6	86400	Y	2.918	71 digits
2	7	108	Y	3.775	58 digits
3	9	800	Y	11.482	219 digits
4	11	460	Y	37.128	276 digits
5	14	22801	Y	692.458	638 digits
16-leaved Rose	18	25740	Y	2.465	

For degrees higher than 10 we get worse timings, in general, due mainly to two causes: (1) the cost of computing the function $f(\beta)$ (i.e. the relationship $\gamma = f(\beta)$); (2) the cost of performing the test (because of the required number of digits). The curve 5 illustrates this situation. Still, in certain cases and even in presence of high degrees, we may get a good performance when a symmetry center is detected without computing $f(\beta)$ (the case of the 16-leaved rose). Furthermore, detecting curves with no central symmetry is faster.

Now the following table shows data corresponding to the algorithm for mirror symmetry.

Curve	$\deg_t(\phi(t))$	Norm	Sym. (Y/N)	Timing	Comments
Descartes' Folium	3	3	Y	3.541	
Epitrochoid	4	288	Y	3.042	
Lissajous Curve	8	20	Y	3.573	
Offset Cubic Curve	8	54	Y	27.097	
Offset Cardioid	8	78732	Y	17.176	
9	10	99	Y	35.085	61 digits, (★)
10	12	6	Y	8.674	(★)
11	13	371	N	40.046	19 digits, (★)
12	16	9	Y	192.194	17 digits, (★)
Sixteen-leaved Rose	18	25740	Y	65.941	28 digits, (★)
Twenty-leaved Rose	22	369512	Y	168.824	34 digits, (★)

In our experiments, we have observed that most of the time is spent in the computation of the expression (18), and in the test of the symmetry axes. In fact, the timing for the last six curves (those with a (★)) corresponds to the computation without explicitly determining (18). This implies that the set of symmetry axes to be tested at the end of the algorithm may include some which is fake; however, in our experiments these fake axes have been always identified by means of high values in the index (which is computed in the test). Now the following table corresponds to the algorithm to detect rotation

symmetry.

Curve	$\deg_t(\phi(t))$	Norm	Sym. (Y/N)	Timing	Comments
Trisectrix	3	3	Y	4.399	
Epitrochoid	4	288	N	4.352	
Three-leaved-rose	4	3	Y	4.992	36 digits
Astroid	6	8	Y	11.123	23 digits
Four-leaved-rose	6	12	Y	6.209	
Lissajous Curve	8	20	N	4.399	
13	9	5	N	178.466	23 digits
Eight-leaved rose	10	140	Y	26.957	28 digits
Twelve-leaved rose	12	1848	Y	129.246	42 digits
Sixteen-leaved rose	16	25740	Y	510.701	55 digits

Finally, in the following table we compare, for some curves defined by a parametrization of the kind studied in Section 6, the timings corresponding to the algorithms for checking central and mirror symmetry in Sections 3 and 4 (denoted “ord”), and the implementation of the ideas in Section 6 (denoted “spec”). One may notice that the latter outperforms the former by a considerable factor.

Curve	$\deg_t(\phi(t))$	Norm	Sym.tested	Timing (ord.)	Timing (spec.)
17	25	504	C	> 1000	1.966
14	20	504	C	> 1000	1.482
15	26	3696	C	> 1000	2.606
Sixteen-leaved rose	18	25740	M	139.886	33.322
Twenty-leaved rose	22	369512	M	502.074	53.867
14	20	504	M	> 1000	2.543
17	25	504	M	> 1000	1.654

References

- [1] Anai H., Weispfenning V. (2000), *Deciding Linear-Trigonometric Problems*, Proceedings ISSAC 2000 pp. 14-22.
- [2] Berner A., Bokeloh M., Wand M., Schilling A., Seidel H.P. (2008), *A Graph-Based Approach to Symmetry Detection*. Symposium on Volume and Point-Based Graphics (2008), pp. 1-8.
- [3] Bokeloh M., Berner A., Wand M., Seidel H.P., Schilling A. (2009), *Symmetry Detection Using Line Features*. Computer Graphics Forum, Vol. 28, No. 2. (2009), pp. 697-706
- [4] Boutin M. (2000), *Numerically Invariant Signature Curves*, International Journal of Computer Vision 40(3), pp. 235-248.
- [5] Calabi E., Olver P.J., Shakiban C., Tannenbaum A., Haker S. (1998), *Differential and Numerically Invariant Signature Curves Applied to Object Recognition*, International Journal of Computer Vision, 26(2), pp. 107-135.
- [6] Farouki R.T. (2008), *Pythagorean-Hodograph Curves*, Springer.
- [7] Gal R., Cohen-Or D. (2006) *Salient geometric features for partial shape matching and similarity*. ACM Transactions on Graphics, Volume 25(1), pp. 130-150.
- [8] Hong H., Schicho J. (1998), *Algorithms for Trigonometric Curves (Simplification, Implicitization, Parameterization)*, Journal of Symbolic Computation, 26, pp. 279-300.
- [9] Huang Z., Cohen F.S. (1996), *Affine-Invariant B-Spline Moments for Curve Matching*, IEE Transactions on Image Processing, Vol. 5, No. 10, pp. 1473-1480.
- [10] Jong T., Pfister G. (2000). *Local Analytic Geometry*, Ed. Vieweg, Advanced Lectures on Mathematics.
- [11] Loy G., Eklundh J. (2006). *Detecting symmetry and symmetric constellations of features*. Proceedings ECCV 2006, 9th European Conference on Computer Vision, pp. 508-521.
- [12] Lebmair P., Richter-Gebert J. (2008), *Rotations, Translations and Symmetry Detection for Complexified Curves*, Computer Aided Geometric Design 25, pp. 707-719.
- [13] Lebmair P. (2009), *Feature Detection for Real Plane Algebraic Curves*, Ph.D. Thesis, Technische Universität München.
- [14] Lei Z., Tasdizen T., Cooper D.B. (1998), *PIMs and Invariant Parts for Shape Recognition*, Proceedings Sixth International Conference on Computer Vision, pp. 827-832.

- [15] Lipman Y., Cheng X., Daubechies I., Funkhouser T. (2010), *Symmetry factored embedding and distance*, ACM Transactions on Graphics (SIGGRAPH 2010).
- [16] Martinet A., Soler C., Holzschuch N., Sillion F. (2006) *Accurate Detection of Symmetries in 3D Shapes*. ACM Trans. Graphics, 25 (2), pp. 439-464.
- [17] Mitra N.J., Guibas L.J., Pauly M. (2006), *Partial and approximate symmetry detection for 3d geometry*, ACM Transactions on Graph. 25 (3), pp. 560-568.
- [18] Perez-Diaz S. (2006), *On the problem of proper reparametrization for rational curves and surfaces*, Computer Aided Geometric Design, Vol. 23, Issue 4, pp. 307-323.
- [19] Podolak J., Shilane P., Golovinskiy A., Rusinkiewicz S., Funkhouser T. (2006), *A Planar-Reflective Symmetry Transform for 3D Shapes*. Proceeding SIGGRAPH 2006, pp. 549-559.
- [20] Pottmann, H. (1984), *On the geometry of higher planet motion*, Mh. Math., 97, pp. 141-156 (in German)
- [21] Petru P., Schicho J. (2000), *Quantifier Elimination for Trigonometric Polynomials by Cylindrical Trigonometric Decomposition*, Journal of Symbolic Computation, pp. 971-983.
- [22] Schnabel R., Wessel R., Wahl R., Klein R. (2008), *Shape Recognition in 3D-Point Clouds*, The 16-th International Conference in Central Europe on Computer Graphics, Visualization and Computer Vision 08.
- [23] Sendra J.R., Winkler F., Perez-Diaz S. (2008), *Rational Algebraic Curves*, Springer-Verlag.
- [24] Sener S., Unel M. (2005), *Affine invariant fitting of algebraic curves using Fourier descriptors*, Pattern Analysis and Applications Vol. 8, pp. 72-83.
- [25] Simari P., Kalogerakis E., Singh K. (2006), *Folding meshes: hierarchical mesh segmentation based on planar symmetry*. Proc. Symp. Geometry Processing, pp. 111-119.
- [26] Suk T., Flusser J. (1993), *Pattern Recognition by Affine Moment Invariants*, Pattern Recognition, Vol. 26, No. 1, pp. 167-174.
- [27] Suk T., Flusser J. (2005), *Affine Normalization of Symmetric Objects*, in Proceedings ACIVS 2005, pp.100-107, Lecture Notes in Computer Science.
- [28] Tarel J.P., Cooper D.B. (2000), *The Complex Representation of Algebraic Curves and Its Simple Exploitation for Pose Estimation and Invariant Recognition*, IEEE Transactions on Pattern Analysis and Machine Intelligence, vol. 22, No. 7, pp. 663-674.
- [29] Tasdizen T., Tarel J.P., Cooper D.B. (2000), *Improving the Stability of Algebraic Curves for Applications*, IEE Transactions on Image Processing, Vol. 9, pp. 405-416.

- [30] Taubin G. (1991), *Estimation of Planar Curves, Surfaces, and Nonplanar Space Curves Defined by Implicit Equations, with Applications to Edge and Range Image Segmentation*, IEEE Trans. Patter Analysis and Machine Intelligence, vol. 13, no. 11, pp. 1115-1138.
- [31] Taubin G., Cooper D.B. (1992), *Object Recognition Based on Moments (or Algebraic) Invariants*, Geometric Invariance in Computer Vision, J.L. Mundy and A.Zisserman, eds., MIT Press, pp. 375-397, 1992.
- [32] Weiss I. (1993), *Noise-Resistant Invariants of Curves*, IEE Transactions on Pattern Analysis and Machine Intelligence, Vol. 15, No. 9, pp. 943-948.
- [33] Wunderlich, W. (1947), *Higher cycloid curves*, Osterr. Ing. Arch., 1, pp. 277-296.
- [34] Wunderlich, W. (1950), *Approximation with higher cycloid curves*, Osterr. Ing. Arch., 4, pp. 3-11.

8 Appendix I: Proof of Lemma 14 in Section 4.2

In order to prove Lemma 14 we need several previous considerations and results. First let $a, b \in \mathbb{R}$ fulfill $a + \mathbf{i}b = \frac{\overline{z''(0)}}{z'(0)}$. Taking into account that $z = x + \mathbf{i}y$, one may see that

$$a = \frac{x'(0)x''(0) + y'(0)y''(0)}{x'^2(0) + y'^2(0)}$$

$$b = \frac{x''(0)y'(0) - x'(0)y''(0)}{x'^2(0) + y'^2(0)}$$

Furthermore, from the expression (16) in Section 4.2 we have that (here we use the above notation):

$$\operatorname{Re}(\gamma) = -\frac{(2 + at)(x'x'' + y'y'') + bt(x'y'' - x''y') + 2a(x'^2 + y'^2) + t(x''^2 + y''^2)}{(tx'' + 2x')^2 + (ty'' + 2y')^2}$$

$$\operatorname{Im}(\gamma) = -\frac{(2 - at)(x'y'' - x''y') + bt(x'x'' + y'y'') + 2b(x'^2 + y'^2)}{(tx'' + 2x')^2 + (ty'' + 2y')^2}$$

For simplicity, in the above expressions we have written t where we should have written β ; we will proceed in the same way along the appendix. Now notice that since γ can always be taken real, from the above expression for $\operatorname{Im}(\gamma)$ we deduce that

$$(2 - at)(x'y'' - x''y') + bt(x'x'' + y'y'') + 2b(x'^2 + y'^2) = 0 \quad (28)$$

is an identity.

Lemma 25 *Under our hypotheses, $b \neq 0$.*

Proof. If $b = 0$, then substituting in (28) we get $x'y'' - x''y' = 0$, which leads to $y' = cx'$, i.e. $y = cx + d$; so, \mathcal{C} is a line, which is excluded by hypothesis. \square

Lemma 26 *There does not exist $t_0 \in \mathbb{C}$ such that $x''(t_0)y'(t_0) - x'(t_0)y''(t_0) = 0$.*

Proof. If such a $t_0 \in \mathbb{C}$ would exist, we might reparametrize the curve with $u = t + t_0$. However, the resulting parametrization would fulfill $b = 0$, contradicting Lemma 25. \square

Then the following result on the choosing of a, b holds.

Lemma 27 *Without loss of generality we can assume $a = 0, b = 2$.*

Proof. We proceed in a constructive way. We want to find a reparametrization of the form

$$t(u) = \frac{m \cdot u + n}{r \cdot u + 1}$$

with $m, n, r \in \mathbb{R}$, $m - nr \neq 0$ such that $\tilde{x} = x(t(u))$, $\tilde{y} = y(t(u))$ fulfills that $a = 0, b = 2$. So, let

$$\tilde{a} = a_{(\tilde{x}, \tilde{y})}(u) = \frac{\tilde{x}'\tilde{x}'' + \tilde{y}'\tilde{y}''}{\tilde{x}'^2 + \tilde{y}'^2}$$

One can check that

$$\frac{\tilde{x}'\tilde{x}'' + \tilde{y}'\tilde{y}''}{\tilde{x}'^2 + \tilde{y}'^2} = \frac{x'(t)x''(t) + y'(t)y''(t)}{x'^2(t) + y'^2(t)}t' + \frac{t''}{t'}$$

Furthermore, $t' = \frac{m-nr}{(ru+1)^2}$ and $t'' = -2r \frac{m-nr}{(ru+1)^3}$. Substituting $u = 0$, we get

$$\tilde{a} = \frac{x'(n)x''(n) + y'(n)y''(n)}{x'^2(n) + y'^2(n)}(m - nr) - 2r \quad (29)$$

Similarly,

$$\tilde{b} = b_{(\tilde{x}, \tilde{y})}(u) = \frac{y'(t)x''(t) - x'(t)y''(t)}{x'^2(t) + y'^2(t)}t'$$

and substituting in $u = 0$, we have

$$\tilde{b} = \frac{y'(n)x''(n) - x'(n)y''(n)}{x'^2(n) + y'^2(n)}(m - nr) \quad (30)$$

So, we want to find $m, n, r \in \mathbb{R}$ such that $\tilde{a} = 0, \tilde{b} = 2$. By imposing these conditions and dividing (29) and (30), we obtain

$$r = \frac{x'(n)x''(n) + y'(n)y''(n)}{y'(n)x''(n) - x'(n)y''(n)}$$

Notice that by Lemma 26, the denominator of the above expression does not vanish for any real value of n . Furthermore, from this equality and (30) it holds that

$$m - nr = \frac{2(x'(n)^2 + y'(n)^2)}{y'(n)x''(n) - x'(n)y''(n)}$$

So, by choosing $n_0 \in \mathbb{R}$ such that the above expressions for r and $m - nr$ are well defined, and $m - n_0r \neq 0$, a reparametrization with the desired properties is found. \square

Now we can prove the following result. Recall here the notation $\gamma = f(\beta)$ (however, we will write $f(t)$, instead).

Lemma 28 *If \mathcal{C} exhibits mirror symmetry and ξ_1, ξ_2 are identically zero, then for every $t \in \mathbb{C}$, it holds that $|z'(t)| \neq 0$.*

Proof. If ξ_2 is identically 0, then from the expression (17) we have $|z'(t)| = u(t)$, where $u(t)$ is a rational function. Writing $z'(t) = x'(t) + \mathbf{i}y'(t)$, there exists an analytic function $v(t)$ fulfilling $x' = u \cdot \cos v$, $y' = u \cdot \sin v$. Using the expression given for γ in the beginning of the appendix, and taking Lemma 27 into account, one can see that

$$f(t) = -\frac{2uu' + 2tu^2v' + t((u')^2 + u^2(v')^2)}{4tuu' + 4u^2 + t^2((u')^2 + u^2(v')^2)}$$

Additionally, since $\text{Im}(\gamma) = 0$, from the expression for $\text{Im}(\gamma)$ at the beginning of this appendix we get that $u^2v' + tuu' + 2u^2 = 0$. By combining this with the above expression for $f(t)$, we have that

$$f(t) = -\frac{u'}{2u + tu'}$$

Now let us argue by contradiction. For this purpose, let $t_0 \in \mathbb{C}$ be a zero of $|z'(t)|$. If $t_0 = 0$, then from the expression (17) in Subsection 4.2 we deduce that $|z'(t)|$ is identically 0, and hence \mathcal{C} is not a curve. So, $t_0 \neq 0$. Also from (17) one may see that if $t = t_0$ is a zero of $|z'(t)|$, then it must be a pole of $f(t)$, which is a rational function. Now since $t = t_0$ is a zero of $u(t)$ (of multiplicity m), then $u(t) = (t - t_0)^m r(t)$, where $r(t)$ is rational and $r(t_0) \neq 0$. Hence, differentiating this expression for $u(t)$, substituting in the above expression for $f(t)$, and eliminating common factors in the numerator and denominator, we get that

$$\lim_{t \rightarrow t_0} -\frac{u'}{2u + tu'} = \lim_{t \rightarrow t_0} -\frac{mr(t) + (t - t_0)r'(t)}{2(t - t_0)r(t) + mtr(t) + t(t - t_0)r'(t)} = \dots = -\frac{1}{t_0}$$

Since $t_0 \neq 0$, we see that the above limit is not equal to infinity, and therefore $t = t_0$ is not a pole of $f(t)$, which is a contradiction. \square

In the sequel, we will use the notation $u(t) = |z'(t)|$, $x' = u \cdot \cos v$, $y' = u \cdot \sin v$ introduced in the proof of the above lemma. Now we are ready to prove Lemma 14.

Proof of Lemma 14. If v is constant then \mathcal{C} is a line, and we have finished. Also, if u is identically 0 then \mathcal{C} cannot be a curve. So, in the sequel we assume that v is not constant, and u is not identically 0. From Lemma 27, it follows that ξ_1 being identically 0 is equivalent to the following identity:

$$2(x'y'' - x''y') + 2t(x'x'' + y'y'') + 4(x'^2 + y'^2) = 0$$

With the above variables u, v , this equation is transformed into

$$v' = -t \frac{u'}{u} - 2,$$

which in turn yields

$$\int \frac{v'}{t} dt = -\ln(C_0^* \cdot t^2 u)$$

Let $R(t) = C_0^* \cdot t^2 u$. Since $R(t)$ is rational, we can write

$$R(t) = C_0 \cdot (t - t_0)^{n_0} \cdots (t - t_p)^{n_p} \cdot (t^2 + a_0 t + b_0)^{m_0} \cdots (t^2 + a_q t + b_q)^{m_q}$$

with $n_i \in \mathbb{Z}$, $m_j \in \mathbb{Z}$, $C_0 \in \mathbb{R}$ and $a_j^2 - 4b_j < 0$, $\forall j \in \{0, 1, \dots, q\}$; furthermore, all the t_j 's are different, and all the terms $t^2 + a_j t + b_j$'s are also different. So, differentiating the logarithm of the above expression and doing some easy computations, we have that

$$v' = n_0 + \cdots + n_p + 2(m_0 + \cdots + m_q) + \frac{n_0 t_0}{t - t_0} + \cdots + \frac{n_p t_p}{t - t_p} - m_0 \frac{a_0 t + 2b_0}{t^2 + a_0 t + b_0} - \cdots - m_q \frac{a_q t + 2b_q}{t^2 + a_q t + b_q}$$

Since we are assuming that v is not constant, we get that

$$v = \arctg \frac{y'}{x'} = I_0 + I_1 + I_2 + I_3$$

where

$$\begin{aligned} I_0 &= C_2 + C_1 t \\ I_1 &= \ln|t - t_0|^{n_0 t_0} \cdots |t - t_p|^{n_p t_p} \\ I_2 &= -\frac{m_0 a_0}{2} \ln|t^2 + a_0 t + b_0| - \cdots - \frac{m_q a_q}{2} \ln|t^2 + a_q t + b_q| \\ I_3 &= -\frac{m_0 a_0}{2} \frac{\left(2 \frac{b_0}{a_0} - a_0\right)}{\sqrt{b_0 - \frac{a_0^2}{4}}} \arctg \left(\frac{t + \frac{a_0}{2}}{\sqrt{b_0 - \frac{a_0^2}{4}}} \right) - \cdots - \frac{m_q a_q}{2} \frac{\left(2 \frac{b_q}{a_q} - a_q\right)}{\sqrt{b_q - \frac{a_q^2}{4}}} \arctg \left(\frac{t + \frac{a_q}{2}}{\sqrt{b_q - \frac{a_q^2}{4}}} \right) \end{aligned}$$

Now observe that v and I_3 are necessarily bounded as t moves in \mathbb{R} . So, $n_0 t_0 = \dots = n_p t_p = 0$. Since all the t_i 's are different, we have that at most one of them can be equal to 0, and therefore there must be at least p elements in $\{n_0, n_1, \dots, n_p\}$ which are 0. W.l.o.g., let us assume that $n_1 = \dots = n_p = 0$. For the same reason (i.e. v and I_3 bounded for $t \in \mathbb{R}$), $C_1 = 0$. As a consequence, we deduce that

$$v' = n_0 + 2(m_0 + \dots + m_q) - m_0 \frac{a_0 t + 2b_0}{t^2 + a_0 t + b_0} - \dots - m_q \frac{a_q t + 2b_q}{t^2 + a_q t + b_q}$$

where $n_0 + 2(m_0 + \dots + m_q) = C_1 = 0$. Substituting $t = 0$ we get that $v'(0) = n_0$. However, since $v' = -t \frac{u'}{u} - 2$ we also deduce that $v'(0) = -2$ (notice that $u(0) \neq 0$ because we assumed $z'(0) \neq 0$, see the beginning of Section 4), and therefore $n_0 = -2$. As a result, we get that $m_0 + \dots + m_q = 1$, and

$$v' = -m_0 \frac{a_0 t + 2b_0}{t^2 + a_0 t + b_0} - \dots - m_q \frac{a_q t + 2b_q}{t^2 + a_q t + b_q}$$

This means that

$$\frac{v'}{t} = \left(\ln \left(C_0 \cdot t^{-2} \cdot (t^2 + a_0 t + b_0)^{m_0} \dots (t^2 + a_q t + b_q)^{m_q} \right) \right)'$$

Therefore,

$$u = C_0 (t^2 + a_0 t + b_0)^{-m_0} \dots (t^2 + a_q t + b_q)^{-m_q}$$

Now since all the m_i 's are integers, and $m_0 + \dots + m_q = 1$, there are two possibilities: (a) all the m_i 's are 0 except, say, $m_0 = 1$; (b) some m_i 's are positive, and others are negative. However, (b) cannot happen because this would contradict Lemma 28. Hence, $u = C_0 (t^2 + a_0 t + b_0)^{-1}$, and

$$v' = -m_0 \frac{a_0 t + 2b_0}{t^2 + a_0 t + b_0}$$

Nevertheless, $a_0 = 0$ because otherwise the primitive of v would contain a logarithm. So, finally we get $u = -\frac{2b_0}{t^2 + b_0} C_0$ and $v' = -\frac{2b_0}{t^2 + b_0}$. So, $u = kv'$. Hence, we get

$$x' = kv' \cos(v)$$

$$y' = kv' \sin(v)$$

Integrating, we deduce that \mathcal{C} is a circle. □