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Representations and symbolic computation of generalized inverses over fields

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Abstract

This paper investigates representations of outer matrix inverses with prescribed range and/or null space in terms of inner inverses. Further, required inner inverses are computed as solutions of appropriate linear matrix equations (LME). In this way, algorithms for computing outer inverses are derived using solutions of appropriately defined LME. Using symbolic solutions to these matrix equations it is possible to derive corresponding algorithms in appropriate computer algebra systems. In addition, we give sufficient conditions to ensure the proper specialization of the presented representations. As a consequence, we derive algorithms to deal with outer inverses with prescribed range and/or null space and with meromorphic functional entries.

Keywords: outer inverse; inner inverse; matrix equation; computer algebra; specialization of matrices; matrices with functional entries.

Mathematics Subject Classification: 15A09

1 Introduction

For a given matrix A, an important characterization of generalized inverses arises from the following Penrose equations with respect to X:

$$(1)\quad AXA = A \qquad (2)\quad XAX = X \qquad (3)\quad (AX)^* = AX \qquad (4)\quad (XA)^* = XA. \tag{1.1}$$

The Moore-Penrose inverse $A^\dagger$ of A is the unique solution to (1.1). The Drazin inverse A^{D} of $A \in \mathbb{C}^{n\times n}$ is the unique matrix $X \in \mathbb{C}^{n\times n}$ which fulfills the matrix equation (2) in conjunction with

$$(1^k)\ \ A^{l+1}X = A^l,\ l \geq \mathrm{ind}(A), \qquad (5)\ \ AX = XA.$$

Here, $\mathrm{ind}(A) = \min\left\{j \mid \mathrm{rank}(A^j) = \mathrm{rank}(A^{j+1})\right\}$ means the index of A. The group inverse $X = A^{\#}$ coincides with the Drazin inverse in the case $\mathrm{ind}(A) = 1$. The set of generalized inverses defined by the equations implied by $\mathcal{S} \subseteq \{1, 2, 3, 4, 1^k, 5\}$, such that the equation (i) is satisfied for each $i \in \mathcal{S}$, is denoted by $A\{\mathcal{S}\}$. Any element from $A\{\mathcal{S}\}$ is termed as $\mathcal{S}$-inverse of A and is denoted by $A^{(\mathcal{S})}$.

Let $\mathcal{R}(\ )$ and $\mathcal{N}(\ )$ denote the range space and the null space of a given matrix, respectively. Arbitrary $X \in A\{S\}$, constrained by $\mathcal{R}(X) = \mathcal{R}(B)$ (resp. $\mathcal{N}(X) = \mathcal{N}(C)$) is termed as $A^{(S)}_{\mathcal{R}(B),*}$ (resp $A^{(S)}_{*,\mathcal{N}(C)}$).

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A selected $X \in A\{S\}$ which fulfils $\mathcal{R}(X) = \mathcal{R}(B)$, $\mathcal{N}(X) = \mathcal{N}(C)$ will be denoted by $A^{(S)}_{\mathcal{R}(B),\mathcal{N}(C)}$. The collection $A\{S\}$ with known $\mathcal{R}(B)$ (resp. $\mathcal{N}(X) = \mathcal{N}(C)$) will be termed as $A\{S\}_{\mathcal{R}(B),*}$ (resp. $A\{S\}_{*,\mathcal{N}(C)}$). Generalized inverses satisfying (2) (resp. (1)) form $\{2\}$-inverses or outer inverses (resp. inner or $\{1\}$-inverses). For other important properties of generalized inverses see [1, 31].

Most important generalized inverses $A^\dagger$, A^D and $A^\#$ are particular outer inverses [31]:

$$A^\dagger = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}, \quad A^D \ = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)}, \quad A^\# = A^{(2)}_{\mathcal{R}(A),N(A)}. \tag{1.2}$$

Representations and numerical algorithms for computing $\{2\}$-inverses have been investigated in a number of researches, among others, those in [2, 4, 5, 14, 22, 33, 34].

Symbolic computation of different varieties of generalized inverses is constituted as an important area of computer algebra and scientific computing. It is well known that numerical algorithms often lack numerical stability and some small quantities are identified as zeros. Also, the discontinuity of the pseudoinverse causes certain problems in numerical computation. During the symbolic implementation, variables are stored in "exact" form without numerical values, so that cumulative round off errors are completely removed [10].

Moreover, algorithms developed for processing matrices in symbolic form, with unevaluated entries, are applicable to certain classes of matrices and to as well as to a much broader class of problems compared to widespread traditional numerical algorithms which are applicable to constant matrices with certain numerical values. Results generated with unassigned symbols can be used in defining various classes of test problems and, consequently, in the attestation of some hypothesis.

On the other hand, symbolic computation has certain disadvantages. Primarily, generating solutions in analytical form occupies a large amount of memory and it is time consuming. For this purpose, simplifications in symbolic computation are important. But, when analytical algorithms become unattainable or unusable it always remains possible to use approximate numerical methods in any particular instance.

Various algorithms for error-free and symbolic construction of various generalized inverses have been proposed and extensively investigated. Algorithms for symbolic computation of matrix generalized inverses can be separated in two different approaches: approach that uses an appropriate representation of the polynomial elements and the approach based upon the matrix interpolation method. Also, effective versions of these algorithms, appropriate for polynomial matrices where only a few polynomial coefficients are nonzero, are developed. The set of one-variable as well as the set of multiple-variable matrices are considered. Effective representations of multiple-variable polynomial matrices are described in [16].

Single-modulus and multiple-modulus residue arithmetic algorithms for the exact pseudoinverse computation of a matrix with rational element were developed in [18]. Various error-free Leverrier-Faddeev-type algorithms, applicable to polynomial matrices, were investigated in [6, 9, 10, 11, 12]. Several extensions of the Greville's partitioning method from [7], that are applicable to matrices whose entries are unevaluated rational expressions and/or polynomials expressions, were identified in [16, 17, 25, 29]. The algorithm based on the LDL^* factorization and aimed to exact computation of $\{1, 2, 3\}, \{1, 2, 4\}$ inverses was proposed in [23]. An efficient algorithm for the exact evaluation of the QDR decomposition and its application in developing corresponding algorithm for symbolic computing $A^{(2)}_{T,S}$ inverses of univariate unevaluated polynomial or rational matrices was given in [26]. Yu and Wang in [35] introduced an algorithm for generating $\{2\}$-inverses of polynomial matrices. In [19], the authors proposed an algorithm which is able to reduce the computation of the Drazin inverse over certain computable fields whose entries are rational functions of finitely many transcendental elements over a complex field into the computation of the Drazin inverse of matrices with multivariate rational entries. The main idea consists in replacing the functions that appear in functional entries by new variables. As a consequence, the computation of generalized inverses of matrices with rational functional entries is reduced to an equivalent but simpler computational problem on matrices whose entries are rational numbers. In this way, the problem of symbolic calculation of generated inversions as well as calculations in matrix algebra is reduced to a simplified form. The key point in this approach is to find sufficient conditions for matrices over a field to ensure that a generalized inverse is stable with respect to the specialization.

Groebner basis approach for generating the Drazin inverse was originated in [21].

Major outcomes of the present manuscript can be highlighted as follows.
(i) An extension of the Urquhart formula from [30] is proposed. This extension gives representations of outer matrix inverses with prescribed range and/or null space in terms of inner inverses.
(ii) An algorithm for obtaining effectively generalized inverses is proposed on the basis of proposed representations. This algorithm consists of two major steps: generate required inner inverse by solving appropriate linear matrix equation (LME) and then multiply the obtained result by appropriate matrix expressions.
(iii) New approach in symbolic computation of outer generalized inverses over the field $\mathbb{K}(\mathbf{x})$ of rational functions with coefficients in the field $\mathbb{K}$ and with respect to variables $\mathbf{x}$ is proposed. The approach is based on solving appropriate LME symbolically.
(iv) Our important topic is symbolic computation of generalized inverses of matrices with functional entries over a field with or without an involution. We show how to reduce the generalized inversion over certain computable fields to a simpler computation on matrices with rational functions as entries. As a consequence, we derive an algorithm to compute, in symbolic form, $\{2\}$-inverses with determined range and/or null space. The characterizations and representations for the Moore-Penrose and the Drazin inverse are derived in particular cases.

The overall organization of sections is as follows. Representations of outer and inner generalized inverses with prescribed range and/or null space in the form $B(CAB)\{1\}C$ are considered in Section 2. Section 3 is aimed to representations and symbolic computation of generalized inverses of matrices which entries are multivariate rational expressions. Section 4 gives some representative examples in symbolic and generic form. Properties of generalized inverses with respect to specializations are considered in Section 5. More precisely, that section investigates the behavior of generalized inverses after the replacement of functional entries $\mathbf{f} = (f_1, \ldots, f_p)$ by unknown variables $\mathbf{x} = (x_1, \ldots, x_p)$ and future value assignments defined after the replacement of the unknowns $\mathbf{x} = (x_1, \ldots, x_p)$ by constant field elements $\mathbf{c} = (c_1, \ldots, c_p)$.

Throughout this paper we will use the following notation. A field of characteristic zero is termed as $\mathbb{K}$, $\mathbf{x} = (x_1, \ldots, x_p)$ is a p-tuple of indeterminates. We will denote by $\mathbb{K}[\mathbf{x}]$ the polynomial ring with coefficients in the field $\mathbb{K}$ and variables $\mathbf{x}$. Similarly, $\mathbb{K}(\mathbf{x})$ is the field of rational functions with coefficients in the field $\mathbb{K}$ and with respect to unknown variables $\mathbf{x}$; i.e. $\mathbb{K}(\mathbf{x})$ is the quotient field of $\mathbb{K}[\mathbf{x}]$. For a field $\mathbb{F}$, say e.g. $\mathbb{F} = \mathbb{K}$ or $\mathbb{F} = \mathbb{K}(\mathbf{x})$, we denote by $\mathbb{F}^{m\times n}$ the set of $m \times n$ matrices over $\mathbb{F}$. Moreover, for $M \in \mathbb{F}^{m\times n}$, we denote by $\mathcal{R}(M)$ and by $\mathcal{N}(M)$ the range and null space of M over $\mathbb{F}$, respectively.

2 Representations of outer generalized inverses over $\mathbb{K}(\mathbf{x})$

The key idea for the design of our algorithms is the use of particular representations of the outer inverses by means of inner inverses. In [24], we provide a complete description of this type of representations over an arbitrary field, that in fact are inspired in [13], [28]. Here, we slightly give a different approach for the case of $\mathbb{K}(\mathbf{x})$ as ground field, although they are applicable for arbitrary fields, and where the precise description of the representations are emphasized. In order to simplify presentation, the notation $\varrho(A_1, \ldots A_k)$ will be used instead of $\mathrm{rank}(A_1) = \cdots = \mathrm{rank}(A_k)$.

We start recalling that $\mathbb{K}(\mathbf{x})^{m\times n}$ is a regular (Von Neumann) ring (see e.g. [24, Lemma 2.1.]). The next result gives a useful correlation between inner inverses and outer inverses with predefined range.

Corollary 2.1. *If $A \in \mathbb{K}(\mathbf{x})^{m\times n}$ and $B \in \mathbb{K}(\mathbf{x})^{n\times k}$ satisfy $\varrho(AB, B)$, then*

$$A\{2\}_{\mathcal{R}(B),*} = B(AB)\{1\}. \tag{2.1}$$

Proof. See Theorem 2.3. in [13] and Theorem 3.3. in [24]. □

Computationally, the direct consequence of Corollary 2.1 is that the outer inverses, with prescribed range, are of the form BU, where $U \in \mathbb{K}(\mathbf{x})^{k\times m}$ is a solution of the LME $(AB)U(AB) = AB$ with respect to $U \in \mathbb{K}(\mathbf{x})^{k\times m}$. However, $(AB)U(AB) = AB$ can be simplified into $BU(AB) = B$. Details are provided in Corollary 2.2.

Corollary 2.2. *If $A \in \mathbb{K}(\mathbf{x})^{m\times n}$ and $B \in \mathbb{K}(\mathbf{x})^{n\times k}$ satisfy $\varrho(AB, B)$, then*

$$A\{2\}_{\mathcal{R}(B),*} = \left\{BU \,|\, U \in \mathbb{K}(\mathbf{x})^{k\times m},\ BUAB = B\right\} \tag{2.2}$$

Proof. It follows from Theorem 2.3. in [13], and using that $\mathbb{K}(\mathbf{x})^{m\times n}$ is regular. □

Theorem 2.1 gives an analogous representation of outer inverses with prescribed range on the basis of the LME $BUCAB = B$, where $C \in \mathbb{K}(\mathbf{x})^{l\times m}$.

Theorem 2.1. *Let $A \in \mathbb{K}(\mathbf{x})^{m\times n}$, $B \in \mathbb{K}(\mathbf{x})^{n\times k}$, $C \in \mathbb{K}(\mathbf{x})^{l\times m}$.*

(a) *The next assertions are equivalent:*

(i) *there is $X \in A\{2\}$ of the form $X := B(CAB)^{(1)}C \in \mathbb{K}(\mathbf{x})^{n\times m}$ such that $\mathcal{R}(X) = \mathcal{R}(B)$, denoted by $A^{(2)}_{\mathcal{R}(B),*}$;*

(ii) *there exists $U \in \mathbb{K}(\mathbf{x})^{k\times l}$ which provides $BUCAB = B$;*

(iia) *there exists $X \in \mathbb{K}(\mathbf{x})^{n\times l}$, $X \in \mathcal{R}(B)$, which provides $XCAB = B$;*

(iii) *$\mathcal{N}(CAB) = \mathcal{N}(B)$;*

(iv) *$B(CAB)^{(1)}CAB = B$;*

(v) *$\varrho(CAB, B)$.*

(b) *In addition, if* **(a)** *holds for A, B, C, then*

$$\begin{aligned} & \{BUC \mid U \in \mathbb{K}(\mathbf{x})^{k\times l},\ BUCAB = B,\ \varrho(CAB, B)\} && (2.3)\\ = & \left\{ B\left((CAB)^{(1)} + Y\left(I_l - CAB(CAB)^{(1)}\right)\right)C \,\middle|\, Y \in \mathbb{K}(\mathbf{x})^{k\times l}\right\} && (2.4)\\ = & \ B(CAB)\{1\}C && (2.5)\\ = & \ A\{2\}_{\mathcal{R}(B),*}. && (2.6) \end{aligned}$$

Proof. **(a)** (i)⇒(ii). Let $X \in \mathbb{K}(\mathbf{x})^{n\times m}$ satisfy $XAX = X$ and $X := B(CAB)^{(1)}C$. Then there exists some $U \in \mathbb{K}(\mathbf{x})^{k\times l}$ satisfying $X = BUC$. Also, $\mathcal{R}(X) = \mathcal{R}(B)$ implies the existence of $W \in \mathbb{K}(\mathbf{x})^{m\times k}$ such that $B = XW$. These two facts further imply $B = XW = XAXW = XAB = BUCAB$.

(ii)⇒(iii). Using known result $\mathcal{N}(B) \subseteq \mathcal{N}(CAB)$ in conjunction with $BUCAB = B$ for some $U \in \mathbb{K}(\mathbf{x})^{k\times l}$, it follows $\mathcal{N}(CAB) \subseteq \mathcal{N}(BUCAB) = \mathcal{N}(B)$, and later $\mathcal{N}(CAB) = \mathcal{N}(B)$.

(iii)⇒(iv). As $\mathcal{N}(CAB) = \mathcal{N}(B)$ initiates $B = VCAB$, for some $V \in \mathbb{K}(\mathbf{x})^{n\times l}$, it follows that

$$B = VCAB = VCAB(CAB)^{(1)}CAB = B(CAB)^{(1)}CAB.$$

(iv)⇒(i). Let $B = B(CAB)^{(1)}CAB$, and set $X = B(CAB)^{(1)}C$. Then $XAX = X$ immediately follows. Now, using $X = B(CAB)^{(1)}C$ and $B = B(CAB)^{(1)}CAB = XAB$ one concludes $X \in A\{2\}_{\mathcal{R}(B)}$.

(ii)⇔(iia). Follows from $X := BU$.

(iv)⇔(v). Evidently.

(b) Since $\mathcal{N}(B) = \mathcal{N}(CAB)$, it follows that $B(CAB)^{(1)}CAB = B$, which implies solvability of the equation $BUCAB = B$. In addition, U is of the general form

$$B^{(1)}B(CAB)^{(1)} + Y - B^{(1)}BYCAB(CAB)^{(1)},$$

which implies that $X := BUC$ is given by (2.4).

The equation $XAX = X$ can be verified straightforward. Clearly, $\mathcal{R}(X) \subseteq R(B)$. On the other hand $XAB = BUCAB = B$ implies $\mathrm{rank}(X) = \mathrm{rank}(B)$ and further $\mathcal{R}(X) = \mathcal{R}(B)$.

Finally, using Urquhart result and [1, Corollary 1, P. 52.], we can obtain

$$\begin{aligned}&\left\{ B\left((CAB)^{(1)} + Y\left(I_l - CAB(CAB)^{(1)}\right)\right) C \,\middle|\, Y \in \mathbb{K}(\mathbf{x})^{k\times l}\right\}\\ &= \left\{ B\left((CAB)^{(1)} + Y - (CAB)^{(1)}CABYCAB(CAB)^{(1)}\right) C \,\middle|\, Y \in \mathbb{K}(\mathbf{x})^{k\times l}\right\}\\ &= B(CAB)\{1\}C.\end{aligned}$$

Finally, after the verification of $\{B(CAB)\{1\}C|\ \mathrm{rank}(B) = \mathrm{rank}(CAB)\} = A\{2\}_{\mathcal{R}(B),*}$ the proof is finished. □

In Theorem 2.1, outer inverses of A, with a prescribed range described by B, are expressed in terms of a third matrix C. Taking into account this, for fixed and arbitrary matrices $A \in \mathbb{K}(\mathbf{x})^{m\times n}$, and $B \in \mathbb{K}(\mathbf{x})^{n\times k}$, and for every $l \in \mathbb{N}$, we introduce the set

$$\mathcal{C}_l(A,B) = \left\{C \in \mathbb{K}(\mathbf{x})^{l\times m} \mid \varrho(CAB, B),\ \mathrm{rank}(B) \le \mathrm{rank}(C)\right\}.$$

as well as

$$\mathfrak{Q}_C = B(CAB)\{1\}C,\ \ C \in \mathcal{C}_l(A,B).$$

In this situation, the following theorem holds.

Theorem 2.2. *The set of outer inverses of $A \in \mathbb{K}(\mathbf{x})^{m\times n}$ defined by the range of $B \in \mathbb{K}(\mathbf{x})^{n\times k}$ is equal to*

$$A\{2\}_{\mathcal{R}(B),*} = \bigcup_{l\ge 1}\ \bigcup_{C\in\mathcal{C}_l(A,B)} \mathfrak{Q}_C. \tag{2.7}$$

Proof. Clearly, Theorem 2.1 leads to $\mathfrak{Q}_C \subseteq A\{2\}_{\mathcal{R}(B),*}$, for each $C \in \mathcal{C}_l(A,B)$. Further, Corollary 2.1 implies $A\{2\}_{\mathcal{R}(B),*} = \mathfrak{Q}_{I_m}$. So,

$$A\{2\}_{\mathcal{R}(B),*} = \mathfrak{Q}_{I_m} \subseteq \bigcup_{l\ge 1}\ \bigcup_{C\in\mathcal{C}_l(A,B)} \mathfrak{Q}_C \subseteq A\{2\}_{\mathcal{R}(B),*},$$

which completes the proof. □

The previous development admits a dual treatment for the case of outer inverses with predefined null space. Theorem 2.3 is dual to Theorem 2.1. Before stating it, it is important mentioning that Theorem 3 proposed in [28, Theorem 5] offers several equivalent characterizations and computationally efficient representations of $A\{2\}_{*,\mathcal{N}(C)}$. Also, the following dual result with respect to Corollary 2.1 also holds.

Corollary 2.3. *If $A \in \mathbb{K}(\mathbf{x})^{m\times n}$ and $C \in \mathbb{K}(\mathbf{x})^{l\times m}$ satisfy $\varrho(CA, C)$, then*

$$A\{2\}_{*,\mathcal{N}(C)} = (CA)\{1\}C. \tag{2.8}$$

Theorem 2.3. *Let $A \in \mathbb{K}(\mathbf{x})^{m\times n}$ and $C \in \mathbb{K}(\mathbf{x})^{l\times m}$ be fixed and $B \in \mathbb{K}(\mathbf{x})^{n\times k}$.*

(a) *The next statements are equivalent*

(i) *there is $X \in A\{2\}$ of the form $X := B(CAB)^{(1)}C \in \mathbb{K}(\mathbf{x})^{n\times m}$ which meets $\mathcal{N}(X) = \mathcal{N}(C)$, denoted by $A^{(2)}_{*,\mathcal{N}(C)}$;*

(ii) *there is $V \in \mathbb{K}(\mathbf{x})^{k\times l}$ such that $CABVC = C$;*

(iia) *there is $X \in \mathbb{K}(\mathbf{x})^{k\times m}$, $X \in \mathcal{N}(C)$, satisfying $CABX = C$;*

(iii) *$\mathcal{R}(CAB) = \mathcal{R}(C)$;*

(iv) *$CAB(CAB)^{(1)}C = C$;*

(v) *$\varrho(CAB, C)$.*

(b) *In addition, if* **(a)** *holds for A, B, C, then*

$$
\begin{aligned}
& \left\{BVC \mid V \in \mathbb{K}(\mathbf{x})^{k\times l},\ CABVC = C,\ \varrho(CAB, C)\right\} && (2.9)\\
= \ & \left\{ B\left((CAB)^{(1)} + \left(I_k - (CAB)^{(1)}CAB\right)Y\right)C \,\middle|\, Y \in \mathbb{K}(\mathbf{x})^{k\times l}\right\} && (2.10)\\
= \ & B(CAB)\{1\}C && (2.11)\\
= \ & A\{2\}_{*,\mathcal{N}(C)}. && (2.12)
\end{aligned}
$$

Also, using Theorem 2.10. in [13], and the fact that every field is a right FP-injective ring (see Def. 2.6. in [13]), one gets the dual version of Corollary 2.2.

Corollary 2.4. *If $A \in \mathbb{K}(\mathbf{x})^{m\times n}$ and $C \in \mathbb{K}(\mathbf{x})^{l\times m}$ satisfy $\mathbb{K}_C = \mathbb{K}_{CA}$, then*

$$A\{2\}_{*,\mathcal{N}(C)} = \left\{CV \mid V \in \mathbb{K}(\mathbf{x})^{n\times l},\ CAVC = C\right\} \tag{2.13}$$

Now, for every $k \in \mathbb{N}$, and for fixed and arbitrary $A \in \mathbb{K}(\mathbf{x})^{m\times n}$, $C \in \mathbb{K}(\mathbf{x})^{l\times m}$ we introduce the set

$$\mathcal{B}_k(A, C) = \left\{B \in \mathbb{K}(\mathbf{x})^{n\times k} \mid \varrho(CAB, C),\ \mathrm{rank}(C) \le \mathrm{rank}(B)\right\}.$$

as well as

$$\mathfrak{Q}_B = B(CAB)\{1\}C, \quad B \in \mathcal{B}_k(A, C).$$

In this situation, Theorem 2.4 is valid.

Theorem 2.4. *Let $A \in \mathbb{K}(\mathbf{x})^{m\times n}$, $C \in \mathbb{K}(\mathbf{x})^{l\times m}$. Then*

$$A\{2\}_{*,\mathcal{N}(C)} = \bigcup_{k\ge 1}\ \bigcup_{B\in\mathcal{B}_k(A,C)} \mathfrak{Q}_B. \tag{2.14}$$

Proof. According to Corollary 2.3 and Theorem 2.3,

$$A\{2\}_{*,\mathcal{N}(C)} = \mathfrak{Q}_{I_n} \subseteq \bigcup_{k\ge 1}\ \bigcup_{B\in\mathcal{B}_k(A,C)} \mathfrak{Q}_B \subseteq A\{2\}_{*,\mathcal{N}(C)},$$

and the proof follows immediately. □

In the essence, we investigate the influence of the matrix $C \in \mathbb{K}(\mathbf{x})^{l\times m}$ with variable dimension l satisfying $\mathrm{rank}(CAB) = \mathrm{rank}(B)$ in Theorem 2.1. Analogously, the influence of the matrix $B \in \mathbb{K}(\mathbf{x})^{n\times k}$ with variable dimension k satisfying $\varrho(CAB, C)$ is considered in Theorem 2.3.

Result for $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ is a consequence of theorems 2.1 and 2.3.

Corollary 2.5. *Let $A \in \mathbb{K}(\mathbf{x})^{m\times n}$, $B \in \mathbb{K}(\mathbf{x})^{n\times k}$ and $C \in \mathbb{K}(\mathbf{x})^{l\times m}$. Then $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ possesses the following representation:*

$$
\begin{aligned}
\left\{A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}\right\} &= \left\{BVC \mid V \in \mathbb{K}(\mathbf{x})^{k\times l},\ CABVC = C,\ \varrho(CAB, B, C)\right\} && (2.15)\\
&= \left\{BUC \mid V \in \mathbb{K}(\mathbf{x})^{k\times l},\ BUCAB = B,\ \varrho(CAB, B, C)\right\} && (2.16)\\
&= \{BUC \mid V \in \mathbb{K}(\mathbf{x})^{k\times l},\ BUCAB = B, CABUC = C,\ \varrho(CAB, B, C) && (2.17)\\
&= \{B(CAB)\{1\}\,C \mid\ \varrho(CAB, B, C)\}. && (2.18)
\end{aligned}
$$

Also, the following notation is useful:

$$\mathfrak{Q}_{B,C} = B(CAB)\{1\}C, \quad B \in \mathcal{B}_k(A, C),\ C \in \mathcal{C}_l(A, B).$$

Corollary 2.6. *The outer inverse* $X := A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ *of* $A \in \mathbb{K}(\mathbf{x})^{m\times n}$ *is represented as*

$$A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = \mathfrak{Q}_{B,C}.$$

¿From the previous results one gets a complete description of the inner and outer inverses.

Corollary 2.7. *Let* $A \in \mathbb{K}(\mathbf{x})^{m\times n}$*, then*

$$A\{1\} = \bigcup_{k\in\mathbb{N}}\bigcup_{l\in\mathbb{N}}\left\{B(CAB)^{(1)}C \mid B \in \mathbb{K}(\mathbf{x})^{n\times k}, C \in \mathbb{K}(\mathbf{x})^{l\times m},\ \varrho(CAB, A)\right\}.$$

Corollary 2.8. *Let* $A \in \mathbb{K}(\mathbf{x})^{m\times n}$ *and* $C \in \mathbb{K}(\mathbf{x})^{l\times m}$*. Then,*

$$A\{2\}_{*,\mathcal{N}(C)} = \bigcup_{k\in\mathbb{N}}\left\{B(CAB)^{(1)}C \mid B \in \mathbb{K}(\mathbf{x})^{n\times k},\ \varrho(CAB, C)\right\}.$$

Corollary 2.9. *Let* $A \in \mathbb{K}(\mathbf{x})^{m\times n}$ *and* $B \in \mathbb{K}(\mathbf{x})^{n\times k}$*. Then,*

$$A\{2\}_{\mathcal{R}(B),*} = \bigcup_{l\in\mathbb{N}}\left\{B(CAB)^{(1)}C \mid C \in \mathbb{K}(\mathbf{x})^{l\times m},\ \varrho(CAB, B)\right\}.$$

Remark 2.1. *Urquhart representation investigated a fixed expression* $X := B(CAB)^{(1)}C$ *and then gives corresponding statements about* X*. Our representations derived in theorems 2.1, 2.3 and Corollary 2.5 offer complete characterizations of outer inverses in terms of* $B(CAB)\{1\}C$*. More precisely, our representations imply that all outer inverses can be presented in the form* $B(CAB)\{1\}C$*.*

3 Symbolic computation of generalized inverses based on LME

In general, presented results offer one specific and efficient computational-algorithmic framework based on solving appropriate equations and then multiplying the obtained solution by appropriate matrices. A selected approach used in solving underlying equations can generate corresponding class of algorithms. Approach based on the usage of Gradient Neural Networks (GNN) and Zeroing Neural Network (ZNN) is very popular and based on the Frobenius on the error matrix which is defined on the basis of an appropriate matrix equation which is being solved. The GNN evolution design is defined upon the Frobenius norm of the matrix corresponding to the equation which is being solved. In [28], starting from theoretical characterizations and representations, the GNN evolution was used in solving underlying matrix equations and proposed a number of algorithms for calculating outer and inner inverses with predefined range and/or null space for matrices over a field. In time-varying case, ZNN evolution is defined using, so called, Zhang functions which represent the underlying equation in matrix, vector or scalar case [8, 36]. Also, the hyperpower iterative method with numerous modifications are defined using the powers of the residual matrix corresponding to the underlying matrix equation [27, 32].

In the present article, an approach based on finding symbolic solutions to underlying matrix equations over an arbitrary field is proposed.

According to the results presented in Theorems 2.1, 2.3 and Corollary 2.5 it is possible to state the Algorithm 1 for computing $B(CAB)^{(1)}C$.

Algorithm 1 Computing outer inverse with prescribed range and/or null space.

Require: $A \in \mathbb{K}(\mathbf{x})^{m\times n}$, $B \in \mathbb{K}(\mathbf{x})^{n\times k}$, $C \in \mathbb{K}(\mathbf{x})^{l\times m}$.

1: Solve symbolically the LME
   $BUCAB = B$ if $\varrho(CAB, B)$ or
   $CABUC = C$ if $\varrho(CAB, C)$.
2: Compute $X := BUC$.
3: Return X.

The output of Algorithm 1 is defined in Corollary 2.5.

Corollary 3.1. *Let $A \in \mathbb{K}(\mathbf{x})^{m\times n}$, $B \in \mathbb{K}(\mathbf{x})^{n\times k}$, $C \in \mathbb{K}(\mathbf{x})^{l\times m}$. Then the output X of Algorithm 1 satisfies*
(1) $X = A^{(2)}_{\mathcal{R}(B),*} \iff \varrho(CAB, B)$;
(2) $X = A^{(2)}_{*,\mathcal{N}(C)} \iff \varrho(CAB, C)$;
(3) $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} \iff \varrho(CAB, B, C)$;
(4) $X = A^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)} \iff \varrho(CAB, B, C, A)$.

Proof. Follows from theorems 2.1, 2.3 and Corollary 2.5. □

The output of Algorithm 1 depending on specific values of A, B, C is defined in Corollary 3.2. For the core-EP inverse, in statement (8) in Corollary 3.2, we refer to [15].

Corollary 3.2. *Let $\mathbb{K}(\mathbf{x})$ be taken as $\mathbb{C}(\mathbf{x})$. The output $X := B(CAB)^{(1)}C$ of Algorithm 1 satisfies:*
(5) $X = A^{\mathrm{D}}$ *if* $B = C = A^k$, $k \geq \mathrm{ind}(A)$.
(6) *The group inverse* $X = A^{\#}$ *if* $B = C = A$.
(7) $X = A^{\dagger}$ *if* $B = C = A^*$ *or when* $BC = A^*$ *is a rank factorization of* A^*.
(8) *The core-EP inverse* $X = A^{\oplus} = A^k \left((A^k)^* A^{k+1}\right)^{(1)} (A^k)^* = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*)}$ *if* $B = A^k$, $C = (A^k)^*$.
(9) *The core inverse* $X = A^{\#\!\!\!\bigcirc} = A\left(A^* A^2\right)^{(1)} A^* = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A^*)}$ *if* $B = A$, $C = A^*$.

The following *Mathematica* function `GinvBC[A_, B_, C_]` is aimed to perform the implementation of Algorithm 1, which requires the computation of $X := BUC$, where $BUCAB = B$ or $CABUC = C$.

```
GinvBC[A_, B_, C_] :=
  Module[{m, n, k, l, U, C1 = C.A.B},
   {n, k} = Dimensions[B]; {l, m} = Dimensions[C];
   If[MatrixRank[C1] < MatrixRank[B] && MatrixRank[C1] < MatrixRank[C], Return[{}]];
   U = Table[Subscript[u, i, j], {i, k}, {j, l}];
   vars = Flatten[U];
   If[MatrixRank[C1] == MatrixRank[B],
        ret = Solve[B.U.C1 == B, vars] // Simplify,
        ret = Solve[C1.U.C == C, vars] // Simplify
   ];
   ret = vars /. Sort[Flatten[ret]];
   U = Table[ret[[(i - 1)*l + j]], {i, k}, {j, l}];
   Return[B.U.C] // Simplify
  ];
```

4 Examples on symbolic computation of outer inverses

This section gives some representative examples in symbolic form. Unknown matrix U is considered in generic form, with arbitrary entries, in order to obtain some general conclusions.

Example 4.1. Consider general nonzero matrices

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \end{bmatrix}, \ C = \begin{bmatrix} c_{1,1} \end{bmatrix}, \ a_{1,1} \neq 0, c_{1,1} \neq 0.$$

(a) If we take nonzero matrices in unevaluated form

$$B_1 = \begin{bmatrix} b_{1,1} \\ b_{2,1} \end{bmatrix}, \quad U = \begin{bmatrix} u_{1,1} \end{bmatrix},$$

then $\text{rank}(CAB_1) = \text{rank}(C) = \text{rank}(B_1) = 1$. So, `GinvBC[A,B`$_1$`,C]` generates family of points (each specialization generates a particular point):

$$\bigcup_{B_1\in\mathcal{B}_1(A,C)} \mathfrak{Q}_{B_1} = \bigcup_{C\in\mathcal{C}_1(A,B)} \mathfrak{Q}_C = \mathfrak{Q}_{B_1,C}$$

$$= A^{(2)}_{\mathcal{R}(B_1),\mathcal{N}(C)} = \begin{bmatrix} \frac{b_{1,1}}{a_{1,1}b_{1,1}+a_{1,2}b_{2,1}} \\ \frac{b_{2,1}}{a_{1,1}b_{1,1}+a_{1,2}b_{2,1}} \end{bmatrix}, \ a_{1,1}b_{1,1}+a_{1,2}b_{2,1} \neq 0.$$

It is important to note that the nonzero outer inverse does not exist in the case $a_{1,1}b_{1,1}+a_{1,2}b_{2,1} = 0$, corresponding to $\text{rank}(CAB_1) = 0$. In general case, `GinvBC[A,B`$_1$`,C]` generates the family

$$\left\{ \frac{b_{1,1}}{a_{1,1}b_{1,1}+a_{1,2}b_{2,1}} + \frac{a_{1,2}}{a_{1,1}} \frac{b_{2,1}}{a_{1,1}b_{1,1}+a_{1,2}b_{2,1}} - \frac{1}{a_{1,1}} = 0 \right\}.$$

So, all the points $A^{(2)}_{\mathcal{R}(B_2),\mathcal{N}(C)}$ belong to the line

$$x + \frac{a_{1,2}}{a_{1,1}}\, y = \frac{1}{a_{1,1}}. \tag{4.1}$$

(b) Consider

$$B_2 = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}, \ U = \begin{bmatrix} u_{1,1} \\ u_{2,1} \end{bmatrix}.$$

(b1) Let us observe the situation $\text{rank}(B_2) = 2$ firstly. If B_2 satisfies $1 = \text{rank}(CAB_2) = \text{rank}(C)$, then `GinvBC[A,B`$_2$`,C]` generates the infinite family (only one line) (each specialization generates a particular line or point):

$$\bigcup_{B_2\in\mathcal{B}_2(A,C)} \mathfrak{Q}_{B_2} = \left\{ \begin{bmatrix} \frac{a_{1,2}b_{1,1}b_{2,2}c_{1,1}u_{1,1}+b_{1,2}(1-a_{1,2}b_{2,1}c_{1,1}u_{1,1})}{a_{1,1}b_{1,2}+a_{1,2}b_{2,2}} \\ \frac{a_{1,1}b_{1,2}b_{2,1}c_{1,1}u_{1,1}+b_{2,2}(1-a_{1,1}b_{1,1}c_{1,1}u_{1,1})}{a_{1,1}b_{1,2}+a_{1,2}b_{2,2}} \end{bmatrix}, \ u_{1,1} \in \mathbb{K} \right\}.$$

If we use the replacements

$$x := \frac{a_{1,2}b_{1,1}b_{2,2}c_{1,1}u_{1,1} + b_{1,2}\left(1 - a_{1,2}b_{2,1}c_{1,1}u_{1,1}\right)}{a_{1,1}b_{1,2}+a_{1,2}b_{2,2}}, y := \frac{a_{1,1}b_{1,2}b_{2,1}c_{1,1}u_{1,1} + b_{2,2}\left(1 - a_{1,1}b_{1,1}c_{1,1}u_{1,1}\right)}{a_{1,1}b_{1,2}+a_{1,2}b_{2,2}},$$

one can verify that the line generated by `GinvBC[A,B`$_2$`,C]` is defined as (4.1). All the points $A^{(2)}_{*,\mathcal{N}(C)}$ belong to this line.

(b2) Our choice in this part is $B_2 = I_2$. Since $\text{rank}(CAI_2) = \text{rank}(CA) = \text{rank}(C)$, in view of Corollary 2.3, `GinvBC[A,`I_2`,C]` produces the infinite family (line)

$$\mathfrak{Q}_{I_2} = A\{2\}_{*,\mathcal{N}(C)} = \begin{cases} \left\{ \begin{bmatrix} c_{1,1}u_{1,1} \\ \frac{1-c_{1,1}a_{1,1}u_{1,1}}{a_{1,2}} \end{bmatrix}, \ u_{1,1} \in \mathbb{C} \right\}, & a_{1,2} \neq 0 \\ \left\{ \begin{bmatrix} \frac{1}{c_{1,1}a_{1,1}} \\ u_{2,1} \end{bmatrix}, \ u_{2,1} \in \mathbb{K} \right\}, & a_{1,2} = 0. \end{cases}$$

In the case $a_{1,2} \neq 0$ we can use $x := c_{1,1}u_{1,1}$. Then $y := \frac{1-c_{1,1}a_{1,1}u_{1,1}}{a_{1,2}} = \frac{1-a_{1,1}x}{a_{1,2}}$, which implies that (x, y) satisfy (4.1).

(c) Further, use

$$B_3 = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \end{bmatrix}, \ U = \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix}.$$

(c1) Let us observe the situations $\operatorname{rank}(B_3) = 2$ firstly. Then $1 = \operatorname{rank}(CAB_3) = \operatorname{rank}(C)$, but not $\operatorname{rank}(CAB_3) < \operatorname{rank}(B_3) = 2$. So, `GinvBC[A,B3,C]` generates the infinite family (line)

$$\bigcup_{B\in\mathcal{B}_3(A,C)} \mathfrak{Q}_B = \left\{ \begin{bmatrix} \frac{a_{1,2}b_{2,3}c_{1,1}(b_{1,1}u_{1,1}+b_{1,2}u_{2,1})-b_{1,3}(a_{1,2}c_{1,1}(b_{2,1}u_{1,1}+b_{2,2}u_{2,1})-1)}{a_{1,1}b_{1,3}+a_{1,2}b_{2,3}} \\ \frac{a_{1,1}b_{1,3}c_{1,1}(b_{2,1}u_{1,1}+b_{2,2}u_{2,1})-b_{2,3}(a_{1,1}c_{1,1}(b_{1,1}u_{1,1}+b_{1,2}u_{2,1})-1)}{a_{1,1}b_{1,3}+a_{1,2}b_{2,3}} \end{bmatrix},\ u_{1,1}, u_{2,1} \in \mathbb{C} \right\}.$$

The replacements

$$x := \frac{a_{1,2}b_{2,3}c_{1,1}\left(b_{1,1}u_{1,1} + b_{1,2}u_{2,1}\right) - b_{1,3}\left(a_{1,2}c_{1,1}\left(b_{2,1}u_{1,1} + b_{2,2}u_{2,1}\right) - 1\right)}{a_{1,1}b_{1,3} + a_{1,2}b_{2,3}}$$
$$y := \frac{a_{1,1}b_{1,3}c_{1,1}\left(b_{2,1}u_{1,1} + b_{2,2}u_{2,1}\right) - b_{2,3}\left(a_{1,1}c_{1,1}\left(b_{1,1}u_{1,1} + b_{1,2}u_{2,1}\right) - 1\right)}{a_{1,1}b_{1,3} + a_{1,2}b_{2,3}}$$

confirms that `GinvBC[A,B3,C]` is again the line (4.1).

(c2) Now, observe the matrix

$$B_{3S} = \begin{bmatrix} 0 & 0 & b_{1,3} \\ 0 & 0 & b_{2,3} \end{bmatrix}$$

which satisfies $\operatorname{rank}(B_{3S}) = 1$. Then $\operatorname{rank}(CAB_{3S}) = \operatorname{rank}(C) = \operatorname{rank}(B_{3S}) = 1$. So, `GinvBC[A,B3S,C]` generates the infinite family (line)

$$\bigcup_{B\in\mathcal{B}_3(A,C)} \mathfrak{Q}_B = \left\{ \begin{bmatrix} \frac{b_{1,3}}{a_{1,1}b_{1,3}+a_{1,2}b_{2,3}} \\ \frac{b_{2,3}}{a_{1,1}b_{1,3}+a_{1,2}b_{2,3}} \end{bmatrix} \right\}.$$

The replacements

$$x := \frac{b_{1,3}}{a_{1,1}b_{1,3} + a_{1,2}b_{2,3}}, \quad y := \frac{b_{1,3}}{a_{1,1}b_{1,3} + a_{1,2}b_{2,3}}$$

confirms that `GinvBC[A,B3S,C]` is again the line (4.1).

General conclusion is that each B_3 of rank 2 generates the same line (4.1). Moreover, each B_3 of rank 1 generates a point, but all these points belong to (4.1).

Example 4.2. **(a)** Consider the symmetric matrix S_5 from [37]:

$$S_5 = \begin{bmatrix} t+1 & t & t & t & t+1 \\ t & t-1 & t & t & t \\ t & t & t+1 & t & t \\ t & t & t & t-1 & t \\ t+1 & t & t & t & t+1 \end{bmatrix}$$

and the following matrices B and C:

$$B = \begin{bmatrix} 2t+1 & t & t \\ t & 2t-1 & t \\ t & t & 2t+1 \\ t & t & t \\ 2t+1 & t & t \end{bmatrix}, \quad C = \begin{bmatrix} t^2+1 & t^2 & t^2 & t^2 & t^2+1 \\ t^2 & t^2-1 & t^2 & t^2 & t^2 \\ t^2 & t^2 & t^2+1 & t^2 & t^2 \end{bmatrix}.$$

The unique solution to $BUCS_5B = B$ is given by

$$U = (CS_5B)^{-1}$$
$$= \begin{bmatrix} \frac{-36t^5-18t^4+6t^3+3t^2+1}{-63t^6-21t^5+41t^4+t^3+10t^2+12t+4} & \frac{t\left(42t^4+57t^3+8t^2+19t+2\right)}{-63t^6-21t^5+41t^4+t^3+10t^2+12t+4} & -\frac{t\left(6t^4+39t^3-10t^2-5t+6\right)}{-63t^6-21t^5+41t^4+t^3+10t^2+12t+4} \\ \frac{-63t^4-21t^3+5t^2+t}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{2\left(63t^4+21t^3+22t^2+10t+2\right)}{63t^5-42t^4+t^3-2t^2-8t-4} & -\frac{7t^2\left(9t^2+3t-2\right)}{63t^5-42t^4+t^3-2t^2-8t-4} \\ \frac{t\left(21t^4+30t^3-8t^2+2t+3\right)}{(t+1)(63t^5-42t^4+t^3-2t^2-8t-4)} & -\frac{t^2\left(77t^3+42t^2+11t+22\right)}{(t+1)(63t^5-42t^4+t^3-2t^2-8t-4)} & \frac{4\left(14t^5+3t^4-4t^3-1\right)}{(t+1)(63t^5-42t^4+t^3-2t^2-8t-4)} \end{bmatrix}.$$

The unique outer inverse $(S_5)^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ of S_5 corresponding to B and C can be generated by the expression `X = GinvBC[`S_5`, B, C]`, and it is equal to

$$(S_5)^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = BUC = B(CS_5B)^{-1}C =$$

$$\begin{bmatrix} \frac{-10t^5+6t^4+3t^3-t^2+t+1}{-63t^5+42t^4-t^3+2t^2+8t+4} & \frac{t(15t^4+t^3+6t^2-2)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t(-15t^4+13t^3-8t^2+4t+2)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t^2(-20t^3+6t^2+3t-1)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{-10t^5+6t^4+3t^3-t^2+t+1}{-63t^5+42t^4-t^3+2t^2+8t+4} \\ \frac{t(-2t^4+3t^3+8t^2-3t-2)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{-66t^5+13t^4-25t^3+2t^2+12t+4}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t(3t^4+t^3+16t^2-8t-4)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t^2(67t^3-9t^2-18t-4)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t(-2t^4+3t^3+8t^2-3t-2)}{63t^5-42t^4+t^3-2t^2-8t-4} \\ \frac{t(-16t^4+13t^3-6t^2+3t+2)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t(39t^4-7t^3+8t^2-4)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{-24t^5+7t^4+15t^3-6t^2+4t+4}{-63t^5+42t^4-t^3+2t^2+8t+4} & \frac{t^2(-31t^3+13t^2+2t-4)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t(-16t^4+13t^3-6t^2+3t+2)}{63t^5-42t^4+t^3-2t^2-8t-4} \\ \frac{t(2t^4-3t^3+t^2-5t+1)}{-63t^5+42t^4-t^3+2t^2+8t+4} & -\frac{t(3t^4+29t^3+6t^2+18t+4)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t(3t^4+t^3-2t^2+10t-4)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t^3(4t^2+33t-1)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t(2t^4-3t^3+t^2-5t+1)}{-63t^5+42t^4-t^3+2t^2+8t+4} \\ \frac{-10t^5+6t^4+3t^3-t^2+t+1}{-63t^5+42t^4-t^3+2t^2+8t+4} & \frac{t(15t^4+t^3+6t^2-2)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t(-15t^4+13t^3-8t^2+4t+2)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{t^2(-20t^3+6t^2+3t-1)}{63t^5-42t^4+t^3-2t^2-8t-4} & \frac{-10t^5+6t^4+3t^3-t^2+t+1}{-63t^5+42t^4-t^3+2t^2+8t+4} \end{bmatrix}.$$

(b) Consider now the function `GinvBC[S`$_5$`,Tanspose[S`$_5$`],Tanspose[S`$_5$`]]` with the aim to compute $S_5^\dagger$. This particular choice corresponds to the assignments $A = S_5$, $B = C = S_5^{\mathrm{T}}$. The matrix system $BUCAB = B$ becomes $S_5^{\mathrm{T}} U S_5^{\mathrm{T}} S_5 S_5^{\mathrm{T}} = S_5^{\mathrm{T}}$. The expression `U = Table[Subscript[u, i, j], {i, k}, {j, l}]` generates the 5×5 matrix $U = [u_{ij}]$ with symbolic entries u_{ij}. Then the solution to the system $S_5^{\mathrm{T}} U S_5^{\mathrm{T}} S_5 S_5^{\mathrm{T}} = S_5^{\mathrm{T}}$ can be obtained using the expression `Solve[`$S_5^{\mathrm{T}} U S_5^{\mathrm{T}} S_5 S_5^{\mathrm{T}} = S_5^{\mathrm{T}}$`, vars] // Simplify` and it is equal to

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} & u_{1,5} \\ u_{2,1} & -\frac{49t^3}{4} - \frac{31t^2}{4} - 3t - 1 & \frac{1}{4}t\left(49t^2+3t+4\right) & -\frac{1}{4}t\left(49t^2+31t+12\right) & \frac{1}{4}\left(t\left(49t^2+10t+3\right) - 4u_{2,1}\right) \\ u_{3,1} & \frac{1}{4}t\left(49t^2+3t+4\right) & -\frac{49t^3}{4} + \frac{25t^2}{4} - 3t + 1 & \frac{1}{4}t\left(49t^2+3t+4\right) & -\frac{1}{4}t\left(49t^2-18t+7\right) - u_{3,1} \\ u_{4,1} & -\frac{1}{4}t\left(49t^2+31t+12\right) & \frac{1}{4}t\left(49t^2+3t+4\right) & -\frac{49t^3}{4} - \frac{31t^2}{4} - 3t - 1 & \frac{1}{4}\left(t\left(49t^2+10t+3\right) - 4u_{4,1}\right) \\ u_{5,1} & \frac{1}{4}\left(t\left(49t^2+10t+3\right) - 4u_{1,2}\right) & \frac{1}{4}\left(t\left(-49t^2+18t-7\right) - 4u_{1,3}\right) & \frac{1}{4}\left(t\left(49t^2+10t+3\right) - 4u_{1,4}\right) & \frac{1}{4}\left(-49t^3+11t^2-3t-4u_{1,1}-4u_{1,5}-4u_{5,1}+1\right) \end{bmatrix}$$

and the result is the Moore-Penrose inverse of S_5, equal to

$$S_5^{\mathrm{T}} U S_5^{\mathrm{T}} = S_5^\dagger = \begin{bmatrix} \frac{1}{4} - \frac{t}{4} & \frac{t}{2} & -\frac{t}{2} & \frac{t}{2} & \frac{1}{4} - \frac{t}{4} \\ \frac{t}{2} & -t-1 & t & -t & \frac{t}{2} \\ -\frac{t}{2} & t & 1-t & t & -\frac{t}{2} \\ \frac{t}{2} & -t & t & -t-1 & \frac{t}{2} \\ \frac{1}{4} - \frac{t}{4} & \frac{t}{2} & -\frac{t}{2} & \frac{t}{2} & \frac{1}{4} - \frac{t}{4} \end{bmatrix}.$$

(c) In this part of the example we consider

$$S_3 = \begin{bmatrix} t+1 & t & t+1 \\ t & t-1 & t \\ t+1 & t & t+1 \end{bmatrix},$$

the matrix B as in the previous cases, but assume that $C := C_1$ is defined as

$$B_1 = \begin{bmatrix} 2t+1 & t & t \\ t & 2t-1 & t \\ t & t & t \end{bmatrix}, \; C_1 = \begin{bmatrix} t^2+1 & t^2 & t^2 \\ t^2 & t^2-1 & t^2 \end{bmatrix}.$$

Now, $\mathrm{rank}(C_1S_3B_1) = \mathrm{rank}(C_1) < \mathrm{rank}(B_1)$. So, `GinvB[S`$_3$`,B`$_1$`,C`$_1$`]` is not applicable. On the other hand, `GinvC[S_3,B_1,C_1]` produces the following outer inverse with prescribed null space:

$$\begin{bmatrix} \frac{t^3+\left(t^3+t^2+t+1\right)u_{1,2}t^2-t+(t+1)\left(t^2+1\right)^2u_{1,1}+1}{2(t^2+1)} & \frac{1}{2}\left((t+1)u_{1,1}t^2+t+(t-1)(t+1)^2u_{1,2}\right) & \frac{t^2\left(-2t+\left(t^3+t^2+t+1\right)u_{1,1}+\left(t^3+t^2+t+1\right)u_{1,2}+1\right)}{2(t^2+1)} \\ -\frac{t\left(t^2+t-1\right)}{t^2+1} & -t-1 & \frac{t^2(2t+1)}{t^2+1} \\ -\frac{-t^3+\left(t^3+t^2+t+1\right)u_{1,2}t^2+t+(t+1)\left(t^2+1\right)^2u_{1,1}-1}{2(t^2+1)} & \frac{1}{2}\left(-(t+1)u_{1,1}t^2+t-(t-1)(t+1)^2u_{1,2}\right) & -\frac{t^2\left(2t+\left(t^3+t^2+t+1\right)u_{1,1}+\left(t^3+t^2+t+1\right)u_{1,2}-1\right)}{2(t^2+1)} \end{bmatrix}.$$

Example 4.3. **(a)** Consider the two-variable rational matrix from [21]:

$$A = \begin{bmatrix} \frac{1}{z_2} & z_1 & 0 \\ 0 & \frac{1}{z_2} & z_1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the index of A is $\text{ind}(A) = 1$, we can ask the group inverse of A in symbolic form by the expression `X = GinvBC[A,A,A]`. This particular choice corresponds to the assignments $B = C = A$. The matrix system $BUCAB = B$ becomes $AUAAA = A$. The expression `U = Table[Subscript[u, i, j], {i, k}, {j, l}]` generates the 3×3 matrix $U = [u_{ij}]$ which contains unassigned symbols u_{ij}. The solution to $AUAAA = A$ can be obtained using the expression `Solve[`$AUAAA = A$`, vars] // Simplify` and it is equal to

$$U = \begin{bmatrix} u_{1,1} & -z_1 z_2\left(2z_2^3+u_{2,2}\right) & u_{1,3} \\ \frac{z_2^3-u_{1,1}}{z_1 z_2} & u_{2,2} & u_{2,3} \\ \frac{u_{1,1}-z_2^3}{z_1^2 z_2^2} & \frac{z_2^3-u_{2,2}}{z_1 z_2} & u_{3,3} \end{bmatrix}.$$

The result $X = AUA$ is the same as in [21]:

$$X = A^{\text{D}} = A^{\#} = \begin{bmatrix} z_2 & -z_1 z_2^2 & -2z_1^2 z_2^3 \\ 0 & z_2 & z_1 z_2^2 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) Further, consider `GinvB[A,Transpose[A],Transpose[A]]`. The general solution to $A^{\text{T}}UA^{\text{T}}AA^{\text{T}}$ is equal to

$$U = \begin{bmatrix} u_{1,1} & \frac{\frac{z_1^2 z_2^5+z_2^3}{z_1^4 z_2^4+z_1^2 z_2^2+1}-u_{1,1}}{z_1 z_2} & \frac{u_{1,1}-\frac{2z_1^2 z_2^5+z_2^3}{z_1^4 z_2^4+z_1^2 z_2^2+1}}{z_1^2 z_2^2} \\ z_1 z_2\left(-\frac{z_2^3}{z_1^4 z_2^4+z_1^2 z_2^2+1}-u_{2,2}\right) & u_{2,2} & \frac{\frac{z_1^2 z_2^5+z_2^3}{z_1^4 z_2^4+z_1^2 z_2^2+1}-u_{2,2}}{z_1 z_2} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix}$$

and the final result is $A^{\text{T}}UA^{\text{T}}$ coincides with the Moore-Penrose inverse of A:

$$A^{\dagger} = \begin{bmatrix} \frac{z_1^2 z_2^3+z_2}{z_1^4 z_2^4+z_1^2 z_2^2+1} & -\frac{z_1 z_2^2}{z_1^4 z_2^4+z_1^2 z_2^2+1} & 0 \\ \frac{z_1^3 z_2^4}{z_1^4 z_2^4+z_1^2 z_2^2+1} & \frac{z_2}{z_1^4 z_2^4+z_1^2 z_2^2+1} & 0 \\ -\frac{z_1^2 z_2^3}{z_1^4 z_2^4+z_1^2 z_2^2+1} & \frac{z_2^2\left(z_2^2 z_1^3+z_1\right)}{z_1^4 z_2^4+z_1^2 z_2^2+1} & 0 \end{bmatrix}.$$

(c) Now, consider

$$B = \begin{bmatrix} z_1 z_2 & 0 \\ z_2 & z_1^2 \\ z_1 z_2 & z_2^3 \end{bmatrix}, \; C = \begin{bmatrix} z_1 & z_2^2 & 0 \\ 0 & z_1^2 & z_1 z_2 \end{bmatrix}.$$

Clearly, the requirement $\operatorname{rank}(CAB) = \operatorname{rank}(B) = \operatorname{rank}(C) = 2 = \operatorname{rank}(A)$ is satisfied. Then `GinvBC[A,B,C]` produces the $\{1,2\}$-inverse

$$A^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)} = \begin{bmatrix} \frac{z_2\left(z_2^4+z_1\right)}{z_2^5+z_2^4-z_1^3z_2^2+z_1} & -\frac{z_1^2z_2^2}{z_2^5+z_2^4-z_1^3z_2^2+z_1} & -\frac{z_2^3\left(z_2^5+z_1z_2+z_1^3\right)}{z_1^2\left(z_2^5+z_2^4-z_1^3z_2^2+z_1\right)} \\ -\frac{z_2^2\left(z_1^3-z_2^3\right)}{z_1\left(z_2^5+z_2^4-z_1^3z_2^2+z_1\right)} & \frac{z_1z_2}{z_2^5+z_2^4-z_1^3z_2^2+z_1} & -\frac{z_2^8-z_1^3\left(z_2^5+z_2^2\right)}{z_1^3\left(z_2^5+z_2^4-z_1^3z_2^2+z_1\right)} \\ -\frac{z_2^4-z_1^3z_2}{z_1^2\left(z_2^5+z_2^4-z_1^3z_2^2+z_1\right)} & -\frac{z_2^2\left(z_1^3-z_2^2(z_2+1)\right)}{z_1\left(z_2^5+z_2^4-z_1^3z_2^2+z_1\right)} & \frac{z_2^3\left(-z_1^5-z_2z_1^3+z_2^2(z_2+1)z_1^2+z_2^4\right)}{z_1^4\left(z_2^5+z_2^4-z_1^3z_2^2+z_1\right)} \end{bmatrix}.$$

(d) Consider the same A, B as in the part **(c)** and

$$C = \begin{bmatrix} z_1 & z_2^2 & 0 \\ 0 & z_1^2 & z_1z_2 \\ z_2 & z_1 & 0 \end{bmatrix}.$$

Since $\operatorname{rank}(CAB) = \operatorname{rank}(B)$ is true, `GinvBC[A, B, C]` produces $A\{2\}_{\mathcal{R}(B),*}$, defined in the generic form

$$\begin{bmatrix} \frac{z_2\left(z_2^4+z_1\right)}{z_2^5+z_2^4-z_1^3z_2^2+z_1} & -\frac{z_1^2z_2^2}{z_2^5+z_2^4-z_1^3z_2^2+z_1} & z_2\left(\left(z_1^2-z_2^3\right)u_{1,1}-\frac{z_2^4+z_1z_2^2+z_1}{z_2^5+z_2^4-z_1^3z_2^2+z_1}\right) \\ -\frac{z_2^2\left(z_1^3-z_2^3\right)}{z_1\left(z_2^5+z_2^4-z_1^3z_2^2+z_1\right)} & \frac{z_1z_2}{z_2^5+z_2^4-z_1^3z_2^2+z_1} & \frac{z_2\left(-\frac{z_2^4+z_1z_2^2+z_1}{z_2^5+z_2^4-z_1^3z_2^2+z_1}+\left(z_1^2-z_2^3\right)u_{1,1}+z_1\left(\frac{z_2z_1^2+z_2^2+z_2+1}{z_2^5+z_2^4-z_1^3z_2^2+z_1}+\frac{z_1\left(z_1^2-z_2^3\right)u_{2,1}}{z_2}\right)\right)}{z_1} \\ -\frac{z_2^4-z_1^3z_2}{z_1^2\left(z_2^5+z_2^4-z_1^3z_2^2+z_1\right)} & -\frac{z_2^2\left(z_1^3-z_2^2(z_2+1)\right)}{z_1\left(z_2^5+z_2^4-z_1^3z_2^2+z_1\right)} & z_1z_2\left(\frac{\left(\frac{z_2z_1^2+z_2^2+z_2+1}{z_2^5+z_2^4-z_1^3z_2^2+z_1}+\frac{z_1\left(z_1^2-z_2^3\right)u_{2,1}}{z_2}\right)z_2^3}{z_1^3}+\frac{\left(z_1^2-z_2^3\right)u_{1,1}-\frac{z_2^4+z_1z_2^2+z_1}{z_2^5+z_2^4-z_1^3z_2^2+z_1}}{z_1}\right) \end{bmatrix}.$$

5 Computing generalized inverses via specializations

In this section we analyse the behavior of generalized inverses when unknown variables included in $\mathbf{x} = (x_1, \ldots, x_p)$ are substituted by field elements $\mathbf{c} = (c_1, \ldots, c_p)$. The results are stated for the case of matrices over the field $\mathbb{K}(\mathbf{x})$. However, the results can be extended for the case of quotient fields of integral domains of characteristic zero.

In the sequel, for a matrix $A(\mathbf{x}) \in \mathbb{K}(\mathbf{x})^{m\times n}$ and $\mathbf{c} \in \mathbb{K}^p$, $A_{|\mathbf{x}\rightsquigarrow\mathbf{c}}$ indicates the specialization of $A(\mathbf{x})$ at $\mathbf{x} = \mathbf{c}$.

5.1 Rank invariance under specialization

The entries of a matrix $A = (a_{ij}) \in \mathbb{K}(\mathbf{x})^{m\times n}$ are rational expressions of polynomials in $\mathbb{K}[\mathbf{x}]$. Thus, each a_{ij} can be represented as

$$a_{ij} = \frac{\operatorname{num}(a_{ij})}{\operatorname{den}(a_{ij})}, \text{ where } \gcd(\operatorname{num}(a_{i,j}), \operatorname{den}(a_{i,j})) = 1.$$

Then, we define the *denominator of the matrix* A as $\operatorname{den}(A) = \operatorname{lcm}\{\operatorname{den}(a_{ij})\}$, where lcm stands for the least common multiple. The *numerator of the matrix* is defined similarly as $\operatorname{num}(A) = \operatorname{lcm}\{\operatorname{num}(a_{ij}) \mid a_{ij} \neq 0\}$.

On the other hand, associated with any matrix we introduce an appropriate polynomial to ensure that, under specializations, the rank is preserved. For this purpose, we recall the notion of square-free part of a polynomial. If $f = \prod_{i=1}^{q} f_i^{k_i} \in \mathbb{K}[\mathbf{x}]$, with f_i irreducible over $\mathbb{K}$, the square-free part of f is

$$\operatorname{SqFree}(f) = \prod_{i=1}^{q} f_i.$$

In this situation, the association is as follows. Let $A \in \mathbb{K}(\mathbf{x})^{m\times n}$ be non-zero. Let U be the upper triangular matrix output by the Gaussian elimination process when applied to A. Let us say that the process generates the sequence of matrices $\{A^{[0]} := A, A^{[1]}, \ldots, A^{[\ell]} := U\}$. The notation $\mathcal{G}(A)$ is used with the meaning

$$\mathcal{G}(A) = \{\mathrm{den}(A^{[i]})\,\mathrm{num}(A^{[i]}) \mid i = 0, \ldots, \ell\}.$$

Then, we define the *rank polynomial* of A as the polynomial

$$\mathrm{RankPol}(A) = \mathrm{SqFree}\left(\prod_{u(\mathbf{x})\in\mathcal{G}(A)} u(\mathbf{x})\right). \tag{5.1}$$

Let

$$\mathbb{K}_A = \{\mathbf{c} \in \mathbb{K}^p \mid \mathrm{RankPol}(A)(\mathbf{a}) \neq 0\} \subset \mathbb{K}^p.$$

The following result ensures that the rank is preserved when specializing with elements in $\mathbb{K}_A$.

Theorem 5.1. *Let $A \in \mathbb{K}(\mathbf{x})^{m\times n}$. For every $\mathbf{c} \in \mathbb{K}_A$ it holds that*

$$\varrho(A_{|\mathbf{x}\rightsquigarrow\mathbf{c}}, A).$$

Proof. Let U be the output of the Gaussian elimination process applied to A. By construction, none denominator appearing through the Gaussian elimination process vanishes at $\mathbf{c}$. Moreover, since the numerators do not vanish either, one has that $U_{|\mathbf{x}\rightsquigarrow\mathbf{c}}$ is the output of the Gaussian elimination process applied to $A_{|\mathbf{x}\rightsquigarrow\mathbf{c}}$. Furthermore, since the first non-zero entries of the non-zero rows of U vanish at $\mathbf{c}$, we get

$$\varrho(A, U, U_{|\mathbf{x}\rightsquigarrow\mathbf{c}}, A_{|\mathbf{x}\rightsquigarrow\mathbf{c}}),$$

which was our initial intention. □

Remark 5.1. *We observe that*

1. *If $\mathbf{c} \in \mathbb{K}_A$, then $\mathrm{den}(A)(\mathbf{c}) \neq 0$, and hence $A_{|\mathbf{x}\rightsquigarrow\mathbf{c}}$ is well-defined.*
2. *For simplicity we have taken $\mathrm{num}(A^{[i]})$ in the definition of $\mathcal{G}(A)$. However, our reasonings are also valid if we take only the numerator of the first non-zero entries of each non-zero row of every $A^{[i]}$.*
3. *If A is not the zero matrix, since $\mathbb{K}(\mathbf{x})$ has infinitely many elements, and $\mathrm{RankPol}(A)$ is not the zero polynomial, one deduces that $\mathbb{K}_A$ has infinitely many elements.*
4. *Let $\{A_j\}_{j\in\{1,\ldots,\ell\}}$ be non zero matrices over $\mathbb{K}(\mathbf{x})$. Taking into account the previous remark, and the fact that $\mathbb{K}^p$ is irreducible, one has that $\mathbb{K}_{A_1} \cap \cdots \cap \mathbb{K}_{A_\ell} \neq \emptyset$, and indeed has infinitely many elements.*

Example 5.1. *Let*

$$A = \begin{bmatrix} \frac{x_1}{x_1+x_2} & x_2x_3 \\ 1 & \frac{x_2}{x_3} \\ \frac{1}{{x_2}^2x_3} & 0 \end{bmatrix} \in \mathbb{C}(x_1,x_2,x_3)^{3\times 2}.$$

The matrix U is

$$U = \begin{bmatrix} \frac{x_1}{x_1+x_2} & x_2x_3 \\ 0 & -\frac{x_2\left({x_3}^2x_1 + {x_3}^2x_2 - x_1\right)}{x_3x_1} \\ 0 & 0 \end{bmatrix}.$$

Therefore,

$$\mathrm{RankPol}(A) = (x_1+x_2)x_2x_3x_1({x_3}^2x_1 + {x_3}^2x_2 - x_1).$$

5.2 Generalized inverses for matrices with rational entries

Our motivation is to extend the specializations of the Moore-Penrose inverse from Proposition 5.1 as well as of the Drazin inverse from [3, 19] to various classes of generalized inverses.

Lemma 5.1. *Let $A \in \mathbb{K}(\mathbf{x})^{m\times n}$, with $\mathbf{x} = (x_1, \dots, x_p)$, and let $\mathbf{c} = (c_1, \dots, c_p) \in \mathbb{K}^p$. If an inner inverse $A^{(1)}$ of A satisfies*

$$\mathrm{den}(A)(\mathbf{c}) \cdot \mathrm{den}(A^{(1)})(\mathbf{c}) \neq 0$$

then the following statement is satisfied:

$$(A^{(1)})_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(A_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}\right)^{(1)}.$$

Proof. According to the assumption, both the specializations $A_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}$ and $(A^{(1)})_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}$ are defined well. Further, it is possible to conclude the following:

$$A_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = (AA^{(1)}A)_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = A_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}(A^{(1)})_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}A_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}.$$

So, $(A^{(1)})_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(A_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}\right)^{(1)}$ and the proof is completed. □

Theorem 5.2. *Let $A \in \mathbb{K}(\mathbf{x})^{m\times n}$, $B \in \mathbb{K}(\mathbf{x})^{n\times k}$, $C \in \mathbb{K}(\mathbf{x})^{l\times m}$, and let $\mathbf{c} = (c_1, \dots, c_p) \in \mathbb{K}^p$. Let $X := B(CAB)^{(1)}C \in \mathbb{K}(\mathbf{x})^{n\times m}$, then the following statements hold.*

(1) *Let $\varrho(CAB, B)$. For every $\mathbf{c} \in \mathbb{K}_{CAB} \cap \mathbb{K}_B$ satisfying*

$$\mathrm{den}((CAB)^{(1)})(\mathbf{c}) \cdot \mathrm{den}(A)(\mathbf{c}) \cdot \mathrm{den}(C)(\mathbf{c}) \neq 0,$$

it holds that

$$X_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(A^{(2)}_{\mathcal{R}(B),*}\right)_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(A_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}\right)^{(2)}_{\mathcal{R}(B_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}),*}.$$

(2) *Let $\varrho(CAB, C)$. For every $\mathbf{c} \in \mathbb{K}_{CAB} \cap \mathbb{K}_C$ satisfying*

$$\mathrm{den}((CAB)^{(1)})(\mathbf{c}) \cdot \mathrm{den}(A)(\mathbf{c}) \cdot \mathrm{den}(B)(\mathbf{c}) \neq 0,$$

it holds that

$$X_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(A^{(2)}_{*,\mathcal{N}(C)}\right)_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(A_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}\right)^{(2)}_{*,\mathcal{N}(C_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}})}.$$

(3) *Let Let $\varrho(CAB, B, C)$. For every $\mathbf{c} \in \mathbb{K}_{CAB} \cap \mathbb{K}_B \cap \mathbb{K}_C$ satisfying*

$$\mathrm{den}((CAB)^{(1)})(\mathbf{c}) \cdot \mathrm{den}(A)(\mathbf{c}) \neq 0,$$

it holds that

$$X_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}\right)_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(A_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}\right)^{(2)}_{\mathcal{R}(B_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}),\mathcal{N}(C_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}})}. \tag{5.2}$$

(4) *Let $\varrho(CAB, B, C, A)$. For every $\mathbf{c} \in \mathbb{K}_{CAB} \cap \mathbb{K}_B \cap \mathbb{K}_C \cap \mathbb{K}_A$ satisfying*

$$\mathrm{den}((CAB)^{(1)})(\mathbf{c}) \neq 0,$$

it holds that

$$X_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(A^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)}\right)_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(A_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}\right)^{(1,2)}_{\mathcal{R}(B_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}),\mathcal{N}(C_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}})}.$$

Proof. We prove statement (1); the reasoning for the other statements is analogous by using Theorem 2.3 and Corollary 2.5.

Let us use the notation $M_A = A_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}$, $M_B = B_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}$, $M_C = C_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}$, $M_{CAB} = (CAB)_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}$. By the hypotheses, $\mathrm{den}(A)(\mathbf{c})\,\mathrm{den}(C)(\mathbf{c}) \neq 0$. So, M_A, and M_C are well-defined. Moreover $\mathbf{c} \in \mathbb{K}_{CAB} \cap \mathbb{K}_B$, and by Remark 5.1 (1) we get that M_{CAB} and M_B are also well defined. Furthermore, $\mathrm{den}((CAB)^{(1)})(\mathbf{c}) \neq 0$. Thus, $((CAB)^{(1)})_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}$ is well-defined,

Using $\varrho(CAB, B)$, by Theorem 2.1, we get $X = A^{(2)}_{\mathcal{R}(B),*}$. Further, since $\mathbf{c} \in \mathbb{K}_{CAB} \cap \mathbb{K}_B$, by Theorem 5.1, it holds that

$$\varrho(M_{CAB}, CAB, B, M_B).$$

On the other hand,

$$\begin{array}{rcll} X_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} & = & M_B((CAB)^{(1)})_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} M_C & \\ & = & M_B(M_{CAB})^{(1)} M_C & \text{(see Lemma 5.1)} \\ & = & M_B(M_C M_A M_B)^{(1)} M_C & \end{array}$$

Thus, applying Theorem Theorem 2.1, we get that

$$X_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = (M_A)^{(2)}_{\mathcal{R}(M_B),*}$$

Therefore,

$$X_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = (A^{(2)}_{\mathcal{R}(B),*})_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = (M_A)^{(2)}_{\mathcal{R}(M_B),*} = \left(A_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}\right)^{(2)}_{\mathcal{R}(B_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}),*}.$$

□

Remark 5.2. *In each statement of Theorem 5.2 appears a different condition for ensuring the corresponding specialization. Nevertheless, one may simplify and give a global conditions for all cases. More precisely, it is sufficient to require that*

$$\mathrm{RankPol}(CAB) \cdot \mathrm{RankPol}(B) \cdot \mathrm{RankPol}(C) \cdot \mathrm{RankPol}(A) \cdot \mathrm{den}((CAB)^{(1)}) \tag{5.3}$$

does not vanish at $\mathbf{c}$*.*

In the last part of the section we consider $(\mathbb{K}, \varphi)$, where $\mathbb{K}$ is a field of characteristic zero and φ is an involutory automorphism of $\mathbb{K}$. In addition, let φ^e the natural extension of φ to the field $\mathbb{K}(\mathbf{x})$. If the field $\mathbb{K}(\mathbf{x})$ is a Moore-Penrose field (MP field shortly, see [20] for the notion of MP field), then it is possible to use the Moore-Penrose inverse instead of arbitrary $\{1\}$-inverse. Then we will need the notation $\mathbb{K}_\varphi = \{x \in \mathbb{K} \mid \varphi(x) = x\}$ and $\mathbb{K}_\varphi^p := \underbrace{\mathbb{K}_\varphi \times \cdots \times \mathbb{K}_\varphi}_{p \text{ times}}$.

Proposition 5.1. ([20, Theorem 13]) *Let $(\mathbb{K}(\mathbf{x}), \varphi^e)$ be an MP field. Let $A \in \mathbb{K}(\mathbf{x})^{m\times n}$, with $\mathbf{x} = (x_1, \ldots, x_p)$, and let $\mathbf{c} \in \mathbb{K}_\varphi^p$ satisfy*

$$\mathrm{den}(A)(\mathbf{c}) \cdot \mathrm{den}(A^\dagger)(\mathbf{c}) \neq 0.$$

Then

$$(A^\dagger)_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}} = \left(A_{|\mathbf{x}\leftrightsquigarrow\mathbf{c}}\right)^\dagger.$$

Corollary 5.1. *Let $A \in \mathbb{K}(\mathbf{x})^{m\times n}$, $B \in \mathbb{K}(\mathbf{x})^{n\times k}$, $C \in \mathbb{K}(\mathbf{x})^{l\times m}$, with $\mathbf{x} = (x_1, \ldots, x_p)$ and let $(\mathbb{K}(\mathbf{x}), \varphi^e)$ be an MP field. Let $\mathbf{c} \in \mathbb{K}_\varphi^p \cap \mathbb{K}_{CAB} \cap \mathbb{K}_B \cap \mathbb{K}_C \cap \mathbb{K}_A$ satisfy*

$$\mathrm{den}\left((CAB)^\dagger\right)(\mathbf{c}) \neq 0. \tag{5.4}$$

Then $X := B(CAB)^\dagger C \in \mathbb{K}(\mathbf{x})^{n\times m}$ satisfies the statements of Theorem 5.2.

5.3 Computing generalized inverses for matrices with functional entries

Algorithms proposed in this subsection are applicable to matrices whose entries are rational expressions of functions. Let $\Omega \subset \mathbb{C}$ be open and connected, and let $\mathrm{Mer}(\Omega)$ denote the meromorphic functions over Ω. Choose the functions $\mathcal{F} = \{f_1(z), \ldots, f_p(z)\} \subset \mathrm{Mer}(\Omega)\}$. The tuple $\mathbf{f} = (f_1(z), \ldots, f_p(z))$ will be shortly denoted as $\mathbf{f}$.

To start with, we assume in the beginning that the elements in $\mathcal{F}$ are algebraically independent over $\mathbb{C}$. In later investigations we will skip this assumption.

Elements in $\mathcal{F}$ are algebraically independent.

We will work with matrices whose entries belong to the field $\mathbb{C}(\mathcal{F})$, that is, matrices whose elements are rational expressions of the elements in $\mathbf{f}$. In addition, we consider the map Rat (called *rationalization*) that converts a matrix with functional entries into a matrix involving rational expressions:

$$\begin{array}{rccl} \mathrm{Rat}: & \mathbb{C}(\mathcal{F})^{m\times n} & \longrightarrow & \mathbb{C}(\mathbf{x})^{m\times n} \\ & A & \longmapsto & \mathrm{Rat}(A) = A_{|_{\mathbf{f}\rightsquigarrow\mathbf{x}}}. \end{array}$$

So, the mapping Rat consists of replacing $f_i(z)$ by x_i in A. In addition, we consider the mapping (called *functionalization*):

$$\begin{array}{rccl} \mathrm{Func}: & \mathbb{C}(\mathbf{x})^{m\times n} & \longrightarrow & \mathbb{C}(\mathcal{F})^{m\times n} \\ & M & \longmapsto & \mathrm{Func}(M) = M_{|_{\mathbf{x}\rightsquigarrow\mathbf{f}}}. \end{array}$$

We observe that, since $\mathcal{F}$ is algebraically independent, Func is well-defined and, indeed, it is the inverse map Rat. The next lemma shows how the inner inverses behave under the map Rat.

Lemma 5.2. *Let $A \in \mathbb{C}(\mathcal{F})^{m\times n}$, with $\mathcal{F}$ algebraically independent. Then, $A^{(1)}$ exists and meets the following characteristics:*

$$\mathrm{Rat}(A^{(1)}) = (\mathrm{Rat}(A))^{(1)};$$
$$\mathrm{Func}(\mathrm{Rat}(A)^{(1)}) = A^{(1)}.$$

Proof. The existence of $A^{(1)}$ is ensured on the basis of the fact that $\mathbb{C}(\mathcal{F})$ is a field (see e.g. Lemma 2.1. in [24]). By definition, $A = AA^{(1)}A$. So, $\mathrm{Rat}(A) = \mathrm{Rat}(AA^{(1)}A) = \mathrm{Rat}(A)\mathrm{Rat}(A^{(1)})\mathrm{Rat}(A)$. Thus, $\mathrm{Rat}(A^{(1)}) \in \mathrm{Rat}(A)\{1\}$ which proves the first statement. For the second statement, we apply the inverse function Func to the equality in the first statement. □

Using Lemma 5.2, and reasoning as in the proof of Theorem 5.2, one gets Theorem 5.3.

Theorem 5.3. *Let $A \in \mathbb{C}(\mathcal{F})^{m\times n}$, $B \in \mathbb{C}(\mathcal{F})^{n\times k}$, $C \in \mathbb{C}(\mathcal{F})^{l\times m}$, with $\mathcal{F}$ algebraically independent. The next statements hold for $X := B(CAB)^{(1)}C$:*

(1) *If* $\varrho\,(\mathrm{Rat}(C)\mathrm{Rat}(A)\mathrm{Rat}(B), \mathrm{Rat}(B))$ *then*

$$X = A^{(2)}_{\mathcal{R}(B),*} = \mathrm{Func}\left(\mathrm{Rat}(A)^{(2)}_{\mathcal{R}(\mathrm{Rat}(C)),*}\right).$$

(2) *If* $\varrho\,(\mathrm{Rat}(C)\mathrm{Rat}(A)\mathrm{Rat}(B), \mathrm{Rat}(C))$ *then*

$$X = A^{(2)}_{*,\mathcal{N}(C)} = \mathrm{Func}\left(\mathrm{Rat}(A)^{(2)}_{*,\mathcal{N}(\mathrm{Rat}(C))}\right).$$

(3) *If* $\varrho\,(\mathrm{Rat}(C)\mathrm{Rat}(A)\mathrm{Rat}(B), \mathrm{Rat}(B), \mathrm{Rat}(C))$ *then*

$$X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = \mathrm{Func}\left(\mathrm{Rat}(A)^{(2)}_{\mathcal{R}(\mathrm{Rat}(B)),\mathcal{N}(\mathrm{Rat}(C))}\right).$$

(4) *If* $\varrho\,(\mathrm{Rat}(C)\mathrm{Rat}(A)\mathrm{Rat}(B), \mathrm{Rat}(B), \mathrm{Rat}(C), \mathrm{Rat}(A))$ *then*

$$X = A^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)} = \mathrm{Func}\left(\mathrm{Rat}(A)^{(1,2)}_{\mathcal{R}(\mathrm{Rat}(B)),\mathcal{N}(\mathrm{Rat}(C))}\right).$$

Proof. Since $\mathcal{F}$ is algebraically independent the rank of a matrix M over $\mathbb{C}(\mathbf{x})$ and the rank of Func(M) are the same. Thus, the proof is a consequence of theorems 2.1, 2.3 and Corollary 2.5. □

Elements in $\mathcal{F}$ are not necessarily algebraically independent.

Let us now treat the case in which the functions in $\mathcal{F}$ could be algebraically dependent. In this case, the function Func is, in general, not well-defined. More precisely, the function Func is not defined on matrices over $\mathbb{C}(\mathbf{x})$ which denominator has non-trivial greatest common divisor with any of the polynomials that provides the algebraic dependency of the elements in $\mathcal{F}$. A second difficulty in this new theoretical frame is that the rank may decrease during the functionalization $|_{\mathbf{x} \leftrightsquigarrow \mathbf{f}}$. Therefore, in order to generalize Theorem 5.3, one needs to ensure that none of denominators during the computational process vanishes at $\mathbf{f}$ as well as that the ranks of involved matrices are preserved. For this purpose, we will use the rank polynomial introduced in (5.1) (see Theorem 5.1).

Lemma 5.3. *Let $A \in \mathbb{C}(\mathcal{F})^{m\times n}$, where $\mathcal{F}$ is not necessarily algebraically independent. Then, $A^{(1)}$ exists. Moreover, if*

$$\mathrm{den}(\mathrm{Rat}(A^{(1)}))(\mathbf{f}) \neq 0$$

then

$$\begin{aligned} \mathrm{Rat}(A^{(1)}) &= (\mathrm{Rat}(A))^{(1)}\,; \\ (\mathrm{Rat}(A)^{(1)})_{|_{\mathbf{x} \leftrightsquigarrow \mathbf{f}}} &= A^{(1)}. \end{aligned}$$

Theorem 5.4. *Let $A \in \mathbb{C}(\mathcal{F})^{m\times n}$, $B \in \mathbb{C}(\mathcal{F})^{n\times k}$, $C \in \mathbb{C}(\mathcal{F})^{l\times m}$, where $\mathcal{F}$ is not necessarily an algebraic independent set. Let us assume that*

$$\begin{aligned} \mathrm{RankPol}(\mathrm{Rat}(C)\mathrm{Rat}(A)\mathrm{Rat}(B))(\mathbf{f}) \cdot \mathrm{RankPol}(\mathrm{Rat}(B))(\mathbf{f}) \cdot \mathrm{RankPol}(\mathrm{Rat}(C))(\mathbf{f}) \cdot \\ \cdot \mathrm{RankPol}(\mathrm{Rat}(A))(\mathbf{f}) \cdot \mathrm{den}(\mathrm{Rat}((CAB)^{(1)})(\mathbf{f}) \quad \neq \quad 0. \end{aligned} \tag{5.5}$$

The next statements hold for $X := B(CAB)^{(1)}C$.

(1) *If* $\varrho\,(\mathrm{Rat}(C)\mathrm{Rat}(A)\mathrm{Rat}(B),\, \mathrm{Rat}(B))$ *then*

$$X = A^{(2)}_{\mathcal{R}(B),*} = \left(\mathrm{Rat}(A)^{(2)}_{\mathcal{R}(\mathrm{Rat}(B)),*}\right)_{|_{\mathbf{x} \leftrightsquigarrow \mathbf{f}}}.$$

(2) *If* $\varrho\,(\mathrm{Rat}(C)\mathrm{Rat}(A)\mathrm{Rat}(B),\, \mathrm{Rat}(C))$ *then*

$$X = A^{(2)}_{*,\mathcal{N}(C)} = \left(\mathrm{Rat}(A)^{(2)}_{*,\mathcal{N}(\mathrm{Rat}(C))}\right)_{|_{\mathbf{x} \leftrightsquigarrow \mathbf{f}}}.$$

(3) *If* $\varrho\,(\mathrm{Rat}(C)\mathrm{Rat}(A)\mathrm{Rat}(B),\, \mathrm{Rat}(B),\, \mathrm{Rat}(C))$ *then*

$$X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)} = \left(\mathrm{Rat}(A)^{(2)}_{\mathcal{R}(\mathrm{Rat}(B)),\mathcal{N}(\mathrm{Rat}(C))}\right)_{|_{\mathbf{x} \leftrightsquigarrow \mathbf{f}}}.$$

(4) *If* $\varrho\,(\mathrm{Rat}(C)\mathrm{Rat}(A)\mathrm{Rat}(B),\, \mathrm{Rat}(B),\, \mathrm{Rat}(C),\, \mathrm{Rat}(A))$ *then*

$$X = A^{(1,2)}_{\mathcal{R}(B),\mathcal{N}(C)} = \left(\mathrm{Rat}(A)^{(1,2)}_{\mathcal{R}(\mathrm{Rat}(B)),\mathcal{N}(\mathrm{Rat}(C))}\right)_{|_{\mathbf{x} \leftrightsquigarrow \mathbf{f}}}.$$

According to Theorems 5.3 we state Algorithm 2 for computing outer inverses of matrices whose entries are rational expressions.

Algorithm 2 Computing outer inverses of matrices of functions.

Require: A subset $\mathcal{F} = \{f_1(z), \ldots, f_p(z))\} \subset \mathrm{Mer}(\Omega)$, not necessarily independent, and $A \in \mathbb{C}(\mathcal{F})^{m\times n}$, $B \in \mathbb{C}(\mathcal{F})^{n\times k}$, $C \in \mathbb{C}(\mathcal{F})^{l\times m}$.

1: Compute the matrices $\mathrm{Rat}(A), \mathrm{Rat}(B), \mathrm{Rat}(C)$ by replacing the function $f_i(z)$ by the variable x_i in A, B, C, for each $i = 1, \ldots, p$. Let $\mathrm{Rat}(A) \in \mathbb{C}(\mathbf{x})^{m\times n}$, $\mathrm{Rat}(B) \in \mathbb{C}(\mathbf{x})^{n\times k}$, $\mathrm{Rat}(C) \in \mathbb{C}(\mathbf{x})^{l\times m}$ be the resulting matrices of the execution of this step.

2: Compute $X := \mathrm{Rat}(B)\,(\mathrm{Rat}(C)\mathrm{Rat}(A)\mathrm{Rat}(B))^{(1)}\,\mathrm{Rat}(C) \in \mathbb{K}(\mathbf{x})^{n\times m}$ applying Algorithm 1.

3: Compute the polynomial $P(\mathbf{x})$ introduced in (5.5).

4: If $P(\mathbf{f}) = 0$ the method fails.

5: Replace in X each variable x_i by the functional entry $f_i(z)$, $i = 1, \ldots, p$. Let $X(\mathbf{f}) = X_{|\mathbf{x}\leftsquigarrow\mathbf{f}} \in \mathbb{C}(\mathcal{F})^{n\times m}$ be the result of this step.

6: Return $X(\mathbf{f})$.

Generalized inverses of matrices over $\mathcal{F}$ with involutions.

In the last part of this section, we consider the involutory automorphism $\varphi : \mathrm{Mer}(\Omega) \mapsto \mathrm{Mer}(\Omega)$, defined as $\varphi(f(z)) = \overline{f(\overline{z})}$, where $\overline{\cdot}$ means the conjugation in $\mathbb{C}$. Further, consider $(\mathbb{C}(z), \overline{\cdot}^e)$, where $\mathbb{C}(z)$ is the field of the complex rational expressions in the complex variable z and the restriction $\varphi_{|\mathbb{C}(z)}$ of φ to $\mathbb{C}(z)$ is defined by

$$\overline{\cdot}^e : \mathbb{C}(z) \mapsto \mathbb{C}(z);\quad \overline{\cdot}^e : R(z) := \frac{\Upsilon(z)}{\Psi(z)} = \frac{\sum_{i=0}^{k_1} a_i z^i}{\sum_{i=0}^{k_2} b_i z^i} \mapsto \overline{R(z)}^e = \frac{\sum_{i=0}^{k_1} \overline{a_i} z^i}{\sum_{i=0}^{k_2} \overline{b_i} z^i}.$$

Considering this theoretical frame, we will be able to approach the outer inverse computation problem by means on Penrose inverses. For this purpose, we will assume in the sequel that the elements in $\mathcal{F}$ are self-adjoint functions (see Def. 7 in [20]). Moreover, $\mathbf{x} = (x_1, \ldots, x_p)$ and $\mathbf{f} = (f_1(z), \ldots, f_p(z))$. Further, observe

$$\mathbb{C}(\mathcal{F}) = \left\{ \frac{\Upsilon(\mathbf{f})}{\Psi(\mathbf{f})} \mid P, Q \in \mathbb{K}[\mathbf{x}],\ Q(\mathbf{f}) \neq 0 \right\} \subset \mathrm{Mer}(\Omega).$$

We investigate matrices with elements taken from $(\mathbb{C}(\mathcal{F}), \overline{\cdot}^e)$. We will treat first the algebraically independent case to deal afterwards the algebraically dependent case.

Elements in $\mathcal{F}$ are algebraically independent.

In this situation, the existence of the Penrose inverse is guaranteed since $(\mathbb{C}(\mathcal{F}), \overline{\cdot})$ and $(\mathbb{C}(\mathcal{F})(\mathbf{x}), \overline{\cdot}^e)$ are MP fields; see Theorems 18 an 19 in [20]. Furthermore, as a consequence of Theorem 21 in [20], if $A \in \mathbb{C}(\mathcal{F})^{m\times n}$, then

$$(\mathrm{Rat}(A)^{\dagger})_{|\mathbf{x}\leftsquigarrow\mathbf{f}} = A^{\dagger}. \tag{5.6}$$

Therefore, the following result holds

Corollary 5.2. *Let $A \in \mathbb{C}(\mathcal{F})^{m\times n}$, $B \in \mathbb{C}(\mathcal{F})^{n\times k}$, $C \in \mathbb{C}(\mathcal{F})^{l\times m}$. Then $X := B(CAB)^{\dagger}C \in \mathbb{K}(\mathbf{x})^{n\times m}$ satisfies the statements of Theorem 5.4.*

Elements in $\mathcal{F}$ are algebraically independent.

Let us now treat the case where the elements in $\mathcal{F}$ are not necessarily independent. In this case, we need to ensure that the denominators appearing throughout the process do not vanish at $\mathcal{F}$. If $A \in \mathbb{C}(\mathcal{F})^{m\times n}$, and

$$\mathrm{den}(\mathrm{Rat}(A)^{\dagger})(\mathbf{f}) \neq 0$$

then Theorem 22 ensures that $A^{\dagger}$ exists and can be computed by (5.6). As a consequence of this fact and of Theorem 5.4 one gets the follows corollary.

Corollary 5.3. *Let $A \in \mathbb{C}(\mathcal{F})^{m\times n}$, $B \in \mathbb{C}(\mathcal{F})^{n\times k}$, $C \in \mathbb{C}(\mathcal{F})^{l\times m}$, where $\mathcal{F}$ is not necessarily an algebraic independent set. Let us assume that*

$$\begin{aligned}\mathrm{RankPol}(\mathrm{Rat}(C)\mathrm{Rat}(A)\mathrm{Rat}(B))(\mathbf{f}) \cdot \mathrm{RankPol}(\mathrm{Rat}(B))(\mathbf{f}) \cdot \mathrm{RankPol}(\mathrm{Rat}(C))(\mathbf{f}) \cdot \\ \cdot \mathrm{RankPol}(\mathrm{Rat}(A))(\mathbf{f}) \cdot \mathrm{den}(\mathrm{Rat}((CAB)^{\dagger})(\mathbf{f}) \neq 0.\end{aligned} \tag{5.7}$$

Then $X := B(CAB)^{\dagger}C \in \mathbb{K}(\mathbf{x})^{n\times m}$ satisfies the statements of Theorem 5.4.

As an application of the previous results, we present an algorithm for computing outer inverses of matrices over $\mathbb{C}(\mathcal{F})$. Algorithm 3 is a generalization of Algorithm 1 from [20]. Its main idea is the replacement of each function f_i included in A by a variable x_i.

Algorithm 3 Computing inner and outer inverses of matrices with functional entries.

Require: Let $\mathcal{F} = \{f_1(z), \ldots, f_p(z)\} \subset \mathrm{Mer}(\Omega)$ be such that the elements in $\mathcal{F}$ are self-adjoint functions (see Def. 7 in [20]), and let $A \in \mathbb{C}(\mathcal{F})^{m\times n}$, $B \in \mathbb{C}(\mathcal{F})^{n\times k}$, $C \in \mathbb{C}(\mathcal{F})^{l\times m}$.

1: Compute $D = CAB$ and simplify entries in D.
2: Compute the matrix $\mathrm{Rat}(D)$ by replacing in D the function $f_i(z)$ by the variable x_i, for each $i = 1, \ldots, p$. Let $\mathrm{Rat}(D) \in \mathbb{C}(\mathcal{F})^{l\times k}$ be the resulting matrix of the execution of this step.
3: Compute $\mathrm{Rat}(D)^{\dagger} := (\mathrm{Rat}(C)\mathrm{Rat}(A)\mathrm{Rat}(B))^{\dagger} \in \mathbb{C}(\mathcal{F})^{k\times l}$.
4: Compute the polynomial $P(\mathbf{x})$ introduced in (5.7).
5: If $P(\mathbf{f}) = 0$ the method fails.
6: Replace in $\mathrm{Rat}(D)^{\dagger}$ the variable x_i by the function $f_i(z)$, for each $i = 1, \ldots, p$ and generate $D^{\dagger} \in \mathbb{C}(\mathcal{F})^{k\times l}$.
7: Compute $X := BD^{\dagger}C$.
8: Return X.

5.4 Examples of specializations

Example 5.2. Consider the two-variable rational matrix from [21]

$$A = \begin{bmatrix} \frac{1}{z_2} & z_1 & 0 \\ 0 & \frac{1}{z_2} & z_1 \\ 0 & 0 & 0 \end{bmatrix}.$$

and the matrices

$$B = \begin{bmatrix} z_1 z_2 & 0 \\ z_2 & z_1^2 \\ z_1 z_2 & z_2^3 \end{bmatrix}, \quad C = \begin{bmatrix} z_1 & z_2^2 & 0 \\ 0 & z_1^2 & z_1 z_2 \end{bmatrix}.$$

Since, $\mathrm{rank}(CAB) = \mathrm{rank}(B)$, the function `GinvBC[A,B,C]` solves $BUCAB = B$ and produces the result

$$X = \begin{bmatrix} \frac{z_2\left(z_2^4+z_1\right)}{z_2^5+z_2^4-z_1^3z_2^2+z_1} & -\frac{z_1^2z_2^2}{z_2^5+z_2^4-z_1^3z_2^2+z_1} & -\frac{z_2^3\left(z_2^5+z_1z_2+z_1^3\right)}{z_1^2\left(z_2^5+z_2^4-z_1^3z_2^2+z_1\right)} \\ -\frac{z_2^2\left(z_1^3-z_2^3\right)}{z_1\left(z_2^5+z_2^4-z_1^3z_2^2+z_1\right)} & \frac{z_1z_2}{z_2^5+z_2^4-z_1^3z_2^2+z_1} & -\frac{z_2^8-z_1^3\left(z_2^5+z_2^2\right)}{z_1^3\left(z_2^5+z_2^4-z_1^3z_2^2+z_1\right)} \\ -\frac{z_2^4-z_1^3z_2}{z_1^2\left(z_2^5+z_2^4-z_1^3z_2^2+z_1\right)} & -\frac{z_2^2\left(z_1^3-z_2^2(z_2+1)\right)}{z_1\left(z_2^5+z_2^4-z_1^3z_2^2+z_1\right)} & \frac{z_2^3\left(-z_1^5-z_2z_1^3+z_2^2(z_2+1)z_1^2+z_2^4\right)}{z_1^4\left(z_2^5+z_2^4-z_1^3z_2^2+z_1\right)} \end{bmatrix}.$$

Let the specialization of `X:=GinvBC[A,B,C]`$\in \left(A^{(2)}_{\mathcal{R}(B),*}\right)_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}$ be defined as $X_{|\mathbf{z}\leftrightsquigarrow\mathbf{c}}$, where $\mathbf{z} = (z_1, z_2)$ and $\mathbf{c} = (1, 1)$. This replacement in *Mathematica* is defined by $X/.\{z_1 \to 1, z_2 \to 1\}$, and it is equal to

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

The polynomial in (5.3) is

$$z_1z_2\left(z_2^5+z_2^4-z_1^3z_2^2+z_1\right)\left(z_1^3z_1^2+z_1^2z_2+z_1^2+z_2^2\right)$$

that does not vanish at $\mathbf{c} = (1, 1)$. After a simple verification, it can be confirmed that the same matrix is generated using
$\texttt{GinvBC}[A/.\{z_1 \to 1, z_2 \to 1\}, B/.\{z_1 \to 1, z_2 \to 1\}, C/.\{z_1 \to 1, z_2 \to 1\}]$.
So, the conclusion is
$\texttt{GinvBC}[\mathrm{A, B, C}]/.\{z_1 \to 1, z_2 \to 1\} = \texttt{GinvBC}[A/.\{z_1 \to 1, z_2 \to 1\}, B/.\{z_1 \to 1, z_2 \to 1\}, C/.\{z_1 \to 1, z_2 \to 1\}]$
or $\texttt{GinvBC}[\mathrm{A, B, C}]_{|_{\mathbf{z} \leftrightsquigarrow \mathbf{c}}} = \texttt{GinvBC}[A_{|_{\mathbf{z} \leftrightsquigarrow \mathbf{c}}}, B_{|_{\mathbf{z} \leftrightsquigarrow \mathbf{c}}}, C_{|_{\mathbf{z} \leftrightsquigarrow \mathbf{c}}}]$.

Example 5.3. Consider the matrices A, B, C with respect to unknown u:

$$A = \begin{bmatrix} \frac{i \sin(u)}{\cos(u)+i\sin(u)} & \frac{i \sin(u)}{\cos(u)-i\sin(u)} & \frac{i \sin(u)}{\cos(u)+\imath\sin(u)} & \frac{\imath \sin(u)}{\cos(u)-\imath\sin(u)} \\ \frac{\cos(u)\sin(u)+\imath\sin(u)}{\cos(u)-\imath\sin(u)} & \frac{\cos(u)\sin(u)+\imath\sin(u)}{\cos(u)+\imath\sin(u)} & \frac{\cos(u)\sin(u)+\imath\sin(u)}{\cos(u)-\imath\sin(u)} & \frac{\cos(u)\sin(u)+\imath\sin(u)}{\cos(u)+\imath\sin(u)} \\ \frac{\imath \sin(u)}{\cos(u)+\imath\sin(u)} & \frac{\cos(u)\sin(u)+\imath\sin(u)}{\cos(u)-\imath\sin(u)} & \frac{\imath \sin(u)}{\cos(u)+\imath\sin(u)} & \frac{\cos(u)\sin(u)+\imath\sin(u)}{\cos(u)-\imath\sin(u)} \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{i \sin(u)}{\cos(u)+\imath\sin(u)} & \frac{\imath \sin(u)}{\cos(u)-\imath\sin(u)} \\ \frac{\cos(u)\sin(u)+\imath\sin(u)}{\cos(u)-\imath\sin(u)} & \frac{\cos(u)\sin(u)+\imath\sin(u)}{\cos(u)+\imath\sin(u)} \\ \frac{i \sin(u)}{\cos(u)+\imath\sin(u)} & \frac{\cos(u)\sin(u)+\imath\sin(u)}{\cos(u)-\imath\sin(u)} \\ \frac{i \sin(u)}{\cos(u)+\imath\sin(u)} & \frac{i \sin(u)}{\cos(u)-\imath\sin(u)} \end{bmatrix}, \quad C = \begin{bmatrix} \frac{\imath \sin(u)}{\cos(u)+\imath\sin(u)} & 0 & \frac{\imath e^u}{\cos(u)+\imath\sin(u)} \\ \frac{\cos(u)\sin(u)+\imath\sin(u)}{\cos(u)-\imath\sin(u)} & 0 & \frac{e^u\sin(u)+\imath\sin(u)}{\cos(u)+\imath\sin(u)} \\ \frac{\imath \sin(u)}{\cos(u)+\imath\sin(u)} & 0 & \frac{\imath e^u}{\cos(u)+\imath\sin(u)} \end{bmatrix},$$

where $\imath$ stands for the imaginary unit. After the replacement $\mathbf{f} \leftrightsquigarrow \mathbf{x}$, given by $\{\cos(u) \to x_1, \sin(u) \to x_2, e^u \to x_3\}$, we obtain the next matrices with rational entries:

$$\mathrm{Rat}(A) = \begin{bmatrix} \frac{\imath x_2}{x_1+\imath x_2} & \frac{\imath x_2}{x_1-ix_2} & \frac{\imath x_2}{x_1+ix_2} & \frac{\imath x_2}{x_1-ix_2} \\ \frac{x_1x_2+\imath x_2}{x_1-\imath x_2} & \frac{x_1x_2+\imath x_2}{x_1+\imath x_2} & \frac{x_1x_2+\imath x_2}{x_1-\imath x_2} & \frac{x_1x_2+\imath x_2}{x_1+\imath x_2} \\ \frac{\imath x_2}{x_1+\imath x_2} & \frac{x_1x_2+\imath x_2}{x_1-\imath x_2} & \frac{\imath x_2}{x_1+\imath x_2} & \frac{x_1x_2+\imath x_2}{x_1-\imath x_2} \end{bmatrix},$$

$$\mathrm{Rat}(B) = \begin{bmatrix} \frac{\imath x_2}{x_1+\imath x_2} & \frac{\imath x_2}{x_1-\imath x_2} \\ \frac{x_1x_2+ix_2}{x_1-ix_2} & \frac{x_1x_2+ix_2}{x_1+\imath x_2} \\ \frac{\imath x_2}{x_1+\imath x_2} & \frac{x_1x_2+\imath x_2}{x_1-\imath x_2} \\ \frac{\imath x_2}{x_1+\imath x_2} & \frac{\imath x_2}{x_1-\imath x_2} \end{bmatrix}, \ \mathrm{Rat}(C) = \begin{bmatrix} \frac{\imath x_2}{x_1+\imath x_2} & 0 & \frac{\imath x_3}{x_1+\imath x_2} \\ \frac{x_1x_2+\imath x_2}{x_1-\imath x_2} & 0 & \frac{x_3x_2+\imath x_2}{x_1+\imath x_2} \\ \frac{\imath x_2}{x_1+\imath x_2} & 0 & \frac{\imath x_3}{x_1+\imath x_2} \end{bmatrix}.$$

Then, one gets that

$$\mathrm{Rat}(X) := \mathrm{Rat}(B)\,(\mathrm{Rat}(C)\mathrm{Rat}(A)\mathrm{Rat}(B))^{\dagger}\,\mathrm{Rat}(C)$$
$$= \begin{bmatrix} \frac{\imath(x_1+\imath x_2)\left(-x_1^2+x_2^2+2(x_1+2\imath)(2x_1+\imath)x_2\right)}{x_1x_2\left(x_1^3+2\imath(x_2+1)x_1^2-x_2(x_2+10)x_1-8\imath x_2\right)} & 0 & \frac{i(x_1+\imath x_2)\left(x_1^2-6\imath x_2x_1-(x_2-4)x_2\right)}{x_1x_2\left(x_1^3+2i(x_2+1)x_1^2-x_2(x_2+10)x_1-8\imath x_2\right)} \\ -\frac{(x_1+\imath)\left(x_1^3+5ix_2x_1^2-3x_2(x_2+4)x_1+ix_2^2(x_2+4)\right)}{x_1x_2\left(x_1^3+2\imath(x_2+1)x_1^2-x_2(x_2+10)x_1-8ix_2\right)} & 0 & \frac{(x_1+\imath)\left(x_1^3+\imath x_2x_1^2+(x_2-12)x_2x_1+\imath x_2^2(x_2+4)\right)}{x_1x_2\left(x_1^3+2\imath(x_2+1)x_1^2-x_2(x_2+10)x_1-8\imath x_2\right)} \\ \frac{(x_2-\imath x_1)\left(x_1^4+\imath(2x_2+3)x_1^3-(x_2(x_2+8)+3)x_1^2-ix_2(x_2+8)x_1+x_2(x_2+4)\right)}{x_1x_2\left(x_1^3+2\imath(x_2+1)x_1^2-x_2(x_2+10)x_1-8\imath x_2\right)} & 0 & -\frac{(x_1+\imath x_2)\left((x_1+ix_2)\left(x_1^2+\imath(x_2+3)x_1-x_2\right)-4ix_2\right)}{x_1x_2\left(x_1^3+2\imath(x_2+1)x_1^2-x_2(x_2+10)x_1-8\imath x_2\right)} \\ \frac{\imath(x_1+ix_2)\left(-x_1^2+x_2^2+2(x_1+2\imath)(2x_1+\imath)x_2\right)}{x_1x_2\left(x_1^3+2i(x_2+1)x_1^2-x_2(x_2+10)x_1-8\imath x_2\right)} & 0 & \frac{\imath(x_1+\imath x_2)\left(x_1^2-6\jmath x_2x_1-(x_2-4)x_2\right)}{x_1x_2\left(x_1^3+2\imath(x_2+1)x_1^2-x_2(x_2+10)x_1-8\imath x_2\right)} \end{bmatrix}.$$

In addition, the polynomial $P(\mathbf{x})$ in (5.7) is

$$x_1x_2x_3\left(-2x_1^3x_2x_3^2+2x_1^2x_2^2x_3^2+2x_1x_2^3x_3^2+x_3^2x_2^4-4x_1^2x_2^2x_3+4x_1x_2^2x_3^2+x_1^2x_2^2-2x_1^2x_2x_3+x_1^2x_3^2+x_2^4+2x_2^3x_3+x_2^2x_3^2+x_1x_3\right)$$
$$\left(x_1^6+2x_1^4x_2^2+x_1^2x_2^4-12x_1^4x_2+20x_1^2x_2^3+4x_1^4+68x_1^2x_2^2-32x_1^2x_2+64x_2^2\right)(x_1+ix_2)(\imath x_2-x_1)$$
$$\left(x_1x_3x_2^2-4\imath x_1^2x_2-2x_1x_2x_3-4\imath x_1^2x_2x_3-2x_1^3x_2x_3+\imath x_1^4x_3-2\imath x_1^2x_2^2-4\imath x_1^2x_3-3x_1^3x_3+x_1x_2^3+2\imath x_2^2x_3-2x_2^2x_1-\imath x_1^2x_2^2x_3+2\imath x_2^3-x_2x_1^3\right)$$
$$\left(\imath x_1^2x_2x_3-2\imath x_1^2x_2-4\imath x_1^2x_3-x_1^3x_3-2\imath x_2^2-2\imath x_2x_3-2x_2^2x_1-2x_1x_2x_3+4x_1x_2+4x_1x_3\right)$$
$$\left(2\imath x_1^3x_2x_3+\imath x_2x_1^3+3\imath x_1^3x_3-\imath x_2^3x_1-\imath x_1x_2^2x_3+x_1^4x_3-x_1^2x_2^2x_3+2\imath x_2^2x_1+2\imath x_1x_2x_3-2x_1^2x_2^2-4x_1^2x_2x_3-4x_1^2x_2-4x_1^2x_3+2x_2^3+2x_3x_2^2\right)$$
$$\left(\imath x_1x_2x_3+\imath x_2^2x_3-\imath x_1x_2+\imath x_1x_3+x_1^2x_3-x_1x_2x_3-x_2^2-x_3x_2\right)\left(2\imath x_1^2x_2+2\imath x_1^2+x_1^3-x_2^2x_1-8\imath x_2-10x_1x_2\right)(\imath+x_1)\left(x_1^2+x_2^2\right)$$

In order to check whether $P(\mathbf{f}) \neq 0$, we observe that $P(\mathbf{f})(\pi/4) \simeq 4.284836934\,10^9 - 1.372886372\,10^{10}\,\imath$ Thus, we get that

$$X := \mathrm{Rat}(X)_{|\mathbf{x} \rightsquigarrow \mathbf{f}} = B(CAB)^{\dagger}C =$$

$$= \begin{bmatrix} \frac{(\sec(u)-\imath\csc(u))\sin(u)\left((\cot(u)-\tan(u)-10\imath)\csc^2(u)+4\sec(u)\right)\tan(u)}{\cot(u)+2\imath(\csc(u)+(5\imath-4\sec(u))\sec(u)+1)-\tan(u)} & 0 & \frac{(\csc(u)+\imath\sec(u))\sec^2(u)(\cos(u)(\cot(u)-6\imath)-\sin(u)+4)}{2(-\imath\cot(2u)+\csc(u)+(5\imath-4\sec(u))\sec(u)+1)} \\ -\frac{(\cos(u)+\imath)\left(\cot(u)\left(\cot^2(u)+5\imath\cot(u)-12\csc(u)-3\right)+\imath(4\csc(u)+1)\right)\sec(u)}{\cos(u)(\cot(u)+\imath)^2+2i(\cot(u)(\cot(u)+5\imath)-4\csc(u))} & 0 & \frac{(\cos(u)+i)\csc^3(u)\sec(u)(\cos(u)-2\imath\cos(2u)+\imath(\sin(u)+6\imath\sin(2u)+2))}{\cos(u)(\cot(u)+\imath)^2+2\imath(\cot(u)(\cot(u)+5\imath)-4\csc(u))} \\ \frac{(1-\imath\cot(u))\left(\cos^2(u)(\cot(u)+\imath)^2-\cot(u)(3\cot(u)+8\imath)+\imath\cos(u)(\cot(u)(3\cot(u)+8\imath)-1)+4\csc(u)+1\right)\sec^2(u)\tan(u)}{\cot(u)+2\imath(\csc(u)+(5\imath-4\sec(u))\sec(u)+1)-\tan(u)} & 0 & \frac{e^{\imath u}\csc(u)\sec(u)(-\imath\cot(u)+3\csc(u)+\imath\tan(u)-\sec(u)(4\sec(u)+\tan(u)-4\imath)+2)}{8\sec^2(u)+\imath(\cos(2u)\csc(u)-10)\sec(u)-2(\csc(u)+1)} \\ \frac{(\sec(u)-\imath\csc(u))\sin(u)\left((\cot(u)-\tan(u)-10\imath)\csc^2(u)+4\sec(u)\right)\tan(u)}{\cot(u)+2\imath(\csc(u)+(5\imath-4\sec(u))\sec(u)+1)-\tan(u)} & 0 & \frac{(\csc(u)+\imath\sec(u))\sec^2(u)(\cos(u)(\cot(u)-6\imath)-\sin(u)+4)}{2(-\imath\cot(2u)+\csc(u)+(5\imath-4\sec(u))\sec(u)+1)}. \end{bmatrix}.$$

Indeed, it can be verified after simplifications that

$$X_1 := B(CAB)^{\dagger}C =$$

$$\begin{bmatrix} \frac{\csc(u)\sec(u)\left(2\cos\left(\frac{u}{2}\right)+2\imath\sin\left(\frac{u}{2}\right)\right)(-\cos(2u)-3\sin(u)+5\imath\sin(2u)+\sin(3u))}{(2+7\imath)\cos\left(\frac{u}{2}\right)+(6-8\imath)\cos\left(\frac{3u}{2}\right)-(4+\imath)\cos\left(\frac{5u}{2}\right)-(7+2\imath)\sin\left(\frac{z}{2}\right)-(8-6\imath)\sin\left(\frac{3u}{2}\right)+(1+4\imath)\sin\left(\frac{5u}{2}\right)} & 0 & \frac{8e^{3\imath u}\left(-2-2\imath e^{iu}+2ie^{3\imath u}+e^{4\imath u}\right)}{\left(-1+e^{2\imath u}\right)\left(1+e^{2\imath u}\right)\left(-4\imath+8e^{\imath u}+2\imath e^{2\imath u}-7e^{3\imath u}+6\imath e^{4\imath u}+e^{5\imath u}\right)} \\ -\frac{\csc(u)\sec(u)\left((6+4\imath)\cos\left(\frac{u}{2}\right)-(4-2\imath)\cos\left(\frac{3u}{2}\right)+(4-7\imath)\cos\left(\frac{5u}{2}\right)-(4+\imath)\cos\left(\frac{7u}{2}\right)+(4+6\imath)\sin\left(\frac{u}{2}\right)-(2-4\imath)\sin\left(\frac{3u}{2}\right)-(7-4\imath)\sin\left(\frac{5u}{2}\right)+(1+4\imath)\sin\left(\frac{7u}{2}\right)\right)}{(2+7\imath)\cos\left(\frac{u}{2}\right)+(6-8\imath)\cos\left(\frac{3u}{2}\right)-(4+\imath)\cos\left(\frac{5u}{2}\right)-(7+2\imath)\sin\left(\frac{u}{2}\right)-(8-6\imath)\sin\left(\frac{3u}{2}\right)+(1+4\imath)\sin\left(\frac{5u}{2}\right)} & 0 & -\frac{4e^{\imath u}\left(-4-8\imath e^{\imath u}-2e^{2\imath u}+3\imath e^{3\imath u}+6e^{4\imath u}+3\imath e^{5\imath u}+2e^{6\imath u}\right)}{\left(-1+e^{2\imath u}\right)\left(1+e^{2\imath u}\right)\left(-4\imath+8e^{\imath u}+2ie^{2\imath u}-7e^{3\imath u}+6\imath e^{4\imath u}+e^{5\imath u}\right)} \\ \frac{(\csc(u)+i\sec(u))\left(3\cos\left(\frac{u}{2}\right)+(10-8i)\cos\left(\frac{3u}{2}\right)-(4+6\imath)\cos\left(\frac{5u}{2}\right)-(1-2\imath)\cos\left(\frac{7u}{2}\right)-3\imath\sin\left(\frac{u}{2}\right)-(8-10\imath)\sin\left(\frac{3u}{2}\right)+(6+4\imath)\sin\left(\frac{5u}{2}\right)+(2-\imath)\sin\left(\frac{7u}{2}\right)\right)}{2\left((2+7\imath)\cos\left(\frac{u}{2}\right)+(6-8\imath)\cos\left(\frac{3u}{2}\right)-(4+\imath)\cos\left(\frac{5u}{2}\right)-(7+2\imath)\sin\left(\frac{u}{2}\right)-(8-6\imath)\sin\left(\frac{3u}{2}\right)+(1+4\imath)\sin\left(\frac{5u}{2}\right)\right)} & 0 & \frac{4e^{4\imath u}\left(4+2\imath e^{\imath u}-3e^{2\imath u}+4\imath e^{3\imath u}+e^{4\imath u}\right)}{\left(-1+e^{2\imath u}\right)\left(1+e^{2\imath u}\right)\left(4+8\imath e^{\imath u}-2e^{2\imath u}-7\imath e^{3\imath u}-6e^{4\imath u}+\imath e^{5\imath u}\right)} \\ \frac{\csc(u)\sec(uz)\left(2\cos\left(\frac{u}{2}\right)+2\imath\sin\left(\frac{u}{2}\right)\right)(-\cos(2u)-3\sin(u)+5\imath\sin(2u)+\sin(3u))}{(2+7\imath)\cos\left(\frac{u}{2}\right)+(6-8\imath)\cos\left(\frac{3u}{2}\right)-(4+\imath)\cos\left(\frac{5u}{2}\right)-(7+2\imath)\sin\left(\frac{u}{2}\right)-(8-6\imath)\sin\left(\frac{3u}{2}\right)+(1+4\imath)\sin\left(\frac{5u}{2}\right)} & 0 & \frac{8e^{3\imath u}\left(-2-2\imath e^{\imath u}+2\imath e^{3\imath u}+e^{4\imath u}\right)}{\left(-1+e^{2\imath u}\right)\left(1+e^{2\imath u}\right)\left(-4\imath+8e^{\imath u}+2\imath e^{2\imath u}-7e^{3\imath u}+6\imath e^{4\imath u}+e^{5\imath u}\right)} \end{bmatrix}.$$

Finally, after simplification, it is possible to verify that $X - X_1$ coincides with the zero matrix.

6 Conclusion

This research investigates representations of outer matrix inverses with prescribed range and null space in terms of inner inverses. More precisely, we explore the relations among the sets $A\{2\}_{\mathcal{R}(B),*}$, $A\{2\}_{*,\mathcal{N}(C)}$ and $A\{2\}_{\mathcal{R}(B),\mathcal{N}(C)}$ in terms of the set $B(CAB)\{1\}C$. Further, required inner inverses are computed as solutions of appropriate LME. More precisely, $(CAB)\{1\}$ is generated as the solution to $BUCAB = B$ under the constraint $\mathrm{rank}(CAB) = \mathrm{rank}(B)$ or the solution to $CABUC = C$ if $\mathrm{rank}(CAB) = \mathrm{rank}(C)$ is satisfied. In this way, algorithms for computing outer inverses are derived using solutions of LME.

The underlying LME can be solved in different ways. Approach used in [28] is based on the usage of matrix the dynamical system arising from Gradient Neural Network (GNN) approach. Here we propose approach based on symbolic solutions to involved LME. Using symbolic solutions to these LME it is possible to derive corresponding algorithms in appropriate computer algebra systems, such as *Mathematica* or *Maple*. Based on two different approaches in the implementation of proposed representations and algorithms, it can be concluded that the presented algorithms are applicable both in symbolic calculation and in numerical calculations.

Alternative possibilities for solving $BUCAB = B$ and $CABUC$ remain open for further research. Efficient solutions to these LME are important in defining efficient algorithm for computing inner and outer generalized inverses with prescribed range and/or null space.

In addition, we study how the above representations, and hence the associated algorithms, behave under specializations. As a consequence of this analysis, we are able to derive algorithms to compute outer inverses of matrices with functional entries satisfying certain conditions.

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